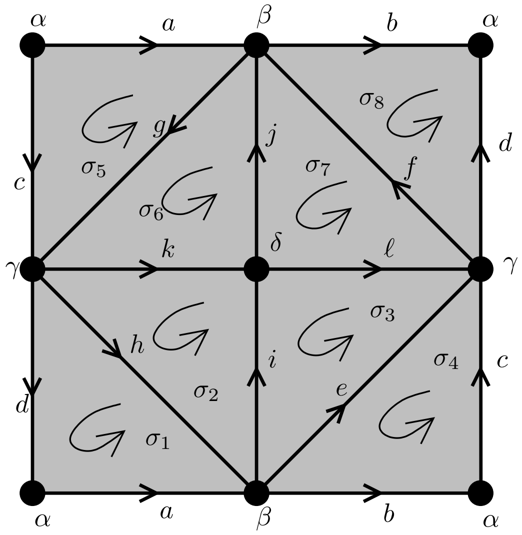
PROBLEM SET 3
To be discussed: 7.11.2018

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next Wednesday lecture after they are distributed, as they will often serve as mental preparation for the material in that lecture. Solutions will be discussed in the Übung.

1. This first problem on simplicial homology is basically recycled from last semester's *Topologie I* course. It's a good problem—if you didn't do it then, you should do it now!

   The following picture shows a simplicial complex $K = (V, S)$ whose associated polyhedron $|K|$ is homeomorphic to the Klein bottle.



   There are four vertices $V = \{\alpha, \beta, \gamma, \delta\}$, twelve 1-simplices labeled by letters $a, \dots, \ell$, and eight 2-simplices labeled σ_i for $i = 1, \dots, 8$. The picture also shows a choice of orientation for each of the 2-simplices[1] (circular arrows represent a cyclic ordering of the vertices) and 1-simplices (arrows point from the first vertex to the last).

   (a) Write down $\partial\sigma_i$ explicitly for each $i = 1, \dots, 8$.

   (b) Prove that $H_2^\Delta(K; \mathbb{Z}_2) \cong \mathbb{Z}_2$, and write down a specific cycle in $C_2^\Delta(K; \mathbb{Z}_2)$ that generates it.

   (c) Prove that $H_2^\Delta(K; \mathbb{Z}) = 0$.
   Hint: Consider how the coefficients of individual 1*-simplices in* $\partial \sum_{i=1}^8 c_i\sigma_i \in C_1(K; \mathbb{Z})$ *are determined. Show that if* $\sum_{i=1}^8 c_i\sigma_i$ *is a cycle, then* $c_1 = c_2$, $c_2 = c_3$ *and so forth, but also* $c_1 + c_8 = 0$.

   (d) Show that the 1-cycle $c + d$ represents a nontrivial homology class $[c + d]$ in both $H_1^\Delta(K; \mathbb{Z})$ and $H_1^\Delta(K; \mathbb{Z}_2)$, but satisfies $2[c + d] = 0 \in H_1^\Delta(K; \mathbb{Z})$ and $[c + d] = 0 \in H_1^\Delta(K; \mathbb{Q})$.

2. The following exercise fills a logical gap that crept in at the end of my lecture on October 24, when I claimed that the isomorphisms $\widetilde{H}_0(S^0; G) \oplus G \cong H_0(S^0; G) \cong H_0(\{\mathrm{pt}\} \amalg \{\mathrm{pt}\}; G) \cong G \oplus G$ imply $\widetilde{H}_0(S^0; G) \cong G$. (In fact, this is a valid argument if G is finitely generated, due to the classification of finitely generated abelian groups, but in general it is not true that $A \oplus G \cong B \oplus G$ implies $A \cong B$.)

[1] Notice however that this does not define an oriented triangulation, as the chosen orientations of neighboring 2-simplices are not always compatible with each other. The Klein bottle does not admit an oriented triangulation.

(a) For any two spaces X and Y with maps $\epsilon^X : X \to \{\mathrm{pt}\}$ and $\epsilon^Y : Y \to \{\mathrm{pt}\}$, show that the natural isomorphism $H_*(X \amalg Y; G) \cong H_*(X;G) \oplus H_*(Y;G)$ identifies $\widetilde{H}_*(X \amalg Y; G)$ with the kernel of the map

$$\epsilon^X_* \oplus \epsilon^Y_* : H_*(X;G) \oplus H_*(Y;G) \to H_*(\{\mathrm{pt}\}; G).$$

Hint: For the inclusions $i^X : X \hookrightarrow X \amalg Y$ and $i^Y : Y \hookrightarrow X \amalg Y$, the unique map $\epsilon : X \amalg Y \to \{\mathrm{pt}\}$ satisfies $\epsilon \circ i^X = \epsilon^X$ and $\epsilon \circ i^Y = \epsilon^Y$.

(b) Apply part (a) in the case $X = Y = \{\mathrm{pt}\}$ to identify $\widetilde{H}_0(S^0; G) \cong \widetilde{H}_0(\{\mathrm{pt}\} \amalg \{\mathrm{pt}\}; G)$ with the kernel of the map

$$\mathbb{1} \oplus \mathbb{1} : G \oplus G \to G : (g,h) \mapsto g + h,$$

and show that the latter is isomorphic to G.

3. Assume $h_* : \mathsf{Top}_{\mathrm{rel}} \to \mathsf{Ab}_{\mathbb{Z}}$ is a functor satisfying all of the Eilenberg-Steenrod axioms for homology theories except possibly the additivity axiom. Given two spaces X and Y, use excision and the long exact sequences of the pairs $(X \amalg Y, X)$ and $(X \amalg Y, Y)$ to prove that for the natural inclusions $i^X : X \hookrightarrow X \amalg Y$ and $i^Y : Y \hookrightarrow X \amalg Y$, the map

$$i^X_* \oplus i^Y_* : h_*(X) \oplus h_*(Y) \to h_*(X \amalg Y) : (x,y) \mapsto i^X_* x + i^Y_* y$$

is an isomorphism. Deduce that h_* does satisfy the additivity axiom for all *finite* disjoint unions.

4. Suppose the following diagram commutes and that both of its rows are exact, meaning $\operatorname{im} f = \ker g$, $\operatorname{im} g' = \ker h'$ and so forth:

$$\begin{array}{ccccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{i} & E \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\varepsilon} \\
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & D' & \xrightarrow{i'} & E'
\end{array}$$

(a) Prove that if α, β, δ and ε are all isomorphisms, then so is γ.
This result is known as the ***five-lemma***.

(b) Here is an application: given an axiomatic homology theory h_* and a map of pairs $f : (X, A) \to (Y, B)$, show that if any two of the induced maps $h_k(X) \xrightarrow{f_*} h_k(Y)$, $h_k(A) \xrightarrow{f_*} h_k(B)$ and $h_k(X, A) \xrightarrow{f_*} h_k(Y, B)$ are isomorphisms for every k, then so is the third.

(c) Given a collection of pairs of spaces $\{(X_\alpha, A_\alpha)\}_{\alpha \in J}$, consider the pair

$$\coprod_{\alpha \in J} (X_\alpha, A_\alpha) := \left(\coprod_{\alpha \in J} X_\alpha, \coprod_{\alpha \in J} A_\alpha \right)$$

with the natural inclusion maps $i^\alpha : (X_\alpha, A_\alpha) \hookrightarrow \coprod_{\beta \in J} (X_\beta, A_\beta)$. Use the five-lemma to prove that for any axiomatic homology theory h_*, the additivity axiom generalizes to pairs, producing an isomorphism

$$\bigoplus_{\alpha \in J} i^\alpha_* : \bigoplus_{\alpha \in J} h_*(X_\alpha, A_\alpha) \xrightarrow{\cong} h_*\Big(\coprod_{\alpha \in J} (X_\alpha, A_\alpha) \Big).$$