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Funktionalanalysis
WiSe 2025–26

Problem Set 6

To be discussed: Thursday, 27.11.2025

Problems marked with $(*)$ should be considered essential, but it is highly recommended that you think through *all* of the problems before the next Thursday lecture.

Convention: *You can assume unless stated otherwise that all functions take values in a fixed finite-dimensional inner product space* $(V, \langle\ ,\ \rangle)$ *over a field* $\mathbb{K}$ *which is either* $\mathbb{R}$ *or* $\mathbb{C}$. *The Lebesgue measure on* $\mathbb{R}^n$ *is denoted by* m.

Problem 1

Assume $f : I \to V$ is a function defined on an interval $I \subset \mathbb{R}$.

(a) $(*)$ Show that f is Lipschitz-continuous with Lipschitz constant $C > 0$[1] if and only if there exist constants $a \in I$ and $v_0 \in V$ and a function $g \in L^\infty(I)$ with $\|g\|_{L^\infty} \leqslant C$ such that $f(x) = v_0 + \int_a^x g(t)\,dt$ for all $x \in I$.
Hint: Start by proving directly that Lipschitz-continuity implies absolute continuity.

(b) Find an explicit example of a function $f : [0,1] \to \mathbb{R}$ that is absolutely but not Lipschitz-continuous.

(c) $(*)$ One says that $f : I \to V$ has a *jump discontinuity* at $x_0 \in I$ if $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ both exist but are not equal. Assume $f \in L^1_{\text{loc}}(I)$ and set $F(x) := \int_a^x f(t)\,dt$ for some constant $a \in I$. Show that if f has a jump discontinuity at x_0, then F is not differentiable at x_0.

(d) Let $\varphi = \chi_{[0,\infty)} : \mathbb{R} \to \mathbb{R}$ denote the characteristic function of $[0, \infty)$. Given an enumeration of the rational numbers $q_1, q_2, q_3, \ldots \in \mathbb{Q}$, show that $f(x) := \sum_{n=1}^\infty \frac{1}{2^n} \varphi(x - q_n)$ defines a real-valued function $f \in L^\infty(\mathbb{R})$ such that for every $n \in \mathbb{N}$, $\lim_{x \to q_n^+} f(x) = f(q_n)$ and $\lim_{x \to q_n^-} f(x) = f(q_n) - \frac{1}{2^n}$.
Hint: For any $n, N \in \mathbb{N}$, there exists a neighborhood $J \subset I$ of q_n such that the function $\varphi_m(x) := \frac{1}{2^m}\varphi(x - q_m)$ is constant on J for all $m \leqslant N$ with the exception of $m = n$. (Why?)

(e) Write down an example of a Lipschitz-continuous function $F : \mathbb{R} \to \mathbb{R}$ that fails to be differentiable on a dense subset of $\mathbb{R}$. (By part (a) and the Lebesgue differentiation theorem, it will still be differentiable almost everywhere.)

Problem 2

Determine the set of Lebesgue points for each of the following functions $f \in L^1_{\text{loc}}(\mathbb{R}^n)$:

(a) $(*)$ The characteristic function of $\mathbb{Q}^n \subset \mathbb{R}^n$

(b) The characteristic function of $\mathbb{R}^n \backslash \mathbb{Q}^n \subset \mathbb{R}^n$

Problem 3

Let $\mathbb{D}^n \subset \mathbb{R}^n$ denote the unit ball, and consider a function of the form $f(x) := \frac{1}{|x|^\alpha}$ on $\mathbb{R}^n \backslash \{0\}$ for some constant $\alpha > 0$.

[1] Recall: $C > 0$ is a Lipschitz constant for $f : I \to V$ if $|f(x) - f(y)| \leqslant C|x - y|$ holds for all $x, y \in I$.

(a) For which values of $\alpha > 0$ does f belong to $L^1_{\text{weak}}(\mathbb{D}^n)$, and for which of these is it also in $L^1(\mathbb{D}^n)$?

(b) For which values of $\alpha > 0$ does f belong to $L^1_{\text{weak}}(\mathbb{R}^n \backslash \mathbb{D}^n)$, and for which of these is it also in $L^1(\mathbb{R}^n \backslash \mathbb{D}^n)$?

Problem 4 $(*)$
Assume (X, μ) is a measure space and λ is another measure that is finite and absolutely continuous with respect to μ. Give a direct proof (without citing the Radon-Nikodým theorem) of the following result: for every $\epsilon > 0$, there exists $\delta > 0$ such that every measurable set $A \subset X$ satisfying $\mu(A) < \delta$ also satisfies $\lambda(A) < \epsilon$.

Problem 5
Our proof of the Radon-Nikodým theorem in lecture established the following slightly stronger result. Assume λ and μ are two σ-finite measures defined on the same σ-algebra on a set X, and consider the space $L^1(X, \lambda + \mu)$ of measurable functions $g : X \to \mathbb{R}$ that satisfy $\int_X |g| \, d(\lambda + \mu) = \int_X |g| \, d\lambda + \int_X |g| \, d\mu < \infty$. The map $g \mapsto \int_X g \, d\lambda$ then defines a bounded linear functional $L^1(X, \lambda + \mu) \to \mathbb{R}$, so by the Riesz representation theorem, there exists a unique $h \in L^\infty(X, \lambda + \mu)$ such that

$$\int_X g \, d\lambda = \int_X hg \, d(\lambda + \mu) \quad \text{for all} \quad g \in L^1(X, \lambda + \mu).$$

Theorem (proved in lecture): $h : X \to \mathbb{R}$ satisfies $0 \leqslant h < 1$ outside of a subset $E \subset X$ with $\mu(E) = 0$, and the resulting function $f := \frac{h}{1-h} : X \backslash E \to [0, \infty)$ satisfies

$$\int_A f \, d\mu \leqslant \lambda(A) \quad \text{for all measurable sets } A \subset X, \tag{1}$$

with equality for all measurable sets $A \subset X$ whenever $\lambda \ll \mu$.
In the following, we consider pairs of measures λ and μ on certain σ-algebras on sets $X \subset \mathbb{R}^n$, where each of λ and μ is either the Lebesgue measure m, the counting measure ν, or the Dirac measure δ.[2] In each case, find the function $f : X \backslash E \to [0, \infty)$ explicitly, determine the collection of measurable subsets $A \subset X$ on which (1) becomes an equality, and determine whether $\lambda \ll \mu$.

(a) $(*)$ $X = \mathbb{R}^n$, $\mu = m$ and $\lambda = \delta$ on the σ-algebra of Lebesgue-measurable sets

(b) $X = \mathbb{R}^n$, $\mu = \delta$ and $\lambda = m$ on the σ-algebra of Lebesgue-measurable sets

(c) $(*)$ $X = \mathbb{Z}^n$, $\mu = \nu$ and $\lambda = \delta$ on the σ-algebra of all subsets

(d) $X = \mathbb{Z}^n$, $\mu = \delta$ and $\lambda = \nu$ on the σ-algebra of all subsets

One last brainteaser:

(e) For $X = \mathbb{R}^n$ with $\mu = \nu$ and $\lambda = m$ defined on the σ-algebra of Lebesgue-measurable sets, show that $\lambda \ll \mu$ but the function f is identically zero, so (1) cannot typically be an equality. What went wrong?

[2] Recall: $\nu(A)$ is the number of elements in A, while $\delta(A)$ is defined to be 1 if $0 \in A$ and 0 otherwise.