PROBLEM SET 9
To be discussed: 20.12.2023

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next week's lectures, as the problems will often serve as mental preparation for the lecture material. Solutions will be discussed in the Übung.

1. Prove that for an additive functor $\mathcal{F} : \mathsf{Mod}^R \to \mathsf{Mod}^R$ on the category Mod^R of modules over a fixed commutative ring R, the following conditions are equivalent:
   (i) $\mathcal{F}$ is exact.
   (ii) For every exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of R-modules, the sequence $\mathcal{F}(A) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(B) \xrightarrow{\mathcal{F}(g)} \mathcal{F}(C)$ is also exact.
   (iii) For every long exact sequence $\ldots \to A_{n-1} \to A_n \to A_{n+1} \to \ldots$ of R-modules, the induced sequence $\ldots \to \mathcal{F}(A_{n-1}) \to \mathcal{F}(A_n) \to \mathcal{F}(A_{n+1}) \to \ldots$ is also exact.

2. The following exercise develops several of the standard tools that are needed for computing the Tor functor. Assume throughout that the objects A, B, G, H are modules over a fixed commutative ring R unless stated otherwise. Prove:
   (a) If A and B are projective, then so is $A \oplus B$.
   (b) There are natural isomorphism $\operatorname{Tor}_n(A \oplus B, G) \cong \operatorname{Tor}_n(A, G) \oplus \operatorname{Tor}_n(B, G)$ for every $n \geqslant 0$.
   (c) There are also natural isomorphisms $\operatorname{Tor}_n(A, G \oplus H) \cong \operatorname{Tor}_n(A, G) \oplus \operatorname{Tor}_n(A, H)$.
   Note: You could deduce this from part (b) and the natural isomorphisms $\operatorname{Tor}_n(A, B) \cong \operatorname{Tor}_n(B, A)$, *but proving symmetry is harder than giving a direct proof of the direct sum property stated here.*
   (d) If $k \in \mathbb{N}$ has the property that multiplication by k defines an injective map $R \to R$, then $\operatorname{Tor}_n(R/kR, G) = 0$ for every $n \geqslant 2$, and $\operatorname{Tor}(R/kR, G)$ is isomorphic to the kernel of the map $G \to G$ defined via multiplication by k.
   (e) Every torsion-free abelian group G is a flat $\mathbb{Z}$-module, i.e. the functor $\otimes G : \mathsf{Ab} \to \mathsf{Ab}$ is exact.
   Hint: If $i : A \hookrightarrow B$ is an injective homomorphism, show that any nontrivial element in the kernel of $i \otimes \mathbb{1} : A \otimes G \to B \otimes G$ also belongs to the kernel of the restriction of this map to $A \otimes G_0 \to B \otimes G_0$ for some finitely-generated subgroup $G_0 \subset G$. If G is torsion-free, what does the classification of finitely-generated abelian groups tell you about G_0?
   Remark: It follows from this exercise that $\operatorname{Tor}^{\mathbb{Z}}(A, G)$ *vanishes whenever G is torsion-free. In light of symmetry, it therefore also vanishes whenever A is torsion-free.*
   (f) If $T(G) \subset G$ denotes the torsion subgroup of G, then the map $\operatorname{Tor}^{\mathbb{Z}}(A, T(G)) \to \operatorname{Tor}^{\mathbb{Z}}(A, G)$ induced by the inclusion $T(G) \hookrightarrow G$ and the functoriality of $\operatorname{Tor}^{\mathbb{Z}}(A, \cdot)$ is an isomorphism.
   Hint: If all else fails, try an exact sequence.

   Comment: Parts (b) and (d) generalize easily from $\operatorname{Tor}_n(\cdot, G)$ *to the left derived functor $L_n\mathcal{F}$ of any right-exact functor $\mathcal{F}$.*

3. The following convenient fact was used prematurely when we discussed the Euler characteristic, but you are now in a position to prove it: for any chain complex C_* of free abelian groups and any field $\mathbb{K}$ of characteristic 0, the canonical map $H_*(C_*) \otimes \mathbb{K} \to H_*(C_* \otimes \mathbb{K})$ is an isomorphism. Where does your proof fail if $\mathbb{K}$ has finite characteristic? What happens, for instance, if $\mathbb{K} = \mathbb{Z}_2$ and $C_* = C_*(\mathbb{RP}^2; \mathbb{Z})$?

4. (a) Prove that for any space X and any abelian group G, $H_1(X; G) \cong H_1(X; \mathbb{Z}) \otimes G$.
   (b) Prove that if X is path-connected and has finite fundamental group, then $H_1(X; \mathbb{Q})$ and $H_1(X, \mathbb{R})$ are trivial.

5. Suppose C_* is a chain complex of free abelian groups such that for some $n \in \mathbb{N}$, $H_n(C_*)$ is finitely generated and satisfies
$$H_n(C_* \otimes \mathbb{Z}_p) \cong H_n(C_*) \otimes \mathbb{Z}_p$$
for every prime $p \in \mathbb{N}$. Prove that $H_{n-1}(C_*)$ is torsion-free.
Remark: We will later show that the hypothesis of this result is satisfied by the singular chain complex $C_*(M;\mathbb{Z})$ of every orientable topological n-manifold M, thus implying that $H_{n-1}(M;\mathbb{Z})$ cannot have torsion. This result is used as one step in the proof that closed simply-connected 3-manifolds must be homotopy equivalent to S^3, an important fact in the background of the Poincaré conjecture.

6. Suppose (C_*, d) and (C^*, ∂) are chain complexes with $C_{-1} = C^{-1} =: C$ which form row -1 and column -1 respectively of a double complex $\{C^i_j\}_{i,j\in\mathbb{Z}}$ as shown in the following diagram:

$$\begin{tikzcd}
\ddots & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \\
\cdots \arrow[r] & C^2_2 \arrow[r, "d^2_2"] \arrow[d, "\partial^2_2"] & C^2_1 \arrow[r, "d^2_1"] \arrow[d, "\partial^2_1"] & C^2_0 \arrow[r, "d^2_0"] \arrow[d, "\partial^2_0"] & C^2 \arrow[r] \arrow[d, "\partial^2"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C^1_2 \arrow[r, "d^1_2"] \arrow[d, "\partial^1_2"] & C^1_1 \arrow[r, "d^1_1"] \arrow[d, "\partial^1_1"] & C^1_0 \arrow[r, "d^1_0"] \arrow[d, "\partial^1_0"] & C^1 \arrow[r] \arrow[d, "\partial^1"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C^0_2 \arrow[r, "d^0_2"] \arrow[d, "\partial^0_2"] & C^0_1 \arrow[r, "d^0_1"] \arrow[d, "\partial^0_1"] & C^0_0 \arrow[r, "d^0_0"] \arrow[d, "\partial^0_0"] & C^0 \arrow[r] \arrow[d, "\partial^0"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C_2 \arrow[r, "d_2"] \arrow[d] & C_1 \arrow[r, "d_1"] \arrow[d] & C_0 \arrow[r, "d_0"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{tikzcd}$$

Here we assume $C^i_j = 0$ whenever $i < -1$ or $j < -1$, and that all rows and columns of the diagram other than row -1 and column -1 are exact sequences. Find an isomorphism
$$H_n(C^*, \partial) \cong H_n(C_*, d)$$
for every $n \in \mathbb{Z}$. Then decide what it should mean to call this isomorphism *natural*, and convince yourself that it is.