PROBLEM SET 9
To be discussed: 10.01.2018

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next Wednesday lecture after they are distributed, as they will often serve as mental preparation for the material in that lecture. We will discuss the solutions in the Übung beforehand.

Scheduling note: No lectures or Übung next week. We meet again in January.

1. Assume R is a commutative ring with unit, and let Mod_R denote the category of R-modules, where the set of morphisms $A \to B$ for two R-modules A and B is defined as

$$\mathrm{Hom}_R(A, B) = \left\{ \Phi \in \mathrm{Hom}(A, B) \;\middle|\; \Phi(\lambda a) = \lambda \Phi(a) \text{ for all } \lambda \in R \text{ and } a \in A \right\}.$$

Here $\mathrm{Hom}(A, B)$ denotes the usual set of group homomorphisms $A \to B$.

(a) Check that $\mathrm{Hom}_R(\cdot, \cdot) : \mathsf{Mod}_R \times \mathsf{Mod}_R \to \mathsf{Mod}_R$ defines a functor, contravariant in the first variable and covariant in the second variable.

(b) The **tensor product** of two R-modules A and B is the R-module

$$A \otimes_R B = (A \otimes B) \big/ \sim$$

where $A \otimes B$ is the usual tensor product of abelian groups and the equivalence relation is defined by $\lambda a \otimes b \sim a \otimes \lambda b$ for all $\lambda \in R$, $a \in A$ and $b \in B$. Check that this defines a functor $\otimes_R : \mathsf{Mod}_R \times \mathsf{Mod}_R \to \mathsf{Mod}_R$, covariant in both variables.

(c) Show that if A is an abelian group and G is an R-module, then $A \otimes G$ and $\mathrm{Hom}(A, G)$ can each be endowed with R-module structures such that the correspondences $A \mapsto A \otimes G$ and $A \mapsto \mathrm{Hom}(A, G)$ define a covariant functor $\otimes G$ and a contravariant functor $\mathrm{Hom}(\,\cdot\,, G)$ from the category of abelian groups Ab to Mod_R.

(d) Given an R-module G, denote by $\otimes_R G$ and $\mathrm{Hom}_R(\,\cdot\,, G)$ the functors $\mathsf{Mod}_R \to \mathsf{Mod}_R$ sending an R-module H to $H \otimes_R G$ or $\mathrm{Hom}_R(H, G)$ respectively. Show that there is a natural transformation from $\otimes G : \mathsf{Ab} \to \mathsf{Mod}_R$ to $(\otimes_R G) \circ (\otimes R) : \mathsf{Ab} \to \mathsf{Mod}_R$ that assigns to every abelian group A an R-module isomorphism

$$A \otimes G \to (A \otimes R) \otimes_R G.$$

Show also that there is a natural transformation from $\mathrm{Hom}(\,\cdot\,, G) : \mathsf{Ab} \to \mathsf{Mod}_R$ to $\mathrm{Hom}_R(\,\cdot\,, G) \circ (\otimes R) : \mathsf{Ab} \to \mathsf{Mod}_R$ that assigns to every abelian group A an R-module isomorphism

$$\mathrm{Hom}(A, G) \to \mathrm{Hom}_R(A \otimes R, G).$$

2. Assume $A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$ is an exact sequence of abelian groups, and G is another abelian group. Note that this is not a full "short exact sequence" since there is no 0 term at the beginning, so in particular we do not assume i to be injective.

(a) Prove that the sequence $0 \longrightarrow \mathrm{Hom}(C, G) \xrightarrow{j^*} \mathrm{Hom}(B, G) \xrightarrow{i^*} \mathrm{Hom}(A, G)$ is exact. If you also assume $0 \to A \to B \to C \to 0$ is a full short exact sequence, what stops you from proving that the dualized sequence is exact at $\mathrm{Hom}(A, G)$?

(b) Prove that the sequence $A \otimes G \xrightarrow{i \otimes \mathbb{1}} B \otimes G \xrightarrow{j \otimes \mathbb{1}} C \otimes G \longrightarrow 0$ is exact.
Hint: Since $\mathrm{im}(i \otimes \mathbb{1}) \subset \ker(j \otimes \mathbb{1})$, $j \otimes \mathbb{1}$ *descends to a homomorphism* $(B \otimes G) \big/ \mathrm{im}(i \otimes \mathbb{1}) \to C \otimes G$. *To prove exactness at the* $B \otimes G$ *term, it suffices to prove that this homomorphism is injective. (Why?) Try to write down a left-inverse* $C \otimes G \to (B \otimes G) \big/ \mathrm{im}(i \otimes \mathbb{1})$.

Remark: This problem can be generalized from abelian groups to R-modules without any change.

3. Use cellular homology to compute $H_*(\mathbb{RP}^n;\mathbb{Z})$ and $H_*(\mathbb{RP}^n;\mathbb{Z}_2)$ for every $n \in \mathbb{N}$.
   Hint: Remember the cell decomposition of S^n that has two k-cells for each $k = 0, \dots, n$?

4. The **complex projective n-space** $\mathbb{CP}^n$ is a compact $2n$-manifold defined as the set of all complex lines through the origin in $\mathbb{C}^{n+1}$, or equivalently,

$$\mathbb{CP}^n = (\mathbb{C}^{n+1}\backslash\{0\})/\sim$$

   where two points $z, z' \in \mathbb{C}^{n+1}\backslash\{0\}$ are equivalent if and only if $z' = \lambda z$ for some $\lambda \in \mathbb{C}$. It is conventional to write elements of $\mathbb{CP}^n$ in so-called *homogeneous coordinates*, meaning the equivalence class represented by $(z_0, \dots, z_n) \in \mathbb{C}^{n+1}$ is written as $[z_0 : \dots : z_n]$. Notice that $\mathbb{CP}^n$ can be partitioned into two disjoint subsets

$$\mathbb{C}^n \cong \{[1 : z_1 : \dots : z_n] \in \mathbb{CP}^n\} \quad \text{and} \quad \mathbb{CP}^{n-1} \cong \{[0 : z_1 : \dots : z_n] \in \mathbb{CP}^n\}.$$

   (a) Show that the partition $\mathbb{CP}^n = \mathbb{C}^n \cup \mathbb{CP}^{n-1}$ gives rise to a cell decomposition of $\mathbb{CP}^n$ with one $2k$-cell for every $k = 0, \dots, n$.

   (b) Compute $H_*(\mathbb{CP}^n;G)$ and $H^*(\mathbb{CP}^n;G)$ for an arbitrary coefficient group G. *Hint: This is easy.*

5. Each of the following spaces is defined as a direct limit in terms of the natural inclusions $\mathbb{F}^m \hookrightarrow \mathbb{F}^n$ for $n \geqslant m$, where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$, and we identify $\mathbb{F}^m$ with the subspace $\mathbb{F}^m \oplus \{0\} \subset \mathbb{F}^n$. In particular, $\mathbb{R}^{m+1} \hookrightarrow \mathbb{R}^{n+1}$ gives rise to inclusions $S^m \hookrightarrow S^n$ and $\mathbb{RP}^m \hookrightarrow \mathbb{RP}^n$, and the complex version gives $\mathbb{CP}^m \hookrightarrow \mathbb{CP}^n$. Use CW-decompositions to compute the homology with integer coefficients for each space:

   (a) $S^\infty = \varinjlim\{S^n\}_{n\in\mathbb{N}}$

   (b) $\mathbb{RP}^\infty = \varinjlim\{\mathbb{RP}^n\}_{n\in\mathbb{N}}$

   (c) $\mathbb{CP}^\infty = \varinjlim\{\mathbb{CP}^n\}_{n\in\mathbb{N}}$

6. Let Σ_g denote the closed, connected and oriented surface of genus $g \geqslant 0$. Show that for any set $\Gamma \subset \Sigma_g$ of $m \geqslant 0$ points, $\chi(\Sigma_g\backslash\Gamma) = 2 - 2g - m$.
   Hint: You may feel free to replace $\Sigma_g\backslash\Gamma$ with something else to which it is homotopy equivalent.