**Course description and syllabus**

## General information

**Instructor:** Prof. Chris Wendl
HU Institute for Mathematics (Rudower Chaussee 25), Room 1.301
`wendl@math.hu-berlin.de`
Office hour: Tuesdays 15:00–16:00

**Course webpage**: `http://www.mathematik.hu-berlin.de/~wendl/Winter2016/DiffGeo1/`

**Lectures**: Tuesdays 9:00–11:00 in 1.013 (Rudower Chaussee 25)
Thursdays 15:00–17:00 in 1.013 (Rudower Chaussee 25)

**Problem classes**: Tuesdays 11:00–13:00 in 1.012 (Rudower Chaussee 25)

**Language**: The course can be offered in German or English depending on the preferences of the students. If you plan to attend but would not be able to follow the course in German, please contact me ahead of time.

**Prerequisites**: Contents of the HU's courses *Analysis I* and *II*, and *Lineare Algebra und Analytische Geometrie I* and *II*
The single most important prerequisite is a solid grasp of calculus for functions $\mathbb{R}^n \to \mathbb{R}^m$, including the statements (if not the proofs) of the inverse and implicit function theorems, and the change of variables formula for integration.

## Short description

An introduction to differential geometry of manifolds: definition of smooth $n$-dimensional manifolds, vector fields, flows and the Lie bracket, tensors, differential forms, integration, Stokes' theorem. Basics on Lie groups, Lie algebras, and the exponential map. Connections and curvature on fiber bundles and vector bundles, distributions, integrability and the Frobenius theorem. Basic Riemannian geometry: curvature tensors, geodesics, the Gauss-Bonnet theorem for surfaces.

## Detailed description

Differential geometry naturally begins with the study of smooth 1-dimensional curves and 2-dimensional surfaces in Euclidean space, but these are only special cases of much more general objects, called smooth $n$-dimensional *manifolds*. Manifolds arise naturally in many branches of mathematics as well as in physics, e.g. as configuration spaces for constrained dynamical systems, or as the curved spacetime in Einstein's theory of gravitation (General Relativity). This course will study manifolds from a fairly general perspective, without limiting the discussion to curves and surfaces, though for the purposes of visualization, most of the examples we consider will be 2-dimensional. The first task is to understand basic notions such as smoothness of maps between manifolds, the derivatives of such maps, tangent vectors, vector fields and the flows that they generate. Tensors are then introduced as a linear algebraic means of encoding local geometric information, and as a special case, we consider differential forms, which define notions of volume on manifolds and can thus be integrated. The first portion of the course culminates with the general version of Stokes' theorem for integrals of differential forms: this is the natural $n$-dimensional generalization of the fundamental theorem of calculus, and also implies the standard vector calculus theorems of Gauss, Green and Stokes.

After this follows a brief interlude on Lie groups, i.e. groups that are also smooth manifolds, the basic examples being the standard subgroups of $\mathrm{GL}(n, \mathbb{R})$. One could teach an entire course about Lie groups, and this is not that course, but we will need to know a few things about them in order to facilitate an elegant approach to differential geometry.

The second half of the course is based on the general notions of vector bundles and fiber bundles, of which several examples (e.g. the set of all tangent vectors on a manifold) will already be familiar from the first half. The need to define derivatives on bundles leads naturally to the notions of parallel transport, connections and covariant differentiation. This raises a natural question as to when covariant derivatives in different directions can be assumed to commute, and the answer requires the introduction of *curvature*, a tensor whose vanishing characterizes the existence of "covariantly constant" vector fields on manifolds. In order to prove this, we introduce smooth distributions on manifolds and prove the Frobenius integrability theorem. The most convincing initial applications of these ideas are in the study of *Riemannian* manifolds: these are manifolds equipped with extra structure so that lengths of paths and angles between them can be defined. We consider geodesics, which define shortest paths between nearby points on Riemannian manifolds, and discuss the geometric meaning of the Riemann curvature tensor in $n$ dimensions, as well as its simpler variant for surfaces, the so-called "Gaussian" curvature. We can then prove one of the most beautiful and fundamental results about 2-dimensional Riemannian manifolds: the Gauss-Bonnet theorem, which relates the sum of the angles in a geodesic triangle to the amount of curvature it encloses, or for the case of compact surfaces without boundary, computes the total curvature in terms of a purely topological invariant, the Euler characteristic.

There are a number of additional topics that we may briefly touch upon if time permits, most of them related to physics: these include Hamiltonian dynamics and symplectic geometry, pseudo-Riemannian manifolds and general relativity, holomorphic vector bundles, principal fiber bundles and gauge theory.

## Literature

The first half of the course will not follow any particular book, but I have listed a few recommendations below and intend to provide regular reading suggestions from each of the first two on the list. Each of the following books can be relied on to cover almost all of what we will discuss in the first half of the course (and also many things that we won't):

- Ilka Agricola and Thomas Friedrich, *Globale Analysis: Differentialformen in Analysis, Geometrie und Physik*, Vieweg 2001 (or the English translation, AMS 2002)
  (several copies available in the HU library, both Lehrbuchsammlung and Freihandbestand)
- John M. Lee, *Introduction to Smooth Manifolds*, Springer GTM 2003
  (online access available via the HU library—the library also has a newer edition from 2013, but without online access)
- Helga Baum, *Eine Einführung in die Differentialgeometrie*, lecture notes available at
  `https://www.mathematik.hu-berlin.de/~baum/Skript/diffgeo1.pdf`
- Frank W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Springer GTM 1983
  (online access available via the HU library)
- Michael Spivak, *A Comprehensive Introduction to Differential Geometry, Volume I*, 3rd edition with corrections, Publish or Perish 2005

Note: Some sections of the books by Lee and Warner noticeably assume more knowledge of topology than we will assume in this course, but students who have not yet taken Topologie I can safely skip those sections and still understand the sections that are relevant to the course.

The second half of the course will mostly follow portions of

- Chris Wendl, *Lecture Notes on Bundles and Connections*

which will be made available on the course webpage. An older version of these notes (written for a course I taught at MIT in Spring 2007) is available at `http://www.mathematik.hu-berlin.de/~wendl/connections.html`. Another good book that focuses specifically on fiber bundles and connections (but not the applications to Riemannian geometry that we'll discuss) is:

- Helga Baum, *Eichfeldtheorie*, Springer 2009 (or second edition 2014)

Finally, here is a brief selection of freely downloadable lecture notes by other authors, most of which I don't know very well, but they also cover many of the topics we will discuss and have been recommended by colleagues:

- Nigel Hitchin, *Differentiable Manifolds*, available at `https://people.maths.ox.ac.uk/hitchin/hitchinnotes/manifolds2012.pdf`
- Thomas Schick, *Kurz-Skript zu "Topologie und Differentialgeometrie 1"*, available at `http://www.uni-math.gwdg.de/schick/teach/diffgeo.pdf`
- Sean Carroll, *Lecture Notes on General Relativity*, available at `https://www.preposterousuniverse.com/grnotes/`
(chapters 2 and 3 present a very readable introduction to manifolds and curvature that is suitable for both mathematicians and physicists)

## Exam and problem sets

Grades in the course will be determined by a three-hour **written exam** in the week following the end of the semester (with a resit option in the week before the beginning of the Summer Semester). Books and notes may be consulted during the exam.

Midway through the semester there will also be a **take-home midterm**. This is like a problem set, but more serious: you will have two weeks to work on it and will be asked to do so alone, not collaboratively. You will receive a grade and feedback on the take-home midterm, and the grade may be used boost your final exam grade as follows:

- 50%–75% = + 0,3
- 75%–100% = + 0,7

**Problem sets** will be handed out weekly on Tuesdays and discussed in the problem class on the following Tuesday. You should come to each problem class prepared to present your solution to at least one of the problems due that week.

## Syllabus

The following week-by-week plan for the lectures is tentative and subject to change.

1. Differentiable manifolds, implicit function theorem, examples, tangent vectors.
2. Tangent maps, vector fields, Lie bracket and commuting flows.
3. Orientability, tensors, index notation, differential forms.

4. Poincaré lemma, Lie derivative, Cartan's formula.
5. Integration, Stokes' theorem, low-dimensional examples.
6. Lie groups and Lie algebras, exponential map.
7. Vector bundles and sections, bundle metrics, orientation.
8. Structure groups, fiber bundles, parallel transport.
9. Linear connections, covariant derivatives, compatibility.
10. Connections on tangent bundles, torsion and symmetry, Riemann normal coordinates.
11. Levi-Cività connection, geodesics, Riemannian manifolds.
12. Integrability and the Frobenius theorem, curvature.
13. Locally flat manifolds, hypersurfaces, second fundamental form and Gaussian curvature.
14. Euler characteristic and the Gauss-Bonnet theorem for surfaces.
15. Sectional curvature, second variation formula, length-minimizing geodesics.
16. (further topics to be decided if time permits)