PROBLEM SET 11
To be discussed: 30.01.2019

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next Wednesday lecture after they are distributed, as they will often serve as mental preparation for the material in that lecture. Solutions will be discussed in the Übung.

1. Use the usual cell decomposition of $\mathbb{RP}^2$ with one cell in each dimension $0, 1, 2$ to compute $H^*(\mathbb{RP}^2; \mathbb{Z})$, then compare this with what you get by applying the universal coefficient theorem to your earlier computation of $H_*(\mathbb{RP}^2; \mathbb{Z})$ (Problem Set 6 #2).

2. The goal of this problem is to understand the role played by 0-chains and 0-cochains in the cross and cup product. Let $\{\mathrm{pt}\}$ denote the 1-point space and, for a path-connected space X, let $[\mathrm{pt}] \in H_0(X) = \mathbb{Z}$ denote the canonical generator, i.e. the homology class represented by any singular 0-simplex $\Delta^0 \to X$. Recall moreover that any coefficient group G has a canonical inclusion $G \hookrightarrow H^0(X; G)$, such that $g \in G$ is identified with $[\varphi] \in H^0(X; G)$ for the cochain $\varphi : C_0(X) \to G$ that sends every singular 0-simplex to g.

   (a) Show that if Y is path-connected, the cross product of any $A \in H_n(X)$ with $[\mathrm{pt}] \in H_0(Y)$ is $A \times [\mathrm{pt}] = i_* A$ for any inclusion map of the form $i : X \hookrightarrow X \times Y : x \mapsto (x, \mathrm{const})$. A similar formula holds for cross products with $[\mathrm{pt}] \in H_0(X)$ if X is path-connected.
   Hint: Remember that $\times$ is induced by a natural chain map $\Phi : C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$, so if you have the right formula for $\Phi : C_n(X) \otimes C_0(Y) \to C_n(X \times Y)$, the relation will become obvious. In general, one can make many choices in defining Φ, but there is an obvious choice that one "should" make when one of the chains is 0-dimensional. Review the construction of Φ via acyclic models to show that this choice is always possible.

   (b) Suppose Ψ associates to every space X a chain map $\Psi : C_*(X) \to C_*(X)$. We will say that Ψ is a **natural chain map** $C_*(X) \to C_*(X)$ if it acts as the identity map on 0-chains and for every continuous map $f : X \to Y$, $\Psi \circ f_* = f_* \circ \Psi$. Use the method of acyclic models to show that any two choices of natural chain maps in this sense are chain homotopic for all X.

   (c) Identify the chain complex $C_*(X \times \{\mathrm{pt}\})$ with $C_*(X)$ via the obvious canonical isomorphism between them, and consider the following two maps:

$$C_*(X \times \{\mathrm{pt}\}) \xrightarrow{\theta} C_*(X) \otimes C_*(\{\mathrm{pt}\}) \xrightarrow{\mathbb{1} \otimes \epsilon} C_*(X) \otimes \mathbb{Z} = C_*(X),$$
$$C_*(X \times \{\mathrm{pt}\}) \xrightarrow{(\pi_X)_*} C_*(X),$$

   where $\pi_X : X \times \{\mathrm{pt}\} \to X$ is the canonical projection, θ is any natural chain homotopy inverse for the natural chain map $\Phi : C_*(X) \otimes C_*(\{\mathrm{pt}\}) \to C_*(X \times \{\mathrm{pt}\})$ as used in the construction of the cross product, and $\epsilon : C_*(\{\mathrm{pt}\}) \to \mathbb{Z}$ is the *augmentation* map, which vanishes on $C_n(\{\mathrm{pt}\})$ for $n \neq 0$ and sends each generator $\sigma \in C_0(\{\mathrm{pt}\})$ to 1. Verify that both of these define natural chain maps, hence by part (b), they are chain homotopic.

   (d) Fix a commutative ring R with unit (denoted by $1 \in R$), and deduce from part (c) that for any space X, the cross product of $\alpha \in H^*(X; R)$ with $1 \in R \subset H^0(\{\mathrm{pt}\}; R)$ satisfies $\alpha \times 1 = \pi_X^* \alpha$.

   (e) Use the naturality formula $(f \times g)^*(\alpha \times \beta) = f^*\alpha \times g^*\beta$ to deduce from part (d) that the cross product of any $\alpha \in H^*(X; R)$ with $1 \in R \subset H^0(Y; R)$ satisfies $\alpha \times 1 = \pi_X^* \alpha$, where $\pi_X : X \times Y \to X$ is the projection. (Similarly, $1 \times \beta = \pi_Y^* \beta$ for $1 \in R \subset H^0(X; R)$ and $\beta \in H^*(Y; R)$.)

   (f) Deduce from the above that for any space X, $1 \in R \subset H^0(X; R)$ acts as the unit with respect to the cup product: $\alpha \cup 1 = \alpha = 1 \cup \alpha$.

3. Recall that the **smash product** $X \wedge Y$ of two pointed spaces (X, x_0) and (Y, y_0) is defined by

$$X \wedge Y := (X \times Y) \Big/ \big((\{x_0\} \times Y) \cup (X \times \{y_0\})\big) \cong (X \times Y)/(X \vee Y).$$

Any two base-point preserving maps $f : (X, x_0) \to (X', x_0')$ and $g : (Y, y_0) \to (Y', y_0')$ then define a product map $f \times g : X \times Y \to X' \times Y'$ which descends continuously to the quotient as a map

$$f \wedge g : X \wedge Y \to X' \wedge Y'.$$

As mentioned in lecture, a popular example of the smash product is furnished by spheres, as there are homeomorphisms

$$S^k \wedge S^\ell \cong S^{k+\ell} \quad \text{for all} \quad k, \ell \geqslant 0.$$

If X and Y are CW-complexes and we choose the base points to be 0-cells in these complexes, then the two subspaces $X \times \{y_0\}$ and $\{x_0\} \times Y$ in $X \times Y$ form an excisive couple, so that the relative cross product and Künneth formula are valid for the pairs $(X, \{x_0\})$ and $(Y, \{y_0\})$. Since $(X, x_0) \times (Y, y_0) = (X \times Y, X \vee Y)$, the Künneth formula now takes the form

$$0 \to \bigoplus_{k+\ell=n} H_k(X, \{x_0\}) \otimes H_\ell(Y, \{y_0\}) \xrightarrow{\times} H_n(X \times Y, X \vee Y)$$
$$\longrightarrow \bigoplus_{k+\ell=n-1} \operatorname{Tor}(H_k(X, \{x_0\}), H_\ell(Y, \{y_0\})) \to 0,$$

which can be identified with

$$0 \to \bigoplus_{k+\ell=n} \widetilde{H}_k(X) \otimes \widetilde{H}_\ell(Y) \xrightarrow{\times} \widetilde{H}_n(X \wedge Y) \longrightarrow \bigoplus_{k+\ell=n-1} \operatorname{Tor}(\widetilde{H}_k(X), \widetilde{H}_\ell(Y)) \to 0$$

since CW-pairs are also "good" pairs in the sense of Lecture 30.

(a) Show that for the cross product on reduced homology as described above and the identification of $S^k \wedge S^\ell$ with $S^{k+\ell}$, if $[S^k] \in \widetilde{H}_k(S^k)$ and $[S^\ell] \in \widetilde{H}_\ell(S^\ell)$ are generators, then $[S^k] \times [S^\ell] \in \widetilde{H}_{k+\ell}(S^{k+\ell})$ is also a generator.

(b) Use the naturality of the Künneth formula to prove that for any two base-point preserving maps $f : S^k \to S^k$ and $g : S^\ell \to S^\ell$, $\deg(f \wedge g) = \deg(f) \cdot \deg(g)$.

(c) Find an alternative proof of the formula in part (b) using the following fact from differential topology: any continuous map $f : S^k \to S^k$ admits a small perturbation to a smooth map such that for almost every point $x \in S^k$, $f^{-1}(x)$ is a finite set of points at which the local degree of f is ± 1. (The latter is an immediate consequence of Sard's theorem.)

(d) Using the definition of cellular chain maps and the cellular cross product, prove that the cellular cross product is natural, i.e. if $f : X \to X'$ and $g : Y \to Y'$ are cellular maps, then the diagram

$$\begin{tikzcd} C_*^{\mathrm{CW}}(X) \otimes C_*^{\mathrm{CW}}(Y) \arrow[r, "\times"] \arrow[d, "f_* \otimes g_*"] & C_*^{\mathrm{CW}}(X \times Y) \arrow[d, "(f \times g)_*"] \\ C_*^{\mathrm{CW}}(X') \otimes C_*^{\mathrm{CW}}(Y') \arrow[r, "\times"] & C_*^{\mathrm{CW}}(X' \times Y') \end{tikzcd}$$

commutes.