Differential Geometry I — Humboldt-Universität zu Berlin
C. Wendl — Winter Semester 2016–17

Topics covered in the lectures

1. Tu. 18.10.2016: general remarks on categories (sets with structure) in algebra/analysis/topology/geometry, definition of topological manifold, examples in dimensions 0, 1 and 2
2. Th. 20.10.2016: definition of manifold with boundary, charts and smooth structures, smooth maps, diffeomorphisms, implicit function theorem, tangent vectors defined as equivalence classes of paths
3. Tu. 25.10.2016: tangent spaces and tangent bundle, the tangent map $Tf : TM \to TN$ of a smooth map $f : M \to N$, smooth vector fields and derivations, definition of the Lie bracket
4. Th. 27.10.2016: proof of correspondence between vector fields and derivations, flow of a vector field, proof that flows exist globally on compact manifolds, statement of the theorem on commuting flows, Lie derivative of a vector field
5. Tu. 01.11.2016: Proof that $L_X Y = [X, Y]$ and the theorem on commuting flows, examples of tensors
6. Th. 03.11.2016: Tensor bundles of type (k, ℓ), smooth tensor fields, index notation and Einstein summation convention, the C^∞-linearity criterion
7. Tu. 08.11.2016: proof of the C^∞-linearity lemma, pullbacks of covariant tensors, integration of 1-forms on 1-manifolds
8. Th. 10.11.2016: existence of partitions of unity (compact case), proof that integral on 1-manifolds is well defined, n-dimensional volume and alternating forms
9. Tu. 15.11.2016: exterior algebra of a vector space, alternating projection and wedge product, differential k-forms and their components, change of variables lemma in dimension n
10. Th. 17.11.2016: proof of change of variables lemma, orientations on n-manifolds and their tangent spaces, volume forms, manifolds with boundary, boundary orientation, existence of the integral on n-manifolds, statement of Stokes' theorem and sketch of 1-dimensional case
11. Tu. 22.11.2016: definition of the exterior derivative, applications of Stokes' theorem in vector calculus (divergence and curl), beginning of the proof of Stokes' theorem
12. Th. 24.11.2016: conclusion of the proof of Stokes' theorem, de Rham cohomology and basic properties, proof of the Poincaré lemma using homotopy invariance of de Rham, Lie derivatives of forms, invariance of forms, statement of Cartan's formula, application to Hamiltonian mechamics (Liouville's theorem for Hamiltonian flows on $\mathbb{R}^{2n}$)
13. Tu. 29.11.2016: definition of topological groups and Lie groups, examples (standard matrix groups and $\mathrm{Isom}(\mathbb{R}^n)$), commutator bracket for matrix groups, Lie algebras in general, matrix exponential and left-invariant vector fields
14. Th. 01.12.2016: existence/uniqueness of the exponential map for general Lie groups, the Lie algebra of left-invariant vector fields, pullbacks/pushforwards of vector fields, proof that the Lie bracket on $T_e G$ matches the commutator bracket for $G \subset \mathrm{GL}(n, \mathbb{F})$
15. Tu. 06.12.2016: smooth vector bundles, transition functions, sections, general examples (TM, T^*M, $T^k_\ell M$, $\Lambda^k T^*M$, $\mathrm{Hom}(E, F)$ and $\mathrm{End}(E)$), example of a nontrivial real line bundle over S^1
16. Th. 08.12.2016: orientations on vector bundles, subbundles, induced/pullback bundles, duals, direct sums, tensor products, exterior product bundles, bundle metrics and proof of existence via partitions of unity

17. Tu. 13.12.2016: vector bundles with G-structures (structure groups), frames and trivializations, classification of real line bundles over S^1, bundle metrics / orientations / volume forms in terms of structure groups

18. Th. 15.12.2016: proof of the Brouwer fixed point theorem via Stokes' theorem (just for fun), inventory of structures and structure groups on real vector bundles, complex structures ($\mathrm{GL}(n, \mathbb{C})$ as a subgroup of $\mathrm{GL}(2n, \mathbb{R})$), smooth fiber bundles and examples (frame bundles, orthonormal frame bundles, unit sphere bundles, the Hopf fibration $S^3 \stackrel{\pi}{\to} S^2$), left/right group actions, structure groups on fiber bundles

19. Tu. 03.01.2017: parallel transport and covariant derivative on fiber bundles, vertical subbundles, definitions of connection as horizontal subbundle $HE \subset TE$ and as connection map $K : TE \to VE$, existence of connections, linearity of connections on vector bundles

20. Th. 05.01.2017: example (vertical subbundle and the trivial connection for the trivial bundle $S^1 \times S^1 \to S^1$), properties of connections on vector bundles: Leibniz rule, coordinate expressions (Christoffel symbols and local connection 1-forms), compatible connections on vector bundles with G-structures, the G-frame bundle, brief sketch of principal G-bundles and existence of principal connections

21. Tu. 10.01.2017: local characterization (via connection 1-forms) of G-compatible connections on vector bundles, induced connections on dual and tensor bundles, Leibniz rules, compatibility with a bundle metric, parallel transport on pullback bundles, tangent bundles and the torsion tensor

22. Th. 12.01.2017: Christoffel symbols $\Gamma^i_{jk} = (\nabla_j \partial_k)^i$ for connections on TM, symmetric connections, geodesics, exponential map $\exp : TM \to M$ with respect to a connection on TM, Riemannian metrics and path length, existence and uniqueness of the Levi-Civita connection, statement of the theorem on Riemann normal coordinates

23. Tu. 17.01.2017: lecture canceled

24. Th. 19.01.2017: proof of the theorem on Riemann normal coordinates, geodesics as critical points of the energy functional (lecture by Viktor Fromm)

25. Tu. 24.01.2017: geodesics as critical points of the length functional with constant speed, induced metrics and geodesics on submanifolds, example of $S^2 \subset (\mathbb{R}^3, g_{\mathrm{Eucl}})$, statement of the Gauss lemma (see Problem Set 12 for proof), proof that nearby points are joined by a geodesic of shortest length

26. Th. 26.01.2017: lecture canceled

27. Tu. 31.01.2017: parallel sections and flat connections on fiber bundles, integrable local frames, distributions and integral submanifolds, Frobenius integrability theorem, the curvature 2-form on a fiber bundle

28. Th. 02.02.2017: curvature on vector bundles, commuting covariant partial derivatives and the Riemann tensor, bundle-valued differential forms, covariant exterior derivative, proof that the curvature 2-form and the Riemann tensor are equivalent

29. Tu. 07.02.2017: locally flat Riemannian metrics and characterization via the Riemann tensor, Gauss and Weingarten maps for surfaces in $\mathbb{R}^3$, Gaussian curvature, statement of the Theorema Egregium, hypersurfaces in $\mathbb{R}^n$ and the second fundamental form

30. Th. 09.02.2017: the Riemann tensor on surfaces, proof of the Theorema Egregium, geodesic curvature, Gauss-Bonnet formula for polygonal regions, triangulation, Euler characteristic and the Gauss-Bonnet formula for closed surfaces

31. Tu. 14.02.2017: computations of Euler characteristic (S^2, $\mathbb{T}^2$, pairs of pants and handles, surfaces of genus g), corollary on surfaces of constant curvature, Chern-Weil theory for Hermitian line bundles (curvature 2-form and the first Chern class), proof of the Gauss-Bonnet formula

32. Th. 16.02.2017: how the first Chern number counts zeroes of sections (the Euler number), transversality of a map to a submanifold, corollaries of the implicit function theorem and Sard's theorem for transverse maps (stated without proof), proof via transversality that the Euler number of an oriented real vector bundle of rank n over a closed n-manifold is well defined, sketch of the proof that you can't embed the Klein bottle in $\mathbb{R}^3$ (just for fun)