PROBLEM SET 9

Suggested reading

Lecture notes (on the website): Chapter 2, *Bundles* (§2.4 and §2.6–2.8)

Problems

1. Assume in the following that all vector spaces are over the field $\mathbb{F}$ (= $\mathbb{R}$ or $\mathbb{C}$) and are finite dimensional, and given a vector space V, denote its dual space by V^*. Recall that by finite-dimensionality, the canonical inclusion $V \to (V^*)^*$ defined by setting $v(\lambda) := \lambda(v)$ for $v \in V$ and $\lambda \in V^*$ is an isomorphism. The word "canonical" will appear several times in the following, and should be understood to mean "can be defined unambiguously for all vector spaces without making any arbitrary choices".

   As sketched in lecture last Thursday, the easiest way to define the tensor product $V \otimes W$ of two vector spaces V and W is as the space of all bilinear maps $V^* \times W^* \to \mathbb{F}$. The operation $V \times W \to V \otimes W : (v, w) \mapsto v \otimes w$ is then defined by $(v \otimes w)(\lambda, \mu) := v(\lambda)w(\mu) := \lambda(v)\mu(w)$.

   (a) Verify that if $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ are bases of V and W respectively, then the mn elements $v_i \otimes w_j \in V \otimes W$ for $i = 1, \ldots, m$ and $j, \ldots, n$ form a basis of $V \otimes W$.

   (b) Show that for any vector space X, there is a canonical isomorphism between the space of linear maps $V \otimes W \to X$ and the space of *bilinear* maps $V \times W \to X$.

   (c) Find a canonical isomorphism between $(V \otimes W) \otimes X$ and $V \otimes (W \otimes X)$ that identifies $(v \otimes w) \otimes x$ with $v \otimes (w \otimes x)$ for every $v \in V$, $w \in W$ and $x \in X$.
   Hint: Identify both spaces with the space of all multilinear maps $V^* \times W^* \times X^* \to \mathbb{F}$.

   (d) By induction, one can now take any finite collection of vector spaces $V_1, \ldots, V_N$ and define $V_1 \otimes \ldots \otimes V_N$ (without worrying about parentheses) as a space of multilinear maps. Extend part (b) to this case, i.e. find a canonical isomorphism between the space of linear maps $V_1 \otimes \ldots \otimes V_N \to X$ and the space of multilinear maps $V_1 \times \ldots \times V_N \to X$ for any vector space X.
   Remark: This isomorphism means that in order to write down a linear map $\Phi : V_1 \otimes \ldots \otimes V_N \to X$, you only have to specify what $\Phi(v_1 \otimes \ldots \otimes v_N) \in X$ is for every N-tuple $(v_1, \ldots, v_n) \in V_1 \times \ldots \times V_N$ and check that the map $(v_1, \ldots, v_n) \mapsto \Phi(v_1 \otimes \ldots \otimes v_n)$ is multilinear.

   (e) Find a canonical isomorphism between $V^* \otimes W$ and $\mathrm{Hom}(V, W)$.

   (f) Recall that $\Lambda^k V \subset \bigotimes^k V$ can be defined as the image of the linear map

   $$\mathrm{Alt} : V \otimes \ldots \otimes V \to V \otimes \ldots \otimes V : v_1 \otimes \ldots \otimes v_k \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^{|\sigma|} v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(k)}.$$

   If $\dim V = n$, then $\Lambda^k V$ has dimension $\frac{n!}{k!(n-k)!}$ and is spanned by so-called k-vectors, meaning elements of the form $v_1 \wedge \ldots \wedge v_k$ for $v_1, \ldots, v_k \in V$. Replacing V with its dual space V^*, one obtains our previous definition of the space of k-forms $\Lambda^k V^*$ on V by identifying $(V^*)^*$ with V. With this understood, find a canonical isomorphism between $\Lambda^k V^*$ and the dual space of $\Lambda^k V$.

   (g) Observe that if V is a complex vector space of dimension n, then it also has the structure of a *real* vector space with dimension $2n$: indeed, the operation of vector addition is the same whether we view V as real or complex, and the real scalar multiplication operation $\mathbb{R} \times V \to V : (c, v) \mapsto cv$ is defined in terms of complex scalar multiplication $\mathbb{C} \times V \to V$ simply by restricting from $\mathbb{C}$ to $\mathbb{R}$. A complex basis $v_1, \ldots, v_n$ of V then gives rise to a real basis $v_1, iv_1, \ldots, v_n, iv_n$. Now if V and W are both complex vector spaces, we can define two versions of their tensor product,

   $$V \otimes_{\mathbb{R}} W \quad \text{and} \quad V \otimes_{\mathbb{C}} W,$$

   the first with V and W both regarded as real vector spaces, the second with both complex. Find a canonical surjective real-linear map $V \otimes_{\mathbb{R}} W \to V \otimes_{\mathbb{C}} W$. What does its kernel look like?
   Hint: Do not think in terms of multilinear maps on dual spaces. See the remark after part (d).

Remark: In every place above where the word "canonical" appeared, a statement about vector spaces can be converted into a similar statement about vector bundles. Thus if E, F, G are vector bundles over the same base M, there exist canonical bundle isomorphisms $(E \otimes F) \otimes G \cong E \otimes (F \otimes G)$, $E^* \otimes F \cong \mathrm{Hom}(E, F)$ and $\Lambda^k E^* \cong (\Lambda^k E)^*$, and if E and F are both complex bundles, there is a canonical surjective real-linear bundle map $E \otimes_{\mathbb{R}} F \to E \otimes_{\mathbb{C}} F$.

2. We've seen that every vector bundle admits a bundle metric, so in particular, every smooth manifold admits a Riemannian metric. Our proof of this used a partition of unity and depended crucially on the fact that positive-definiteness is a "convex" condition: specifically, if g_0 and g_1 are both symmetric positive-definite bilinear forms on the same vector space, then so is $ag_0 + bg_1$ for any pair of real numbers $a, b \geq 0$ with $a + b > 0$. On the other hand, there is a more general notion of bundle metrics that arises prominently in physics: a *pseudo-Riemannian* metric g on a smooth manifold M is a type $(0, 2)$ tensor field $g \in \Gamma(T_2^0 M)$ which is

   (i) *symmetric*: $g(X, Y) = g(Y, X)$ for all $(X, Y) \in TM \oplus TM$;
   (ii) *nondegenerate*: for all $p \in M$ and nonzero vectors $X \in T_p M$, $g(X, \cdot) : T_p M \to \mathbb{R}$ is nonzero.

   In other words, g defines a nondegenerate symmetric bilinear form on every fiber, but it need not be positive definite. Recall that on $\mathbb{R}^n$, every symmetric bilinear form Q can be expressed in terms of the standard Euclidean inner product $\langle\ ,\ \rangle$ as $Q(v, w) = \langle v, \mathbf{A}w \rangle$ for a unique symmetric matrix $\mathbf{A}$, which is invertible if and only if Q is nondegenerate. We say that the *signature* of Q is (p, q), where p and q are the numbers of positive and negative eigenvalues of $\mathbf{A}$ respectively. Equivalently, p and q are the largest dimensions of any subspaces on which Q is positive- or negative-definite respectively; in these terms, we see that the signature is also well defined for a pseudo-Riemannian metric on a manifold M, and it is constant on any connected component of M. For $\dim M = n$, a Riemannian metric is now just a pseudo-Riemannian metric of signature $(n, 0)$. We call the metric *Lorentzian* if it has signature $(n-1, 1)$ (or $(1, n-1)$, depending on your sign conventions). Metrics of the latter type in the case $n = 4$ are the main geometric structure that describes gravity on the "spacetime" manifold in Einstein's general theory of relativity; the fact that some directions $X \in TM$ satisfy $g(X, X) > 0$ and others satisfy $g(X, X) < 0$ corresponds to the distinction between "spacelike" and "timelike" directions in spacetime.

   But there is a caveat: since pseudo-Riemannian metrics are not positive definite, our existence proof via partitions of unity does not work, and in fact, such metrics do not always exist. We now show that S^2 does not admit a Lorentzian metric.

   (a) Show that every real line bundle over the 2-dimensional disk $\mathbb{D}^2 := \{x \in \mathbb{R}^2 \mid |x| \leq 1\}$ is trivial.
   Hint: Given such a bundle $E \to \mathbb{D}^2$, pick a bundle metric g and look at the submanifold $\{v \in E \mid g(v, v) = 1\}$ in the total space. How many connected components can this submanifold have?
   (b) Show that every real line bundle over S^2 is trivial.
   Hint: Cover S^2 with a pair of subsets $\mathcal{U}_+, \mathcal{U}_- \subset S^2$ which are both disks such that $\mathcal{U}_+ \cap \mathcal{U}_-$ is an annulus. Part (a) provides trivializations of your bundle over $\mathcal{U}_+$ and $\mathcal{U}_-$ separately. Can you now construct a nowhere zero section?
   (c) Show that if g is a Lorentzian metric on S^2, then $TS^2 \cong \ell_+ \oplus \ell_-$ for a pair of line bundles $\ell_\pm \to S^2$. This contradicts something we proved on Problem Set 8; why?
   You may use the following fact without proof: if $\{\mathbf{A}_\tau\}_{\tau \in \mathbb{R}^k}$ is a smooth family of symmetric n-by-n matrices and $\lambda \in \mathbb{R}$ is an eigenvalue of $\mathbf{A}_0$ with multiplicity 1, then for τ sufficiently close to 0, there exist smooth families $\{\lambda_\tau \in \mathbb{R}\}$ and $\{\mathbf{v}_\tau \in \mathbb{R}^n \setminus \{0\}\}$ such that $\lambda_0 = \lambda$ and $\mathbf{A}_\tau \mathbf{v}_\tau = \lambda_\tau \mathbf{v}_\tau$ for every τ. This can be proved via the implicit function theorem (given sufficient cleverness).

3. (a) Show that every real vector bundle is isomorphic to its dual bundle. *Hint: Use a bundle metric.*
   (b) Show that a complex vector bundle E is trivial if and only if its dual is trivial.
   (c) Show that if E is any complex line bundle, then $E \otimes E^*$ is a trivial complex line bundle.

   Remark: It is not true in general for a complex bundle E that $E \cong E^*$. For instance if E is $TS^2 \to S^2$ endowed with any complex structure, then E and E^* are not isomorphic. We will hopefully learn enough about the Euler number before the end of the semester to be able to prove this.