PROBLEM SET 7
To be discussed: 6.12.2023

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next week's lectures, as the problems will often serve as mental preparation for the lecture material. Solutions will be discussed in the Übung.

1. Each of the following spaces can be defined as a direct limit in terms of the natural inclusions $\mathbb{F}^m \hookrightarrow \mathbb{F}^n$ for $n \geqslant m$, where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$, and we identify $\mathbb{F}^m$ with the subspace $\mathbb{F}^m \oplus \{0\} \subset \mathbb{F}^n$. In particular, $\mathbb{R}^{m+1} \hookrightarrow \mathbb{R}^{n+1}$ gives rise to inclusions $S^m \hookrightarrow S^n$ and $\mathbb{RP}^m \hookrightarrow \mathbb{RP}^n$, and the complex version gives $\mathbb{CP}^m \hookrightarrow \mathbb{CP}^n$. Use cell decompositions to compute the homology with integer coefficients for each space:
   (a) $S^\infty = \varinjlim\{S^n\}_{n\in\mathbb{N}}$
   (b) $\mathbb{RP}^\infty = \varinjlim\{\mathbb{RP}^n\}_{n\in\mathbb{N}}$
   (c) $\mathbb{CP}^\infty = \varinjlim\{\mathbb{CP}^n\}_{n\in\mathbb{N}}$

2. Suppose $\{X_\alpha, \varphi_{\beta\alpha}\}$ is a direct system in a category $\mathscr{C}$ over a directed set $(I, \prec)$, with the property that for some $\alpha_0 \in I$, $\varphi_{\gamma\beta} \in \operatorname{Mor}(X_\beta, X_\gamma)$ is an isomorphism for every $\beta, \gamma \in I$ with $\beta \succ \alpha_0$ and $\gamma \succ \alpha_0$. For each $\alpha \in I$, choose $\gamma \in I$ such that $\gamma \succ \alpha$ and $\gamma \succ \alpha_0$, and define
$$\varphi_\alpha := \varphi_{\gamma\alpha_0}^{-1} \circ \varphi_{\gamma\alpha} \in \operatorname{Mor}(X_\alpha, X_{\alpha_0}).$$
   (a) Prove that the morphism φ_α does not depend on the choice of the element $\gamma \in I$.
   (b) Prove that $\{X_{\alpha_0}, \varphi_\alpha\}$ is a target of the system.
   (c) Prove that $\{X_{\alpha_0}, \varphi_\alpha\}$ also satisfies the universal property in the definition of a direct limit, hence $X_{\alpha_0} = \varinjlim\{X_\alpha\}$.

3. Suppose $\{X_\alpha, \varphi_{\beta\alpha}\}$ is a direct system of topological spaces such that each X_α is a subspace of some fixed topological space X, $\beta \succ \alpha$ if and only if $X_\alpha \subset X_\beta$, and the maps $\varphi_{\beta\alpha} : X_\alpha \to X_\beta$ in this case are the natural inclusions. Identify $\varinjlim\{X_\alpha\}$ with $\coprod_\alpha X_\alpha \big/ \sim$, using the equivalence relation
$$X_\alpha \ni x \sim y \in X_\beta \qquad \Leftrightarrow \qquad \varphi_{\gamma\alpha}(x) = \varphi_{\gamma\beta}(y) \text{ for some } \gamma \in I \text{ with } \gamma \succ \alpha, \gamma \succ \beta.$$
   The disjoint union of the inclusions $X_\alpha \hookrightarrow \bigcup_{\beta\in I} X_\beta$ then descends to the quotient as a bijection
$$\varinjlim\{X_\alpha\} \to \bigcup_{\alpha\in I} X_\alpha,$$
   but the following example shows that this need not be a homeomorphism in general. Let $I = (0,1)$ and consider the family of sets $X_t = \{0\} \cup (t, 1] \subset \mathbb{R}$ for $t \in I$, ordered by inclusion. The union of these sets is $[0,1]$, but show that the topological space $\varinjlim\{X_t\}$ is not connected.

4. Let $\{X_\alpha\}_{\alpha\in I}$ denote the family of all countable subsets of S^1, ordered by inclusion as in Problem 3.
   (a) Show that this forms a direct system of topological spaces whose direct limit is S^1, with its usual topology.
   (b) Show that $H_*(\varinjlim\{X_\alpha\}; \mathbb{Z}) \not\cong \varinjlim\{H_*(X_\alpha; \mathbb{Z})\}$.

5. Direct limits are a special case of a more general notion in category theory called *colimits*. In order to express the definition, recall (cf. Problem Set 1 #4) that every pre-ordered set $(I, \prec)$ can be encoded as a category $\mathscr{I}$ in which I is the set of objects, while the set of morphisms $\alpha \to \beta$ has exactly one element whenever $\alpha \prec \beta$, and is otherwise empty. From this perspective, a direct system $\{X_\alpha, \varphi_{\beta\alpha}\}$ over $(I, \prec)$ in the category $\mathscr{C}$ is the same thing as a (covariant) functor $\mathcal{F} : \mathscr{I} \to \mathscr{C}$ that sends each object $\alpha \in I$ to the object X_α and each morphism $\alpha \to \beta$ of $\mathscr{I}$ to the morphism $\varphi_{\beta\alpha} : X_\alpha \to X_\beta$. To define targets and direct limits in this language, one can identify each object Y of $\mathscr{C}$ with the "constant" functor $\mathcal{Y} : \mathscr{I} \to \mathscr{C}$ that sends every $\alpha \in I$ to Y and every morphism $\alpha \to \beta$ of $\mathscr{I}$ to the identity morphism $Y \to Y$. Note that if $\mathcal{X}, \mathcal{Y} : \mathscr{I} \to \mathscr{C}$ are two such contant functors associated to objects X, Y respectively in $\mathscr{C}$, then a natural transformation from $\mathcal{X}$ to $\mathcal{Y}$ must associate to every $\alpha \in I$ the same morphism $X \to Y$, and conversely, every morphism $X \to Y$ determines a natural transformation $\mathcal{X} \to \mathcal{Y}$. A target $\{Y, f_\alpha\}$ of the system $\{X_\alpha, \varphi_{\beta\alpha}\}$ is now the same thing as a natural transformation $T_Y : \mathcal{F} \to \mathcal{Y}$, assigning to each object α of $\mathscr{I}$ the morphism $T_Y(\alpha) := f_\alpha : X_\alpha \to Y$, and in this language, a target $T_X : \mathcal{F} \to \mathcal{X}$ is called universal (and is thus a direct limit of the system) if for every target $T_Y : \mathcal{F} \to \mathcal{Y}$, there is a unique natural transformation $\Phi : \mathcal{X} \to \mathcal{Y}$ such that $T_Y = \Phi \circ T_X$.

   Having expressed the definition of a direct limit in this form, the whole discussion still makes sense if one replaces the category $\mathscr{I}$ associated with the directed set $(I, \prec)$ by an arbitrary[1] category $\mathscr{A}$. For reasons that may become clearer when you look at the examples below, a functor $\mathcal{F} : \mathscr{A} \to \mathscr{C}$ is then often referred to as a *diagram* in $\mathscr{C}$ over $\mathscr{A}$. A target is again simply a natural transformation $T_X : \mathcal{F} \to \mathcal{X}$ to the constant functor $\mathcal{X} : \mathscr{A} \to \mathscr{C}$ determined by some object X of $\mathscr{C}$, and it is called a **colimit** of the diagram if it satisfies the universal property described above. In this case one often writes

   $$X = \operatorname{colim} \mathcal{F},$$

   though it is important to keep in mind that the colimit consists of not just the object X but also the morphisms $T_X(\alpha) : \mathcal{F}(\alpha) \to X$ associated to each object α of $\mathscr{A}$. As with direct limits, colimits are not guaranteed to exist, and they are also not generally unique, but the universal property guarantees that they are unique up to canonical isomorphisms whenever they exist.

   (a) If $\mathscr{A}$ is a category whose objects form a set J and whose morphisms consist of only the identity morphism on each object, then a diagram $\mathscr{A} \to \mathscr{C}$ is simply a collection $\{X_\alpha\}_{\alpha \in J}$ of objects in $\mathscr{C}$, and a colimit of such a diagram is called a **coproduct** of the collection. Flesh out the details of the following statement: coproducts in the categories Top and $\mathsf{Top}_{\text{rel}}$ are disjoint unions, and coproducts in Ab, $\mathsf{Ab}_{\mathbb{Z}}$ and Chain are direct sums.

   (b) Give a concrete description of coproducts in the categories Top_* (pointed spaces) and Grp (not necessarily abelian groups).
   Hint: Both answers are constructions that were introduced in Topologie I.

   (c) If $\mathscr{A}$ contains only two objects α, β and its morphisms consist only of the identity morphisms on α, β plus exactly two morphisms $\alpha \to \beta$, then a diagram $\mathcal{F} : \mathscr{A} \to \mathscr{C}$ can be described as a pair of objects in $\mathscr{C}$ with a pair of morphisms

   $$X \overset{f}{\underset{g}{\rightrightarrows}} Y,$$

   and a colimit of such a diagram is called a **coequalizer**. Give an explicit description of coequalizers in the categories Top and Ab.
   Hint: Use quotients.

   (d) Prove: If $\mathscr{C}$ is a category in which coproducts and coequalizers always exist, then every direct system in $\mathscr{C}$ has a direct limit.
   Hint: Special cases of this yield the explicit descriptions of direct limits in Top *and* Ab *that we saw in lecture.*

[1] I say "arbitrary," but in practice, $\mathscr{A}$ is almost always taken to be a *small* category, meaning that its objects form an honest set, rather than a proper class. In many important special cases, $\mathscr{A}$ contains only finitely many objects, and there are already interesting examples (as in Problem 5(c)) in which it has only two.