PROBLEM SET 1
To be discussed: 25.10.2023

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next week's lectures, as the problems will often serve as mental preparation for the lecture material. Solutions will be discussed in the Übung.

1. Consider categories $\mathsf{Ab}_{\mathbb{Z}}$ and Chain, defined as follows:

   - Objects of $\mathsf{Ab}_{\mathbb{Z}}$ are $\mathbb{Z}$-graded abelian groups $G_* = \bigoplus_{n\in\mathbb{Z}} G_n$, and morphisms from G_* to H_* are group homomorphisms $\Phi : G_* \to H_*$ satisfying $\Phi(G_n) \subset H_n$ for every $n \in \mathbb{Z}$.
   - Objects of Chain are chain complexes (C_*, ∂), meaning $\mathbb{Z}$-graded abelian groups $C_* = \bigoplus_{n\in\mathbb{Z}} C_n$ endowed with homomorphisms $\partial : C_* \to C_*$ that satisfy $\partial(C_n) \subset C_{n-1}$ for each $n \in \mathbb{Z}$ and $\partial^2 = 0$. Morphisms from (A_*, ∂_A) to (B_*, ∂_B) are chain maps, meaning homomorphisms $\Phi : A_* \to B_*$ with $\Phi(A_n) \subset B_n$ for each $n \in \mathbb{Z}$ and $\Phi \circ \partial_A = \partial_B \circ \Phi$.

   Recall that the homology of a chain complex (C_*, ∂) is defined in general as the graded abelian group $H_*(C_*, \partial) = \bigoplus_{n\in\mathbb{Z}} H_n(C_*, \partial)$ where $H_n(C_*, \partial) = \ker \partial_n \big/ \operatorname{im} \partial_{n+1}$, with the restriction of $\partial : C_* \to C_*$ to $C_n \to C_{n-1}$ denoted by ∂_n.

   (a) Show that H_* defines a functor from Chain to $\mathsf{Ab}_{\mathbb{Z}}$ in a natural way. How does this functor act on morphisms of Chain?

   (b) Recall that for two chain maps Φ, Ψ from (A_*, ∂_A) to (B_*, ∂_B), a **chain homotopy** from Φ to Ψ is a homomorphism $h : A_* \to B_*$ satisfying $h(A_n) \subset B_{n+1}$ for all n and

   $$\partial_B \circ h + h \circ \partial_A = \Psi - \Phi.$$

   Show that the existence of chain homotopies defines an equivalence relation on the set of chain maps. (The resulting equivalence classes are called **chain homotopy classes**, and we say Φ and Ψ are **chain homotopic** if there exists a chain homotopy between them.)

   (c) We can now define Chain^h as the category whose objects are the same as in Chain, but with morphisms defined as chain homotopy classes of chain maps. Show that H_* also defines a functor from Chain^h to $\mathsf{Ab}_{\mathbb{Z}}$.

2. One can speak of "functors of multiple variables" in much the same way as with functions. Show for instance that on the category Ab of abelian groups and homomorphisms,

   $$\operatorname{Hom} : \mathsf{Ab} \times \mathsf{Ab} \to \mathsf{Ab}$$

   defines a functor that is contravariant in the first variable and covariant in the second, assigning to each pair of abelian groups (G, H) the group $\operatorname{Hom}(G, H)$ of homomorphisms $G \to H$.

3. For a pointed space (X, p), recall that the *Hurewicz homomorphism*[1]

   $$h : \pi_1(X, p) \to H_1(X; \mathbb{Z})$$

   sends each element $[\gamma] \in \pi_1(X, p)$ represented by a path $\gamma : I \to X$ with $\gamma(0) = \gamma(1) = p$ to the homology class represented by the singular 1-cycle $\gamma : \Delta^1 \to X$, defined by identifying the unit interval $I := [0, 1]$ with the standard 1-simplex $\Delta^1 := \{(t_0, t_1) \in I^2 \mid t_0 + t_1 = 1\}$. Let Top_* denote the category of pointed spaces with base-point preserving continuous maps, so that we can regard both π_1 and $H_1(\cdot; \mathbb{Z})$ as functors from Top_* to the category Grp of groups with homomorphisms. (Note that the base point is irrelevant for the definition of $H_1(\cdot; \mathbb{Z})$, which actually takes values in the smaller subcategory of *abelian* groups, but these details are unimportant for now.) In this context, show that the Hurewicz homomorphism defines a natural transformation from π_1 to $H_1(\cdot; \mathbb{Z})$.

[1] See Problem Set 9 #3 from last semester's *Topologie I* class, which will be discussed in the Übung on 18.10.2023.

4. Suppose $\mathscr{A}$ is a category whose objects form a set X, such that for each pair $x, y \in X$, the set of morphisms $\mathrm{Mor}(x, y)$ contains either exactly one element or none. We can turn this into a binary relation by writing $x \bowtie y$ for every pair such that $\mathrm{Mor}(x, y) \neq \varnothing$.

   (a) What properties does the relation $\bowtie$ need to have in order for it to define a category in the way indicated above?

   (b) If $\mathscr{B}$ is another category whose objects form a set Y with morphisms determined by a binary relation $\bowtie$ as indicated above, what properties does a map $f : X \to Y$ need to have in order for it to define a functor from $\mathscr{A}$ to $\mathscr{B}$?

5. In any category $\mathscr{C}$, each object X has an **automorphism group** (also called **isotropy** group) $\mathrm{Aut}(X)$, consisting of all the isomorphisms in $\mathrm{Mor}(X, X)$. A **groupoid** is a category in which all morphisms are also isomorphisms.

   (a) Show that if $\mathscr{G}$ is a groupoid and Grp denotes the usual category of groups with homomorphisms, there exists a contravariant functor from $\mathscr{G}$ to Grp that assigns to each object X of $\mathscr{G}$ its automorphism group $\mathrm{Aut}(X)$. How does this functor act on morphisms $X \to Y$? Could you alternatively define it as a *covariant* functor? Conclude either way that whenever X and Y are isomorphic objects in $\mathscr{G}$ (meaning there exists an isomorphism in $\mathrm{Mor}(X, Y)$), the groups $\mathrm{Aut}(X)$ and $\mathrm{Aut}(Y)$ are isomorphic.

   (b) Given a topological space X and two points x, y, let $\mathrm{Mor}(x, y)$ denote the set of homotopy classes (with fixed end points) of paths $I := [0, 1] \to X$ from x to y, and define a composition function $\mathrm{Mor}(x, y) \times \mathrm{Mor}(y, z) \to \mathrm{Mor}(x, z) : (\alpha, \beta) \mapsto \alpha \cdot \beta$ by the usual notion of concatenation of paths. Show that this notion of morphisms defines a groupoid whose objects are the points in X.[2] In this case, what are the automorphism groups $\mathrm{Aut}(x)$ and the isomorphisms $\mathrm{Aut}(y) \to \mathrm{Aut}(x)$ given by the functor in part (a)?

6. For a fixed field $\mathbb{K}$, let $\mathsf{Vec}_{\mathbb{K}}$ denote the category of finite-dimensional vector spaces over $\mathbb{K}$ with $\mathbb{K}$-linear maps as morphisms.

   (a) Show that there is a covariant functor $\boldsymbol{\Delta}^2$ from $\mathsf{Vec}_{\mathbb{K}}$ to itself, assigning to each $V \in \mathsf{Vec}_{\mathbb{K}}$ the dual of its dual space $(V^*)^*$. Describe how this functor acts on morphisms.

   (b) Let Id denote the identity functor on $\mathsf{Vec}_{\mathbb{K}}$, which sends each object and morphism to itself. Construct a natural transformation from Id to $\boldsymbol{\Delta}^2$ that assigns to every $V \in \mathsf{Vec}_{\mathbb{K}}$ a vector space isomorphism $V \to (V^*)^*$.

   (c) Every complex vector space $V \in \mathsf{Vec}_{\mathbb{C}}$ has a **conjugate** space $\bar{V} \in \mathsf{Vec}_{\mathbb{C}}$, defined as the same set with the same notion of vector addition but with scalar multiplication conjugated: in other words, if for each $v \in V$ we denote the same element in $\bar{V}$ by $\bar{v}$, then scalar multiplication on $\bar{V}$ is defined for $\lambda \in \mathbb{C}$ by
   $$\lambda \bar{v} := \overline{\bar{\lambda} v}.$$
   Show that there is a covariant functor $\mathsf{Vec}_{\mathbb{C}} \to \mathsf{Vec}_{\mathbb{C}}$ sending each complex vector space to its conjugate, and describe how it acts on morphisms.

   (d) Notice that for $V \in \mathsf{Vec}_{\mathbb{C}}$, the map $V \to \bar{V} : v \mapsto \bar{v}$ is not a morphism in $\mathsf{Vec}_{\mathbb{C}}$, as it is complex *antilinear*. Of course V and $\bar{V}$ are both complex vector spaces of the same dimension, so they are *always* isomorphic. Prove however that—in contrast to the case of the double dual in part (b)—there exists no natural transformation from the identity to the conjugation functor that provides a complex-linear isomorphism $V \to \bar{V}$ for every $V \in \mathsf{Vec}_{\mathbb{C}}$.
   Hint: If such a natural transformation exists, what will it imply about the specific morphism $V \to V : v \mapsto iv$?

[2] It is called the **fundamental groupoid** of X.