



Problem Set 10

To be discussed: Thursday, 8.01.2026

Problems marked with (*) should be considered essential, but it is highly recommended that you think through *all* of the problems before the next Thursday lecture.

Problem 1

Show that every function $f \in L^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$ represents a tempered distribution $\Lambda_f \in \mathcal{S}'(\mathbb{R}^n)$ defined by $\Lambda_f(\varphi) := \int_{\mathbb{R}^n} f\varphi \, dm$, and the resulting inclusion $L^p(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ is continuous.

Hint: Use the continuity of the inclusions $\mathcal{S}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ and the natural injection $L^p \hookrightarrow (L^q)^*$ for $\frac{1}{p} + \frac{1}{q} = 1$.

Problem 2

Recall that the support $\text{supp}(\varphi) \subset \Omega$ of a function $\varphi : \Omega \rightarrow V$ defined on some open subset $\Omega \subset \mathbb{R}^n$ is the closure of the set of points $x \in \Omega$ at which $\varphi(x) \neq 0$. The support $\text{supp}(\Lambda)$ of a distribution $\Lambda \in \mathcal{D}'(\Omega)$ is defined as the intersection of all closed subsets $\mathcal{U} \subset \Omega$ such that

$$\varphi \in \mathcal{D}(\Omega) \text{ with } \text{supp}(\varphi) \cap \mathcal{U} = \emptyset \quad \Rightarrow \quad \Lambda(\varphi) = 0.$$

- (a) Show that for a locally integrable function $f \in L^1_{\text{loc}}(\Omega)$ and the associated distribution $\Lambda_f \in \mathcal{D}'(\Omega)$, one has

$$\text{supp}(\Lambda_f) \subset \text{supp}(f),$$

and the two sets are equal if f is continuous.

- (b) Find an example of a function $f \in L^1_{\text{loc}}(\Omega)$ such that $\text{supp}(\Lambda_f) \neq \text{supp}(f)$.

Hint: Λ_f does not change if f is modified on a set of measure zero.

- (c) Show that for any distribution $\Lambda \in \mathcal{D}'(\Omega)$ and any multi-index α , $\text{supp}(\partial^\alpha \Lambda) \subset \text{supp}(\Lambda)$.

- (d) What is the support of the Dirac delta function $\delta \in \mathcal{D}'(\mathbb{R}^n)$?

- (e) (*) There is a standard result in the theory of distributions stating that every distribution on \mathbb{R}^n with support $\{0\} \subset \mathbb{R}^n$ is a finite linear combination of derivatives of the delta function. Use this as a black box to show that a tempered distribution has support $\{0\} \subset \mathbb{R}^n$ if and only if its Fourier transform is represented by a polynomial function on \mathbb{R}^n .

Problem 3 (*)

Recall from Problem Set 9 #1 the Cauchy-Riemann operator $\bar{\partial} := \partial_x + i\partial_y$, defined on complex-valued functions of one complex variable $z = x + iy \in \mathbb{C}$, which can equivalently be regarded as complex-valued functions of the *two real* variables $(x, y) \in \mathbb{R}^2$. In the following, we identify \mathbb{C} in this way with \mathbb{R}^2 and endow it with the Lebesgue measure m . Consider the complex-valued function

$$f(z) := \frac{1}{2\pi z},$$

which is defined almost everywhere on \mathbb{C} since $\{0\} \subset \mathbb{C}$ is a set of measure zero.

- (a) Show that $f \in L^1_{\text{loc}}(\mathbb{C})$ and f represents a tempered distribution $\Lambda_f \in \mathcal{S}'(\mathbb{C})$, but f is in neither $L^1(\mathbb{C})$ nor $L^2(\mathbb{C})$.
- (b) Prove that in the sense of distributions, $\bar{\partial}f = \delta$. This is equivalent to showing that for every Schwartz function $\varphi \in \mathcal{S}(\mathbb{C})$,

$$-\int_{\mathbb{C}} f \bar{\partial}\varphi \, dm = \varphi(0).$$

Hint: $f \bar{\partial}\varphi$ will be a Lebesgue-integrable function, so you can rewrite the integral as the limit as $\epsilon \rightarrow 0$ of integrals over the complement of the ϵ -ball $B_\epsilon(0) \subset \mathbb{C}$ around the origin. There are various ways to compute the integral over $\mathbb{C} \setminus B_\epsilon(0)$, e.g. you could switch to polar coordinates and use Fubini's theorem plus the fundamental theorem of calculus, or you could apply some version of Stokes' theorem.

- (c) Let us write Fourier transforms of functions of $z = x + iy \in \mathbb{C}$ as functions of the complex variable $\zeta = p + iq \in \mathbb{C}$. Show that in the sense of tempered distributions, the Fourier transform $\hat{f} \in \mathcal{S}'(\mathbb{C})$ of f satisfies the equation

$$2\pi i \zeta \hat{f} = 1.$$

- (d) Show that the tempered distribution $\hat{f} \in \mathcal{S}'(\mathbb{C})$ in part (c) is represented by the locally integrable function

$$\hat{f}(\zeta) = \frac{1}{2\pi i \zeta}.$$

Use as a black box the result mentioned in Problem 2(e).