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Funktionalanalysis
WiSe 2025–26

Problem Set 13

To be discussed: Thursday, 12.02.2026

Problems marked with $(*)$ should be considered essential, but it is highly recommended that you think through *all* of the problems before the next Thursday lecture.

Problem 1
Show that the following conditions on an unbounded operator $X \supset \mathcal{D} \xrightarrow{T} Y$ are equivalent:

(i) T is closable;

(ii) The closure $\bar{\Gamma}_T$ of $\Gamma_T \subset X \oplus Y$ is also the graph of an operator;

(iii) The conditions $(x, y) \in \bar{\Gamma}_T$ and $(x, y') \in \bar{\Gamma}_T$ imply $y = y'$;

(iv) For every $x \in \bar{\mathcal{D}} \subset X$, there exists at most one element $y \in Y$ arising as a limit of sequences Tx_n with $x_n \in \mathcal{D}$ converging to x.

Problem 2
Assume $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces, $\mathcal{D} \subset X$ is a subspace, and $X \supset \mathcal{D} \xrightarrow{T} Y$ is a linear operator, possibly unbounded, and not necessarily closed. Prove:

(a) If T is closed, then so is the operator $\mathcal{D} \to Y : x \mapsto Tx + Ax$ for every bounded operator $A \in \mathscr{L}(X, Y)$.

(b) T is closed if and only if the so-called *graph norm* $\|x\|_T := \|x\|_X + \|Tx\|_Y$ on $\mathcal{D}$ is complete.

(c) $(*)$ Assuming T is closed, it is injective with closed image if and only if there exists a constant $c > 0$ such that $\|Tx\|_Y \geqslant c\|x\|_X$ for all $x \in \mathcal{D}$.

Now assume $X = Y$ is a complex Banach space.

(d) Show that for every $\lambda \in \mathbb{C}$ such that $\lambda - T : \mathcal{D} \to X$ is bijective, T is closed if and only if the resolvent operator $R_\lambda(T) : X \to X : x \mapsto (\lambda - T)^{-1}x$ is bounded.

Next, assume additionally that T is closed. Let us call $\lambda \in \mathbb{C}$ an *approximate eigenvalue* of T if there exists a sequence $x_n \in \mathcal{D}$ such that $\|x_n\|_X = 1$ and $(\lambda - T)x_n \to 0$, and say that λ belongs to the *residual spectrum* of T if the image of $\lambda - T : \mathcal{D} \to X$ is not dense. Prove:

(e) $(*)$ If $\lambda \in \sigma(T)$ is not in the residual spectrum of T, then it is an approximate eigenvalue.

(f) $(*)$ Every approximate eigenvalue of T is in $\sigma(T)$.

Problem 3
Let $\mathrm{AC}^2([0,1])$ denote the space of absolutely continuous complex-valued functions $f(t)$ on $[0,1]$ whose derivatives (defined almost everywhere) are in $L^2([0,1])$. Given the domains

$$\begin{aligned}
\mathcal{D}_0 &:= \mathrm{AC}^2([0,1]), \\
\mathcal{D}_1 &:= \left\{ f \in \mathrm{AC}^2([0,1]) \;\middle|\; f(0) = 0 \right\}, \\
\mathcal{D}_2 &:= \left\{ f \in \mathrm{AC}^2([0,1]) \;\middle|\; f(0) = f(1) = 0 \right\},
\end{aligned}$$

consider for $j = 0, 1, 2$ the unbounded operators $L^2([0,1]) \supset \mathcal{D}_j \xrightarrow{T_j} L^2([0,1])$ defined by $T_j := i\partial_t = i\frac{d}{dt}$. Prove:

(a) (∗) All three domains are dense in $L^2([0,1])$, and all three operators are closed.

(b) Every $\lambda \in \mathbb{C}$ is an eigenvalue of T_0, thus $\sigma(T_0) = \mathbb{C}$.

(c) Every $\lambda \in \mathbb{C}$ is in the resolvent set of T_1, and $(\lambda - T_1)^{-1} : L^2([0,1]) \to \mathcal{D}_1$ sends $g \in L^2([0,1])$ to the function $f(t) := i\int_0^t e^{-i\lambda(t-s)}g(s)\,ds$. In particular, $\sigma(T_1) = \varnothing$.[1]

(d) T_2 is symmetric, but not self-adjoint.

(e) Every $\lambda \in \mathbb{C}$ is in the residual spectrum of T_2, hence $\sigma(T_2) = \mathbb{C}$.

Problem 4 (∗)
Fix an L^2-function $P : [0,1] \to \mathbb{R}$ and define $\mathcal{D}$ to be the vector space of C^1-functions $x : [0,1] \to \mathbb{C}$ such that $x(0) = x(1) = 0$ and the derivative $\dot{x}$ belongs to the space $\mathrm{AC}^2([0,1])$ from Problem 3, so every $x \in \mathcal{D}$ has an almost everywhere defined second derivative $\ddot{x} \in L^2([0,1])$. Setting $Tx := \ddot{x} + Px$, show that $L^2([0,1]) \supset \mathcal{D} \xrightarrow{T} L^2([0,1])$ is an unbounded self-adjoint operator.
Hint: Interpret the condition defining the domain of T^* in terms of weak derivatives, and then look at Problem Set 9 #4.[2]

Problem 5
Suppose $\mathcal{H} \supset \mathcal{D} \xrightarrow{A} \mathcal{H}$ is a closed unbounded operator such that $\langle x, Ax\rangle \geqslant 0$ for all $x \in \mathcal{D}$. Prove that if A has no residual spectrum, then $\sigma(A)$ contains no negative real numbers.

Problem 6
On a semifinite measure space (X, μ) with a measurable function $F : X \to \mathbb{C}$, we consider the unbounded multiplication operator

$$L^2(X,\mu) \supset \mathcal{D}_F \xrightarrow{T_F} L^2(X,\mu) : u \mapsto Fu,$$

with domain $\mathcal{D}_F := \{u \in L^2(X,\mu) \mid Fu \in L^2(X,\mu)\}$.

(a) Show that $\mathcal{D}_F \subset L^2(X,\mu)$ is dense and T_F is a closed operator.

(b) Show that the spectrum $\sigma(T_F)$ is the essential range of $F : X \to \mathbb{C}$ (cf. Problem Set 11 #4), and F can therefore be modified on a set of measure zero so that $F(X) \subset \sigma(T_F)$ without loss of generality.

(c) Assuming $F(X) \subset \sigma(T_F)$, we can define a functional calculus

$$\mathscr{B}(\sigma(T_F)) \to \mathscr{L}(L^2(X,\mu)) : f \mapsto f(T_F) := T_{f\circ F},$$

where $\mathscr{B}(\sigma(T_F))$ is the space of bounded Borel-measurable functions $\sigma(T_F) \to \mathbb{C}$. Show that if $f_n \in \mathscr{B}(\sigma(T_F))$ is a sequence converging pointwise to $f(\lambda) := \lambda$ and satisfying the bound $|f_n| \leqslant |f|$ for all n, then $f_n(T_F)u \xrightarrow{L^2} T_F u$ for all $u \in \mathcal{D}_F$.

[1] The invertibility of $\lambda - T_1$ can also be deduced from general principles without writing down an explicit formula. The essential question is: given $g \in L^2([0,1])$ and $\lambda \in \mathbb{C}$, how many absolutely continuous functions $f : [0,1] \to \mathbb{C}$ satisfy the initial value problem $f'(t) = H(t, f(t)) := -i[\lambda f(t) - g(t)]$ with $f(0) = 0$? Intuitively, the Picard-Lindelöf theorem suggests that the answer must be exactly one, though strictly speaking, the theorem does not apply here since g cannot be assumed continuous. But since $H(t,x) = -i(\lambda x - g(t))$ is Lipschitz continuous with respect to x, the usual proof can be adapted for this case.

[2] One can also extract from this hint most of a proof that the space $\mathrm{AC}^2([0,1])$ in Problem 3 is equivalent to the Sobolev space $W^{1,2}((0,1))$, and the domain in Problem 4 can similarly be related to the Sobolev space $W^{2,2}((0,1))$. See Remark 19.16 in the lecture notes.