0 Intro

0.1 CM

target X Riemannian mfd, $G_{\mu\nu}$, $\dim X = d$

trajectory $x : I \to X$ $\qquad I \subset \mathbb{R}$

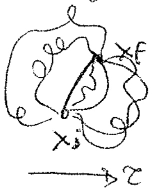
$\tau \to x(\tau) \in X$, $x(\tau) = (x^0 \dots x^d)$ $\qquad \tau \in I$ proper time

free particle : $S(x(\tau)) = \int d\tau\, G_{\mu\nu}\, \partial_\tau X^\mu \partial_\tau X^\nu$ length of worldline

eom: $\frac{\delta S}{\delta x} = 0 \longrightarrow$ worldline is a geodesic

0.2 QM

Sum over worldlines



bc: say $x(\tau_0) = x_i$, $x(\tau_1) = x_f$, $I = [\tau_0, \tau_1]$

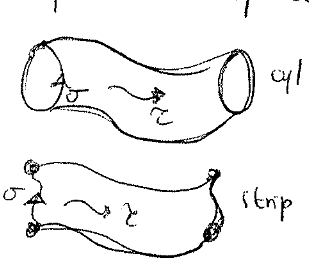
then PI sums over all paths with a probability measure e^{-S}

$K(x_i, x_f) = \int Dx\, e^{-\frac{1}{\hbar} S}$ Euclidean propagator

One can also take periodic trajectories : $x : S^1 \longrightarrow X$, radius β $\leadsto$ thermal field theory

0.3 String theory

points are replaced by extended 1d objects : strings $\longrightarrow$ closed ○ open ∼



trajectory is given by a map

$x : I \times S^1 = \Sigma \longrightarrow X$ $\qquad \Sigma$, worldsheet

$(\tau, \sigma) \longrightarrow x(\tau, \sigma) = (x^1 \dots x^d)$

the action is the Polyakov action (surface of WS) : $S_P = \frac{1}{2\pi\alpha'} \int_\Sigma d\tau d\sigma \sqrt{h}\, G_{\mu\nu} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu$

$\mu, \nu = 1 \dots d$ $\qquad h = \det h_{\alpha\beta}$, metric on Σ. $\qquad = S_{NG} = \frac{1}{2\pi\alpha'} \int_\Sigma d\tau d\sigma \sqrt{\det(G_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu)}$

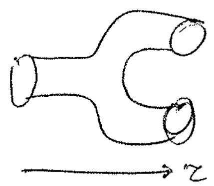
$\alpha, \beta = 1, 2$ / $\tau\ \sigma$ $\qquad \alpha' = l_s^2$, $l_s \sim l_p \sim 10^{-33}$ cm "stringy-ness"

eom: minimal surface.

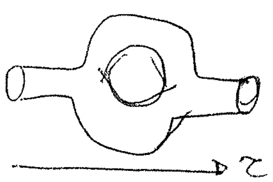
parameterize $S^1 = [0, \pi]$. Closed string : $x(\tau, 0) = x(\tau, \pi)$ $\forall \tau \in I$.

open string : $x(\tau, 0), x(\tau, \pi) \in M \subset X$ DD. NN, DN, ND.

Strings interact by joining and splitting. Strength of interaction : g_s.

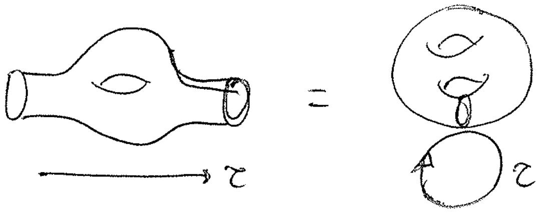


g_s (joining $\tau \to -\tau$)

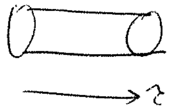
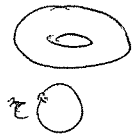


g_s^2 $\qquad$ n interactions g_s^n

Consider periodic configurations



Feynmandiagrams for closed strings/ ~~[illegible]~~
time evolution of periodic string processes are
closed oriented RS Σ_g. Weight g_s^{2g-2}

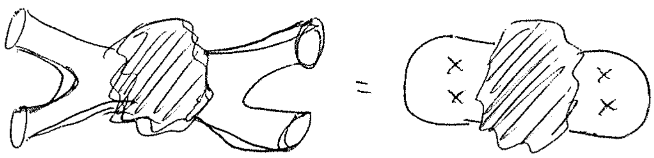
eg:  g_s^0 =  g_s^0 ;  g_s^{-2}

at its core closed ST is the study of maps $x: \Sigma_g \longrightarrow X$

Quantization: sum over all paths and interactions $\Rightarrow$ sum over g and for each g, sum over metrics (connected RS)

$$\rightsquigarrow \quad F = \sum_{g=0}^{\infty} g_s^{2g-2} F_g \quad , \quad F_g = \int_{\mathcal{M}_g} d\tau \int Dh_{\alpha\beta}\, Dx\, e^{-S_P(\tau)}$$

In general one considers operators, ie functions of the fields evaluated at pts $p_1 \dots p_n \in \Sigma_g$ and computes correlation functions. Operator-state correspondence: $|\phi\rangle = \lim_{z,\bar z \to 0} \phi(z,\bar z)|0\rangle$ for CFT.



$$\int_{\mathcal{M}_{g,n}} d\tau \int Dh_{\alpha\beta}\, Dx\, e^{-S_P} \prod_{i=1}^{n} O_i$$

$= \ldots g_s^{-2+n} + \ldots g_s^{n} + \ldots g_s^{2+n} + \ldots$

NB: F depends on both α' stringyness. Same as $\hbar$ in QM. $\alpha' \to 0$, $l_s \to 0$, point particles
g_s coupling.

$\rightsquigarrow$ ST is a consistent deformation of QM.

Ingredients: CFT aka "matter" $\oplus$ 2d gravity $\oplus$ ghost CFT $\xrightarrow{\text{Magic}}$ $\boxed{\text{ST}}$
x... h..

different CFTs lead to different strings and symmetries of the CFT lead to additional geometric structure on X.

examples:
- bosonic ST $S_P(x,h)$ D=26
- non-critical strings, minimal models, ...
- Superstrings D=10.
  type I, type IIA, IIB ; Heterotic SO(32), Heterotic $E_8 \times E_8$

String compactification $X_{10} = X \times \mathbb{R}^{1,3}$. For pheno X special holonomy ($\subset SO(3)$) $\Rightarrow CY_3$
type IIA, IIB on X Kähler ($\mathcal{N}=(2,2)$), Ricci-flat (CFT), 6d $\rightsquigarrow$ A-model $\equiv$ GW theory
B-model

1 2d TQFT

1.0 The most invariant way of constructing QFT$\to S$ is to specify properties of correlation functions.

A theory depends on position of operators and background: topology Σ, metric $h^{\alpha\beta}$, cplx str, orientation

TQFT: 1 Corr. func are indep of metric on Σ ie position of ops.

$\phi_i(z,\bar z)$. $\langle \phi_{i_1} \dots \phi_{i_n} \rangle_{\Sigma_g}$ depends only on labels and g.

2 Corr. func factorize by the insertion of a cplt set of states $|\phi_i\rangle \in \mathcal{H}_{phys}$

$\sum_{ij} |\phi_i\rangle \eta^{ij} \langle \phi_j| = 1$, $\eta^{ij} = (\eta_{ij})^{-1}$, $\eta_{ij} = \langle \phi_i \phi_j \rangle_0$

$\langle \prod_{n=1}^{s} \phi_{i_n} \rangle_\Sigma = \sum_{j,k} \langle \phi_{i_1} \dots \phi_{i_r} \phi_j \rangle_{\Sigma_1} \eta^{jk} \langle \phi_k \phi_{i_{r+1}} \dots \phi_{i_s} \rangle_{\Sigma_2}$ (trivial) | (non-trivial) $= \sum_{j,k} \eta^{jk} (-1)^{F_j} \langle \phi_j \phi_k \dots \rangle_{\Sigma'}$ (, Fermion number of ϕ_j) $\Sigma' = g-1$.

NB: 1 $\Rightarrow T_{\alpha\beta} = \delta S / \delta h^{\alpha\beta}$ decouples from correlation functions.

TQFT is realized by a nilpotent BRST operator Q / $Q^2 = 0$. Q is anticommuting (fermionic)
A TQFT realized by Q is of "cohomological type" or "Witten-type".

props:
- Q fermionic, $Q^2 = 0$.
- Physical operators are Q-closed: $\{Q, \phi_i\} = 0$.
- the Q-symm is not broken $\leadsto$ symm vacuum. $Q|0\rangle = 0$. $\Rightarrow \phi_i \sim \phi_i + \{Q, \Lambda\}$
  - $\to$ physical observables are in the cohomology of Q. $\mathcal{H}_{phys} = \frac{\ker Q}{\operatorname{Im} Q}$.
- $T_{\alpha\beta} = \frac{\delta S}{\delta h^{\alpha\beta}} = \{Q, G_{\alpha\beta}\}$ the energy-momentum tensor is Q-exact, for some operator $G_{\alpha\beta}$.

$\to$ indeed ★ $\langle 0 | \phi_{i_1} \dots \phi_{i_j} \{Q, \Lambda\} \dots \phi_{i_n} | 0 \rangle$, $\{Q,\Lambda\} = Q\Lambda + \Lambda Q$, each term vanishes separately $\langle 0 | \phi_{i_1} \dots Q \Lambda \dots \phi_{i_n} | 0 \rangle = 0$.

this trivially implies $\frac{\delta}{\delta h^{\alpha\beta}} \langle \phi_{i_1} \dots \phi_{i_s} \rangle = i \int D X \prod_{n=1}^{s} \phi_{i_n} \frac{\delta S}{\delta h^{\alpha\beta}} e^{-S}$

$= i \langle \prod \phi_{i_n} \{Q, G_{\alpha\beta}\} \rangle = 0$.

obs: An easy way to ensure prop 4 is ~~least~~ to ask

$S = \int_\Sigma \{Q, V\}$, Q-exact Lagrangian. $\leadsto$ in PI $\exp it \{Q, \int_\Sigma V\}$

$\leadsto \frac{d}{dt} \langle \phi_{i_1} \dots \phi_{i_s} \rangle = 0$ using ★

Correlators are independent of t

$\to$ can be computed exactly in $t \to \infty$ classical limit

Define the operator algebra

def: $C_{ijk} = \langle \phi_i \phi_j \phi_k \rangle_0$

the algebra of phys observables is given by OPE: $\phi_i(z)\phi_j(w) = \lim_{w\to z} \phi_i(z)\phi_j(w) = \sum_k C_{ij}^k \phi_k$, $C_{ij}^k = \eta^{lk} C_{ijl}$

Notice that $\langle \phi_i \phi_j \phi_k \phi_l \rangle_0 = \sum_{m,n} \langle \phi_i \phi_j \phi_m \rangle_0 \eta^{mn} \langle \phi_n \phi_k \phi_l \rangle_0$

$= \sum_m C_{ij}^m C_{mkl} = \sum C_{ik}^m C_{mjl}$.

The operator algebra is a commutative associative ring. $\phi_i = (C_i)^j_k$.

prop Descent equation

$d\phi^{(n)} = \{Q, \phi^{(n+1)}\}$. n-forms . $\rightarrow$ non-local ops : $\oint \phi_i^{(1)}$; $\int_\Sigma \phi_i^{(2)}$

1.1 2d TCFT

We look for special pts (critical) in the space of all TQFTs . $\Rightarrow$ Topological Conformal Field Theories.

NB: metric independence $\Rightarrow$ conformal inv. However, we can still ask for tracelessness of $T_{\alpha\beta}$.

TQFT is TCFT if $G^\alpha_\alpha = 0 \Rightarrow T^\alpha_\alpha = 0$, before restricting to cohomology.

property: in a TCFT Q is split : $Q = Q_L + Q_R$, $Q_L^2 = Q_R^2 = \{Q_L, Q_R\} = 0$.

Charges are integrals of currents

$$Q_L = \oint \frac{dz}{2\pi i} Q(z) \quad \text{holomorphic spin 1 . ; } \quad Q_R = \oint \frac{d\bar z}{2\pi i} \bar Q(\bar z)$$

Construct then $T(z) = \{Q_L, G(z)\}$; $\bar T, \bar G$ $\quad$ T,G spin 2

$Q(z) = -[Q_L, J(z)]$; $\bar J$ $\quad$ J spin 1 $\quad$ Q-exact currents.

Spin ie conformal weight is measured by OPE (Taylor exp of product of operators valued in algebra)

$$T(z)T(w) = \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{regular} \rightsquigarrow \text{spin } 2$$

$$T(z)J(w) = \frac{J(w)}{(z-w)^2} + \frac{\partial J}{z-w} + \text{regular} \rightsquigarrow \text{spin } 1.$$

So, we have two sectors of currents $\{T(z), G(z), Q(z), J(z)\}$ and their anti-holo c.

Expand in modes: $A(z) = \sum_{n\in\mathbb{Z}} A_n z^{-n-h}$, $A_n = \oint \frac{dz}{2\pi i} z^{n+h-1}$, $(A_n)^\dagger = A_{-n}$ hermiticity. $(Q_0 = Q_L)$

Then you can think of spin being measured by L_0 on states. Op-state correspondence

$L_0|\phi_i\rangle = h_i|\phi_i\rangle \rightsquigarrow (h_i, \bar h_i)$ $\quad \left(T\phi = \frac{h_i\phi}{(z-w)^2} + \frac{\partial\phi}{z-w} + \dots\right)$

$J_0|\phi_i\rangle = q_i|\phi_i\rangle \rightsquigarrow (q_i, \bar q_i)$ $\quad 0 \le q_i \le d$.

Now we write the algebra satisfied by the currents : twisted $\mathcal{N}=2$ SCA :

$[L_m, L_n] = (m-n)L_{m+n}$ $\qquad [J_m, J_n] = d\, m\, \delta_{n+m,0}$ $\qquad d \in \mathbb{Z} \cdot \hat c$. (Anomalous U(1))

$[L_m, G_n] = (m-n)G_{m+n}$ $\qquad [J_m, G_n] = -G_{m+n}$ $\qquad$ We have 2 copies $\rightarrow$ twisted $\mathcal{N}=(2,2)$ SCA.

$[L_m, Q_n] = -nQ_{m+n}$ $\qquad [J_m, Q_n] = Q_{m+n}$

$\{G_m, Q_n\} = L_{m+n} + nJ_{m+n} + \frac{1}{2}dm(m+1)\delta_{m+n,0}$; $[L_m, J_n] = -nJ_{m+n} - \frac{1}{2}dm(m+1)\delta_{m+n,0}$, ...

def: primaries : highest weight states : $G_n|\phi_i\rangle = 0$ $(\Rightarrow L_n|\phi_i\rangle = J_n|\phi_i\rangle = 0, n>0)$

Since states are in cohom we can choose a representative / $G_0|\phi_i\rangle = L_0|\phi_i\rangle = 0 \rightsquigarrow (0,0)$ Chiral primaries

We now study physical models that exhibit this structure, namely twisted $\mathcal{N}=(2,2)$ SCFT.

2 Topological Sigma models

Sigma model: QFT of maps $\Phi : M \to X$

2.0 geometry

For us: $M = \Sigma_g$ RS. Coords $z, \bar{z}$; X in principle any almost cplx mfd but to make contact with physics ($\mathcal{N}=(2,2)$ SCFT) we have additional structure. Fix $\dim X = d$.

- $\mathcal{N}=(2,2) \rightarrow$ Kähler.
- CFT $\rightarrow c_1(TX) = 0$.

Start with X ~~cplx~~ real d-fold. Metric g_{IJ}, $I,J = 1..2d$ real

$$\Phi : \Sigma_g \rightarrow X$$

$(z,\bar{z}) \rightarrow (\phi^1(z,\bar{z}) \ldots \phi^d(z,\bar{z}))$ local real coords on X

Let $K, \bar{K}$ be canonical, anti-canonical line bundle on Σ.

Fermions: $\quad \psi_+^I \in \Gamma(K^{1/2} \otimes \Phi^*(TX))$, $\quad \psi_-^I \in \Gamma(\bar{K}^{1/2} \otimes \Phi^*(TX))$

Now introduce complex indices: $i,j = 1 \ldots d$; $\bar{i},\bar{j} = 1 \ldots d$. $\rightarrow \phi^i$; $\phi^{\bar{i}} = \bar{\phi}^i$ (cplx conj)

Kähler: $TX_{\mathbb{C}} = TX_{\mathbb{R}} \otimes \mathbb{C} = T^{1,0}X \oplus T^{0,1}X \equiv TX \oplus \overline{TX}$

Fermions: $\quad \psi_+^i \in \Gamma(K^{1/2} \otimes \Phi^*(TX)) \qquad \psi_-^i \in \Gamma(\bar{K}^{1/2} \otimes \Phi^*(TX))$

$\qquad \psi_+^{\bar{i}} \in \Gamma(K^{1/2} \otimes \Phi^*(\overline{TX})) \qquad \psi_-^{\bar{i}} \in \Gamma(\bar{K}^{1/2} \otimes \Phi^*(\overline{TX}))$

Metric: $\quad g_{IJ}$ must be real: $\left| \begin{array}{l} g_{\bar{i}\bar{j}} = (g_{ij})^* \\ g_{i\bar{j}} = (g_{\bar{i}j})^* \end{array} \right.$

and Hermitian: $g_{ij} = g_{\bar{i}\bar{j}} = 0 \rightarrow$ only $g_{i\bar{j}} = (g_{\bar{i}j})^*$ survives.

Kähler: $g_{i\bar{j}} = 2i\,\partial_i \partial_{\bar{j}} K$, K Kähler potential $\rightsquigarrow$ Kähler metric

Introduce Kähler form: $g = g_{i\bar{j}}\, d\phi^i \wedge d\bar{\phi}^j \in H^{1,1}(X,\mathbb{R})$ / $dg = 0$

Complexified Kähler form: $k = g + iB \in H^{1,1}(X,\mathbb{C})$, $B \in H^{1,1}(X,\mathbb{R})$ B-field.

2.1 $\mathcal{N}=(2,2)$ SCFT on $\Sigma = \mathbb{C}$. (Nonlinear σ-model with target space X) NB: CY: $R_{ij} = 0 \leftrightarrow c_1(TX) = 0$.
$\chi(X) = 2(h^{1,1}(X) - h^{2,1}(X))$

$$S = 2t \int d^2z \left(g_{i\bar{j}}\, \bar{\partial}\phi^i \partial\bar{\phi}^j + g_{i\bar{j}}\, \partial\phi^i \bar{\partial}\bar{\phi}^j + g_{i\bar{j}}\, \psi_-^{\bar{j}} D \psi_-^i + g_{i\bar{j}}\, \psi_+^{\bar{j}} \bar{D} \psi_+^i + R_{i\bar{i}j\bar{j}}\, \psi_+^i \psi_+^{\bar{i}} \psi_-^j \psi_-^{\bar{j}} \right)$$

$$D = \partial + \Gamma^{\bar{i}}_{\bar{j}\bar{k}} \partial\bar{\phi}^j, \quad \bar{D} = \bar{\partial} + \Gamma^i_{jk} \bar{\partial}\phi^j$$

susy: $\quad \delta\phi^i = i\alpha_- \psi_+^i + i\alpha_+ \psi_-^i$, $\quad \alpha_- \in \Gamma(K^{1/2})$, $\alpha_+ \in \Gamma(\bar{K}^{1/2})$

$\vdots$

The algebraic structure encoded in this theory is that of the $\mathcal{N}=(2,2)$ SCA.

two sets of $\mathcal{N}=2$ currents : $\{T, G^-, G^+, J\}$ and their anti-holo counterpart.

h	2	3/2	3/2	1
q	0	-1	$+1$	0

physical states : (anti-) chiral primaries : $G^\pm \phi$ = regular : $h = \frac{1}{2}q$, $L_n|\phi_i\rangle = J_n|\phi_i\rangle = 0 \quad n>0$
$G^\mp_r|\phi_i\rangle = 0, \quad r>0$.

(anti-) chiral : $G^\pm\phi$ = regular. : $h \geq \frac{1}{2}q$

NB : On general grounds $\phi_i(z)\phi_j(w) = \sum_k C_{ij}^k (z-w)^{h_k - h_i - h_j} \chi_k(w)$, χ not chiral-primary.
charge conservation: $q_i + q_j = q_k$, $h_k \geq \frac{q_k}{2}$

$$\Rightarrow h_i + h_j - h_k = \frac{q_i}{2} + \frac{q_j}{2} - h_k = \frac{q_k}{2} - \left|\frac{q_k}{2}\right| \leq 0 .$$

$\Rightarrow$ regular OPE.

Then the restriction to chiral primaries looks like chiral ring of TQFT.
ie there is a 1:1 mapping between chiral primaries and eq classes of operators in TQFT.

2.2 Topological twist : make a TCFT out of $\mathcal{N}=(2,2)$ SCFT

twisted $\mathcal{N}=2$

	T	G	Q	J
h	2	2	1	1
q	0	-1	$+1$	0

$\mathcal{N}=2$

T	G^+	G^-	J
2	3/2	3/2	1
0	1	-1	0

$L_0|\phi\rangle = h|\phi\rangle$
$J_0|\phi\rangle = q|\phi\rangle$

twist $\rightarrow$ SCALAR SUPERCHARGE!

top twist : $L_0 \rightarrow L_0 \pm \frac{1}{2}J_0 \quad (T_{new} = T_{old} \pm \frac{1}{2}\partial J)$

$+$: G^+ (2) $\equiv$ "G", G^- (1) $\equiv$ "Q" — cplx conj in correlation func.

$-$: G^+ (1) $\equiv Q$, G^- (2) G

2 ways to get twisted $\mathcal{N}=2$ from $\mathcal{N}=2$.

However, we have $\mathcal{N}=(2,2)$: $\left| \begin{array}{l} L_0 \rightarrow L_0 \pm \frac{1}{2}J_0 \\ \bar{L}_0 \rightarrow \bar{L}_0 \pm \frac{1}{2}\bar{J}_0 \end{array}\right.$ $\rightarrow$ 4 twists

$$\boxed{\begin{array}{cc} - & - \\ - & + \end{array}} \begin{array}{l} A(X) \\ B(X) \end{array}$$
$$\begin{array}{cc} + & + \\ + & - \end{array}$$

Flip cplx str. (cplx conj)

the top-twisted $\mathcal{N}=(2,2)$ SCFTs, $A(X)$ and $B(X)$ are TCFTs : top sigma models.

2.3 $A(X)$,

$(G^+, \bar{G}^+) = (Q_L, Q_R) \rightarrow$ Fermions : $\psi^i_+ \in \Gamma(\Phi^*(TX)) \equiv \chi^i$, $\psi^i_- \in \Gamma(\bar{K} \otimes \Phi^*(TX)) \equiv \psi^i$
$\psi^{\bar{i}}_+ \in \Gamma(K \otimes \Phi^*(\overline{TX})) \equiv \psi^{\bar{i}}$, $\psi^{\bar{i}}_- \in \Gamma(\Phi^*(\overline{TX})) \equiv \chi^{\bar{i}}$

Currents : $Q(z) = g_{i\bar{j}}\chi^i \partial\phi^{\bar{j}}$ $\quad J(z) = g_{i\bar{j}}\chi^i\psi^{\bar{j}}$
$T(z) = g_{i\bar{j}}\partial\phi^i\partial\phi^{\bar{j}} + g_{i\bar{j}}\partial\psi^{\bar{j}} D\chi^i \quad G(z) = g_{i\bar{j}}\psi^{\bar{j}}\partial\phi^i$

$$S_A = 2t\int d^2z\left(\frac{1}{2}g_{IJ}\partial\phi^I\bar{\partial}\phi^J + i\psi^{\bar{i}} \bar{D}\chi^i g_{\bar{i}i} + i\psi^i D\chi^{\bar{i}} g_{\bar{i}i} - R_{i\bar{i}j\bar{j}}\psi^i\psi^{\bar{i}}\chi^j\chi^{\bar{j}}\right)$$
$$= it\int d^2z \{Q, V\} + t\int \Phi^*(g)$$

$$S_A = it\int_\Sigma d^2z\,\{Q,V\} + t\int_\Sigma \Phi^*(g) \qquad V = g_{i\bar{j}}(\psi^{\bar{j}}\bar{\partial}\phi^i + \partial\phi^{\bar{j}}\psi^i)$$

$$= S_A' + t\int_\Sigma \Phi^*(g) \qquad g = \text{Kähler form.}$$

<u>Obs</u>: - S_A' is exact $\to$ changes in metric decouple

- $\frac{\partial}{\partial t}\langle\dots\rangle = 0 \to$ the theory is independent of t.

- $t\int_\Sigma \Phi^*(g)$ only depends on the Kähler class (we sum over β...)

$$= t\int_{\Phi_A(\Sigma)=\beta} g \qquad \beta\in H_2^0(X,\mathbb{Z})$$

Recall $K = g + iB \in H^2(X,\mathbb{C})$. Say $K = \sum_\alpha t^\alpha K_\alpha$, $t^\alpha = \int_{C^\alpha} K$ Kähler parameters

$\beta = \sum_\alpha n_\alpha C^\alpha$ $\quad n_\alpha$ instanton sectors ($h^{1,1}$ of them)

$$\to t\int_\beta K = t\langle K,\beta\rangle = \sum_\alpha t^\alpha n_\alpha$$

In PI $e^{-t\int_\beta K}$. Then def $q_\alpha = e^{-t^\alpha}$, $Q^\beta = \prod_\alpha q_\alpha^{n_\alpha}$

Now we want to construct local observables to compute correlators

<u>local</u> $\Rightarrow$ we must use scalars ie $\psi^i, \psi^{\bar{j}}, \phi^i$ not allowed (we would need the metric)

Then define $W_A(z,\bar{z}) = A_{i_1\dots i_p,\bar{j}_1\dots\bar{j}_q}(\phi)\,\chi^{i_1}\dots\chi^{i_p}\chi^{\bar{j}_1}\dots\chi^{\bar{j}_q}$

From the algebra one has $\{Q,\phi^i\} = \chi^i$, $\{Q,\chi^i\} = 0 \Rightarrow$ identify $\chi^i = d\phi^i$

$\Rightarrow W_A \in H^{p,q}(X)$. Local operators of the A-model are 1:1 with cohomology classes of X.

Moreover, $\{Q, W_A\} = W_{dA} \Rightarrow$ We find <u>$Q = d$</u> de Rham operator

From using the relations of the algebra/explicit expressions for the currents we get

$$Q_L \longleftrightarrow \partial$$
$$G_0 \longleftrightarrow \partial^\dagger$$
$$J_0 \longleftrightarrow \deg \quad (W_A \text{ has charge } p+q)$$

Finally, $Q = Q_L + Q_R = \partial + \bar{\partial} = d$.

Note that $L_0 = \{Q, G_0\} = \partial\partial^\dagger + \partial^\dagger\partial = \Delta_\partial \quad (\Delta = 2\Delta_\partial = 2\Delta_{\bar{\partial}})$

This implies $\quad$ Q-exact operators : Q-exact forms $A = \partial B$.

physical chiral operators : $Q_L|\phi_i\rangle = 0$ are closed forms $\partial A = 0$.

and chiral primaries: $G_0|\phi\rangle = L_0|\phi\rangle = Q_L|\phi\rangle = 0 \rightarrow \left| \begin{array}{l} \partial A = 0 \\ \partial^\dagger A = 0 \end{array} \right.$ harmonic forms.

The Hilbert space of the A-model is isomorphic to the de Rham cohomology. $\mathcal{H}_{phys} \simeq H_d^A$

Now we can compute correlation functions.

<u>Recall</u>: $\frac{\partial}{\partial t}\langle \ldots \rangle = 0 \Rightarrow$ we can evaluate correlation functions in the weak coupling limit, where $t \to \infty$, $\mathrm{Re}\, t > 0$.

$$\langle \prod W_{A_i} \rangle = \sum_\beta \langle \prod W_{A_i} \rangle_\beta , \qquad \langle \prod W_{A_i} \rangle_\beta = Q^{t\beta} \int_{M_\beta} D\phi D\chi D\psi\, e^{-it\{Q, \int_\Sigma V\}} \prod W_{A_i}$$

M_β is the component of field config space for maps Φ of homology class β.

Take $t \to \infty$, $\mathrm{Re}\, t > 0 \rightsquigarrow$ saddle point expansion around classical config:

Classical config minimizes the action: $V = 0 \Rightarrow \underline{\bar\partial \phi^i = \partial \phi^{\bar\imath} = 0}$ holomorphic maps ie WS instantons.

After proper treatment of 1-loop determinants $(= +1)$:

$$\int_{M_\beta} \xrightarrow{t \to \infty} \int_{\mathcal{M}_\beta}$$ $\mathcal{M}_\beta$ is the moduli space of holomorphic maps of degree β.

Now, we understand how to compute, we need to make sure $\langle \ldots \rangle \neq 0$. How many insertions?

	ϕ	χ	ψ	Q
q	0	1	-1	0

Charge conservation!

<u>def</u>: $a_\beta = \#$ of χ zero modes ie dim space of solutions of $\bar D\chi^i = D\chi^{\bar\imath} = 0$

$b_\beta = \#$ of ψ zero modes.

$w_\beta = a_\beta - b_\beta$ is a topological invariant. $= \mathrm{ind}\, D$ "twisted Dirac operator" computed by Riemann-Roch.

turns out $= 2d(1-g) + \int_\Sigma \cancel{\Phi^*(C_1(X))}$ CY and is independent of β.

By looking at the Lagrangian one shows that $D\chi = 0$ is the linearization of the instanton eq $\bar\partial\phi = 0$ $\rightsquigarrow$ the space of zero modes is the space of deformations of the embedding of Σ in X ie $T\mathcal{M}_\beta$.

Heuristically, this is true because of $\chi^i = d\phi^i$

$\Rightarrow w_\beta = \mathrm{vir\,dim}\, \mathcal{M}_\beta$, $a_\beta = \dim_{\mathbb{R}} T\mathcal{M}_\beta$

In sufficiently generic situation $b_\beta = 0$, $w_\beta = a_\beta$. In general as long as a_β is constant then also b_β is constant $\rightsquigarrow$ the space of ψ zero modes varies as the fibers of a vector bundle V on $\mathcal{M}_\beta$.

thus we have selection rule

$\sum_i \deg(W_{A_i}) = \sum_i (p_i+q_i) = 2d(1-g)$ to have non-vanishing correlator.

Consider now H cycle and $A = PD(H)$ / W_A has δ-function support on H.

Clearly $(p+q) = \mathrm{Codim}\, H$. Pick points $P_i \in \Sigma$, and require $\Phi(P_i) \in H_i$ ~ constraints

$$\langle W_{A_1}(P_1) \dots W_{A_s}(P_s)\rangle_\beta = Q^{t\beta} \int_{M_\beta} D\phi\, D\psi\, D\chi\, e^{it\{Q, \int V\}} \prod W_{A_i}(P_i)$$

$$\xrightarrow{t\to\infty} Q^{t\beta} \int_{\mathcal{M}_\beta} ev_1^*(A_1)\wedge \dots \wedge ev_s^*(A_s) \quad , \quad ev_i : \mathcal{M}_\beta \to X, \quad \beta \to W_\beta(P)$$

the requirement $\Phi(P_i) \in H_i$ imposes $\sum_i (p_i+q_i)$ constraints

But $\dim \mathcal{M}_\beta$ (= virdim here) $= 2d(1-g)$ and the selection rule tells us $\sum_i (p_i+q_i) = 2d(1-g)$

$\Rightarrow$ we are integrating δ functions over a 0-dim space $\Rightarrow$ counting points

$$= Q^{t\beta} \# \mathcal{M}_\beta .$$

Life is not always generic, it can be argued $\#\mathcal{M}_\beta \rightsquigarrow \int_{\mathcal{M}_\beta} \chi(V)$. <u>NB</u>: true for all d.
but for $g>1$ there will be no maps since virdim < 0.

Let us present one particular result on $CY_3 \to \underline{d=3}$, genus 0.

<u>def</u> $\phi_A = A_{i\bar{j}} \chi^i \chi^{\bar{j}}$. Selection rule $\Rightarrow$ 3 insertions as $3 \cdot 2 = 6(1-0)$.

$$\langle \phi_{A_1}^{(P_1)} \phi_{A_2} \phi_{A_3} \rangle = \langle \phi_{A_1}\phi_{A_2}\phi_{A_3}\rangle_0 + \sum_\beta \langle \phi_{A_1}\phi_{A_2}\phi_{A_3}\rangle_\beta$$

(hiding points)

$\beta = 0$: the image of the sphere is just a point $\to$ Kähler volume

$$= K_{ABC} + \sum_\beta Q^{t\beta} N_{0,\beta} \prod \int_\beta A_i$$

where $K_{ABC} = \int_X A_1 \wedge A_2 \wedge A_3$ since $\mathcal{M}_0 = X \to$ classical intersection number

$N_{0,\beta}$ are rational numbers that count holomorphic maps of degree $\beta > 0 \rightsquigarrow$ GW

Notice that $\langle \phi_{A_1}\phi_{A_2}\phi_{A_3}\rangle = C_{123}$ chiral ring of TQFT ~ quantum cohomology ring

It is convenient to put this info (Yukawa coupling) into a generating function

$$F_0(t) = -\frac{1}{3!} K_{ABC}\, t^{A_1} t^{A_2} t^{A_3} + \sum_\beta N_{0,\beta} Q^{t\beta} \quad \text{prepotential}$$

prop: $$\frac{\partial^3 F_0}{\partial t^{A_1}\partial t^{A_2}\partial t^{A_3}} = C_{123} .$$

2.4 Results of top string

Coupling gravity → higher genus corrections

$$F_1 = \frac{1}{2}\int \frac{d^2\tau}{\tau}\, \mathrm{tr}\left((-1)^{J_0+\bar J_0} J_0 \bar J_0\, q^{L_0} \bar q^{\bar L_0}\right) \quad q = e^{2\pi i \tau} \quad \text{torus (Cecotti-Vafa)}$$

<u>NB</u>: For $d=3$ the selection rule is $\sum_i \deg(W_{A_i}) = 6g-6$. Just like bosonic ST.

→ one construct topological string amplitudes in formal analogy to bosonic strings

Since $T = \{Q_0, G\}$, after twisting it's obvious to identify G^- with the b-ghost.

$$\rightarrow \quad F_g = \int_{\bar{\mathcal{M}}_g} \left\langle \prod_{k=1}^{3g-3} (G^-, \mu_k)(\bar G^-, \bar\mu_k) \right\rangle, \quad (G,\mu) = \int_\Sigma G\mu \quad g>1.$$

As there are $3g-3$ Beltrami's (and $3g-3$ $\bar\mu$), we find the selection rule

$(6-2d)(1-g) = 0 \rightsquigarrow CY_3$ is special.

$$= c_g\, \chi(X) + \sum_\beta N_{g,\beta} Q^{t\beta}, \quad c_g = \frac{(-1)^{g+1} B_{2g} B_{2g-2}}{4g(2g-2)(2g-2)!}$$