Lectures on Symplectic Field Theory

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Preface

This book is a slightly expanded version of the lecture notes I produced for a two-semester course taught at University College London in 2015–16, for Ph.D. students with a background in basic symplectic geometry and interest in symplectic topology and/or geometric analysis. I say "slightly expanded," although the reader will quickly notice that most individual chapters contain far more material than can reasonably fit into a two-hour lecture. In reality, much of that material was only sketched or mentioned in passing during lectures, and I ended up using the notes to discuss everything that I would like to have explained if I'd had unlimited time. This includes relatively detailed discussions of several important technical points (e.g. the definition of spectral flow, generic transversality in symplectizations, the punctured Riemann-Roch formula, finite energy and asymptotics with arbitrary stable Hamiltonian structures) which are either incompletely covered by the existing literature or, in my opinion, simply more difficult to learn from other sources than they should be. For topics that are on the other hand well covered elsewhere, I have usually not felt obliged to explain every detail, but have tried always to provide adequate references.

One of the interesting features of SFT is that its foundations are—at the time of this writing—not yet complete. When the original "propaganda paper" [**EGH00**] appeared in 2000, it was widely believed that the technical details would be filled in within a few years, and several papers introducing important applications of SFT to contact topology were written under this assumption. Since then, a certain realization has set in that the results in those papers cannot truly be regarded as "theorems" in the sense of mathematics, and it has become less socially acceptable to preface statements of results with caveats of the form, "this theorem is dependent on the foundations of SFT". At the same time, the need for a robust perturbation scheme to achieve transversality in SFT spawned the development of a whole new approach to infinite-dimensional differential geometry, the *polyfold* project [**Hof06**], which is intended for much more general applications but is not yet finished. Opinions vary among symplectic topologists as to how unsatisfied we should all be with this state of affairs, and what could be done about it—among other things, one could make an entire course out of the discussion of such issues, but I have not chosen to do that. My approach is instead to develop the *classical*[1] analysis of pseudoholomorphic curves in symplectizations and symplectic cobordisms, to explain how this would lead to a theory of algebraic contact invariants if transversality for multiple covers were not an issue, and then to use the tools and insights gained from this

[1] For the purposes of this discussion, the word "classical" may be defined as "not involving the words *polyfold*, *virtual* or *Kuranishi*".

discussion to prove *rigorous mathematical theorems* about contact manifolds. Typically, such theorems can be regarded informally as consequences of computations in a (not yet well-defined) theory called SFT, but in a rigorous sense, they are actually consequences of the methods used in those computations. Examples covered in these notes include distinguishing tight contact structures on the 3-torus that are homotopic but not isomorphic (Lecture 10), and the nonexistence of symplectic fillings or symplectic cobordisms between certain pairs of contact manifolds (Lecture 16). The choice of applications is of course biased somewhat toward my own research interests.

Prerequisites. The stated target audience for the lecture course was "Ph.D. students in differential geometry or related fields who are not afraid of analysis". More precisely, the notes assume some knowledge of the following topics:

- Differential geometry: manifolds and vector bundles, differential forms and Stokes' theorem, connections, basic familiarity with symplectic manifolds
- Functional analysis: linear operators on Banach spaces, basics of Sobolev spaces, Fredholm operators
- Differential topology: smooth mapping degree, intersection numbers, Sard's theorem
- Algebraic topology: fundamental group, homology and cohomology of manifolds, Poincaré duality, first Chern class, homological intersection numbers

The following topics are not considered formal prerequisites, but some knowledge of them is likely in any case to be helpful to the reader, who may want to have a good reference for them (as suggested below) within arm's reach:

- Contact manifolds (e.g. Geiges [**Gei08**])
- Differential calculus on Banach spaces and Banach manifolds (e.g. these two books by Lang: [**Lan93**] and [**Lan99**])
- Closed pseudoholomorphic curves (e.g. McDuff-Salamon [**MS04**] or my other book in preparation [**Wend**])
- Floer homology (e.g. Salamon [**Sal99**] or Audin-Damian [**AD14**])

Acknowledgements. I would like to thank the students who sat through the course that gave rise to these notes, and in particular Alexandru Cioba and Agustín Moreno for their assistance in editing the first several lectures. My understanding of Taubes's approach to the Riemann-Roch formula (explained in Lecture 5) and its generalization to the punctured case emerged in part from discussions with Chris Gerig, and I am grateful also to Tim Perutz for helpful hints about Weitzenböck formulas, and Patrick Massot for patient discussions of singular integral operators and elliptic regularity. Thanks also to Michael Hutchings and Janko Latschev for helping me understand the combinatorial factors in Lecture 12, to Jo Nelson for helpful comments on coefficients and orbifold singularities, and to Sam Lisi and Barney Bramham for advice on the Floer C_ε space.

About the current version

At the time of posting this on the arXiv, Lectures 14, 15 and 16 each consist of messy handwritten notes that have not yet been typed up, but will eventually appear in the published version of the book. The main goal for those lectures is to carry out some explicit computations of the torsion invariant introduced at the end of Lecture 13, and to explain the consequences for filling and cobordism obstructions, including for instance the classic result that overtwistedness implies vanishing contact homology and thus obstructs fillability. In keeping with the spirit of the book, the theorems about torsion in Lecture 16 will need to be understood with the usual caveat that they depend on the unfinished foundations of SFT, but part of the point is also to extract complete and *rigorous* proofs of the important consequences regarding symplectic fillings. Lectures 14 and 15 are more technical in nature, in the spirit of Lectures 2 through 9 except that they deal with topics that are only relevant in low-dimensional settings (and thus significantly increase the power of the theory in those settings). Aside from dealing with topics that are valuable in their own right, they specifically precede Lecture 16 because they introduce techniques that will be used in the computations in that lecture.

As far as the rest of the manuscript is concerned, I have tried to produce something that is relatively well polished, but I admit I have not tried quite as diligently for that as I do with most of my research papers. Trying to produce another one of these lectures every week while teaching the course was a formidable task, and I had more time to be careful with it in some weeks than in others. I have since gone back and reworked some portions, but not all, so I apologize for any sloppiness that I may have failed so far to expunge. All comments and corrections are welcome,[2] and may be sent to `wendl@math.hu-berlin.de`. Updates on the publication of the book will be posted periodically on my website at

https://www.mathematik.hu-berlin.de/~wendl/publications.html#notes

[2] especially if those corrections are received before the book goes to press

LECTURE 1

Introduction

Contents

Symplectic field theory is a general framework for defining invariants of contact manifolds and symplectic cobordisms between them via counts of "asymptotically cylindrical" pseudoholomorphic curves. In this first lecture, we'll summarize some of the historical background of the subject, and then sketch the basic algebraic formalism of SFT.

1.1. In the beginning, Gromov wrote a paper

Pseudoholomorphic curves first appeared in symplectic geometry in a 1985 paper of Gromov [Gro85]. The development was revolutionary for the field of symplectic topology, but it was not unprecedented: a few years before this, Donaldson had demonstrated the power of using elliptic PDEs in geometric contexts to define invariants of smooth 4-manifolds (see [DK90]). The PDE that Gromov used was a slight generalization of one that was already familiar from complex geometry.

Recall that if M is a smooth $2n$-dimensional manifold, an **almost complex structure** on M is a smooth linear bundle map $J : TM \to TM$ such that $J^2 = -\mathbb{1}$. This makes the tangent spaces of M into complex vector spaces and thus induces an orientation on M; the pair (M, J) is called an **almost complex manifold**. In this context, a **Riemann surface** is an almost complex manifold of real dimension 2 (hence complex dimension 1), and a **pseudoholomorphic curve** (also called **J-holomorphic**) is a smooth map

$$u : \Sigma \to M$$

satisfying the **nonlinear Cauchy-Riemann equation**

$$Tu \circ j = J \circ Tu, \tag{1.1}$$

where (Σ, j) is a Riemann surface and (M, J) is an almost complex manifold (of arbitrary dimension). The almost complex structure J is called **integrable** if M is admits the structure of a complex manifold such that J is multiplication by i in holomorphic coordinate charts. By a basic theorem of the subject, every almost complex structure in real dimension two is integrable, hence one can always find local coordinates (s, t) on neighorhoods in Σ such that

$$j\partial_s = \partial_t, \qquad j\partial_t = -\partial_s.$$

In these coordinates, (1.1) takes the form

$$\partial_s u + J(u)\partial_t u = 0.$$

The fundamental insight of [**Gro85**] was that solutions to the equation (1.1) capture information about symplectic structures on M whenever they are related to J in the following way.

DEFINITION 1.1. Suppose (M, ω) is a symplectic manifold. An almost complex structure J on M is said to be **tamed** by ω if

$$\omega(X, JX) > 0 \quad \text{for all } X \in TM \text{ with } X \neq 0.$$

Additionally, J is **compatible** with ω if the pairing

$$g(X, Y) := \omega(X, JY)$$

defines a Riemannian metric on M.

We shall denote by $\mathcal{J}(M)$ the space of all smooth almost complex structures on M, with the C^∞_{loc}-topology, and if ω is a symplectic form on M, let

$$\mathcal{J}_\tau(M, \omega), \mathcal{J}(M, \omega) \subset \mathcal{J}(M)$$

denote the subsets consisting of almost complex structures that are tamed by or compatible with ω respectively. Notice that $\mathcal{J}_\tau(M, \omega)$ is an open subset of $\mathcal{J}(M)$, but $\mathcal{J}(M, \omega)$ is not. A proof of the following may be found in [**Wend**, §2.2], among other places.

PROPOSITION 1.2. *On any symplectic manifold (M, ω), the spaces $\mathcal{J}_\tau(M, \omega)$ and $\mathcal{J}(M, \omega)$ are each nonempty and contractible.* □

Tameness implies that the **energy** of a J-holomorphic curve $u : \Sigma \to M$,

$$E(u) := \int_\Sigma u^*\omega,$$

is always nonnegative, and it is strictly positive unless u is constant. Notice moreover that if the domain Σ is closed, then $E(u)$ depends only on the cohomology class $[\omega] \in H^2_{\text{dR}}(M)$ and the homology class

$$[u] := u_*[\Sigma] \in H_2(M),$$

so in particular, any family of J-holomorphic curves in a fixed homology class satisfies a uniform energy bound. This basic observation is one of the key facts behind Gromov's compactness theorem, which states that moduli spaces of closed curves in a fixed homology class are compact up to "nodal" degenerations.

The most famous application of pseudoholomorphic curves presented in [**Gro85**] is Gromov's *nonsqueezing theorem*, which was the first known example of an obstruction for embedding symplectic domains that is subtler than the obvious obstruction defined by volume. The technology introduced in [**Gro85**] also led directly to the development of the *Gromov-Witten invariants* (see [**MS04**, **RT95**, **RT97**]), which follow the same pattern as Donaldson's earlier smooth 4-manifold invariants; they use counts of J-holomorphic curves to define invariants of symplectic manifolds up to symplectic deformation equivalence.

Here is another sample application from [**Gro85**]. We denote by

$$A \cdot B \in \mathbb{Z}$$

the intersection number between two homology classes $A, B \in H_2(M)$ in a closed oriented 4-manifold M.

THEOREM 1.3. *Suppose (M, ω) is a closed and connected symplectic 4-manifold with the following properties:*

(i) (M, ω) does not contain any symplectic submanifold $S \subset M$ that is diffeomorphic to S^2 and satisfies $[S] \cdot [S] = -1$.

(ii) (M, ω) contains two symplectic submanifolds $S_1, S_2 \subset M$ which are both diffeomorphic to S^2, satisfy

$$[S_1] \cdot [S_1] = [S_2] \cdot [S_2] = 0,$$

and have exactly one intersection point with each other, which is transverse and positive.

Then (M, ω) is symplectomorphic to $(S^2 \times S^2, \sigma_1 \oplus \sigma_2)$, where for $i = 1, 2$, the σ_i are area forms on S^2 satisfying

$$\int_{S^2} \sigma_i = \langle [\omega], [S_i] \rangle.$$

SKETCH OF THE PROOF. Since S_1 and S_2 are both symplectic submanifolds, one can choose a compatible almost complex structure J on M for which both of them are the images of embedded J-holomorphic curves. One then considers the moduli spaces $\mathcal{M}_1(J)$ and $\mathcal{M}_2(J)$ of equivalence classes of J-holomorphic spheres homologous to S_1 and S_2 respectively, where any two such curves are considered equivalent if one is a reparametrization of the other (in the present setting this just means they have the same image). These spaces are both manifestly nonempty, and one can argue via Gromov's compactness theorem for J-holomorphic curves that both are compact. Moreover, an infinte-dimensional version of the implicit function theorem implies that both are smooth 2-dimensional manifolds, carrying canonical orientations, hence both are diffeomorphic to closed surfaces. Finally, one uses *positivity of intersections* to show that every curve in $\mathcal{M}_1(J)$ intersects every curve in $\mathcal{M}_2(J)$ exactly once, and this intersection is always transverse and positive; moreover, any two curves in the same space $\mathcal{M}_1(J)$ or $\mathcal{M}_2(J)$ are either identical or disjoint. It follows that both moduli spaces are diffeomorphic to S^2, and both consist of smooth families of J-holomorphic spheres that foliate M, hence defining a diffeomorphism

$$\mathcal{M}_1(J) \times \mathcal{M}_2(J) \to M$$

that sends (u_1, u_2) to the unique point in the intersection $\operatorname{im} u_1 \cap \operatorname{im} u_2$. This identifies M with $S^2 \times S^2$ such that each of the submanifolds $S^2 \times \{*\}$ and $\{*\} \times S^2$ are symplectic. The latter observation can be used to determine the symplectic form up to deformation, so that by the Moser stability theorem, ω is determined up to isotopy by its cohomology class $[\omega] \in H^2_{\mathrm{dR}}(S^2 \times S^2)$, which depends only on the evaluation of ω on $[S^2 \times \{*\}]$ and $[\{*\} \times S^2] \in H_2(S^2 \times S^2)$. □

For a detailed exposition of the above proof of Theorem 1.3, see [**Wene**, Theorem E].

1.2. Hamiltonian Floer homology

Throughout the following, we write

$$S^1 := \mathbb{R}/\mathbb{Z},$$

so maps on S^1 are the same as 1-periodic maps on $\mathbb{R}$. One popular version of the *Arnold conjecture* on symplectic fixed points can be stated as follows. Suppose (M, ω) is a closed symplectic manifold and $H : S^1 \times M \to \mathbb{R}$ is a smooth function. Writing $H_t := H(t, \cdot) : M \to \mathbb{R}$, H determines a 1-periodic time-dependent Hamiltonian vector field X_t via the relation[1]

$$\omega(X_t, \cdot) = -dH_t. \tag{1.2}$$

CONJECTURE 1.4 (Arnold conjecture). *If all 1-periodic orbits of X_t are nondegenerate, then the number of these orbits is at least the sum of the Betti numbers of M.*

Here a 1-periodic orbit $\gamma : S^1 \to M$ of X_t is called **nondegenerate** if, denoting the flow of X_t by φ^t, the linearized time 1 flow

$$d\varphi^1(\gamma(0)) : T_{\gamma(0)}M \to T_{\gamma(0)}M$$

does not have 1 as an eigenvalue. This can be thought of as a Morse condition for an action functional on the loop space whose critical points are periodic orbits; like Morse critical points, nondegenerate periodic orbits occur in isolation. To simplify our lives, let's restrict attention to *contractible* orbits and also assume that (M, ω) is **symplectically aspherical**, which means

$$[\omega]|_{\pi_2(M)} = 0.$$

Then if $C^\infty_{\mathrm{contr}}(S^1, M)$ denotes the space of all smoothly contractible smooth loops in M, the **symplectic action functional** can be defined by

$$\mathcal{A}_H : C^\infty_{\mathrm{contr}}(S^1, M) \to \mathbb{R} : \gamma \mapsto -\int_{\mathbb{D}} \bar{\gamma}^*\omega + \int_{S^1} H_t(\gamma(t))\, dt,$$

[1] Elsewhere in the literature, you will sometimes see (1.2) without the minus sign on the right hand side. If you want to know why I strongly believe that the minus sign belongs there, see [**Wenc**], but to some extent this is just a personal opinion.

where $\bar{\gamma} : \mathbb{D} \to M$ is any smooth map on the closed unit disk $\mathbb{D} \subset \mathbb{C}$ satisfying

$$\bar{\gamma}(e^{2\pi i t}) = \gamma(t),$$

and the symplectic asphericity condition guarantees that $\mathcal{A}_H(\gamma)$ does not depend on the choice of $\bar{\gamma}$.

EXERCISE 1.5. Regarding $C^\infty_{\mathrm{contr}}(S^1, M)$ as a Fréchet manifold with tangent spaces $T_\gamma C^\infty_{\mathrm{contr}}(S^1, M) = \Gamma(\gamma^* TM)$, show that the first variation of the action functional $\mathcal{A}_H$ is

$$d\mathcal{A}_H(\gamma)\eta = \int_{S^1} [\omega(\dot{\gamma}, \eta) + dH_t(\eta)]\, dt = \int_{S^1} \omega(\dot{\gamma} - X_t(\gamma), \eta)\, dt$$

for $\eta \in \Gamma(\gamma^* TM)$. In particular, the critical points of $\mathcal{A}_H$ are precisely the contractible 1-periodic orbits of X_t.

A few years after Gromov's introduction of pseudoholomorphic curves, Floer proved the most important cases of the Arnold conjecture by developing a novel version of infinite-dimensional Morse theory for the functional $\mathcal{A}_H$. This approach mimicked the homological approach to Morse theory which has since been popularized in books such as [**AD14**, **Sch93**], but was apparently only known to experts at the time. In *Morse homology*, one considers a smooth Riemannian manifold (M, g) with a Morse function $f : M \to \mathbb{R}$, and defines a chain complex whose generators are the critical points of f, graded according to their Morse index. If we denote the generator corresponding to a given critical point $x \in \mathrm{Crit}(f)$ by $\langle x \rangle$, the boundary map on this complex is defined by

$$\partial \langle x \rangle = \sum_{\mathrm{ind}(y) = \mathrm{ind}(x) - 1} \# \left(\mathcal{M}(x, y)/\mathbb{R} \right) \langle y \rangle,$$

where $\mathcal{M}(x, y)$ denotes the moduli space of negative gradient flow lines $u : \mathbb{R} \to M$, satisfying $\partial_s u = -\nabla f(u(s))$, $\lim_{s \to -\infty} u(s) = x$ and $\lim_{s \to +\infty} u(s) = y$. This space admits a natural $\mathbb{R}$-action by shifting the variable in the domain, and one can show that for generic choices of f and the metric g, $\mathcal{M}(x, y)/\mathbb{R}$ is a finite set whenever $\mathrm{ind}(x) - \mathrm{ind}(y) = 1$. The real magic however is contained in the following statement about the case $\mathrm{ind}(x) - \mathrm{ind}(y) = 2$:

PROPOSITION 1.6. *For generic choices of f and g and any two critical points $x, y \in \mathrm{Crit}(f)$ with $\mathrm{ind}(x) - \mathrm{ind}(y) = 2$, $\mathcal{M}(x, y)/\mathbb{R}$ is homeomorphic to a finite collection of circles and open intervals whose end points are canonically identified with the finite set*

$$\partial \overline{\mathcal{M}}(x, y) := \bigcup_{\mathrm{ind}(z) = \mathrm{ind}(x) - 1} \mathcal{M}(x, z) \times \mathcal{M}(z, y).$$

We say that $\mathcal{M}(x, y)$ has a natural **compatification** $\overline{\mathcal{M}}(x, y)$, which has the topology of a compact 1-manifold with boundary, and its boundary is the set of all **broken flow lines** from x to y, cf. Figure 1.1. This set of broken flow lines is precisely what is counted if one computes the $\langle y \rangle$ coefficient of $\partial^2 \langle x \rangle$, hence we deduce

$$\partial^2 = 0$$

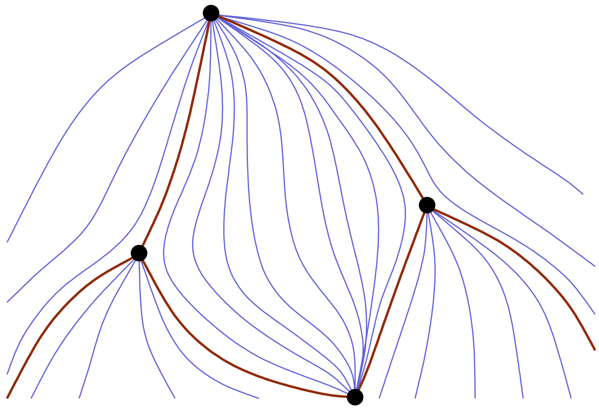

FIGURE 1.1. One-parameter families of gradient flow lines on a Riemannian manifold degenerate to broken flow lines.

as a consequence of the fact that compact 1-manifolds always have zero boundary points when counted with appropriate signs.[2] The homology of the resulting chain complex can be denoted by $HM_*(M\,;\,g,f)$ and is called the **Morse homology** of M. The well-known Morse inequalities can then be deduced from a fundamental theorem stating that $HM_*(M\,;\,g,f)$ is, for generic f and g, isomorphic to the singular homology of M.

With the above notion of Morse homology understood, Floer's approach to the Arnold conjecture can now be summarized as follows:

Step 1: Under suitable technical assumptions, construct a homology theory

$$HF_*(M,\omega\,;\,H,\{J_t\}),$$

depending *a priori* on the choices of a Hamiltonian $H : S^1 \times M \to \mathbb{R}$ with all 1-periodic orbits nondegenerate, and a generic S^1-parametrized family of ω-compatible almost complex structures $\{J_t\}_{t\in S^1}$. The generators of the chain complex are the critical points of the symplectic action functional $\mathcal{A}_H$, i.e. 1-periodic orbits of the Hamiltonian flow, and the boundary map is defined by counting a suitable notion of gradient flow lines connecting pairs of orbits (more on this below).

Step 2: Prove that $HF_*(M,\omega) := HF_*(M,\omega\,;\,H,\{J_t\})$ is a *symplectic invariant*, i.e. it depends on ω, but not on the auxiliary choices H and $\{J_t\}$.

Step 3: Show that if H and $\{J_t\}$ are chosen to be time-independent and H is also C^2-small, then the chain complex for $HF_*(M,\omega\,;\,H,\{J_t\})$ is isomorphic (with a suitable grading shift) to the chain complex for Morse homology $HM_*(M\,;\,g,H)$ with $g := \omega(\cdot, J_t\cdot)$. The isomorphism between $HM_*(M\,;\,g,H)$ and singular homology thus implies that the Floer complex must have at least as many generators (i.e. periodic orbits) as there are generators of $H_*(M)$, proving the Arnold conjecture.

[2] Counting with signs presumes that we have chosen suitable orientations for the moduli spaces $\mathcal{M}(x,y)$, and this can always be done. Alternatively, one can avoid this issue by counting modulo 2 and thus define a homology theory with $\mathbb{Z}_2$ coefficients.

The implementation of Floer's idea required a different type of analysis than what is needed for Morse homology. The moduli space $\mathcal{M}(x, y)$ in Morse homology is simple to understand as the (generically transverse) intersection between the unstable manifold of x and the stable manifold of y with respect to the negative gradient flow. Conveniently, both of those are finite-dimensional manifolds, with their dimensions determined by the Morse indices of x and y. We will see in Lecture 3 that no such thing is true for the symplectic action functional: to the extent that $\mathcal{A}_H$ can be thought of as a Morse function on an infinite-dimensional manifold, its Morse index and its Morse "co-index" at every critical point are both infinite, hence the stable and unstable manifolds are not nearly as nice as finite-dimensional manifolds, providing no reason to expect that their intersection should be. There are additional problems since $C^\infty_{\text{contr}}(S^1, M)$ does not have a Banach space topology: in order to view the negative gradient flow of $\mathcal{A}_H$ as an ODE and make use of the usual local existence/uniqueness theorems (as in [**Lan99**, Chapter IV]), one would have to extend to $\mathcal{A}_H$ to a smooth function on a suitable Hilbert manifold with a Riemannian metric. There is a very limited range of situations in which one can do this and obtain a reasonable formula for $\nabla\mathcal{A}_H$, e.g. [**HZ94**, §6.2] explains the case $M = \mathbb{T}^{2n}$, in which $\mathcal{A}_H$ can be defined on the Sobolev space $H^{1/2}(S^1, \mathbb{R}^{2n})$ and then studied using Fourier series. This approach is very dependent on the fact that the torus $\mathbb{T}^{2n}$ is a quotient of $\mathbb{R}^{2n}$; for general symplectic manifolds (M, ω), one cannot even define $H^{1/2}(S^1, M)$ since functions of class $H^{1/2}$ on S^1 need not be continuous ($H^{1/2}$ is a "Sobolev borderline case" in dimension one).

One of the novelties in Floer's approach was to refrain from viewing the gradient flow as an ODE in a Banach space setting, but instead to write down a formal version of the gradient flow equation and regard it as an elliptic PDE. To this end, let us regard $C^\infty_{\text{contr}}(S^1, M)$ formally as a manifold with tangent spaces

$$T_\gamma C^\infty_{\text{contr}}(S^1, M) := \Gamma(\gamma^* TM),$$

choose a formal Riemannian metric on this manifold (i.e. a smoothly varying family of L^2 inner products on the spaces $\Gamma(\gamma^*TM)$) and write down the resulting equation for the negative gradient flow. A suitable Riemannian metric can be defined by choosing a smooth S^1-parametrized family of compatible almost complex structures

$$\{J_t \in \mathcal{J}(M, \omega)\}_{t \in S^1},$$

abbreviated in the following as $\{J_t\}$, and setting

$$\langle \xi, \eta \rangle_{L^2} := \int_{S^1} \omega(\xi(t), J_t \eta(t))\, dt$$

for $\xi, \eta \in \Gamma(\gamma^*TM)$. Exercise 1.5 then yields the formula

$$d\mathcal{A}_H(\gamma)\eta = \langle J_t(\dot{\gamma} - X_t(\gamma)), \eta \rangle_{L^2},$$

so that it seems reasonable to define the so-called *unregularized* gradient of $\mathcal{A}_H$ by

$$\nabla\mathcal{A}_H(\gamma) := J_t(\dot{\gamma} - X_t(\gamma)) \in \Gamma(\gamma^*TM). \tag{1.3}$$

Let us also think of a path $u : \mathbb{R} \to C^\infty_{\text{contr}}(S^1, M)$ as a map $u : \mathbb{R} \times S^1 \to M$, writing $u(s, t) := u(s)(t)$. The negative gradient flow equation $\partial_s u + \nabla\mathcal{A}_H(u(s)) = 0$ then

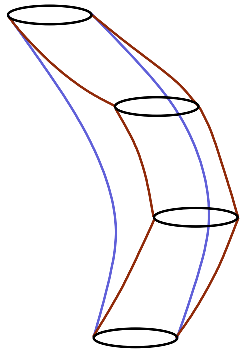

FIGURE 1.2. A family of smooth Floer trajectories can degenerate into a broken Floer trajectory.

becomes the elliptic PDE

$$\partial_s u + J_t(u)\left(\partial_t u - X_t(u)\right) = 0. \tag{1.4}$$

This is called the **Floer equation**, and its solutions are often called **Floer trajectories**. The relevance of Floer homology to our previous discussion of pseudoholomorphic curves should now be obvious. Indeed, the resemblance of the Floer equation to the nonlinear Cauchy-Riemann equation is not merely superficial—we will see in Lecture 6 that the former can always be viewed as a special case of the latter. In any case, one can use the same set of analytical techniques for both: elliptic regularity theory implies that Floer trajectories are always smooth, Fredholm theory and the implicit function theorem imply that (under appropriate assumptions) they form smooth finite-dimensional moduli spaces. Most importantly, the same "bubbling off" analysis that underlies Gromov's compactness theorem can be used to prove that spaces of Floer trajectories are compact up to "breaking", just as in Morse homology (see Figure 1.2)—this is the main reason for the relation $\partial^2 = 0$ in Floer homology.

We should mention one complication that does not arise either in the study of closed holomorphic curves or in finite-dimensional Morse theory. Since the gradient flow in Morse homology takes place on a closed manifold, it is obvious that every gradient flow line asymptotically approaches critical points at both $-\infty$ and $+\infty$. The following example shows that in the infinite-dimensional setting of Floer theory, this is no longer true.

EXAMPLE 1.7. Consider the Floer equation on $M := S^2 = \mathbb{C} \cup \{\infty\}$ with $H := 0$ and J_t defined as the standard complex structure i for every t. Then the orbits of X_t are all constant, and a map $u : \mathbb{R} \times S^1 \to S^2$ satisfies the Floer equation if and only if it is holomorphic. Identifying $\mathbb{R} \times S^1$ with $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ via the biholomorphic map $(s,t) \mapsto e^{2\pi(s+it)}$, a solution u approaches periodic orbits as $s \to \pm\infty$ if and only if the corresponding holomorphic map $\mathbb{C}^* \to S^2$ extends continuously (and therefore holomorphically) over 0 and ∞. But this is not true for every holomorphic map $\mathbb{C}^* \to S^2$, e.g. take any entire function $\mathbb{C} \to \mathbb{C}$ that has an essential singularity at ∞.

EXERCISE 1.8. Show that in the above example with an essential singularity at ∞, the symplectic action $\mathcal{A}_H(u(s,\cdot))$ is unbounded as $s \to \infty$.

EXERCISE 1.9. Suppose $u : \mathbb{R} \times S^1 \to M$ is a solution to the Floer equation with $\lim_{s\to\pm\infty} u(s,\cdot) = \gamma_\pm$ uniformly for a pair of 1-periodic orbits $\gamma_\pm \in \mathrm{Crit}(\mathcal{A}_H)$. Show that

(1.5)
$$\mathcal{A}(\gamma_-) - \mathcal{A}(\gamma_+) = \int_{\mathbb{R}\times S^1} \omega(\partial_s u, \partial_t u - X_t(u))\, ds\, dt = \int_{\mathbb{R}\times S^1} \omega(\partial_s u, J_t(u)\partial_s u)\, ds\, dt.$$

The right hand side of (1.5) is manifestly nonnegative since J_t is compatible with ω, and it is strictly positive unless $\gamma_- = \gamma_+$. It is therefore sensible to call this expression the **energy** $E(u)$ of a Floer trajectory. The following converse of Exercise 1.9 plays a crucial role in the compactness theory for Floer trajectories, as it guarantees that all the "levels" in a broken Floer trajectory are asymptotically well behaved. We will prove a variant of this result in the SFT context (see Prop. 1.23 below) in Lecture 9.

PROPOSITION 1.10. *If $u : \mathbb{R} \times S^1 \to M$ is a Floer trajectory with $E(u) < \infty$ and all 1-periodic orbits of X_t are nonegenerate, then there exist orbits $\gamma_-, \gamma_+ \in \mathrm{Crit}(\mathcal{A}_H)$ such that $\lim_{s\to\pm\infty} u(s,\cdot) = \gamma_\pm$ uniformly.* □

REMARK 1.11. It should be emphasized again that we have assumed $[\omega]|_{\pi_2(M)} = 0$ throughout this discussion; Floer homology can also be defined under more general assumptions, but several details become more complicated.

For nice comprehensive treatments of Hamiltonian Floer homology—unfortunately not always with the same sign conventions as used here—see [**Sal99**, **AD14**]. Note that this is only one of a few "Floer homologies" that were introduced by Floer in the late 80's: the others include *Lagrangian intersection Floer homology* [**Flo88a**] (which has since evolved into the *Fukaya category*, see [**Sei08**]), and *instanton homology* [**Flo88c**], an extension of Donaldson's gauge-theoretic smooth 4-manifold invariants to dimension three. The development of new Floer-type theories has since become a major industry.

1.3. Contact manifolds and the Weinstein conjecture

A Hamiltonian system on a symplectic manifold (W, ω) is called **autonomous** if the Hamiltonian $H : W \to \mathbb{R}$ does not depend on time. In this case, the Hamiltonian vector field X_H defined by
$$\omega(X_H, \cdot) = -dH$$
is time-independent and its orbits are confined to level sets of H. The images of these orbits on a given regular level set $H^{-1}(c)$ depend on the geometry of $H^{-1}(c)$ but not on H itself, as they are the integral curves (also known as **characteristics**) of the **characteristic line field** on $H^{-1}(c)$, defined as the unique direction spanned by a vector X such that $\omega(X, Y) = 0$ for all Y tangent to $H^{-1}(c)$. In 1978, Weinstein [**Wei78**] and Rabinowitz [**Rab78**] proved that certain kinds of regular level sets in symplectic manifolds are guaranteed to admit closed characteristics, hence implying

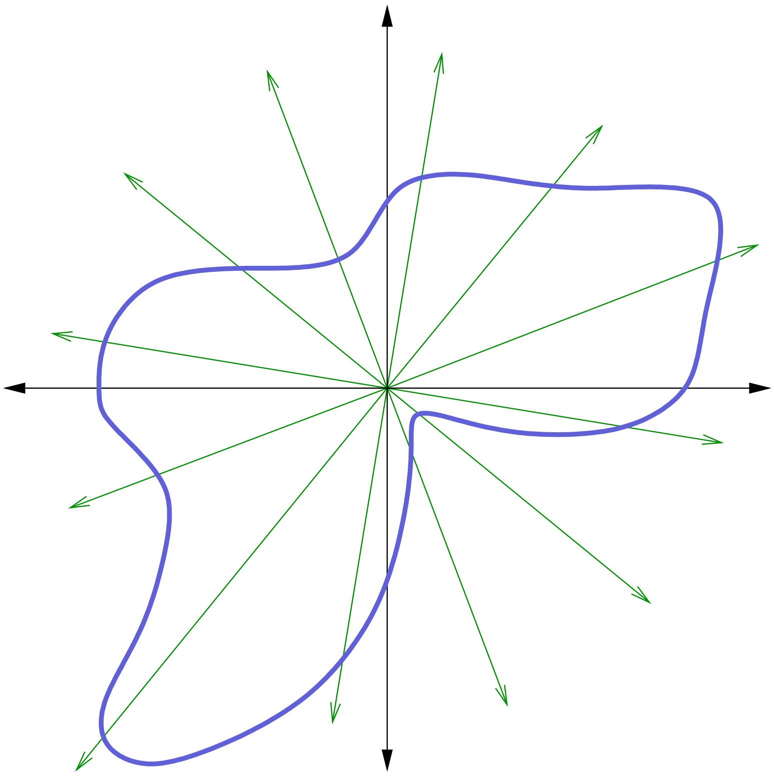

FIGURE 1.3. A star-shaped hypersurface in Euclidean space

the existence of periodic Hamiltonian orbits. In particular, this is true whenever $H^{-1}(c)$ is a *star-shaped* hypersurface in the standard symplectic $\mathbb{R}^{2n}$ (see Figure 1.3).

The following symplectic interpretation of the star-shaped condition provides both an intuitive reason to believe Rabinowitz's existence result and motivation for the more general conjecture of Weinstein. In any symplectic manifold (W, ω), a **Liouville vector field** is a smooth vector field V that satisfies

$$\mathcal{L}_V \omega = \omega.$$

By Cartan's formula for the Lie derivative, the dual 1-form λ defined by $\lambda := \omega(V, \cdot)$ satisfies $d\lambda = \omega$ if and only if V is a Liouville vector field; moreover, λ then also satisfies $\mathcal{L}_V \lambda = \lambda$, and it is referred to as a **Liouville form**. A hypersurface $M \subset (W, \omega)$ is said to be of **contact type** if it is transverse to a Liouville vector field defined on a neighborhood of M.

EXAMPLE 1.12. Using coordinates $(q_1, p_1, \dots, q_n, p_n)$ on $\mathbb{R}^{2n}$, the standard symplectic form is written as

$$\omega_{\text{std}} := \sum_{j=1}^{n} dp_j \wedge dq_j,$$

and the Liouville form $\lambda_{\mathrm{std}} := \frac{1}{2}\sum_{j=1}^{n}(p_j\,dq_j - q_j\,dp_j)$ is dual to the radial Liouville vector field

$$V_{\mathrm{std}} := \frac{1}{2}\sum_{j=1}^{n}\left(p_j\frac{\partial}{\partial p_j} + q_j\frac{\partial}{\partial q_j}\right).$$

Any star-shaped hypersurface is therefore of contact type.

EXERCISE 1.13. Suppose (W,ω) is a symplectic manifold of dimension $2n$, $M \subset W$ is a smoothly embedded and oriented hypersurface, V is a Liouville vector field defined near M and $\lambda := \omega(V,\cdot)$ is the dual Liouville form. Define a 1-form on M by $\alpha := \lambda|_{TM}$.

(a) Show that V is positively transverse to M if and only if α satisfies

$$\alpha \wedge (d\alpha)^{n-1} > 0. \tag{1.6}$$

(b) If V is positively transverse to M, choose $\epsilon > 0$ sufficiently small and consider the embedding

$$\Phi : (-\epsilon,\epsilon)\times M \hookrightarrow W : (r,x) \mapsto \varphi_V^r(x),$$

where φ_V^t denotes the time t flow of V. Show that

$$\Phi^*\lambda = e^r\alpha,$$

hence $\Phi^*\omega = d(e^r\alpha)$.

The above exercise presents any contact-type hypersurface $M \subset (W,\omega)$ as one member of a smooth 1-parameter family of contact-type hypersurfaces $M_r := \varphi_V^r(M) \subset W$, each canonically identified with M such that $\omega|_{TM_r} = e^r\,d\alpha$. In particular, the characteristic line fields on M_r are the same for all r, thus the existence of a closed characteristic on any of these implies that there also exists one on M. This observation has sometimes been used to prove such existence theorems, e.g. it is used in [**HZ94**, Chapter 4] to reduce Rabinowitz's result to an "almost existence" theorem based on symplectic capacities. This discussion hopefully makes the following conjecture seem believable.

CONJECTURE 1.14 (Weinstein conjecture, symplectic version). *Any closed contact-type hypersurface in a symplectic manifold admits a closed characteristic.*

Weinstein's conjecture admits a natural rephrasing in the language of contact geometry. A 1-form α on an oriented $(2n-1)$-dimensional manifold M is called a (positive) **contact form** if it satisfies (1.6), and the resulting co-oriented hyperplane field

$$\xi := \ker\alpha \subset TM$$

is then called a (positive and co-oriented) **contact structure**.[3] We call the pair (M,ξ) a **contact manifold**, and refer to a diffeomorphism $\varphi : M \to M'$ as a

[3]The adjective "positive" refers to the fact that the orientation of M agrees with the one determined by the volume form $\alpha\wedge(d\alpha)^{n-1}$; we call α a *negative* contact form if these two orientations disagree. It is also possible in general to define contact structures without co-orientations, but contact structures of this type will never appear in these notes; for our purposes, the co-orientation is *always* considered to be part of the data of a contact structure.

contactomorphism from (M, ξ) to (M', ξ') if φ_* maps ξ to ξ' and also preserves the respective co-orientations. Equivalently, if ξ and ξ' are defined via contact forms α and α' respectively, this means

$$\varphi^*\alpha' = f\alpha \quad \text{for some } f \in C^\infty(M, (0, \infty)).$$

Contact topology studies the category of contact manifolds (M, ξ) up to contactomorphism. The following basic result provides one good reason to regard ξ rather than α as the geometrically meaningful data, as the result holds for contact *structures*, but not for contact *forms*.

THEOREM 1.15 (Gray's stability theorem). *If M is a closed $(2n-1)$-dimensional manifold and $\{\xi_t\}_{t\in[0,1]}$ is a smooth 1-parameter family of contact structures on M, then there exists a smooth 1-parameter family of diffeomorphisms $\{\varphi_t\}_{t\in[0,1]}$ such that $\varphi_0 = \mathrm{Id}$ and $(\varphi_t)_*\xi_0 = \xi_t$.*

PROOF. See [**Gei08**, §2.2] or [**Wend**, Theorem 1.6.12]. □

A corollary is that while the contact form α induced on a contact-type hypersurface $M \subset (W, \omega)$ via Exercise 1.13 is not unique, its induced contact structure is unique up to isotopy. Indeed, the space of all Liouville vector fields transverse to M is very large (e.g. one can add to V any sufficiently small Hamiltonian vector field), but it is *convex*, hence any two choices of the induced contact form α on M are connected by a smooth 1-parameter family of contact forms, implying an isotopy of contact structures via Gray's theorem.

EXERCISE 1.16. If α is a nowhere zero 1-form on M and $\xi = \ker\alpha$, show that α is contact if and only if $d\alpha|_\xi$ defines a symplectic vector bundle structure on $\xi \to M$. Moreover, the orientation of ξ determined by this symplectic bundle structure is compatible with the co-orientation determined by α and the orientation of M for which $\alpha \wedge (d\alpha)^{n-1} > 0$.

The following definition is based on the fact that since $d\alpha|_\xi$ is nondegenerate when α is contact, $\ker d\alpha \subset TM$ is always 1-dimensional and transverse to ξ.

DEFINITION 1.17. Given a contact form α on M, the **Reeb vector field** is the unique vector field R_α that satisfies

$$d\alpha(R_\alpha, \cdot) \equiv 0, \qquad \text{and} \qquad \alpha(R_\alpha) \equiv 1.$$

EXERCISE 1.18. Show that the flow of any Reeb vector field R_α preserves both $\xi = \ker\alpha$ and the symplectic vector bundle structure $d\alpha|_\xi$.

CONJECTURE 1.19 (Weinstein conjecture, contact version). *On any closed contact manifold (M, ξ) with contact form α, the Reeb vector field R_α admits a periodic orbit.*

To see that this is equivalent to the symplectic version of the conjecture, observe that any contact manifold $(M, \xi = \ker\alpha)$ can be viewed as the contact-type hypersurface $\{0\} \times M$ in the open symplectic manifold

$$(\mathbb{R} \times M, d(e^r\alpha)),$$

called the **symplectization** of (M, ξ).

EXERCISE 1.20. Recall that on any smooth manifold M, there is a tautological 1-form λ that locally takes the form $\lambda = \sum_{j=1}^{n} p_j\, dq_j$ in any choice of local coordinates $(q_1, \ldots, q_n)$ on a neighbood $\mathcal{U} \subset M$, with $(p_1, \ldots, p_n)$ denoting the induced coordinates on the cotangent fibers over $\mathcal{U}$. This is a Liouville form, with $d\lambda$ defining the canonical symplectic structure of T^*M. Now if $\xi \subset TM$ is a co-oriented hyperplane field on M, consider the submanifold

$$S_\xi M := \left\{ p \in T^*M \;\middle|\; \ker p = \xi \text{ and } p(X) > 0 \text{ for any } X \in TM \text{ pos. transverse to } \xi \right\}.$$

Show that ξ is contact if and only if $S_\xi M$ is a symplectic submanifold of $(T^*M, d\lambda)$, and the Liouville vector field on T^*M dual to λ is tangent to $S_\xi M$. Moreover, if ξ is contact, then any choice of contact form for ξ determines a diffeomorphism of $S_\xi M$ to $\mathbb{R} \times M$ identifying the Liouville form λ along $S_\xi M$ with $e^r \alpha$.

REMARK 1.21. Exercise 1.20 shows that up to symplectomorphism, our definition of the symplectization of (M, ξ) above actually depends only on ξ and not on α.

In 1993, Hofer [**Hof93**] introduced a new approach to the Weinstein conjecture that was based in part on ideas of Gromov and Floer. Fix a contact manifold (M, ξ) with contact form α, and let

$$\mathcal{J}(\alpha) \subset \mathcal{J}(\mathbb{R} \times M)$$

denote the nonempty and contractible space of all almost complex structures J on $\mathbb{R} \times M$ satisfying the following conditions:

(1) The natural translation action on $\mathbb{R} \times M$ preserves J;
(2) $J\partial_r = R_\alpha$ and $JR_\alpha = -\partial_r$, where r denotes the canonical coordinate on the $\mathbb{R}$-factor in $\mathbb{R} \times M$;
(3) $J\xi = \xi$ and $d\alpha(\cdot, J\cdot)|_\xi$ defines a bundle metric on ξ.

It is easy to check that any $J \in \mathcal{J}(\alpha)$ is compatible with the symplectic structure $d(e^r\alpha)$ on $\mathbb{R} \times M$. Moreover, if $\gamma : \mathbb{R} \to M$ is any periodic orbit of R_α with period $T > 0$, then for any $J \in \mathcal{J}(\alpha)$, the so-called **trivial cylinder**

$$u : \mathbb{R} \times S^1 \to \mathbb{R} \times M : (s, t) \mapsto (Ts, \gamma(Tt))$$

is a J-holomorphic curve. Following Floer, one version of Hofer's idea would be to look for J-holomorphic cylinders that satisfy a finite energy condition as in Prop. 1.10 forcing them to approach trivial cylinders asymptotically—the existence of such a cylinder would then imply the existence of a closed Reeb orbit and thus prove the Weinstein conjecture. The first hindrance is that the "obvious" definition of energy in this context,

$$\int_{\mathbb{R} \times S^1} u^* d(e^r \alpha),$$

is not the right one: this integral is infinite if u is a trivial cylinder. To circumvent this, notice that every $J \in \mathcal{J}(\alpha)$ is also compatible with any symplectic structure of the form

$$\omega_\varphi := d(e^{\varphi(r)}\alpha),$$

where φ is a function chosen freely from the set

$$\mathcal{T} := \left\{ \varphi \in C^\infty(\mathbb{R}, (-1,1)) \;\middle|\; \varphi' > 0 \right\}. \tag{1.7}$$

Essentially, choosing ω_φ means identifying $\mathbb{R} \times M$ with a subset of the bounded region $(-1,1) \times M$, in which trivial cylinders have finite symplectic area. Since there is no preferred choice for the function φ, we define the **Hofer energy**[4] of a J-holomorphic curve $u : \Sigma \to \mathbb{R} \times M$ by

$$E(u) := \sup_{\varphi \in \mathcal{T}} \int_\Sigma u^* \omega_\varphi. \tag{1.8}$$

This has the desired property of being finite for trivial cylinders, and it is also nonnegative, with strict positivity whenever u is not constant.

Another useful observation from [**Hof93**] was that if the goal is to find periodic orbits, then we need not restrict our attention to J-holomorphic *cylinders* in particular. One can more generally consider curves defined on an arbitrary *punctured* Riemann surface

$$\dot{\Sigma} := \Sigma \setminus \Gamma,$$

where (Σ, j) is a closed connected Riemann surface and $\Gamma \subset \Sigma$ is a finite set of punctures. For any $\zeta \in \Gamma$, one can find coordinates identifying some punctured neighborhood of ζ biholomorphically with the closed punctured disk

$$\dot{\mathbb{D}} := \mathbb{D} \setminus \{0\} \subset \mathbb{C},$$

and then identify this with either the positive or negative half-cylinder

$$Z_+ := [0, \infty) \times S^1, \qquad Z_- := (-\infty, 0] \times S^1$$

via the biholomorphic maps

$$Z_+ \to \dot{\mathbb{D}} : (s,t) \mapsto e^{-2\pi(s+it)}, \qquad Z_- \to \dot{\mathbb{D}} : (s,t) \mapsto e^{2\pi(s+it)}.$$

We will refer to such a choice as a (positive or negative) **holomorphic cylindrical coordinate** system near ζ, and in this way, we can present $(\dot{\Sigma}, j)$ as a *Riemann surface with cylindrical ends*, i.e. the union of some compact Riemann surface with boundary with a finite collection of half-cylinders $Z_\pm$ on which j takes the standard form $j\partial_s = \partial_t$. Note that the standard cylinder $\mathbb{R} \times S^1$ is a special case of this, as it can be identified biholomorphically with $S^2 \setminus \{0, \infty\}$. Another important special case is the plane, $\mathbb{C} = S^2 \setminus \{\infty\}$.

If $u : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$ is a J-holomorphic curve and $\zeta \in \Gamma$ is one of its punctures, we will say that u is **positively/negatively asymptotic** to a T-periodic Reeb orbit $\gamma : \mathbb{R} \to M$ at ζ if one can choose holomorphic cylindrical coordinates $(s,t) \in Z_\pm$ near ζ such that

$$u(s,t) = \exp_{(Ts, \gamma(Tt))} h(s,t) \quad \text{for } |s| \text{ sufficiently large},$$

[4] Strictly speaking, the energy defined in (1.8) is not identical to the notion introduced in [**Hof93**] and used in many of Hofer's papers, but it is equivalent to it in the sense that uniform bounds on either notion of energy imply uniform bounds on the other.

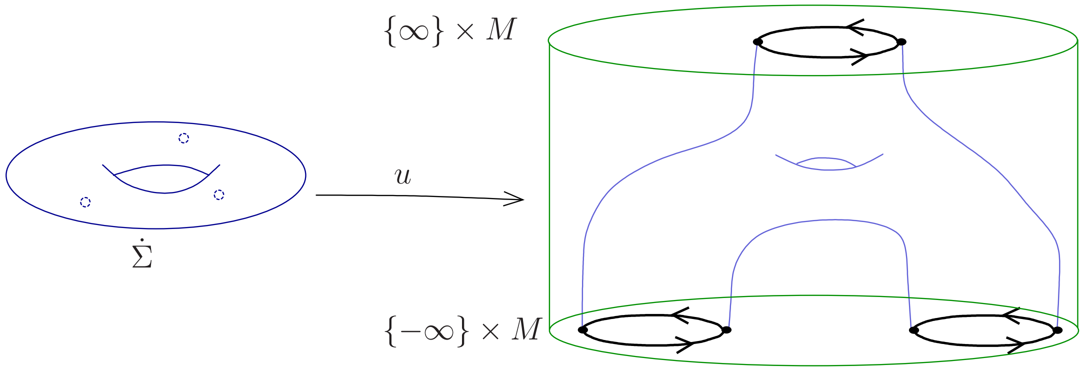

FIGURE 1.4. An asymptotically cylindrical holomorphic curve in a symplectization, with genus 1, one positive puncture and two negative punctures.

where $h(s,t)$ is a vector field along the trivial cylinder satisfying $h(s,\cdot) \to 0$ uniformly as $|s| \to \infty$, and the exponential map is defined with respect to any $\mathbb{R}$-invariant choice of Riemannian metric on $\mathbb{R}\times M$. We say that $u : (\dot{\Sigma}, j) \to (\mathbb{R}\times M, J)$ is **asymptotically cylindrical** if it is (positively or negatively) asymptotic to some closd Reeb orbit at each of its punctures. Note that this partitions the finite set of punctures $\Gamma \subset \Sigma$ into two subsets,

$$\Gamma = \Gamma^+ \cup \Gamma^-,$$

the *positive* and *negative* punctures respectively, see Figure 1.4.

EXERCISE 1.22. Suppose $u : (\dot{\Sigma}, j) \to (\mathbb{R}\times M, J)$ is an asymptotically cylindrical J-holomorphic curve, with the asymptotic orbit at each puncture $\zeta \in \Gamma^\pm$ denoted by γ_ζ, having period $T_\zeta > 0$. Show that

$$\sum_{\zeta\in\Gamma^+} T_\zeta - \sum_{\zeta\in\Gamma^-} T_\zeta = \int_{\dot{\Sigma}} u^* d\alpha \ge 0,$$

with equality if and only if the image of u is contained in that of a trivial cylinder. In particular, u must have at least one positive puncture unless it is constant. Show also that $E(u)$ is finite and satisfies an upper bound determined only by the periods of the positive asymptotic orbits.

The following analogue of Prop. 1.10 will be proved in Lecture 9. For simplicity, we shall state a weakened version of what Hofer proved in [**Hof93**], which did not require any nondegeneracy assumption. A T-periodic Reeb orbit $\gamma : \mathbb{R} \to M$ is called **nondegenerate** if the Reeb flow φ_α^t has the property that its linearization along the contact bundle (cf. Exercise 1.18),

$$d\varphi_\alpha^T(\gamma(0))|_{\xi_{\gamma(0)}} : \xi_{\gamma(0)} \to \xi_{\gamma(0)}$$

does not have 1 as an eigenvalue. Note that since R_α is not time-dependent, closed Reeb orbits are never completely isolated—they always exist in S^1-parametrized families—but these families are isolated in the nondegenerate case.

PROPOSITION 1.23. *Suppose (M, ξ) is a closed contact manifold, with a contact form α such that all closed Reeb orbits are nondegenerate. If $u : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$ is a J-holomorphic curve with $E(u) < \infty$ on a punctured Riemann surface such that none of the punctures are removable, then u is asymptotically cylindrical.* □

The main results in [**Hof93**] state that under certain assumptions on a closed contact 3-manifold (M, ξ), namely if either ξ is *overtwisted* (as defined in [**Eli89**]) or $\pi_2(M) \neq 0$, one can find for any contact form α on (M, ξ) and any $J \in \mathcal{J}(\alpha)$ a finite-energy J-holomorphic plane. By Proposition 1.23, this implies the existence of a contractible periodic Reeb orbit and thus proves the Weinstein conjecture in these settings.

1.4. Symplectic cobordisms and their completions

After the developments described in the previous three sections, it seemed natural that one might define invariants of contact manifolds via a Floer-type theory generated by closed Reeb orbits and counting asymptotically cylindrical holomorphic curves in symplectizations. This theory is what is now called SFT, and its basic structure was outlined in a paper by Eliashberg, Givental and Hofer [**EGH00**] in 2000, though some of its analytical foundations remain unfinished in 2016. The term "field theory" is an allusion to "topological quantum field theories," which associate vector spaces to certain geometric objects and morphisms to cobordisms between those objects. Thus in order to place SFT in its proper setting, we need to introduce symplectic cobordisms between contact manifolds.

Recall that if M_+ and M_- are smooth oriented closed manifolds of the same dimension, an oriented cobordism from M_- to M_+ is a compact smooth oriented manifold W with oriented boundary

$$\partial W = -M_- \sqcup M_+,$$

where $-M_-$ denotes M_- with its orientation reversed. Given positive contact structures $\xi_\pm$ on $M_\pm$, we say that a symplectic manifold (W, ω) is a **symplectic cobordism from** (M_-, ξ_-) **to** (M_+, ξ_+) if W is an oriented cobordism[5] from M_- to M_+ such that both components of ∂W are contact-type hypersurfaces with induced contact structures isotopic to $\xi_\pm$. Note that our chosen orientation conventions imply in this case that the Liouville vector field chosen near ∂W must point *outward* at M_+ and *inward* at M_-; we say in this case that M_+ is a symplectically **convex** boundary component, while M_- is symplectically **concave**. As important special cases, (W, ω) is a **symplectic filling** of (M_+, ξ_+) if $M_- = \emptyset$, and it is a **symplectic cap** of (M_-, ξ_-) if $M_+ = \emptyset$. In the literature, fillings and caps are sometimes also referred to as *convex fillings* or *concave fillings* respectively.

The contact-type condition implies the existence of a Liouville form λ near ∂W with $d\lambda = \omega$, such that by Exercise 1.13, neighborhoods of M_+ and M_- in W can be identified with the collars (see Figure 1.5)

$$(-\epsilon, 0] \times M_+ \quad \text{or} \quad [0, \epsilon) \times M_-$$

[5] We assume of course that W is assigned the orientation determined by its symplectic form.

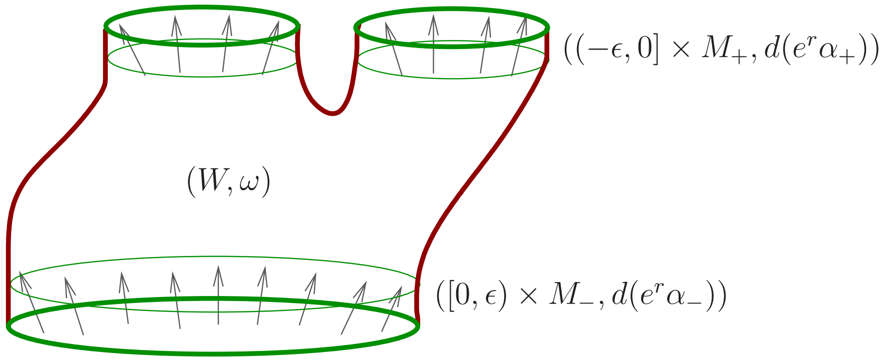

FIGURE 1.5. A symplectic cobordism with concave boundary (M_-, ξ_-) and convex boundary (M_+, ξ_+), with symplectic collar neighborhoods defined by flowing along Liouville vector fields near the boundary.

respectively for sufficiently small $\epsilon > 0$, with λ taking the form

$$\lambda = e^r \alpha_\pm,$$

where $\alpha_\pm := \lambda|_{TM_\pm}$ are contact forms for $\xi_\pm$. The **symplectic completion** of (W, ω) is the noncompact symplectic manifold $(\widehat{W}, \hat{\omega})$ defined by attaching cylindrical ends to these collar neighborhoods (Figure 1.6):

$$(1.9)\qquad \begin{aligned}(\widehat{W}, \hat{\omega}) = ((-\infty, 0] \times M_-, d(e^r\alpha_-)) \cup_{M_-} (W, \omega)\\ \cup_{M_+} ([0, \infty) \times M_+, d(e^r\alpha_+)) .\end{aligned}$$

In this context, the symplectization $(\mathbb{R} \times M, d(e^r\alpha))$ is symplectomorphic to the completion of the **trivial symplectic cobordism** $([0, 1] \times M, d(e^r\alpha))$ from $(M, \xi = \ker \alpha)$ to itself. More generally, the object in the following easy exercise can also sensibly be called a trivial symplectic cobordism:

EXERCISE 1.24. Suppose (M, ξ) is a closed contact manifold with contact form α, and $f_\pm : M \to \mathbb{R}$ is a pair of functions with $f_- < f_+$ everywhere. Show that the domain

$$\left\{(r, x) \in \mathbb{R} \times M \;\middle|\; f_-(x) \le r \le f_+(x)\right\} \subset \mathbb{R} \times M$$

defines a symplectic cobordism from (M, ξ) to itself, with a global Liouville form $\lambda = e^r\alpha$ inducing contact forms $e^{f_-}\alpha$ and $e^{f_+}\alpha$ on its concave and convex boundaries respectively.

We say that (W, ω) is an **exact symplectic cobordism** or **Liouville cobordism** if the Liouville form λ can be extended from a neighborhood of ∂W to define a global primitive of ω on W. Equivalently, this means that ω admits a global Liouville vector field that points inward at M_- and outward at M_+. An **exact filling** of (M_+, ξ_+) is an exact cobordism whose concave boundary is empty. Observe that if (W, ω) is exact, then its completion $(\widehat{W}, \hat{\omega})$ also inherits a global Liouville form.

EXERCISE 1.25. Use Stokes' theorem to show that there is no such thing as an exact symplectic cap.

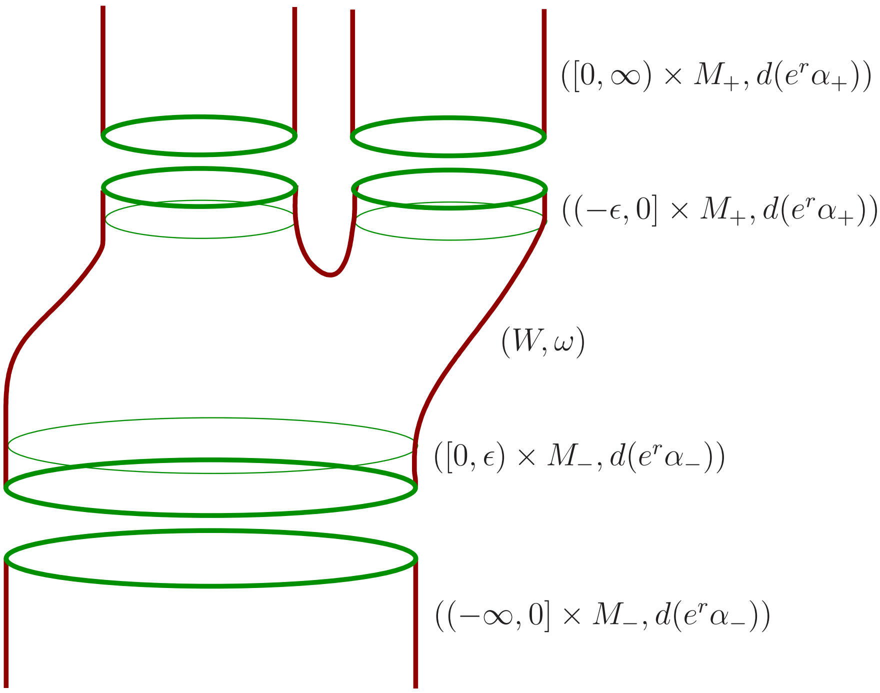

FIGURE 1.6. The completion of a symplectic cobordism

The above exercise hints at an important difference between cobordisms in the *symplectic* as opposed to the *oriented smooth* category: symplectic cobordisms are not generally reversible. If W is an oriented cobordism from M_- to M_+, then reversing the orientation of W produces an oriented cobordism from M_+ to M_-. But one cannot simply reverse orientations in the symplectic category, since the orientation is determined by the symplectic form. For example, many obstructions to the existence of symplectic fillings of given contact manifolds are known—some of them defined in terms of SFT—but we do not know any obstructions at all to symplectic caps, in fact it is known that all contact 3-manifolds admit them.

The definitions for holomorphic curves in symplectizations in the previous section generalize to completions of symplectic cobordisms in a fairly straightforward way since these completions look exactly like symplectizations outside of a compact subset. Define

$$\mathcal{J}(W,\omega,\alpha_+,\alpha_-)\subset\mathcal{J}(\widehat{W})$$

as the space of all almost complex structures J on $\widehat{W}$ such that

$$J|_W\in\mathcal{J}(W,\omega),\qquad J|_{[0,\infty)\times M_+}\in\mathcal{J}(\alpha_+)\quad\text{and}\quad J|_{(-\infty,0]\times M_-}\in\mathcal{J}(\alpha_-).$$

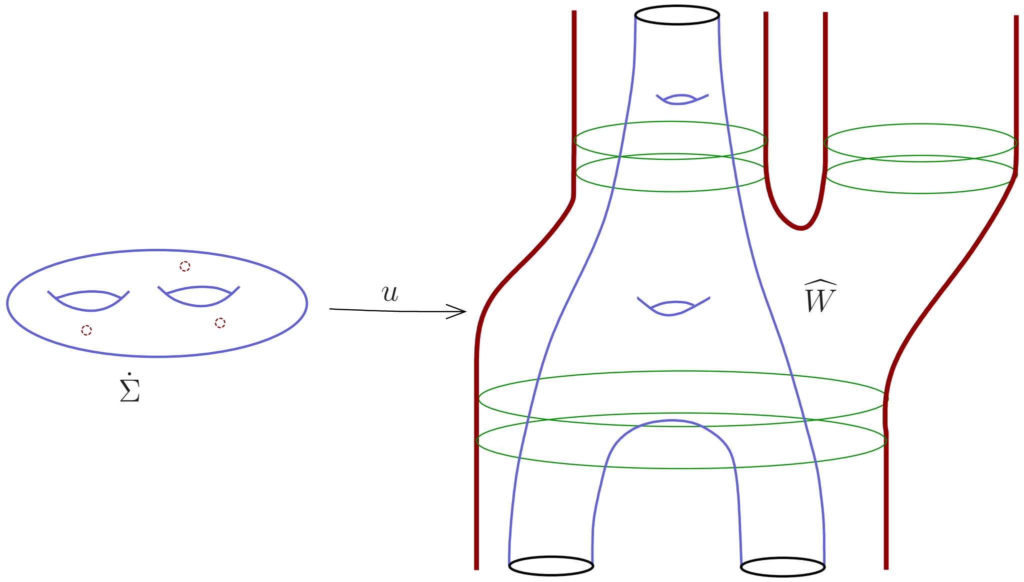

FIGURE 1.7. An asymptotically cylindrical holomorphic curve in a completed symplectic cobordism, with genus 2, one positive puncture and two negative punctures.

Occasionally it is useful to relax the compatibility condition on W to tameness,[6] i.e. $J|_W \in \mathcal{J}_\tau(W, \omega)$, producing a space that we shall denote by

$$\mathcal{J}_\tau(W, \omega, \alpha_+, \alpha_-) \subset \mathcal{J}(\widehat{W}).$$

As in Prop. 1.2, both of these spaces are nonempty and contractible. We can then consider asymptotically cylindrical J-holomorphic curves

$$u : (\dot{\Sigma} = \Sigma \setminus (\Gamma^+ \cup \Gamma^-), j) \to (\widehat{W}, J),$$

which are proper maps asymptotic to closed orbits of $R_{\alpha_\pm}$ in $M_\pm$ at punctures in $\Gamma^\pm$, see Figure 1.7.

One must again tinker with the symplectic form on $\widehat{W}$ in order to define a notion of energy that is finite when we need it to be. We generalize (1.7) as

$$\mathcal{T} := \left\{ \varphi \in C^\infty(\mathbb{R}, (-1,1)) \;\middle|\; \varphi' > 0 \text{ and } \varphi(r) = r \text{ near } r = 0 \right\},$$

and associate to each $\varphi \in \mathcal{T}$ a symplectic form $\hat{\omega}_\varphi$ on $\widehat{W}$ defined by

$$\hat{\omega}_\varphi := \begin{cases} d(e^{\varphi(r)}\alpha_+) & \text{on } [0,\infty) \times M_+, \\ \omega & \text{on } W, \\ d(e^{\varphi(r)}\alpha_-) & \text{on } (-\infty, 0] \times M_-. \end{cases}$$

One can again check that every $J \in \mathcal{J}(W, \omega, \alpha_+, \alpha_-)$ or $\mathcal{J}_\tau(W, \omega, \alpha_+, \alpha_-)$ is compatible with or, respectively, tamed by $\hat{\omega}_\varphi$ for every $\varphi \in \mathcal{T}$. Thus it makes sense to

[6] It seems natural to wonder whether one could not also relax the conditions on the cylindrical ends and require $J|_{\xi_\pm}$ to be tamed by $d\alpha_\pm|_{\xi_\pm}$ instead of compatible with it. I do not currently know whether this works, but in later lectures we will see some reasons to worry that it might not.

define the **energy** of $u : (\dot{\Sigma}, j) \to (\widehat{W}, J)$ by

$$E(u) := \sup_{\varphi \in \mathcal{T}} \int_{\dot{\Sigma}} u^* \hat{\omega}_\varphi.$$

It will be a straightforward matter to generalize Proposition 1.23 and show that finite energy implies asymptotically cylindrical behavior in completed cobordisms.

EXERCISE 1.26. Show that if (W, ω) is an exact cobordism, then every asymptotically cylindrical J-holomorphic curve in $\widehat{W}$ has at least one positive puncture.

1.5. Contact homology and SFT

We can now sketch the algebraic structure of SFT. We shall ignore or suppress several pesky details that are best dealt with later, some of them algebraic, others analytical. Due to analytical problems, some of the "theorems" that we shall (often imprecisely) state in this section are not yet provable at the current level of technology, though we expect that they will be soon. We shall use quotation marks to indicate this caveat wherever appropriate.

The standard versions of SFT all define homology theories with varying levels of algebraic structure which are meant to be invariants of a contact manifold (M, ξ). The chain complexes always depend on certain auxiliary choices, including a nondegenerate contact form α and a generic $J \in \mathcal{J}(\alpha)$. The generators consist of formal variables q_γ, one for each[7] closed Reeb orbit γ. In the most straightforward generalization of Hamiltonian Floer homology, the chain complex is simply a graded $\mathbb{Q}$-vector space generated by the variables q_γ, and the boundary map is defined by

$$\partial_{\mathrm{CCH}} q_\gamma = \sum_{\gamma'} \# \left(\mathcal{M}(\gamma, \gamma')/\mathbb{R} \right) q_{\gamma'},$$

where $\mathcal{M}(\gamma, \gamma')$ is the moduli space of J-holomorphic cylinders in $\mathbb{R} \times M$ with a positive puncture asymptotic to γ and a negative puncture asymptotic to γ', and the sum ranges over all orbits γ' for which this moduli space is 1-dimensional. The count $\# \left(\mathcal{M}(\gamma, \gamma')/\mathbb{R} \right)$ is rational, as it includes rational weighting factors that depend on combinatorial information and are best not discussed right now.[8]

"THEOREM" 1.27. *If α admits no contractible Reeb orbits, then $\partial_{\mathrm{CCH}}^2 = 0$, and the resulting homology is independent of the choices of α with this property and generic $J \in \mathcal{J}(\alpha)$.*

The invariant arising from this result is known as **cylindrical contact homology**, and it is sometimes quite easy to work with when it is well defined, though it has the disadvantage of not always being defined. Namely, the relation $\partial_{\mathrm{CCH}}^2 = 0$ can fail if α admits contractible Reeb orbits, because unlike in Floer homology, the compactification of the space of cylinders $\mathcal{M}(\gamma, \gamma')$ generally includes objects that are not broken cylinders. In fact, the objects arising in the "SFT compactification"

[7] Actually I should be making a distinction here between "good" and "bad" Reeb orbits, but let's discuss that later; see Lecture 11.

[8] Similar combinatorial factors are hidden behind the symbol "#" in our definitions of ∂_{CH} and $\mathbf{H}$, and will be discussed in earnest in Lecture 12.

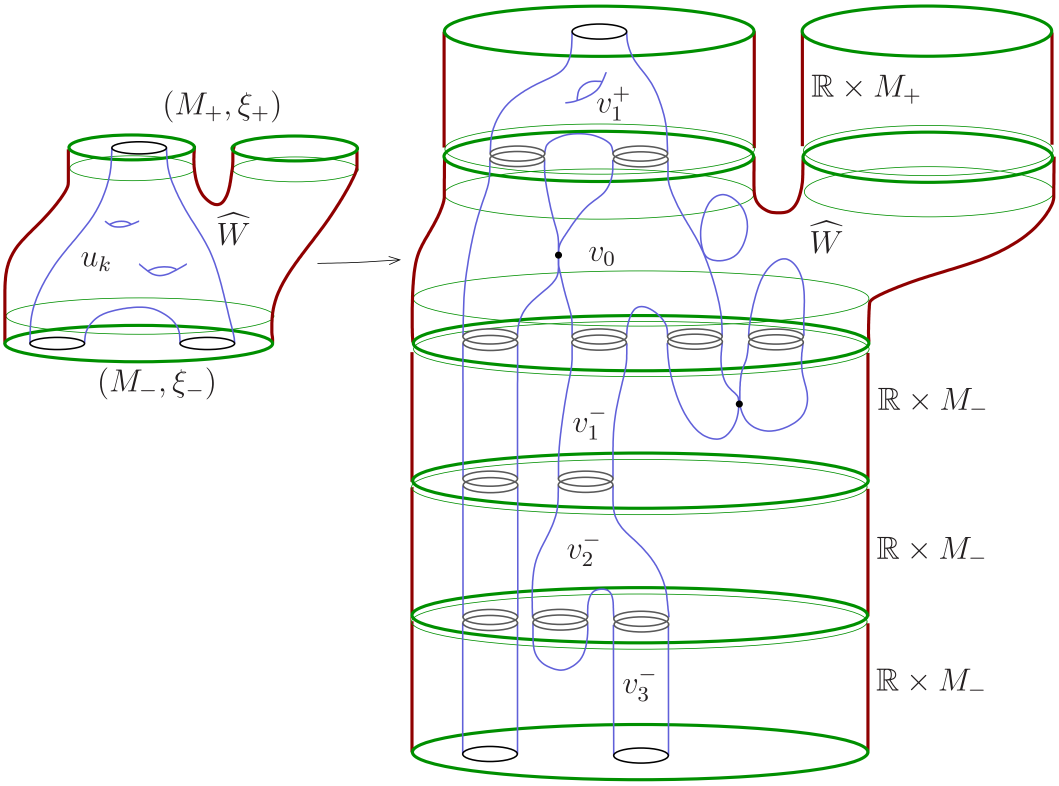

FIGURE 1.8. Degeneration of a sequence u_k of finite energy punctured holomorphic curves with genus 2, one positive puncture and two negative punctures in a symplectic cobordism. The limiting holomorphic building $(v_1^+, v_0, v_1^-, v_2^-, v_3^-)$ in this example has one upper level living in the symplectization $\mathbb{R} \times M_+$, a main level living in $\widehat{W}$, and three lower levels, each of which is a (possibly disconnected) finite-energy punctured nodal holomorphic curve in $\mathbb{R} \times M_-$. The building has arithmetic genus 2 and the same numbers of positive and negative punctures as u_k.

of moduli spaces of finite-energy curves in completed cobordisms can be quite elaborate, see Figure 1.8. The combinatorics of the situation are not so bad however if the cobordism is exact, as is the case for a symplectization: Exercise 1.26 then prevents curves without positive ends from appearing. The only possible degenerations for cylinders then consist of broken configurations whose levels each have *exactly one positive puncture* and arbitrary negative punctures; moreover, all but one of the negative punctures must eventually be capped off by planes, which is why "Theorem" 1.27 holds in the absence of planes.

If planes do exist, then one can account for them by defining the chain complex as an *algebra* rather than a vector space, producing the theory known as **contact homology**. For this, the chain complex is taken to be a graded unital algebra over

$\mathbb{Q}$, and we define

$$\partial_{\mathrm{CH}} q_\gamma = \sum_{(\gamma_1,\dots,\gamma_m)} \# \left(\mathcal{M}(\gamma;\gamma_1,\dots,\gamma_m)/\mathbb{R}\right) q_{\gamma_1}\dots q_{\gamma_m},$$

with $\mathcal{M}(\gamma;\gamma_1,\dots,\gamma_m)$ denoting the moduli space of punctured J-holomorphic spheres in $\mathbb{R}\times M$ with a positive puncture at γ and m negative punctures at the orbits $\gamma_1,\dots,\gamma_m$, and the sum ranges over all integers $m \ge 0$ and all m-tuples of orbits for which the moduli space is 1-dimensional. The action of ∂_{CH} is then extended to the whole algebra via a graded Leibniz rule

$$\partial_{\mathrm{CH}}(q_\gamma q_{\gamma'}) := (\partial_{\mathrm{CH}} q_\gamma)\, q_{\gamma'} + (-1)^{|\gamma|} q_\gamma\, (\partial_{\mathrm{CH}} q_{\gamma'})\,.$$

The general compactness and gluing theory for genus zero curves with one positive puncture now implies:

"THEOREM" 1.28. *$\partial_{\mathrm{CH}}^2 = 0$, and the resulting homology is (as a graded unital $\mathbb{Q}$-algebra) independent of the choices α and J.*

Maybe you've noticed the pattern: in order to accommodate more general classes of holomorphic curves, we need to add more algebraic structure. The **full SFT** algebra counts all rigid holomorphic curves in $\mathbb{R}\times M$, including all combinations of positive and negative punctures and all genera. Here is a brief picture of what it looks like. Counting all the 1-dimensional moduli spaces of J-holomorphic curves modulo $\mathbb{R}$-translation in $\mathbb{R}\times M$ produces a formal power series

$$\mathbf{H} := \sum \# \left(\mathcal{M}_g(\gamma_1^+,\dots,\gamma_{m_+}^+;\gamma_1^-,\dots,\gamma_{m_-}^-)\Big/\mathbb{R}\right) q_{\gamma_1^-}\dots q_{\gamma_{m_-}^-} p_{\gamma_1^+}\dots p_{\gamma_{m_+}^+} \hbar^{g-1},$$

where the sum ranges over all integers $g, m_+, m_- \ge 0$ and tuples of orbits, $\hbar$ and p_γ (one for each orbit γ) are additional formal variables, and

$$\mathcal{M}_g(\gamma_1^+,\dots,\gamma_{m_+}^+;\gamma_1^-,\dots,\gamma_{m_-}^-)$$

denotes the moduli space of J-holomorphic curves in $\mathbb{R}\times M$ with genus g, m_+ positive punctures at the orbits $\gamma_1^+,\dots,\gamma_{m_+}^+$, and m_- negative punctures at the orbits $\gamma_1^-,\dots,\gamma_{m_-}^+$. We can regard $\mathbf{H}$ as an operator on a graded algebra $\mathfrak{W}$ of formal power series in the variables $\{p_\gamma\}$, $\{q_\gamma\}$ and $\hbar$, equipped with a graded bracket operation that satisfies the quantum mechanical commutation relation

$$[p_\gamma, q_\gamma] = \kappa_\gamma \hbar,$$

where κ_γ is a combinatorial factor that is best ignored for now. Note that due to the signs that accompany the grading, odd elements $\mathbf{F}\in\mathfrak{W}$ need not satisfy $[\mathbf{F},\mathbf{F}] = 0$, and $\mathbf{H}$ itself is an odd element, thus the following statement is nontrivial; in fact, it is the algebraic manifestation of the general compactness and gluing theory for punctured holomorphic curves in symplectizations.

"THEOREM" 1.29. *$[\mathbf{H},\mathbf{H}] = 0$, hence by the graded Jacobi identity, $\mathbf{H}$ determines an operator*

$$D_{\mathrm{SFT}} : \mathfrak{W} \to \mathfrak{W} : \mathbf{F} \mapsto [\mathbf{H},\mathbf{F}]$$

satisfying $D_{\mathrm{SFT}}^2 = 0$. The resulting homology depends on (M,ξ) but not on the auxiliary choices α and J.

It takes some time to understand how pictures such as Figure 1.8 translate into algebraic relations like $[\mathbf{H}, \mathbf{H}] = 0$, but this is a subject we'll come back to. There is also an intermediate theory between contact homology and full SFT, called **rational SFT**, which counts only genus zero curves with arbitrary positive and negative punctures. Algebraically, it is obtained from the full SFT algebra as a "semiclassical approximation" by discarding higher-order factors of $\hbar$ so that the commutation bracket in $\mathfrak{W}$ becomes a graded Poisson bracket. We will discuss all of this in Lecture 12.

1.6. Two applications

We briefly mention two applications that we will be able to establish rigorously using the methods developed in this book. Since SFT itself is not yet well defined in full generality, this sometimes means using SFT for inspiration while proving corollaries via more direct methods.

1.6.1. Tight contact structures on $\mathbb{T}^3$. The 3-torus $\mathbb{T}^3 = S^1 \times S^1 \times S^1$ with coordinates (t, θ, ϕ) admits a sequence of contact structures

$$\xi_k := \ker\left(\cos(2\pi kt)\, d\theta + \sin(2\pi kt)\, d\phi\right),$$

one for each $k \in \mathbb{N}$. These cannot be distinguished from each other by any classical invariants, e.g. they all have the same Euler class, in fact they are all homotopic as co-oriented 2-plane fields. Nonetheless:

THEOREM 1.30. *For $k \neq \ell$, $(\mathbb{T}^3, \xi_k)$ and $(\mathbb{T}^3, \xi_\ell)$ are not contactomorphic.*

We will be able to prove this in Lecture 10 by rigorously defining and computing cylindrical contact homology for a suitable choice of contact forms on $(\mathbb{T}^3, \xi_k)$.

1.6.2. Filling and cobordism obstructions. Consider a closed connected and oriented surface Σ presented as $\Sigma_+ \cup_\Gamma \Sigma_-$, where $\Sigma_\pm \subset \Sigma$ are each (not necessarily connected) compact surfaces with a common boundary Γ. By an old result of Lutz [**Lut77**], the 3-manifold $S^1 \times \Sigma$ admits a unique isotopy class of S^1-invariant contact structures ξ_Γ such that the loops $S^1 \times \{z\}$ are positively/negatively transverse to ξ_Γ for $z \in \mathring{\Sigma}_\pm$ and tangent to ξ_Γ for $z \in \Gamma$. Now for each $k \in \mathbb{N}$, define

$$(V_k, \xi_k) := (S^1 \times \Sigma, \xi_\Gamma)$$

where $\Sigma = \Sigma_+ \cup_\Gamma \Sigma_-$ is chosen such that Γ has k connected components, Σ_- is connected with genus zero, and Σ_+ is connected with positive genus (see Figure 1.9).

THEOREM 1.31. *The contact manifolds (V_k, ξ_k) do not admit any symplectic fillings. Moreover, if $k > \ell$, then there exists no exact symplectic cobordism from (V_k, ξ_k) to (V_ℓ, ξ_ℓ).*

For these examples, one can use explicit constructions from [**Wen13**, **Avd**] to show that non-exact cobordisms from (V_k, ξ_k) to (V_ℓ, ξ_ℓ) do exist, and so do exact cobordisms from (V_ℓ, ξ_ℓ) to (V_k, ξ_k), thus both the directionality of the cobordism relation and the distinction between exact and non-exact are crucial. The proof of the theorem, due to the author with Latschev and Hutchings [**LW11**], uses a

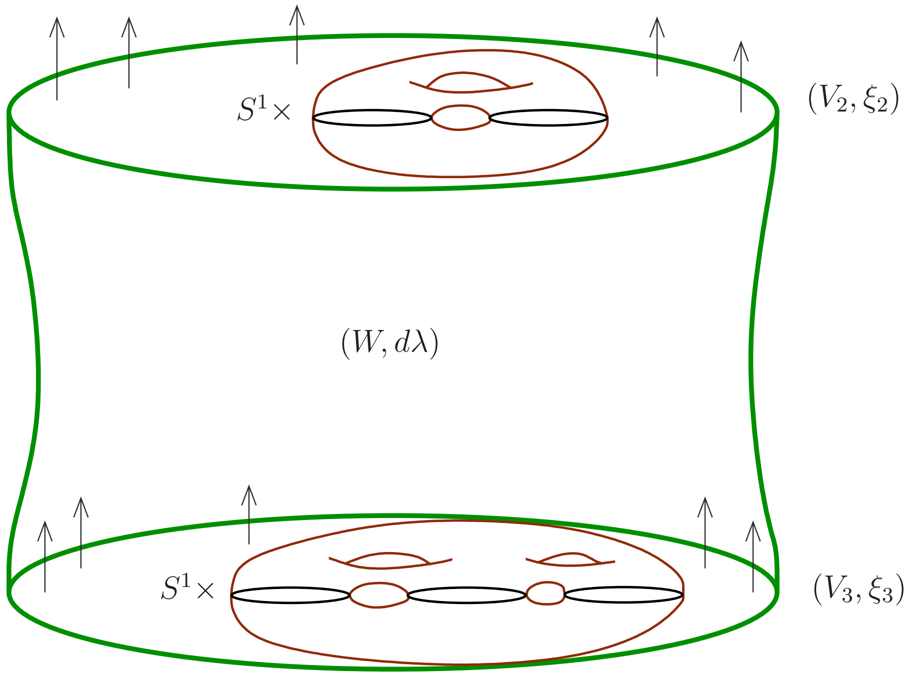

FIGURE 1.9. This exact symplectic cobordism does not exist.

numerical contact invariant based on the full SFT algebra—in particular, the curves that cause this phenomenon have multiple positive ends and are thus not seen by contact homology. We will introduce the relevant numerical invariant in Lecture 13 and compute it for these examples in Lecture 16.

LECTURE 2

Basics on holomorphic curves

Contents

In this lecture we begin studying the analysis of J-holomorphic curves. The coverage will necessarily be a bit sparse in some places, but more detailed proofs of everything in this lecture can be found in [**Wend**].

2.1. Linearized Cauchy-Riemann operators

In order to motivate the study of linear Cauchy-Riemann type operators, we begin with a formal discussion of the nonlinear Cauchy-Riemann equation and its linearization.

Fix a Riemann surface (Σ, j) and almost complex manifold (W, J), and suppose that we wish to understand the structure of some space of the form

$$\{u : \Sigma \to W \mid Tu \circ j = J \circ Tu \text{ plus further conditions}\}, \tag{2.1}$$

where the "further conditions" (which we will for now leave unspecified) may impose constraints on e.g. the regularity of u, as well as its boundary and/or asymptotic behavior. The standard approach in global analysis can be summarized as follows:

Step 1: Construct a smooth Banach manifold $\mathcal{B}$ of maps $u : \Sigma \to W$ such that all the solutions we're interested in will be elements of $\mathcal{B}$. The tangent spaces $T_u\mathcal{B}$ are then Banach spaces of sections of u^*TW.

Step 2: Construct a smooth Banach space bundle $\mathcal{E} \to \mathcal{B}$ such that for each $u \in \mathcal{B}$, the fiber $\mathcal{E}_u$ is a Banach space of sections of the vector bundle

$$\overline{\text{Hom}}_{\mathbb{C}}(T\Sigma, u^*TW) \to \Sigma$$

of complex-antilinear bundle maps $(T\Sigma, j) \to (u^*TW, J)$. Since our purpose is to study a first-order PDE, we need the sections in $\mathcal{E}_u$ to be "one step less regular" than the maps in $\mathcal{B}$, e.g. if $\mathcal{B}$ consists of maps of Sobolev class $W^{k,p}$, then the sections in $\mathcal{E}_u$ should be of class $W^{k-1,p}$.

Step 3: Show that

$$\bar{\partial}_J : \mathcal{B} \to \mathcal{E} : u \mapsto Tu + J(u) \circ Tu \circ j$$

defines a smooth section of $\mathcal{E} \to \mathcal{B}$, whose zero set is precisely the space of solutions (2.1).

Step 4: Show that under suitable assumptions (e.g. on regularity and asymptotic behavior), one can arrange such that for every $u \in \bar{\partial}_J^{-1}(0)$, the **linearization** of $\bar{\partial}_J$,

$$D\bar{\partial}_J(u) : T_u\mathcal{B} \to \mathcal{E}_u$$

is a Fredholm operator and is generically surjective. (In geometric terms, this would mean that $\bar{\partial}_J$ is *transverse to the zero section.*)

Step 5: Using the implicit function theorem in Banach spaces (see [**Lan93**]), the surjectivity of $D\bar{\partial}_J(u)$ implies that $\bar{\partial}_J^{-1}(0)$ is a smooth finite-dimensional manifold, with its tangent space at each $u \in \bar{\partial}_J^{-1}(0)$ canonically identified with $\ker D\bar{\partial}_J(u)$, hence the dimension of $\bar{\partial}_J^{-1}(0)$ near u equals the Fredholm index of $D\bar{\partial}_J(u)$.

Without worrying about the fact that these are actually not Banach spaces, and some Sobolev completion is needed, let us assume, for simplicity, that the bundle $\mathcal{E} \to \mathcal{B}$ has as base the space $\mathcal{B} = C^\infty(\Sigma, W)$ and the fiber over $u \in \mathcal{B}$ is given by $\mathcal{E}_u = \overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TW)$. The linearization of the section $\overline{\partial}_J$ at a point $u \in \bar{\partial}_J^{-1}(0)$ should then take the form

$$\mathbf{D}_u : \Gamma(u^*TW) \to \Omega^{0,1}(\Sigma, u^*TW),$$

where the right hand side denotes the space of u^*TW-valued $(0,1)$-forms on Σ, or equivalently, smooth sections of $\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TW) = T^{0,1}\Sigma \otimes_{\mathbb{C}} u^*TW$, where $T^{0,1}\Sigma$ denotes the $(0,1)$-part of the complexified cotangent bundle.

In order to compute such a linearization, we need to make a choice of "connection" on the bundle $\mathcal{E}$. Choose a connection ∇ on W, and recall the fact that this naturally induces a connection on the bundles $T^{0,1}\Sigma \otimes_{\mathbb{C}} u^*TW$ and $\mathrm{End}(u^*TW)$ by setting $\nabla(\alpha \otimes s) = \alpha \otimes \nabla s$ and $(\nabla J)s = \nabla(Js) - J\nabla s$, for $s \in \Gamma(u^*TW)$, $J \in \mathrm{End}(u^*TW)$ and $\alpha \in \Gamma(T^{0,1}\Sigma)$. We shall make the ansatz that for any smooth 1-parameter family of maps $u_\rho : \Sigma \to W$ for $\rho \in (-\epsilon, \epsilon)$ and a section $\eta_\rho \in \mathcal{E}_{u_\rho}$ along the path (i.e a section of the pullback bundle of $\mathcal{E}$ under the map $(-\epsilon, \epsilon) \to \mathcal{B}$ mapping ρ to u_ρ), the connection takes the form

$$(\nabla_\rho \eta_\rho)X = \nabla_\rho(\eta_\rho(X)),$$

for $X \in T\Sigma$, where this expression should be interpreted as the pullback connection under the map displayed above. The tensorial property of connections implies that $\nabla_\rho \eta_\rho$ does not depend on the connection at the values ρ for which $\eta_\rho = 0$.

Given $u \in \overline{\partial}_J^{-1}(0)$ and η in $T_u\mathcal{B} = \Gamma(u^*TW)$, take a one-parameter family $u_\rho \in \mathcal{B}$ with $u_0 = u$ and $\partial_\rho u_\rho|_{\rho=0} = \eta$. We then have that

$$\mathbf{D}_u\eta = \nabla_\rho \left(\bar{\partial}_J(u_\rho)\right)\Big|_{\rho=0} = \nabla_\rho(Tu_\rho + J(u_\rho) \circ Tu_\rho \circ j)\big|_{\rho=0}.$$

Since $\bar{\partial}_J u = 0$, this is independent of the connection, and we may therefore choose ∇ to be symmetric.

Fix a point $z \in \Sigma$ and choose local holomorphic coordinates $s+it$ around it. The symmetry of the connection implies $\nabla_\rho \partial_s u_\rho|_{\rho=0} = \nabla_s \partial_\rho u_\rho|_{\rho=0} = \nabla_s \eta$, and similarly for the variable t. Observing also that $\nabla_\eta J = \nabla_{\partial_\rho u_\rho|_{\rho=0}} J = \nabla_\rho \left(J(u_\rho)\right)|_{\rho=0}$, and using the above ansatz, we obtain

$$(\mathbf{D}_u \eta)\partial_s = \nabla_\rho(\partial_s u_\rho + J(u_\rho) \circ \partial_t u_\rho)|_{\rho=0} = \nabla_s \eta + J(u)\nabla_t \eta + (\nabla_\eta J)\partial_t u$$

Since $\mathbf{D}_u \eta$ is an antilinear map, and $\partial_t = j\partial_s$, it is therefore determined by its action on ∂_s. One can check that the operator on the right hand side below is also antilinear, and thus removing the ∂_s, we obtain

$$\boxed{\mathbf{D}_u \eta = \nabla \eta + J(u) \circ \nabla \eta \circ j + (\nabla_\eta J) \circ Tu \circ j.} \tag{2.2}$$

DEFINITION 2.1. Fix a complex vector bundle E over a Riemann surface (Σ, j). A (real) linear **Cauchy-Riemann type operator** on E is a real-linear first-order differential operator

$$\mathbf{D} : \Gamma(E) \to \Omega^{0,1}(\Sigma, E)$$

such that for every $f \in C^\infty(\Sigma, \mathbb{R})$ and $\eta \in \Gamma(E)$,

$$\mathbf{D}(f\eta) = (\bar{\partial} f)\eta + f\mathbf{D}\eta, \tag{2.3}$$

where $\bar{\partial} f$ denotes the complex-valued $(0,1)$-form $df + i\,df \circ j$.

Observe that $\mathbf{D}$ is complex linear if and only if the Leibniz rule (2.3) also holds for all smooth complex-valued functions f, not just real-valued. It is a standard result in complex geometry that choosing a complex-linear Cauchy-Riemann type operator $\mathbf{D}$ on E is equivalent to endowing it with the structure of a *holomorphic* vector bundle, where local sections η are defined to be holomorphic if and only if $\mathbf{D}\eta = 0$. Indeed, every holomorphic bundle comes with a canonical Cauchy-Riemann operator that is expressed as $\bar{\partial}$ in holomorphic trivializations, and in the other direction, the equivalence follows from a local existence result for solutions to the equation $\mathbf{D}\eta = 0$, proved in §2.5 below.[1]

EXERCISE 2.2. If $\mathbf{D}$ is a linear Cauchy-Riemann type operator on E, prove that every other such operator is of the form $\mathbf{D} + A$ where $A : E \to \overline{\mathrm{Hom}_{\mathbb{C}}(T\Sigma, E)}$ is a smooth linear bundle map. Using this, show that in suitable local trivializations over a subset $\mathcal{U} \subset \Sigma$ identified biholomorphically with an open set in $\mathbb{C}$, every Cauchy-Riemann type operator $\mathbf{D}$ takes the form

$$\mathbf{D} = \bar{\partial} + A : C^\infty(\mathcal{U}, \mathbb{C}^m) \to C^\infty(\mathcal{U}, \mathbb{C}^m),$$

where $\bar{\partial} = \partial_s + i\partial_t$ in complex coordinates $z = s + it$ and $A \in C^\infty(\mathcal{U}, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^m))$.

EXERCISE 2.3. Verify that the linearized operator $\mathbf{D}_u$ of (2.2) is a real-linear Cauchy-Riemann type operator.

[1] This statement about the existence of holomorphic vector bundle structures is true when the base is a Riemann surface, but not if it is a higher-dimensional complex manifold. In higher dimensions there are obstructions, see e.g. [Kob87].

2.2. Some useful Sobolev inequalities

In this section, we review a few general properties of Sobolev spaces that are essential for applications in nonlinear analysis. The results stated here are explained in more detail in Appendix A.

Throughout this section we consider functions with values in $\mathbb{C}$ unless otherwise specified, and defined on an open domain $\mathcal{U}$ in either $\mathbb{R}^n$ or a quotient of $\mathbb{R}^n$ on which the Lebesgue measure is well defined. Certain regularity assumptions must generally be placed on the boundary of $\overline{\mathcal{U}}$ in order for all the results stated below to hold; we will ignore this detail except to mention that the necessary assumptions are satisfied for the two classes of domains that we are most interested in, which are

$$\mathcal{U} = \mathring{\mathbb{D}} \subset \mathbb{C},$$
$$\mathcal{U} = (0, L) \times S^1 \subset \mathbb{C}/\mathbb{Z}, \qquad 0 < L \le \infty.$$

Here $\mathbb{D}$ denotes the closed unit disk and $\mathring{\mathbb{D}}$ is its interior. Certain results will be specified to hold only for *bounded* domains, which means in practice that they hold on $\mathring{\mathbb{D}}$ and $(0, L) \times S^1$ for any $L > 0$, but not on $(0, \infty) \times S^1$.

Recall that for $p \in [1, \infty)$ we define the L^p norm of a measurable function $f : \mathcal{U} \to \mathbb{R}^m$ to be

$$\|f\|_{L^p} = \left(\int_{\mathcal{U}} |f|^p \right)^{1/p}.$$

For the space L^∞ we define the norm to be the essential supremum of f over $\mathcal{U}$. Denote by

$$C_0^\infty(\mathcal{U}) \subset C^\infty(\mathcal{U})$$

the set of smooth functions with compact support in $\mathcal{U}$. We say a function f has a **weak j-th partial derivative** g if the *integration by parts* formula holds for all $\varphi \in C_0^\infty(\mathcal{U})$:

$$\int_{\mathcal{U}} g\varphi = -\int_{\mathcal{U}} f\, \partial_j \varphi.$$

Equivalently, this means that g is a partial derivative of f **in the sense of distributions** (see e.g. [LL01]). Higher order weak partial derivatives are defined similarly: recall that for a multiindex $\alpha = (i_1, \ldots i_n)$ we denote

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}},$$

where $|\alpha| := \sum_j i_j$. We then write $\partial^\alpha f = g$ if for all $\varphi \in C_0^\infty(\mathcal{U})$,

$$\int_{\mathcal{U}} g\varphi = (-1)^{|\alpha|} \int_{\mathcal{U}} f\, \partial^\alpha \varphi.$$

Now we may define $W^{k,p}(\mathcal{U})$ to be the set of functions on $\mathcal{U}$ with weak partial derivatives up to order k lying in L^p, and define the norm of such a function by:

$$\|f\|_{W^{k,p}} = \sum_{|\alpha| \le k} \|\partial^\alpha f\|_{L^p}.$$

As $W^{k,p}(\mathcal{U})$ can be regarded as a subset of a k-fold product of $L^p(\mathcal{U})$, it is a Banach space, and it is reflexive and separable for $1 < p < \infty$.

While the Sobolev spaces $W^{k,p}(\mathcal{U})$ are generally defined on *open* domains, we often consider the closure $\overline{\mathcal{U}}$ as the domain for spaces of differentiable functions $C^k(\overline{\mathcal{U}})$ and $C^\infty(\overline{\mathcal{U}})$. For instance, $C^k(\overline{\mathcal{U}})$ is the Banach space of k-times differentiable functions on $\mathcal{U}$ whose derivatives up to order k are bounded and uniformly continuous on $\mathcal{U}$; note that uniform continuity implies the existence of continuous extensions to the closure $\overline{\mathcal{U}}$. Given suitable regularity assumptions for the boundary of $\overline{\mathcal{U}}$, one can show (with some effort) that $C^k(\overline{\mathcal{U}})$ is precisely the set of functions which admit k-times differentiable extensions to some open set containing $\overline{\mathcal{U}}$.

The following two results are special cases of the more general Theorems A.6 and A.9 in Appendix A, proofs of which may be found e.g. in [**AF03**].

PROPOSITION 2.4 (Sobolev embedding theorem). *Assume $1 \le p < \infty$, $kp > n$ and $d \ge 0$ is an integer. Then there exists a continuous inclusion*

$$W^{k+d,p}(\mathcal{U}) \hookrightarrow C^d(\overline{\mathcal{U}}),$$

which is compact if $\mathcal{U}$ is bounded. □

PROPOSITION 2.5 (Rellich-Kondrachov compactness theorem). *If $1 \le p < \infty$ and $\mathcal{U}$ is bounded, then the natural inclusion*

$$W^{k+1,p}(\mathcal{U}) \hookrightarrow W^{k,p}(\mathcal{U})$$

is compact. □

EXERCISE 2.6. Show that Proposition 2.5 fails in general for unbounded domains, e.g. for $\mathbb{R}$.

The next three results for the case $kp > n$ are proved in §A.2 as corollaries of the Sobolev embedding theorem.

PROPOSITION 2.7 (Banach algebra property). *Suppose $1 \le p < \infty$, $kp > n$ and $0 \le m \le k$. Then the product pairing $(f, g) \mapsto fg$ defines a continuous bilinear map*

$$W^{k,p}(\mathcal{U}) \times W^{m,p}(\mathcal{U}) \to W^{m,p}(\mathcal{U}).$$

In particular, $W^{k,p}(\mathcal{U})$ is a Banach algebra. □

The continuity statements above translate into inequalities between the norms in the respective spaces. For example, continuous inclusions $W^{k+d,p} \hookrightarrow C^d$ and $W^{k+1,p} \hookrightarrow W^{k,p}$ respectively imply that

$$\|f\|_{C^d} \le c\|f\|_{W^{k+d,p}}$$
$$\|f\|_{W^{k,p}} \le c\|f\|_{W^{k+1,p}}$$

for some constants $c > 0$ which may depend on d, k, p or $\mathcal{U}$, but not f. Similarly, the Banach algebra property implies

$$\|fg\|_{W^{m,p}} \le c\|f\|_{W^{k,p}}\|g\|_{W^{m,p}},$$

where again, the constant c is independent of g and f.

We state the next result only for the case of bounded domains; it does have an extension to unbounded domains, but the statement becomes more complicated (cf. Theorem A.12). Given an open set $\Omega \subset \mathbb{R}^n$, we denote

$$W^{k,p}(\mathcal{U}, \Omega) := \left\{ u \in W^{k,p}(\mathcal{U}, \mathbb{R}^n) \;\middle|\; \overline{u(\mathcal{U})} \subset \Omega \right\}.$$

Note that this is an open subset if $kp > n$, due to the Sobolev embedding theorem.

PROPOSITION 2.8 (C^k-continuity property). *Assume $1 \le p < \infty$, $kp > n$, $\mathcal{U}$ is bounded and $\Omega \subset \mathbb{R}^n$ is an open set. Then the map*

$$C^k(\Omega, \mathbb{R}^N) \times W^{k,p}(\mathcal{U}, \Omega) \to W^{k,p}(\mathcal{U}, \mathbb{R}^N) : (f, u) \mapsto f \circ u$$

is well defined and continuous. □

REMARK 2.9. Though we will not yet use it in this lecture, Propositions 2.4, 2.7 and 2.8 are the essential conditions needed in order to define smooth Banach manifold structures on spaces of $W^{k,p}$-smooth maps from one manifold to another, cf. [**Elĭ67**, **Pal68**]. This only works under the condition $kp > n$, as the smooth category is not well equipped to deal with discontinuous maps!

The following rescaling result will be needed for nonlinear regularity arguments; see Theorem A.15 in Appendix A for a proof.

PROPOSITION 2.10. *Assume $p \in [1, \infty)$ and $k \in \mathbb{N}$ satisfy $kp > n$, let $\mathring{\mathbb{D}}^n$ denote the open unit ball in $\mathbb{R}^n$, and for each $f \in W^{k,p}(\mathring{\mathbb{D}}^n)$ and $\epsilon \in (0, 1]$, define $f_\epsilon \in W^{k,p}(\mathring{\mathbb{D}}^n)$ by*

$$f_\epsilon(x) := f(\epsilon x).$$

Then there exist constants $C > 0$ and $r > 0$ such that for every $f \in W^{k,p}(\mathring{\mathbb{D}}^n)$,

$$\|f_\epsilon - f(0)\|_{W^{k,p}(\mathring{\mathbb{D}}^n)} \le C\epsilon^r \|f - f(0)\|_{W^{k,p}(\mathring{\mathbb{D}}^n)} \quad \textit{for all } \epsilon \in (0, 1].$$

□

EXERCISE 2.11. Working on a 2-dimensional domain with $kp > 2$, prove directly that for any multiindex α of positive degree k,

$$\|\partial^\alpha f_\epsilon\|_{L^p(\mathring{\mathbb{D}})} \le \epsilon^{k-2/p} \|\partial^\alpha f\|_{L^p(\mathring{\mathbb{D}})}$$

for $f \in W^{k,p}(\mathring{\mathbb{D}})$. Find examples (e.g. in $W^{1,2}(\mathring{\mathbb{D}})$) to show that no estimate of the form

$$\|\partial^\alpha f_\epsilon\|_{L^p(\mathring{\mathbb{D}})} \le C_\epsilon \|f - f(0)\|_{W^{k,p}(\mathring{\mathbb{D}})}$$

with $\lim_{\epsilon \to 0^+} C_\epsilon = 0$ is possible when $kp \le 2$.

2.3. The fundamental elliptic estimate

We will make considerable use of the fact that the linear first-order differential operator

$$\bar{\partial} := \partial_s + i\partial_t : C^\infty(\mathbb{C}, \mathbb{C}) \to C^\infty(\mathbb{C}, \mathbb{C})$$

is **elliptic**. There is no need to discuss here precisely what ellipticity means in full generality (see [**Wend**, §2.B] if you're curious about this); in practice, the main consequence is the following pair of analytical results.

THEOREM 2.12. *If* $1 < p < \infty$*, then* $\bar{\partial} : W^{1,p}(\mathring{\mathbb{D}}) \to L^p(\mathring{\mathbb{D}})$ *admits a bounded right inverse* $T : L^p(\mathring{\mathbb{D}}) \to W^{1,p}(\mathring{\mathbb{D}})$.

THEOREM 2.13. *If* $1 < p < \infty$ *and* $k \in \mathbb{N}$*, then there exists a constant* $c > 0$ *such that for all* $f \in W_0^{k,p}(\mathring{\mathbb{D}})$*,*

$$\|f\|_{W^{k,p}} \le c\|\bar{\partial} f\|_{W^{k-1,p}}.$$

Here $W_0^{k,p}(\mathring{\mathbb{D}})$ denotes the $W^{k,p}$-closure of $C_0^\infty(\mathring{\mathbb{D}})$, the latter being space of smooth functions on $\mathring{\mathbb{D}}$ with compact support.

The complete proofs of the two theorems above are rather lengthy, and we shall refer to [**Wend**, §2.6 and 2.A] for the details, but we can at least explain why they hold in the case $p = 2$. First, it is straightforward to show that the function $K \in L^1_{\text{loc}}(\mathbb{C})$ defined by

$$K(z) = \frac{1}{2\pi z}$$

is a **fundamental solution** for the equation $\bar{\partial} u = f$, meaning it satisfies

$$\bar{\partial} K = \delta$$

in the sense of distributions, where δ denotes the Dirac δ-function. Hence for any $f \in C_0^\infty(\mathbb{C})$, one finds a smooth solution $u : \mathbb{C} \to \mathbb{C}$ to the equation $\bar{\partial} u = f$ as the convolution

$$u(z) = (K * f)(z) := \int_{\mathbb{C}} K(z - \zeta) f(\zeta)\, d\mu(\zeta),$$

where $d\mu(\zeta)$ denotes the Lebesgue measure with respect to the variable $\zeta \in \mathbb{C}$. It is not hard to show from this formula that whenever $f \in C_0^\infty$, $K * f$ has decaying behavior at infinity (see [**Wend**, Lemma 2.6.13]). Thus if $u \in C_0^\infty$ and $\bar{\partial} u = f$, it follows that $u - K * f$ is a holomorphic function on $\mathbb{C}$ that decays at infinity, hence $u \equiv K * f$. Since $C_0^\infty(\mathring{\mathbb{D}})$ is dense in $L^p(\mathring{\mathbb{D}})$ for all $p < \infty$, Theorem 2.12 now follows from the claim that for all $f \in C_0^\infty(\mathring{\mathbb{D}})$, there exist estimates of the form

$$\|K * f\|_{L^p(\mathring{\mathbb{D}})} \le c\|f\|_{L^p(\mathring{\mathbb{D}})}, \qquad \|\partial_j (K * f)\|_{L^p(\mathring{\mathbb{D}})} \le c\|f\|_{L^p(\mathring{\mathbb{D}})}, \tag{2.4}$$

with $\partial_j = \partial_s$ or ∂_t for $j = 1, 2$ respectively, and the constant $c > 0$ independent of f.

EXERCISE 2.14. Use Theorem 2.12 and the remarks above to prove Theorem 2.13 for the case $k = 1$ with $f \in C_0^\infty(\mathring{\mathbb{D}})$, then extend it to $f \in W_0^{1,p}(\mathring{\mathbb{D}})$ by a density argument. Then extend it to the general case by differentiating both f and $\bar{\partial} f$.

The first estimate in (2.4) is not too hard if you remember your introductory measure theory class: it follows from a general "potential inequality" for convolution operators (see [**Wend**, Lemma 2.6.10]), similar to Young's inequality, the key points being that K is locally of class L^1 and $\mathring{\mathbb{D}}$ has finite measure. For the second inequality, observe that $\bar{\partial}(K * f) = f$, and the rest of the first derivative of $K * f$ is determined by $\partial(K * f)$, where

$$\partial := \partial_s - i\partial_t.$$

Differentiating K in the sense of distributions provides a formula for $\partial(K * f)$ as a principal value integral, namely

$$\partial(K * f)(z) = -\frac{1}{\pi} \lim_{\epsilon \to 0^+} \int_{|\zeta - z| \ge \epsilon} \frac{f(\zeta)}{(z-\zeta)^2}\, d\mu(\zeta).$$

This is a so-called **singular integral operator**: it is similar to our previous convolution operator, but more difficult to handle because the kernel $\frac{1}{z^2}$ is not of class L^1_{loc} on $\mathbb{C}$. The proof of the estimate $\|\partial(K * f)\|_{L^p} \le c\|f\|_{L^p}$ for all $f \in C_0^\infty(\mathring{\mathbb{D}})$ follows from a rather difficult general estimate on singular integral operators, known as the *Calderón-Zygmund inequality*, cf. [**Wend**, §2.A] and the references therein. The good news however is that the first step in that proof is not hard: that is the case $p = 2$.

As is the case for all elliptic operators with constant coefficients, the L^2-estimate on the fundamental solution of $\bar\partial$ admits an easy proof using Fourier transforms:

PROPOSITION 2.15. *For all $f \in C_0^\infty(\mathbb{C})$, we have $\|\partial(K * f)\|_{L^2} = \|f\|_{L^2}$.*

PROOF. A sufficiently nice function $u : \mathbb{C} \to \mathbb{C}$ is related to its Fourier transform $\hat{u} : \mathbb{C} \to \mathbb{C}$ by

$$u(z) = \int_{\mathbb{C}} \hat{u}(\zeta) e^{2\pi i (z \cdot \zeta)}\, d\mu(\zeta)$$

and thus satisfies the identities

$$\widehat{\bar\partial u}(\zeta) = 2\pi i \zeta \hat{u}(\zeta),$$
$$\widehat{\partial u}(\zeta) = 2\pi i \bar\zeta \hat{u}(\zeta).$$

Since $u = K * f$ we have $\hat{u} = \hat{K}\hat{f}$, and since $\bar\partial K = \delta$, we have $2\pi i \zeta \hat{K} = 1$. Hence we may apply Plancharel's theorem to deduce

$$\begin{aligned}\|\partial(K * f)\|_{L^2} &= \|\partial u\|_{L^2} = \|\widehat{\partial u}\|_{L^2} = \|2\pi i \bar\zeta \hat{u}\|_{L^2} = \|2\pi i \bar\zeta \hat{K}\hat{f}\|_{L^2} \\ &= \left\| \frac{\bar\zeta}{\zeta} 2\pi i \zeta \hat{K}\hat{f} \right\|_{L^2} = \left\| \frac{\bar\zeta}{\zeta} \hat{f} \right\|_{L^2} = \|\hat{f}\|_{L^2} = \|f\|_{L^2}.\end{aligned}$$

□

2.4. Regularity

We will now use the estimate $\|u\|_{W^{k,p}} \le c\|\bar\partial u\|_{W^{k-1,p}}$ from the previous section to prove three types of results about solutions to Cauchy-Riemann type equations:

(1) All solutions of reasonable Sobolev-type regularity are smooth.
(2) Any collection of solutions satisfying uniform bounds in certain Sobolev norms also locally satisfy uniform C^∞-bounds.
(3) All reasonable Sobolev-type topologies on spaces of solutions are (locally) equivalent to the C^∞-topology.

In the following,

$$\mathbb{D}_r \subset \mathbb{C}$$

denotes the closed disk of radius $r > 0$, and $\mathring{\mathbb{D}}_r$ denotes its interior. Note that functions of class $C^\infty(\mathbb{D}_r)$ are assumed to be smooth up to the boundary (or equivalently, on some open neighborhood of $\mathbb{D}_r$ in $\mathbb{C}$), not just on $\mathring{\mathbb{D}}_r$.

2.4.1. The linear case. Recall from Exercise 2.2 that every linear Cauchy-Riemann type operator on a vector bundle of complex rank n locally takes the form $\bar\partial + A$, where $\bar\partial = \partial_s + i\partial_t$, and A is a smooth function with values in $\mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$. Using the Sobolev embedding theorem, the following result implies by induction that solutions $u \in W^{1,p}$ to the equation $(\bar\partial + A)u = 0$ are always smooth.

THEOREM 2.16 (Linear regularity). *Assume $1 < p < \infty$ and $k, m \in \mathbb{N}$.*

(1) If $u \in W^{k,p}(\mathring{\mathbb{D}})$ satisfies $\bar\partial u \in W^{m,p}(\mathring{\mathbb{D}})$, then u is in $W^{m+1,p}$ on every compact subset of $\mathring{\mathbb{D}}$.

(2) Suppose $f_\nu \in W^{m,p}(\mathring{\mathbb{D}})$ is a sequence converging in the $W^{m,p}$-topology to $f \in W^{m,p}(\mathring{\mathbb{D}})$ as $\nu \to \infty$, and $u_\nu \in W^{k,p}(\mathring{\mathbb{D}})$ is a sequence with $\bar\partial u_\nu = f_\nu$.

(a) If there exist uniform bounds on $\|u_\nu\|_{W^{k,p}}$ and $\|f_\nu\|_{W^{m,p}}$ over $\mathring{\mathbb{D}}$ as $\nu \to \infty$, then $\|u_\nu\|_{W^{m+1,p}}$ is also uniformly bounded on every compact subset of $\mathring{\mathbb{D}}$.

(b) If the sequence u_ν is $W^{k,p}$-convergent on $\mathring{\mathbb{D}}$ to a function $u \in W^{k,p}(\mathring{\mathbb{D}})$ satisfying $\bar\partial u = f$, then it is also $W^{m+1,p}$-convergent on every compact subset of $\mathring{\mathbb{D}}$.

PROOF. We begin by proving statement (2a), assuming that statement (1) is already known, hence $u_\nu \in W^{m+1,p}_{\mathrm{loc}}(\mathring{\mathbb{D}})$ since $f_\nu \in W^{m,p}(\mathring{\mathbb{D}})$. Assume $m = k$, since there is otherwise nothing to prove. Then by induction, it suffices to show that uniform bounds on $\|u_\nu\|_{W^{k,p}(\mathring{\mathbb{D}})}$ and $\|f_\nu\|_{W^{k,p}(\mathring{\mathbb{D}})}$ imply a uniform bound on $\|u_\nu\|_{W^{k+1,p}(\mathring{\mathbb{D}}_r)}$ for any given $r < 1$; equivalently, this would mean there is a uniform bound on $\|\partial_j u_\nu\|_{W^{k,p}(\mathring{\mathbb{D}}_r)}$ for $j = 1, 2$. In order to apply the elliptic estimate, we need to work with functions with compact support in $\mathring{\mathbb{D}}$, thus choose a smooth bump function

$$\beta \in C_0^\infty(\mathring{\mathbb{D}}, [0, 1])$$

that satisfies $\beta|_{\mathbb{D}_r} \equiv 1$. We then have $\beta\,\partial_j u_\nu \in C_0^\infty(\mathring{\mathbb{D}})$, so by Theorem 2.13,

$$\begin{aligned} \|\partial_j u_\nu\|_{W^{k,p}(\mathring{\mathbb{D}}_r)} &\le \|\beta\,\partial_j u_\nu\|_{W^{k,p}(\mathring{\mathbb{D}})} \le c\left\|\bar\partial\left(\beta\,\partial_j u_\nu\right)\right\|_{W^{k-1,p}(\mathring{\mathbb{D}})} \\ &\le c\|(\bar\partial\beta)(\partial_j u_\nu)\|_{W^{k-1,p}} + c\|\beta\,\bar\partial(\partial_j u_\nu)\|_{W^{k-1,p}}. \end{aligned} \tag{2.5}$$

The first term on the right hand side is uniformly bounded since $\bar\partial\beta$ is smooth and $\|u_\nu\|_{W^{k,p}}$ is uniformly bounded. To control the second term, we differentiate the equation $\bar\partial u_\nu = f_\nu$, giving

$$\bar\partial(\partial_j u_\nu) = \partial_j f_\nu.$$

This also has a uniformly bounded $W^{k-1,p}$-norm since $\|f_\nu\|_{W^{k,p}}$ is uniformly bounded. Since β is smooth, this bounds the second term on the right hand side of (2.5) as $\nu \to \infty$, and we are done.

Statement (2b) follows by a similar argument bounding $\|\partial_j(u - u_\nu)\|_{W^{k,p}(\mathring{\mathbb{D}}_r)}$ in terms of $\|u - u_\nu\|_{W^{k,p}(\mathring{\mathbb{D}})}$ and $\|f - f_\nu\|_{W^{k,p}(\mathring{\mathbb{D}})}$; we leave the details as an exercise.

Lastly, we prove statement (1), where again it suffices to assume $\bar{\partial} u = f \in W^{k,p}(\mathring{\mathbb{D}})$ and show that $u|_{\mathring{\mathbb{D}}_r} \in W^{k+1,p}(\mathring{\mathbb{D}}_r)$ for some $r < 1$. The idea is to use the same argument that was used for statement (2a), but with the partial derivatives $\partial_j u$ replaced by the **difference quotients**

$$D_j^h u(z) := \frac{u(z + he_j) - u(z)}{h}, \qquad j = 1, 2,$$

where $e_1 := \partial_s$, $e_2 := \partial_t$, and the role of the index $\nu \to \infty$ is now played by the parameter $h \in \mathbb{R} \setminus \{0\}$ approaching 0. Note that if $u \in W^{k,p}(\mathring{\mathbb{D}})$, then $\beta\, D_j^h u$ is a well-defined function on $\mathring{\mathbb{D}}$ for all $|h| \neq 0$ sufficiently small and belongs to $W_0^{k,p}(\mathring{\mathbb{D}})$. The analogue of (2.5) in this context is then

$$\begin{aligned}\|D_j^h u\|_{W^{k,p}(\mathring{\mathbb{D}}_r)} \le \|\beta D_j^h u\|_{W^{k,p}(\mathring{\mathbb{D}})} &\le c \left\| \bar{\partial} \left(\beta\, D_j^h u\right) \right\|_{W^{k-1,p}(\mathring{\mathbb{D}})} \\ &\le c \|(\bar{\partial}\beta)(D_j^h u)\|_{W^{k-1,p}} + c\|\beta\, \bar{\partial}(D_j^h u)\|_{W^{k-1,p}}.\end{aligned}$$

The first term is bounded independently of h since $\partial_j u \in W^{k-1,p}(\mathring{\mathbb{D}})$, implying a uniform $W^{k-1,p}$-bound on $D_j^h u$ as $h \to 0$. To control the second term, we can apply the operator D_j^h to the equation $\bar{\partial} u = f$, giving

$$\bar{\partial}(D_j^h u) = D_j^h(\bar{\partial} u) = D_j^h f.$$

This satisfies a $W^{k-1,p}$-bound that is uniform in h since $\partial_j f \in W^{k-1,p}(\mathring{\mathbb{D}})$, so we conclude that for all $|h|$ sufficiently small,

$$\|D_j^h u\|_{W^{k,p}(\mathring{\mathbb{D}}_r)} \le c$$

for some constant $c > 0$ that does not change as $h \to 0$. By a standard application of the Banach-Alaoglu theorem (cf. [**Eva98**, §5.8.2]), this implies the existence of a sequence $h_\nu \to 0$ for which $D_j^{h_\nu} u$ is $W^{k,p}$-convergent on $\mathring{\mathbb{D}}_r$, and its limit is necessarily $\partial_j u$, which therefore belongs to $W^{k,p}$. Indeed, if $k = 0$, the uniform L^p-bound on $D_j^{h_\nu} u$ over $\mathring{\mathbb{D}}_r$ for any sequence $h_\nu \to 0$ gives rise to a weakly L^p-convergent subsequence via the Banach-Alaoglu theorem. The limit of this subsequence belongs to $L^p(\mathring{\mathbb{D}}_r)$, and it is straightforward to show using the definition of weak derivatives that this limit is $\partial_j u$. One finds the same result for any $k \in \mathbb{N}$ by applying this argument to higher-order derivatives of $\partial_j u$. The conclusion is that u is in $W^{k+1,p}$ on $\mathring{\mathbb{D}}_r$, since u and both of its first partial derivatives belong to $W^{k,p}$. ☐

EXERCISE 2.17. Show that all three parts of Theorem 2.16 continue to hold if the operator $\bar{\partial}$ is replaced by $\bar{\partial} + A$ or $\bar{\partial} + A_\nu$, where $A, A_\nu \in C^\infty(\mathbb{D}, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n))$ with $A_\nu \to A$ in C^∞ as $\nu \to \infty$.

EXERCISE 2.18. Use Theorem 2.16(1) to extend Theorem 2.12 to the existence of a bounded right inverse for

$$\bar{\partial} : W^{k,p}(\mathring{\mathbb{D}}) \to W^{k-1,p}(\mathring{\mathbb{D}}).$$

Hint: For any $R > 1$, there exists a bounded linear extension operator $E : W^{k,p}(\mathring{\mathbb{D}}) \to W^{k,p}(\mathring{\mathbb{D}}_R)$ with the property $(Ef)|_{\mathring{\mathbb{D}}} = f$ for all $f \in W^{k,p}(\mathring{\mathbb{D}})$; see Theorem A.4 and Corollary A.5.

The above exercise can be used to improve the first part of Theorem 2.16 to cover weak solutions of class L^1_{loc}. We start with a classical result about "weakly holomorphic" functions:

LEMMA 2.19. *If $u \in L^1(\mathring{\mathbb{D}})$ satisfies $\bar{\partial} u = 0$ in the sense of distributions, then u is smooth and holomorphic.*

PROOF. Taking real and imaginary parts, it suffices to prove that the same statement holds for the Laplace equation. By mollification, any weakly harmonic function can be approximated in L^1 with smooth harmonic functions. The latter satisfy the mean value property, which behaves well under L^1-convergence, so the result follows from the mean value characterization of harmonic functions; see [**Wend**, Lemma 2.6.26] for more details. $\square$

LEMMA 2.20. *Suppose $1 < p < \infty$, $k \in \mathbb{N}$, and $u \in L^1(\mathring{\mathbb{D}})$ is a weak solution to $\bar{\partial} u = f$ for some $f \in W^{k,p}(\mathring{\mathbb{D}})$. Then u is of class $W^{k+1,p}$ on every compact subset of $\mathring{\mathbb{D}}$.*

PROOF. Let $T : W^{k,p}(\mathring{\mathbb{D}}) \to W^{k+1,p}(\mathring{\mathbb{D}})$ denote a bounded right inverse of $\bar{\partial} : W^{k+1,p}(\mathring{\mathbb{D}}) \to W^{k,p}(\mathring{\mathbb{D}})$ as provided by Exercise 2.18. Then $u - Tf \in L^1(\mathring{\mathbb{D}})$ is a weak solution to $\bar{\partial}(u - Tf) = 0$ and is thus smooth by Lemma 2.19. In particular, $u - Tf$ restricts to $\mathring{\mathbb{D}}_r$ for every $r < 1$ as a function of class $W^{k+1,p}$, implying that u also has a restriction in $W^{k+1,p}(\mathring{\mathbb{D}}_r)$. $\square$

COROLLARY 2.21 (Weak linear regularity). *Suppose $1 < p < \infty$. Then given $A \in C^\infty(\mathbb{D}, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n))$, every weak solution $u \in L^p(\mathring{\mathbb{D}}, \mathbb{C}^n)$ of $(\bar{\partial} + A)u = 0$ is smooth on $\mathring{\mathbb{D}}$.* $\square$

2.4.2. The nonlinear case. Locally, every J-holomorphic curve can be regarded as a map $u : \mathring{\mathbb{D}} \to \mathbb{C}^n$ satisfying $u(0) = 0$ and

$$\bar{\partial}_J u := \partial_s u + J(u)\partial_t u = 0,$$

where J is a smooth almost complex structure on $\mathbb{C}^n$ satisfying $J(0) = i$. Theorem 2.16 now has the following analogue.

THEOREM 2.22 (Nonlinear regularity). *Assume $1 < p < \infty$ and $k \in \mathbb{N}$ satisfy $kp > 2$, and fix a smooth almost complex structure J on $\mathbb{C}^n$ with $J(0) = i$.*

(1) Every map $u \in W^{k,p}(\mathring{\mathbb{D}}, \mathbb{C}^n)$ satisfying $u(0) = 0$ and $\bar{\partial}_J u = 0$ is smooth on $\mathring{\mathbb{D}}$.
(2) Suppose J_ν is a sequence of smooth almost complex structures on $\mathbb{C}^n$ converging in C^∞_{loc} to J as $\nu \to \infty$, and $u_\nu \in W^{k,p}(\mathring{\mathbb{D}}, \mathbb{C}^n)$ is a sequence of smooth maps satisfying $\bar{\partial}_{J_\nu} u_\nu = 0$.
   (a) If the maps u_ν are uniformly $W^{k,p}$-bounded on $\mathring{\mathbb{D}}$, then they are also uniformly C^m-bounded on compact subsets of $\mathring{\mathbb{D}}$ for every $m \in \mathbb{N}$.
   (b) If the sequence u_ν is $W^{k,p}$-convergent on $\mathring{\mathbb{D}}$ to a smooth map $u : \mathring{\mathbb{D}} \to \mathbb{C}^n$, then it is also C^∞-convergent on every compact subset of $\mathring{\mathbb{D}}$.

Our proof of this will follow much the same outline as the proof of Theorem 2.16, and indeed, one could use exactly the same argument if J were identically equal to i (in which case the theorem can also be deduced from complex analysis). The reason it works in the general case is that if we zoom in on a sufficiently small neighborhood of the origin in $\mathbb{C}^n$, then J can be viewed as a C^∞-small perturbation of i. To make this precise, we shall use the following rescaling trick.

Associate to any smooth almost complex structure J on $\mathbb{C}^n$ the function

$$Q := i - J \in C^\infty(\mathbb{C}^n, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)).$$

In terms of Q, the equation $\partial_s u + J(u)\partial_t u = 0$ then becomes

$$\bar{\partial} u - (Q \circ u)\partial_t u = 0, \tag{2.6}$$

where we are regarding $Q \circ u$ as a function $\mathring{\mathbb{D}} \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$. Given constants $R \geq 1$ and $\epsilon \in (0, 1]$, associate to J and u the functions

$$\begin{aligned} \widehat{J} : \mathbb{C}^n &\to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n), & \widehat{J}(p) &:= J(p/R), \\ \widehat{Q} : \mathbb{C}^n &\to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n), & \widehat{Q}(p) &:= Q(p/R) = i - \widehat{J}(p), \\ \hat{u} : \mathring{\mathbb{D}} &\to \mathbb{C}^n, & u(z) &:= Ru(\epsilon z). \end{aligned} \tag{2.7}$$

Now u satisfies (2.6) if and only if $\hat{u}$ satisfies

$$\bar{\partial}\hat{u} - (\widehat{Q} \circ \hat{u})\partial_t \hat{u} = 0. \tag{2.8}$$

The rescaled almost complex structure has the convenient feature that if $J(0) = i$, then $\widehat{J}$ can be made arbitrarily C^∞-close to i on the unit disk

$$\mathbb{D}^{2n} \subset \mathbb{C}^n$$

by choosing R sufficiently large, which means $\|\widehat{Q}\|_{C^m(\mathbb{D}^{2n})}$ can be made arbitrarily small for every $m \in \mathbb{N}$. If u is also continuous and satisfies $u(0) = 0$, then after fixing some large value for R, we can also choose $\epsilon \in (0, 1]$ sufficiently small to ensure $\overline{u(\mathring{\mathbb{D}})} \subset \mathring{\mathbb{D}}^{2n}$ and make $\|\widehat{Q} \circ \hat{u}\|_{C^0(\mathbb{D})}$ arbitrarily small. By Propositions 2.8 and 2.10, we can similarly arrange for $\|\widehat{Q} \circ \hat{u}\|_{W^{k,p}}$ to be arbitrarily small if u is of class $W^{k,p}$ with $kp > 2$, and the same will hold for $\|\widehat{Q}_\nu \circ \hat{u}_\nu\|_{W^{k,p}}$ when ν is large if $\|u_\nu\|_{W^{k,p}}$ is uniformly bounded and $u_\nu(0) \to 0$. Here of course we abbreviate $Q_\nu := i - J_\nu$ and $\widehat{Q}_\nu(p) := Q_\nu(p/R)$. The effect is to make equations such as (2.8) $W^{k,p}$-close to the linear equation $\bar{\partial}\hat{u} = 0$ if $\epsilon > 0$ and $R > 0$ are sufficiently small and large respectively.

The price we pay for this rescaling is that if we are able to prove e.g. a uniform bound on the norms $\|\hat{u}_\nu\|_{W^{k,p}(\mathring{\mathbb{D}})}$ for some sequence u_ν, then the resulting $W^{k+1,p}$-bound for u_ν will be valid only on $\mathring{\mathbb{D}}_\epsilon$, a very small ball about the origin. But this is good enough for obtaining estimates over all compact subsets of $\mathring{\mathbb{D}}$: indeed, we can always reparametrize $u : \mathring{\mathbb{D}} \to \mathbb{C}^n$ to put the origin at some other point and prove suitable estimates near that point, appealing in the end to the fact that any compact subset of $\mathring{\mathbb{D}}$ is covered by a finite union of small disks about points.

The need to use this rescaling trick is one of a few reasons why the condition $kp > 2$ is needed in Theorem 2.22, while it was irrelevant in the linear case.

PROOF OF THEOREM 2.22. We will prove statement (2a) and leave the rest as exercises.

By the remarks above, it suffices to prove that if $u_\nu : \mathring{\mathbb{D}} \to \mathbb{C}^n$ are smooth J_ν-holomorphic curves satisfying a uniform bound in $W^{k,p}(\mathring{\mathbb{D}})$, then for some $r < 1$, the rescaled $\widehat{J}_\nu$-holomorphic curves $\hat{u}_\nu : \mathring{\mathbb{D}} \to \mathbb{C}^n$ defined as in (2.7) satisfy a uniform $W^{k+1,p}$-bound on $\mathring{\mathbb{D}}_r$. In fact, it suffices to prove that every subsequence of u_ν has a further subsequence for which this is true. Indeed, if the bound for the whole sequence did not exist, then we would be able to find a subsequence with norms blowing up to infinity, and no further subsequence of this subsequence could satisfy a uniform bound. With this understood, we can appeal to the fact that $W^{k,p}$-bounded sequences are also C^0-bounded for $kp > 2$ and thus replace u_ν with a subsequence (still denoted by u_ν) such that, after a suitable change of coordinates on $\mathbb{C}^n$,

$$u_\nu(0) \to 0.$$

Our goal is then to show that for a suitable choice of the rescaling parameters ϵ and R, this subsequence admits a uniform bound on $\|\partial_j \hat{u}_\nu\|_{W^{k,p}(\mathring{\mathbb{D}}_r)}$ for $j = 1, 2$.

The argument begins exactly the same as in the linear case: choose a smooth bump function

$$\beta \in C_0^\infty(\mathring{\mathbb{D}}, [0, 1])$$

that satisfies $\beta|_{\mathbb{D}_r} \equiv 1$. We then have $\beta\, \partial_j \hat{u}_\nu \in C_0^\infty(\mathring{\mathbb{D}})$, so by Theorem 2.13,

$$\|\partial_j \hat{u}_\nu\|_{W^{k,p}(\mathring{\mathbb{D}}_r)} \le \|\beta\, \partial_j \hat{u}_\nu\|_{W^{k,p}(\mathring{\mathbb{D}})} \le c \left\|\bar{\partial}\left(\beta\, \partial_j \hat{u}_\nu\right)\right\|_{W^{k-1,p}(\mathring{\mathbb{D}})} . \tag{2.9}$$

Instead of rewriting $\bar{\partial}(\beta\, \partial_j \hat{u}_\nu)$ as a sum of two terms, let us derive a PDE satisfied by $\beta\, \partial_j \hat{u}_\nu$. Differentiating the equation $\bar{\partial}\hat{u}_\nu - (\widehat{Q}_\nu \circ \hat{u}_\nu)\partial_t \hat{u}_\nu = 0$ gives

$$\bar{\partial}(\partial_j \hat{u}_\nu) = \partial_j(\bar{\partial}\hat{u}_\nu) = (d\widehat{Q}_\nu \circ \hat{u}_\nu)\,(\partial_j \hat{u}_\nu, \partial_t \hat{u}_\nu) + (\widehat{Q}_\nu \circ \hat{u}_\nu)\partial_j \partial_t \hat{u}_\nu,$$

thus $\beta\, \partial_j \hat{u}_\nu$ satisfies

$$\begin{aligned} \bar{\partial}(\beta\, \partial_j \hat{u}_\nu) - (\widehat{Q}_\nu {\circ} \hat{u}_\nu)\partial_t(\beta\, \partial_j \hat{u}_\nu) & \\ = \beta(d\widehat{Q}_\nu \circ \hat{u}_\nu)(\partial_j \hat{u}_\nu, \partial_t \hat{u}_\nu) &+ \left(\bar{\partial}\beta - (\widehat{Q}_\nu \circ \hat{u}_\nu)\partial_t \beta\right) \partial_j \hat{u}_\nu \\ = (d\widehat{Q}_\nu \circ \hat{u}_\nu)(\beta\, \partial_j \hat{u}_\nu, \partial_t \hat{u}_\nu) &+ \left(\bar{\partial}\beta - (\widehat{Q}_\nu \circ \hat{u}_\nu)\partial_t \beta\right) \partial_j \hat{u}_\nu, \end{aligned} \tag{2.10}$$

and combining this with (2.9) gives

$$\begin{aligned} \|\beta\, \partial_j \hat{u}_\nu\|_{W^{k,p}} \le c\left\|(\widehat{Q}_\nu \circ \hat{u}_\nu)\partial_t(\beta\, \partial_j \hat{u}_\nu)\right\|_{W^{k-1,p}} &+ c\left\|(d\widehat{Q}_\nu \circ \hat{u}_\nu)(\beta\, \partial_j \hat{u}_\nu, \partial_t \hat{u}_\nu)\right\|_{W^{k-1,p}} \\ &+ c\left\|\left(\bar{\partial}\beta - (\widehat{Q}_\nu \circ \hat{u}_\nu)\partial_t \beta\right) \partial_j \hat{u}_\nu\right\|_{W^{k-1,p}} . \end{aligned} \tag{2.11}$$

In order to find bounds for the three terms on the right, recall that using Propositions 2.8 and 2.10 and the assumption $u_\nu(0) \to 0$, we can suppose

$$\left\|\widehat{Q}_\nu \circ \hat{u}_\nu\right\|_{W^{k,p}} \le \delta$$

for sufficiently large ν, where $\delta > 0$ is a constant that may be assumed arbitrarily small via suitable choices of the rescaling parameters ϵ and R. This provides a uniform bound on the third term in (2.11), as there is also a continuous product pairing $W^{k,p} \times W^{k-1,p} \to W^{k-1,p}$ by Prop. 2.7, giving an estimate of the form

$$\left\| \left(\bar{\partial}\beta - (\widehat{Q}_\nu \circ \hat{u}_\nu)\partial_t\beta \right) \partial_j \hat{u}_\nu \right\|_{W^{k-1,p}} \le c \left\| \left(\bar{\partial}\beta - (\widehat{Q}_\nu \circ \hat{u}_\nu)\partial_t\beta \right) \right\|_{W^{k,p}} \cdot \|\partial_j \hat{u}_\nu\|_{W^{k-1,p}} \le c' \|\hat{u}_\nu\|_{W^{k,p}} \le c''.$$

For the first term on the right side of (2.11), the product pairing similarly gives

$$\left\| (\widehat{Q}_\nu \circ \hat{u}_\nu)\partial_t(\beta\, \partial_j \hat{u}_\nu) \right\|_{W^{k-1,p}} \le c \left\| \widehat{Q}_\nu \circ \hat{u}_\nu \right\|_{W^{k,p}} \cdot \|\partial_t(\beta\, \partial_j \hat{u}_\nu)\|_{W^{k-1,p}} \le c\delta \|\beta\, \partial_j \hat{u}_\nu\|_{W^{k,p}}.$$

Finally, since $J_\nu \to J$ in C^{k+1} on compact subsets, we are also free to assume after adjusting the rescaling parameters that

$$\|d\widehat{Q}_\nu \circ \hat{u}_\nu\|_{W^{k,p}} \le \delta,$$

so we can apply the product pairing $W^{k,p} \times W^{k-1,p} \to W^{k-1,p}$ twice to estimate

$$\begin{aligned} \left\| (d\widehat{Q}_\nu \circ \hat{u}_\nu)(\beta\, \partial_j \hat{u}_\nu, \partial_t \hat{u}_\nu) \right\|_{W^{k-1,p}} &\le c \left\| d\widehat{Q}_\nu \circ \hat{u}_\nu \right\|_{W^{k,p}} \cdot \|\beta\, \partial_j \hat{u}_\nu\|_{W^{k,p}} \cdot \|\partial_t \hat{u}_\nu\|_{W^{k-1,p}} \\ &\le c\delta \|\beta\, \partial_j \hat{u}_\nu\|_{W^{k,p}} \cdot \|\hat{u}_\nu\|_{W^{k,p}} \\ &\le cc'\delta \|\beta\, \partial_j \hat{u}_\nu\|_{W^{k,p}} =: c''\delta \|\beta\, \partial_j \hat{u}_\nu\|_{W^{k,p}}. \end{aligned}$$

Combining the three estimates for the right hand side of (2.11) now gives

$$\|\beta\, \partial_j \hat{u}_\nu\|_{W^{k,p}} \le c + c\delta \|\beta\, \partial_j \hat{u}_\nu\|_{W^{k,p}},$$

so after adjusting the scaling parameters R and ϵ to ensure $c\delta < 1$, we obtain the uniform bound

$$\|\beta\, \partial_j \hat{u}_\nu\|_{W^{k,p}} \le \frac{c}{1 - c\delta}.$$

This provides the desired uniform bound on $\|\partial_j \hat{u}_\nu\|_{W^{k,p}(\mathring{\mathbb{D}}_r)}$. □

EXERCISE 2.23. Use an analogous argument via difference quotients to prove statement (1) in Theorem 2.22. *Hint: If you're anything like me, you might get stuck trying to estimate the second term in the difference quotient analogue of* (2.11)*. The difficulty is that this expression was derived using the chain rule for derivatives, and there is no similarly simple chain rule for difference quotients. The trick is to remember that difference quotients only differ from the corresponding derivatives by a remainder term. The remainder will produce an extra term in the difference quotient version of* (2.11)*, but the extra term can be bounded.*

2.5. Linear local existence and applications

The following lemma can be applied in the case $A \in C^\infty(\mathbb{D}, \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n))$ to prove the aforementioned standard fact that complex-linear Cauchy-Riemann type operators induce holomorphic structures on vector bundles. The version with weakened regularity will be applied below to prove a useful "unique continuation" result about solutions to $(\bar{\partial} + A)f = 0$ in the real-linear case.

LEMMA 2.24. *Assume $2 < p < \infty$ and $A \in L^p(\mathring{\mathbb{D}}, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n))$. Then for sufficiently small $\epsilon > 0$, the problem*

$$\begin{aligned} \bar{\partial} u + Au &= 0 \\ u(0) &= u_0 \end{aligned}$$

has a solution $u \in W^{1,p}(\mathring{\mathbb{D}}_\epsilon, \mathbb{C}^n)$.

REMARK 2.25. Note that $u : \mathring{\mathbb{D}}_\epsilon \to \mathbb{C}^n$ in the above statement is only a *weak* solution to $\bar{\partial} u + Au = 0$, as it is not necessarily differentiable, but by the Sobolev embedding theorem, it is at least continuous.

PROOF OF LEMMA 2.24. The main idea is that if we take $\epsilon > 0$ sufficiently small, then the restriction of $\bar{\partial} + A$ to $\mathring{\mathbb{D}}_\epsilon$ can be regarded as a small perturbation of $\bar{\partial}$ in the space of bounded linear operators $W^{1,p} \to L^p$. Since the latter has a bounded right inverse by Theorem 2.12, the same will be true for the perturbation.

Since $p > 2$, the Sobolev embedding theorem implies that functions $u \in W^{1,p}$ are also continuous and bounded by $\|u\|_{W^{1,p}}$, thus we can define a bounded linear operator

$$\Phi : W^{1,p}(\mathring{\mathbb{D}}) \to L^p(\mathring{\mathbb{D}}) \times \mathbb{C}^n : u \mapsto (\bar{\partial} u, u(0)).$$

Theorem 2.12 implies that this operator is also surjective and has a bounded right inverse, namely

$$L^p(\mathring{\mathbb{D}}) \times \mathbb{C}^n \to W^{1,p}(\mathring{\mathbb{D}}) : (f, u_0) \mapsto Tf - Tf(0) + u_0,$$

where $T : L^p(\mathring{\mathbb{D}}) \to W^{1,p}(\mathring{\mathbb{D}})$ is a right inverse of $\bar{\partial}$. Thus any operator sufficiently close to Φ in the norm topology also has a right inverse. Now define $\chi_\epsilon : \mathbb{D} \to \mathbb{R}$ to be the function that equals 1 on $\mathbb{D}_\epsilon$ and 0 outside of it, and let

$$\Phi_\epsilon : W^{1,p}(\mathring{\mathbb{D}}) \to L^p(\mathring{\mathbb{D}}) \times \mathbb{C}^n : u \mapsto ((\bar{\partial} + \chi_\epsilon A)u, u(0)).$$

To see that this is a bounded operator, it suffices to check that $W^{1,p} \to L^p : u \mapsto Au$ is bounded if $A \in L^p$; indeed,

$$\|Au\|_{L^p} \le \|A\|_{L^p} \|u\|_{C^0} \le c\|A\|_{L^p} \|u\|_{W^{1,p}},$$

again using the Sobolev embedding theorem. Now by this same trick, we find

$$\|\Phi_\epsilon u - \Phi u\| = \|\chi_\epsilon A u\|_{L^p(\mathring{\mathbb{D}})} \le c\|A\|_{L^p(\mathring{\mathbb{D}}_\epsilon)} \|u\|_{W^{1,p}(\mathring{\mathbb{D}})},$$

thus $\|\Phi_\epsilon - \Phi\|$ is small if ϵ is small, and it follows that in this case Φ_ϵ is surjective. Our desired solution is therefore the restriction of any $u \in \Phi_\epsilon^{-1}(0, u_0)$ to $\mathring{\mathbb{D}}_\epsilon$. □

Here is a corollary, which says that every solution to a real-linear Cauchy-Riemann type equation looks locally like a holomorphic function in some *continuous* local trivialization.

THEOREM 2.26 (Similarity principle). *Suppose $A : \mathbb{D} \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$ is smooth and $u : \mathring{\mathbb{D}} \to \mathbb{C}^n$ satisfies the equation $\bar{\partial} u + Au = 0$ with $u(0) = 0$. Then for sufficiently small $\epsilon > 0$, there exist maps $\Phi \in C^0(\mathbb{D}_\epsilon, \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n))$ and $f \in C^\infty(\mathring{\mathbb{D}}_\epsilon, \mathbb{C}^n)$ such that*

$$u(z) = \Phi(z) f(z), \qquad \bar{\partial} f = 0, \qquad \text{and} \qquad \Phi(0) = \mathbb{1}.$$

PROOF. After shrinking the domain if necessary, we may assume without loss of generality that the smooth solution $u : \mathring{\mathbb{D}} \to \mathbb{C}^n$ is bounded. Choose a map $C : \mathbb{D} \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$ satisfying $C(z)u(z) = A(z)u(z)$ and $|C(z)| \le |A(z)|$ for almost every $z \in \mathbb{D}$. Then $C \in L^\infty(\mathring{\mathbb{D}}, \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n))$ and u is a weak solution to $(\bar\partial + C)u = 0$. Note that since we do not know anything about the zero set of u, we cannot assume C is continuous, but we have no trouble assuming $C \in L^p(\mathring{\mathbb{D}})$ for every $p > 2$.

Since $\bar\partial + C$ is now complex linear, we can use Lemma 2.24 to find a complex basis of $W^{1,p}$-smooth weak solutions to $(\bar\partial + C)v = 0$ on $\mathring{\mathbb{D}}_\epsilon$ that define the standard basis of $\mathbb{C}^n$ at 0, and these solutions are continuous by the Sobolev embedding theorem. This gives rise to a map $\Phi \in C^0(\mathring{\mathbb{D}}_\epsilon, \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n))$ that satisfies $(\bar\partial + C)\Phi = 0$ in the sense of distributions and $\Phi(0) = \mathbb{1}$. Since Φ is continuous, we can assume without loss of generality that $\Phi(z)$ is invertible everywhere on $\mathring{\mathbb{D}}_\epsilon$. Setting $f := \Phi^{-1}u : \mathring{\mathbb{D}}_\epsilon \to \mathbb{C}^n$, the Leibniz rule then implies

$$0 = (\bar\partial + C)u = (\bar\partial + C)(\Phi f) = \left[(\bar\partial + C)\Phi\right] f + \Phi(\bar\partial f) = \Phi(\bar\partial f),$$

thus $\bar\partial f = 0$, and f is smooth by Lemma 2.19. □

COROLLARY 2.27 (Unique continuation). *Suppose* $\mathbf{D}$ *is a linear Cauchy-Riemann type operator on a vector bundle* E *over a connected Riemann surface, and* $\eta \in \Gamma(E)$ *satisfies* $\mathbf{D}\eta = 0$. *Then either* η *is identically zero or its zeroes are isolated.*

The similarity principle also has many nice applications for the nonlinear Cauchy-Riemann equation. Here is another "unique continuation" type result for the nonlinear case.

PROPOSITION 2.28. *Suppose* J *is a smooth almost complex structure on* $\mathbb{C}^n$ *and* $u, v : \mathring{\mathbb{D}} \to \mathbb{C}^n$ *are smooth* J*-holomorphic curves such that* $u(0) = v(0) = 0$ *and* u *and* v *have matching partial derivatives of all orders at* 0. *Then* $u \equiv v$ *on a neighborhood of* 0.

PROOF. Let $h = v - u : \mathring{\mathbb{D}} \to \mathbb{C}^n$. We have

$$\partial_s u + J(u(z))\partial_t u = 0 \tag{2.12}$$

and

$$\begin{aligned}
\partial_s v + J(u(z))\partial_t v &= \partial_s v + J(v(z))\partial_t v + \left[J(u(z)) - J(v(z))\right]\partial_t v \\
&= -\left[J(u(z) + h(z)) - J(u(z))\right]\partial_t v \\
&= -\left(\int_0^1 \frac{d}{dt} J(u(z) + th(z))\, dt\right)\partial_t v \qquad (2.13)\\
&= -\left(\int_0^1 dJ(u(z) + th(z)) \cdot h(z)\, dt\right)\partial_t v =: -A(z)h(z),
\end{aligned}$$

where the last step defines a smooth family of linear maps $A(z) \in \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$. Subtracting (2.12) from (2.13) gives the linear equation

$$\partial_s h(z) + \bar J(z)\partial_t h(z) + A(z)h(z) = 0,$$

where $\bar J(z) := J(u(z))$. This is a linear Cauchy-Riemann type equation on a trivial complex vector bundle over $\mathring{\mathbb{D}}$ with complex structure $\bar J(z)$ on the fiber at z. The similarity principle thus implies $h(z) = \Phi(z)f(z)$ near 0 for some holomorphic function $f(z) \in \mathbb{C}^n$ and some continuous map $\Phi(z) \in \mathrm{GL}(2n, \mathbb{R})$ representing a change of trivialization. Now if h has vanishing derivatives of all orders at 0, Taylor's formula implies

$$\lim_{z \to 0} \frac{|\Phi(z)f(z)|}{|z|^k} = 0$$

for all $k \in \mathbb{N}$, so f must also have a zero of infinite order and thus $f \equiv 0$. □

2.6. Simple curves and multiple covers

We now prove a global result about the structure of closed J-holomorphic curves. In Lecture 6 we will be able to generalize it in a straightforward way for punctured holomorphic curves with asymptotically cylindrical behavior.

THEOREM 2.29. *Assume (Σ, j) is a closed connected Riemann surface, (W, J) is a smooth almost complex manifold and $u : (\Sigma, j) \to (W, J)$ is a nonconstant pseudoholomorphic curve. Then there exists a factorization $u = v \circ \varphi$, where*

- *$\varphi : (\Sigma, j) \to (\Sigma', j')$ is a holomorphic map of positive degree to another closed and connected Riemann surface (Σ', j');*
- *$v : (\Sigma', j') \to (W, J)$ is a pseudoholomorphic curve which is embedded except at a finite set of critical points and self-intersections.*

Note that holomorphic maps $(\Sigma, j) \to (\Sigma', j')$ of degree 1 are always diffeomorphisms, so the factorization $u = v \circ \varphi$ in this case is just a reparametrization, and u is then called a **simple** curve. In all other cases, $k := \deg(\varphi) \geq 2$ and φ is in general a branched cover; we then call u a **k-fold branched cover** of the simple curve v.

The main idea in the proof is to construct Σ' (minus some punctures) explicitly as the image of u after removing finitely many singular points, so that we can take v to be the inclusion $\Sigma' \hookrightarrow W$. The map $\varphi : \Sigma \to \Sigma'$ is then uniquely determined. In order to carry out this program, we need some information on what the image of u can look like near each of its singularities. These come in two types, each type corresponding to one of the lemmas below, both of which should seem immediately plausible if your intuition comes from complex analysis.

LEMMA 2.30 (Intersections). *Suppose $u : (\Sigma, j) \to (W, J)$ and $v : (\Sigma', j') \to (W, J)$ are two nonconstant pseudoholomorphic curves with an intersection $u(z) = v(z')$. Then there exist neighborhoods $z \in \mathcal{U} \subset \Sigma$ and $z' \in \mathcal{U}' \subset \Sigma'$ such that*

$$\text{either} \quad u(\mathcal{U}) = v(\mathcal{U}') \qquad \text{or} \qquad u(\mathcal{U} \setminus \{z\}) \cap v(\mathcal{U}') = u(\mathcal{U}) \cap v(\mathcal{U}' \setminus \{z'\}) = \emptyset.$$

□

LEMMA 2.31 (Branching). *Suppose $u : (\Sigma, j) \to (W, J)$ is a nonconstant pseudoholomorphic curve and $z_0 \in \Sigma$ is a critical point of u. Then a neighborhood $\mathcal{U} \subset \Sigma$ of z_0 can be biholomorphically identified with the unit disk $\mathbb{D} \subset \mathbb{C}$ such that*

$$u(z) = v(z^k) \quad \text{for} \quad z \in \mathbb{D} = \mathcal{U},$$

where $k \in \mathbb{N}$, and $v : \mathbb{D} \to W$ is an injective J-holomorphic map with no critical points except possibly at the origin. □

These two local results follow from a well-known formula of Micallef and White [**MW95**] describing the local behavior of J-holomorphic curves near critical points and their intersections. The proof of that theorem is analytically quite involved, but one can also use an easier "approximate" version, which is proved in [**Wend**, §2.14]. Since both are closely related to the phenomenon of unique continuation, you will not be surprised to learn that the similarity principle plays a role in the proof: the main idea is again to exploit the fact that locally J is always a small perturbation of i, hence the local behavior of J-holomorphic curves is also similar to the integrable case.

PROOF OF THEOREM 2.29. Let $\mathrm{Crit}(u) = \{z \in \Sigma \mid du(z) = 0\}$ denote the set of critical points, and define $\Delta \subset \Sigma$ to be the set of all points $z \in \Sigma$ such that there exists $z' \in \Sigma$ and neighborhoods $z \in \mathcal{U} \subset \Sigma$ and $z' \in \mathcal{U}' \subset \Sigma$ with $u(z) = u(z')$ but $u(\mathcal{U} \setminus \{z\}) \cap u(\mathcal{U}' \setminus \{z'\}) = \emptyset$.

The lemmas quoted above imply that both of these sets are discrete. Both are therefore finite, and the set $\dot{\Sigma}' = u(\Sigma \setminus (\mathrm{Crit}(u) \cup \Delta)) \subset W$ is then a smooth submanifold of W with J-invariant tangent spaces, so it inherits a natural complex structure j' for which the inclusion $(\dot{\Sigma}', j') \hookrightarrow (W, J)$ is pseudoholomorphic. We shall now construct a new Riemann surface (Σ', j') from which $(\dot{\Sigma}', j')$ is obtained by removing a finite set of points. Let $\widehat{\Delta} = (\mathrm{Crit}(u) \cup \Delta)/\sim$, where two points in $\mathrm{Crit}(u) \cup \Delta$ are defined to be equivalent whenever they have neighborhoods in Σ with identical images under u. Then for each $[z] \in \widehat{\Delta}$, the branching lemma provides an injective J-holomorphic map $u_{[z]}$ from the unit disk $\mathbb{D}$ onto the image of a neighborhood of z under u. We define (Σ', j') by

$$\Sigma' = \dot{\Sigma}' \cup_{\Phi} \left(\bigsqcup_{[z] \in \widehat{\Delta}} \mathbb{D} \right),$$

where the gluing map Φ is the disjoint union of the maps $u_{[z]} : \mathbb{D} \setminus \{0\} \to \dot{\Sigma}'$ for each $[z] \in \widehat{\Delta}$; since this map is holomorphic, the complex structure j' extends from $\dot{\Sigma}'$ to Σ'. Combining the maps $u_{[z]} : \mathbb{D} \to W$ with the inclusion $\dot{\Sigma}' \hookrightarrow W$ now defines a pseudoholomorphic map $v : (\Sigma', j') \to (W, J)$ which restricts to $\dot{\Sigma}'$ as an embedding and otherwise has at most finitely many critical points and double points. Moreover, the restriction of u to $\Sigma \setminus (\mathrm{Crit}(u) \cup \Delta)$ defines a holomorphic map to $(\dot{\Sigma}', j')$ which extends by removal of singularities to a proper holomorphic map $\varphi : (\Sigma, j) \to (\Sigma', j')$ such that $u = v \circ \varphi$. Its holomorphicity implies that it has positive degree. □

LECTURE 3

Asymptotic operators

Contents

We now begin with the analysis of the particular class of J-holomorphic curves that are important in SFT. The next three lectures will focus on the linearized problem, the goal being to prove that this linearization is Fredholm and to compute its index. Using this along with the implicit function theorem and the Sard-Smale theorem (on genericity of smooth nonlinear Fredholm maps), we will later be able to show that moduli spaces of asymptotically cylindrical J-holomorphic curves are smooth finite-dimensional manifolds under suitable genericity assumptions.

3.1. The linearization in Morse homology

Since Morse homology is the prototype for all Floer-type theories, we can gain useful intuition by recalling how the analysis works for the linearization of the gradient flow problem in Morse theory. The basic features of the problem were discussed already in §1.2.

Assume (M, g) is a closed n-dimensional Riemannian manifold, $f : M \to \mathbb{R}$ is a smooth function, and for two critical points $x_+, x_- \in \mathrm{Crit}(f)$, consider the moduli space of parametrized gradient flow lines

$$\mathcal{M}(x_-, x_+) := \left\{ u \in C^\infty(\mathbb{R}, M) \;\middle|\; \dot{u} + \nabla f(u) = 0,\ \lim_{s \to \pm\infty} u(s) = x_\pm \right\}.$$

The map $\mathcal{M}(x_-, x_+) \to M : u \mapsto u(0)$ gives a natural identification of $\mathcal{M}(x_-, x_+)$ with the intersection between the unstable manifold of x_- and the stable manifold of x_+ for the negative gradient flow, and we say the pair (g, f) is **Morse-Smale** if f is Morse and this intersection is transverse, in which case $\mathcal{M}(x_-, x_+)$ is a smooth manifold with

$$\dim \mathcal{M}(x_-, x_+) = \mathrm{ind}(x_-) - \mathrm{ind}(x_+).$$

This can all be proved using finite-dimensional differential topology, but since that approach does not work in the study of Floer trajectories or holomorphic curves in symplectizations, let us instead see how one proves it using nonlinear functional analysis. For more details on the following discussion, see [Sch93].

Following the strategy laid out in §2.1, $\mathcal{M}(x_-, x_+)$ can be identified with the zero set of a smooth section

$$\boldsymbol{\sigma} : \mathcal{B} \to \mathcal{E} : u \mapsto \dot{u} + \nabla f(u),$$

where $\mathcal{B}$ is a Banach manifold of maps $u : \mathbb{R} \to M$ satisfying $\lim_{s \to \pm\infty} u(s) = x_\pm$, and $\mathcal{E} \to \mathcal{B}$ is a smooth Banach space bundle whose fibers $\mathcal{E}_u$ contain $\Gamma(u^*TM)$. The linearization $D\boldsymbol{\sigma}(u) : T_u\mathcal{B} \to \mathcal{E}_u$ of this section at a zero $u \in \boldsymbol{\sigma}^{-1}(0)$ defines a first-order linear differential operator

$$\mathbf{D}_u : \Gamma(u^*TM) \to \Gamma(u^*TM)$$

which takes the form

$$\mathbf{D}_u \eta = \nabla_s \eta + \nabla_\eta \nabla f$$

for any choice of symmetric connection ∇ on M. Taking suitable Sobolev completions of $\Gamma(u^*TM)$, we are therefore led to consider bounded linear operators[1] of the form

$$\mathbf{D}_u = \nabla_s + \nabla\nabla f : W^{k,p}(u^*TM) \to W^{k-1,p}(u^*TM) \tag{3.1}$$

for $k \in \mathbb{N}$ and $1 < p < \infty$, and the first task is to prove that whenever x_+ and x_- satisfy the Morse condition, this is a Fredholm operator of index $\operatorname{ind} \mathbf{D}_u = \operatorname{ind}(x_-) - \operatorname{ind}(x_+)$.

Choose coordinates near x_+ in which g looks like the standard Euclidean inner product at x_+. This induces a trivialization of u^*TM over $[T, \infty)$ for $T > 0$ sufficiently large, and we are free to assume that the connection ∇ is the standard one determined by these coordinates on $[T, \infty)$. Using the trivialization to identify sections $\beta \in \Gamma(u^*TM)$ over $[T, \infty)$ with maps $f : [T, \infty) \to \mathbb{R}^n$, $\mathbf{D}_u$ now acts on f as

$$(\mathbf{D}_u f)(s) = \partial_s f(s) + A(s) f(s), \tag{3.2}$$

where $A(s) \in \mathbb{R}^{n \times n}$ is the matrix of the linear transformation $dX(s) : \mathbb{R}^n \to \mathbb{R}^n$, with $X(s) \in \mathbb{R}^n$ being the coordinate representation of $\nabla f(u(s)) \in T_{u(s)}M$. As $s \to \infty$, the zeroth-order term in this expression converges to a symmetric matrix

$$A_+ := \lim_{s \to \infty} A(s),$$

which is the coordinate representation of the Hessian $\nabla^2 f(x_+)$. Any choice of coordinates near x_- produces a similar formula for $\mathbf{D}_u$ over $(-\infty, -T]$, $A(s)$ converging as $s \to -\infty$ to another symmetric matrix A_- representing $\nabla^2 f(x_-)$. Both the Morse condition and the dimension $\operatorname{ind}(x_-) - \operatorname{ind}(x_+)$ can now be expressed entirely

[1] We are ignoring an analytical subtlety: since $u^*TM \to \mathbb{R}$ has no canonical trivialization and $\mathbb{R}$ is noncompact, it is not completely obvious what the definition of the Sobolev space $W^{k,p}(u^*TM)$ should be. We will return to this issue in a more general context in the next lecture.

in terms of these two matrices: $x_\pm$ is Morse if and only if $A_\pm$ is invertible, and the Fredholm index of $\mathbf{D}_u$ will then be

$$\operatorname{ind}(x_-) - \operatorname{ind}(x_+) = \dim E^-(A_-) - \dim E^-(A_+),$$

where for any symmetric matrix A we denote by $E^-(A)$ the direct sum of all its eigenspaces with negative eigenvalue. The main linear functional analytic result underlying Morse homology can now be stated as follows (cf. [**Sch93**]):

PROPOSITION 3.1. *Assume $k \in \mathbb{N}$ and $1 < p < \infty$. Suppose $E \to \mathbb{R}$ is a smooth vector bundle with trivializations fixed in neighborhoods of $-\infty$ and $+\infty$, and $\mathbf{D} : W^{k,p}(E) \to W^{k-1,p}(E)$ is a first-order differential operator which asymptotically takes the form (3.2) near $\pm\infty$ with respect to the chosen trivializations, where $A(s)$ is a smooth family of n-by-n matrices with well-defined asymptotic limits $A_\pm := \lim_{s\to\pm\infty} A(s)$ which are symmetric. If A_+ and A_- are also invertible, then $\mathbf{D}$ is Fredholm and*

$$\operatorname{ind}(\mathbf{D}) = \dim E^-(A_-) - \dim E^-(A_+). \tag{3.3}$$

□

REMARK 3.2. The hypothesis that $A_\pm$ is invertible in Prop. 3.1 cannot be lifted: indeed, suppose $\mathbf{D}$ is Fredholm but e.g. A_+ has 0 in its spectrum. Then one can easily perturb $A(s)$ and hence A_+ in two distinct ways producing two distinct values of $\dim E^-(A_+)$, pushing the zero eigenvalue either up or down. This produces two perturbed Fredholm operators that have different indices according to (3.3), but they also belong to a continuous family of Fredholm operators, and must therefore have the same index, giving a contradiction.

The formula (3.3) makes sense of course because $E^-(A_\pm)$ are both finite-dimensional vector spaces, but in Floer-type theories we typically encounter critical points with infinite Morse index. With this in mind, it is useful to note that (3.3) can be rewritten without explicitly referencing $E^-(A_+)$ or $E^-(A_-)$. Indeed, choose a continuous path of symmetric matrices $\{B_t\}_{t\in[-1,1]}$ connecting $B(-1) := A_-$ to $B(1) := A_+$. The spectrum of B_t varies continuously with t in the following sense: one can choose a family of continuous functions

$$\{\lambda_j : [-1,1] \to \mathbb{R}\}_{j\in I}$$

for the index set $I = \{1,\dots,n\}$ such that for every $t \in [-1,1]$, the set of eigenvalues of B_t counted with multiplicity is $\{\lambda_j(t)\}_{j\in I}$. The **spectral flow** from A_- to A_+ is then defined as a signed count of the number of paths of eigenvalues that cross from one side of zero to the other, namely (cf. Theorem 3.3)

$$\mu^{\text{spec}}(A_-, A_+) := \#\left\{j \in I \;\middle|\; \lambda_j(-1) < 0 < \lambda_j(1)\right\} - \#\left\{j \in I \;\middle|\; \lambda_j(-1) > 0 > \lambda_j(1)\right\}.$$

The index formula (3.3) now becomes

$$\operatorname{ind}(\mathbf{D}) = \mu^{\text{spec}}(A_-, A_+).$$

This description of the index has the advantage that it could potentially make sense and give a well-defined integer even if $A_\pm$ were symmetric operators on an infinite-dimensional Hilbert space: they might both have infinitely many positive and negative eigenvalues, but only finitely many that change sign along a path from A_- to A_+. We will make this discussion precise in the next section.

3.2. Spectral flow

We will see in §3.3 that in Floer-type theories, the role of the symmetric linear transformation $T_x M \to T_x M$ defined by the Hessian $\nabla^2 f(x)$ of a Morse function $f : M \to \mathbb{R}$ at a critical point is played by a certain class of symmetric operators on the space of loops $\eta : S^1 \to \mathbb{R}^{2n}$, namely operators of the form

$$(\mathbf{A}\eta)(t) := -J_0\, \partial_t \eta(t) - S(t)\eta(t), \tag{3.4}$$

where J_0 denotes the standard complex structure on $\mathbb{R}^{2n} = \mathbb{C}^n$, and $S : S^1 \to \operatorname{End}(\mathbb{R}^{2n})$ is a smooth loop of symmetric matrices. The goal of this section is to define a notion of spectral flow for operators of this type. Regarding $\mathbf{A}$ as an unbounded linear operator on $L^2(S^1, \mathbb{R}^{2n})$ with dense domain $H^1(S^1, \mathbb{R}^{2n})$, we will see that its spectrum consists of isolated real eigenvalues with finite multiplicity. We shall prove:

THEOREM 3.3. *Assume $\{S_s : S^1 \to \operatorname{End}(\mathbb{R}^{2n})\}_{s \in [-1,1]}$ is a smooth family of loops of symmetric matrices, and consider the corresponding 1-parameter family of unbounded linear operators*

$$\mathbf{A}_s = -J_0 \partial_t - S_s(t) : L^2(S^1, \mathbb{R}^{2n}) \supset H^1(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n}).$$

Then there exists a set of continuous functions

$$\{\lambda_j : [-1, 1] \to \mathbb{R}\}_{j \in \mathbb{Z}}$$

such that for every $s \in [-1, 1]$, the spectrum of $\mathbf{A}_s$ consists of the numbers $\{\lambda_j(s)\}_{j \in \mathbb{Z}}$, each of which is an eigenvalue with finite multiplicity equal to the number of times it is repeated as j varies in $\mathbb{Z}$.

Moreover, if additionally $\mathbf{A}_- := \mathbf{A}_{-1}$ and $\mathbf{A}_+ := \mathbf{A}_1$ both have trivial kernel, then the number $\mu^{\text{spec}}(\mathbf{A}_-, \mathbf{A}_+) \in \mathbb{Z}$ defined by

$$\#\left\{ j \in \mathbb{Z} \;\middle|\; \lambda_j(-1) < 0 < \lambda_j(1) \right\} - \#\left\{ j \in \mathbb{Z} \;\middle|\; \lambda_j(-1) > 0 > \lambda_j(1) \right\}$$

is well defined and depends only on $\mathbf{A}_-$ and $\mathbf{A}_+$.

We will start by giving a more abstract definition of spectral flow as an intersection number between a path of symmetric index 0 Fredholm operators and the subvariety of noninvertible operators. This relies on the general fact that spaces of operators with kernel and cokernel of fixed finite dimensions form smooth finite-codimensional submanifolds in the Banach space of all bounded linear operators. We explain this fact in §3.2.1, and then specialize to the case of symmetric index 0 operators to define the abstract version of spectral flow in §3.2.2. In §3.2.3, we show that the spectra of such operators vary continuously under small perturbations, and in §3.2.4 we specialize further to operators of the form (3.4) and explain how to interpret the abstract definition of spectral flow in terms of eigenvalues crossing the origin in $\mathbb{R}$, leading to a proof of Theorem 3.3.

Spectral flow can be defined more generally for certain classes of self-adjoint elliptic partial differential operators, see e.g. [**APS76**, **RS95**], and standard proofs of its existence typically rely on perturbation results as in [**Kat95**] for the spectra of self-adjoint operators. In the following presentation, we have chosen to avoid making explicit use of self-adjointness and instead focus on the Fredholm property; in this way the discussion is mostly self-contained and, in particular, does not require any results from [**Kat95**].

3.2.1. Geometry in the space of Fredholm operators. Fix a field

$$\mathbb{F} := \mathbb{R} \text{ or } \mathbb{C}.$$

Given Banach spaces X and Y over $\mathbb{F}$, denote by $\mathscr{L}_{\mathbb{F}}(X,Y)$ the Banach space of bounded $\mathbb{F}$-linear maps from X to Y, with $\mathscr{L}_{\mathbb{F}}(X) := \mathscr{L}_{\mathbb{F}}(X,X)$, and let

$$\operatorname{Fred}_{\mathbb{F}}(X,Y) \subset \mathscr{L}_{\mathbb{F}}(X,Y)$$

denote the open subset consisting of Fredholm operators. Recall that an operator $\mathbf{T} \in \mathscr{L}_{\mathbb{F}}(X,Y)$ is **Fredholm** if its image is closed,[2] and its kernel and cokernel (i.e. the quotient $\operatorname{coker} \mathbf{T} := Y/\operatorname{im} \mathbf{T}$) are both finite dimensional. Its **index** is defined as

$$\operatorname{ind}_{\mathbb{F}}(\mathbf{T}) := \dim_{\mathbb{F}} \ker \mathbf{T} - \dim_{\mathbb{F}} \operatorname{coker} \mathbf{T} \in \mathbb{Z}.$$

The index defines a continuous and thus locally constant function $\operatorname{Fred}_{\mathbb{F}}(X,Y) \to \mathbb{Z}$, and for each $i \in \mathbb{Z}$, we shall denote

$$\operatorname{Fred}^i_{\mathbb{F}}(X,Y) := \left\{ \mathbf{T} \in \operatorname{Fred}_{\mathbb{F}}(X,Y) \;\middle|\; \operatorname{ind}(\mathbf{T}) = i \right\}.$$

We will often have occasion to use the following general construction. Given $\mathbf{T}_0 \in \operatorname{Fred}_{\mathbb{F}}(X,Y)$, one can choose splittings into closed linear subspaces

$$X = V \oplus K, \qquad Y = W \oplus C$$

such that $K = \ker \mathbf{T}_0$, $W = \operatorname{im} \mathbf{T}_0$, the quotient projection $\pi_C : Y \to \operatorname{coker} \mathbf{T}_0$ restricts to $C \subset Y$ as an isomorphism, and $\mathbf{T}_0|_V$ defines an isomorphism from V to W. Using these splittings, any other $\mathbf{T} \in \operatorname{Fred}_{\mathbb{F}}(X,Y)$ can be written in block form as

$$\mathbf{T} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix},$$

with $\mathbf{T}_0$ itself written in this way as $\begin{pmatrix} \mathbf{A}_0 & 0 \\ 0 & 0 \end{pmatrix}$ for some Banach space isomorphism $\mathbf{A}_0 : V \to W$. Let $\mathcal{O} \subset \operatorname{Fred}_{\mathbb{F}}(X,Y)$ denote the open neighborhood of $\mathbf{T}_0$ for which the block $\mathbf{A}$ is invertible, and define a map

$$\Phi : \mathcal{O} \to \operatorname{Hom}_{\mathbb{F}}(\ker \mathbf{T}_0, \operatorname{coker} \mathbf{T}_0) : \mathbf{T} \mapsto \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}. \tag{3.5}$$

LEMMA 3.4. *The map Φ in (3.5) is smooth, and holomorphic in the case $\mathbb{F} = \mathbb{C}$, and its derivative at $\mathbf{T}_0$ defines a surjective bounded linear operator $\mathscr{L}_{\mathbb{F}}(X,Y) \to$ $\operatorname{Hom}_{\mathbb{F}}(\ker \mathbf{T}_0, \operatorname{coker} \mathbf{T}_0)$ of the form*

$$d\Phi(\mathbf{T}_0)\mathbf{H} = \pi_C \mathbf{H}|_{\ker \mathbf{T}_0},$$

[2] It is not strictly necessary to require that $\operatorname{im} \mathbf{T} \subset Y$ be closed, as this follows from the finite-dimensionality of the kernel and cokernel, cf. [**AA02**, Cor. 2.17].

where π_C denotes the natural projection $Y \to \operatorname{coker} \mathbf{T}_0$. Moreover, there exists a smooth function $\Psi : \mathcal{O} \to \mathscr{L}_{\mathbb{F}}(X)$ such that for every $\mathbf{T} \in \mathcal{O}$, $\Psi(\mathbf{T}) : X \to X$ maps $\ker \Phi(\mathbf{T}) \subset \ker \mathbf{T}_0$ isomorphically to $\ker \mathbf{T}$.

PROOF. Smoothness, holomorphicity[3] and the formula for the derivative are easily verified from the given formula for Φ; in particular, since the blocks $\mathbf{B}$ and $\mathbf{C}$ both vanish for $\mathbf{T} = \mathbf{T}_0$, we have

$$d\Phi(\mathbf{T}_0) : \mathscr{L}_{\mathbb{F}}(X, Y) \to \operatorname{Hom}_{\mathbb{F}}(K, C)$$
$$\begin{pmatrix} \mathbf{A}' & \mathbf{B}' \\ \mathbf{C}' & \mathbf{D}' \end{pmatrix} \mapsto \mathbf{D}'.$$

The map $\Psi : \mathcal{O} \to \mathscr{L}_{\mathbb{F}}(X) = \mathscr{L}_{\mathbb{F}}(V \oplus K)$ is defined by

$$\Psi(\mathbf{T}) = \begin{pmatrix} \mathbb{1} & -\mathbf{A}^{-1}\mathbf{B} \\ 0 & \mathbb{1} \end{pmatrix}.$$

For each $\mathbf{T}$, this is an isomorphism; indeed, its inverse is given by

$$\Psi(\mathbf{T})^{-1} = \begin{pmatrix} \mathbb{1} & \mathbf{A}^{-1}\mathbf{B} \\ 0 & \mathbb{1} \end{pmatrix}.$$

Then $\mathbf{T}\Psi(\mathbf{T}) = \begin{pmatrix} \mathbf{A} & 0 \\ \mathbf{C} & \Phi(\mathbf{T}) \end{pmatrix}$, and since $\mathbf{A}$ is invertible, $\ker \mathbf{T}\Psi(\mathbf{T}) = \{0\} \oplus \ker \Phi(\mathbf{T})$. □

PROPOSITION 3.5. *For each $i \in \mathbb{Z}$ and each nonnegative integer $k \geq i$, the subset*

$$\operatorname{Fred}^{i,k}_{\mathbb{F}}(X, Y) := \left\{ \mathbf{T} \in \operatorname{Fred}^i_{\mathbb{F}}(X, Y) \;\middle|\; \dim_{\mathbb{F}} \ker \mathbf{T} = k \text{ and } \dim_{\mathbb{F}} \operatorname{coker} \mathbf{T} = k - i \right\}$$

admits the structure of a smooth (and complex-analytic if $\mathbb{F} = \mathbb{C}$) finite-codimensional Banach submanifold of $\mathscr{L}_{\mathbb{F}}(X, Y)$, with

$$\operatorname{codim}_{\mathbb{F}} \operatorname{Fred}^{i,k}_{\mathbb{F}}(X, Y) = k(k - i).$$

PROOF. Applying the implicit function theorem to the map Φ from Lemma 3.4 endows a neighborhood of $\mathbf{T}_0$ in $\Phi^{-1}(0) \subset \operatorname{Fred}_{\mathbb{F}}(X, Y)$ with the structure of a smooth Banach submanifold with

$$\operatorname{codim}_{\mathbb{F}} \Phi^{-1}(0) = \dim_{\mathbb{F}} \operatorname{Hom}_{\mathbb{F}}(\ker \mathbf{T}_0, \operatorname{coker} \mathbf{T}_0) = k(k - i).$$

If $\mathbb{F} = \mathbb{C}$, then Φ is also holomorphic and $\Phi^{-1}(0)$ is thus a complex-analytic submanifold near $\mathbf{T}_0$. Now observe that for every $\mathbf{T} \in \mathcal{O}$,

$$\dim_{\mathbb{F}} \ker \mathbf{T} = \dim_{\mathbb{F}} \ker \Phi(\mathbf{T}) \leq \dim_{\mathbb{F}} \ker \mathbf{T}_0 = k,$$

with equality if and only if $\Phi(\mathbf{T}) = 0$, hence, since the index is locally constant, we get $\Phi^{-1}(0) = \operatorname{Fred}^{i,k}_{\mathbb{F}}(X, Y)$ in a neighborhood of $\mathbf{T}_0$. □

[3] Holomorphicity in this infinite-dimensional setting means the same thing as usual: $\mathscr{L}_{\mathbb{C}}(X, Y)$ and $\operatorname{Hom}_{\mathbb{C}}(\ker \mathbf{T}_0, \operatorname{coker} \mathbf{T}_0)$ both have natural complex structures if $\mathbf{T}_0 \in \operatorname{Fred}_{\mathbb{C}}(X, Y)$, and we require $d\Phi(\mathbf{T})$ to commute with them for all $\mathbf{T} \in \mathcal{O}$.

For real-linear operators of index 0, one can use Prop. 3.5 to define the following "relative" invariant. Given two Banach space isomorphisms $\mathbf{T}_\pm : X \to Y$ that lie in the same connected component of $\mathrm{Fred}_\mathbb{R}(X,Y)$, define

$$\mu^{\mathrm{spec}}_{\mathbb{Z}_2}(\mathbf{T}_-, \mathbf{T}_+) \in \mathbb{Z}_2$$

as the parity of the number of times that a generic smooth path $[-1,1] \to \mathrm{Fred}^0_\mathbb{R}(X,Y)$ from $\mathbf{T}_-$ to $\mathbf{T}_+$ passes through operators with nontrivial kernel. This is well defined due to the following consequences of standard transversality theory (see Exercise 3.6): first, generic paths $\{\mathbf{T}(t) \in \mathrm{Fred}^0_\mathbb{R}(X,Y)\}_{t\in[-1,1]}$ are transverse to $\mathrm{Fred}^{0,k}_\mathbb{R}(X,Y)$ for every $k \in \mathbb{N}$, which implies via the codimension formula in Prop. 3.5 that they never intersect $\mathrm{Fred}^{0,k}_\mathbb{R}(X,Y)$ for $k \geq 2$, and their intersections with $\mathrm{Fred}^{0,1}_\mathbb{R}(X,Y)$ are transverse and thus isolated. Second, transversality also holds for generic homotopies

$$[0,1] \times [-1,1] \to \mathrm{Fred}^0_\mathbb{R}(X,Y) : (s,t) \mapsto \mathbf{T}_s(t)$$

with fixed end points between any pair of generic paths $\mathbf{T}_0(t)$ and $\mathbf{T}_1(t)$, so that the set of intersections with $\mathrm{Fred}^{0,k}_\mathbb{R}(X,Y)$ is again empty for $k \geq 2$ and forms a smooth 1-dimensional submanifold in $[0,1] \times [-1,1]$ for $k = 1$. This submanifold, moreover, is disjoint from $[0,1] \times \{-1,1\}$ since $\mathbf{T}_s(\pm 1) = \mathbf{T}_\pm$, and it is also compact since the set of $\mathbf{T} \in \mathrm{Fred}^0_\mathbb{R}(X,Y)$ with nontrivial kernel is a closed subset. We therefore obtain a compact 1-dimensional cobordism between the intersection sets of $\mathbf{T}_0$ and $\mathbf{T}_1$ respectively with $\mathrm{Fred}^{0,1}_\mathbb{R}(X,Y)$, implying that the count of intersections modulo 2 does not depend on the choice of generic path.

EXERCISE 3.6. Convince yourself that the standard results (as in e.g. [**Hir94**, §3.2] about generic transversality of intersections between smooth maps $f : M \to N$ and submanifolds $A \subset N$ continue to hold—with minimal modifications to the proofs—when N is an infinite-dimensional Banach manifold and $A \subset N$ has finite codimension.

EXERCISE 3.7. For matrices $A_\pm \in \mathrm{GL}(n,\mathbb{R})$, show that $\mu^{\mathrm{spec}}_{\mathbb{Z}_2}(A_-, A_+) = 0$ if and only if $\det A_+$ and $\det A_-$ have the same sign.

3.2.2. Symmetric operators of index zero. We now add the following assumptions to the setup from the previous subsection:

- Y is a Hilbert space $\mathcal{H}$ over $\mathbb{F}$, with inner product denoted by $\langle\ ,\ \rangle_\mathcal{H}$;
- X is an $\mathbb{F}$-linear subspace $\mathcal{D} \subset \mathcal{H}$, carrying a Banach space structure for which the inclusion $\mathcal{D} \hookrightarrow \mathcal{H}$ is a compact linear operator.

The notation $\mathcal{D} = X$ is motivated by the fact that if $\mathbf{T} \in \mathscr{L}_\mathbb{F}(\mathcal{D},\mathcal{H})$, then we can also regard $\mathbf{T}$ as an **unbounded operator** on $\mathcal{H}$ with domain $\mathcal{D}$ and thus consider the spectrum of $\mathbf{T}$, see §3.2.3 below.

Since $\mathcal{H}$ is a Hilbert space, the space $\mathscr{L}_\mathbb{F}(\mathcal{H})$ of bounded linear operators from $\mathcal{H}$ to itself contains a distinguished closed linear subspace

$$\mathscr{L}^{\mathrm{sym}}_\mathbb{F}(\mathcal{H}) \subset \mathscr{L}_\mathbb{F}(\mathcal{H}),$$

consisting of self-adjoint operators. For operators that are bounded from $\mathcal{D}$ to $\mathcal{H}$ but not necessarily defined or bounded on $\mathcal{H}$, there is also the space of **symmetric** operators

$$\mathscr{L}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{D},\mathcal{H}) := \left\{ \mathbf{T} \in \mathscr{L}_{\mathbb{F}}(\mathcal{D},\mathcal{H}) \;\middle|\; \langle x, \mathbf{T}y\rangle_{\mathcal{H}} = \langle \mathbf{T}x, y\rangle_{\mathcal{H}} \text{ for all } x,y\in\mathcal{D} \right\}.$$

Important examples of symmetric operators are those which are self-adjoint (see Remark 3.11 below), though for our purposes, it will suffice to restrict attention to symmetric operators that are also Fredholm with index 0. It turns out that the space of symmetric operators in $\mathrm{Fred}^{0,1}_{\mathbb{F}}(\mathcal{D},\mathcal{H})$ is a canonically co-oriented hypersurface in $\mathscr{L}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{D},\mathcal{H})$, so that the invariant $\mu^{\mathrm{spec}}_{\mathbb{Z}_2}(\mathbf{T}_-,\mathbf{T}_+)$ defined above has a natural integer-valued lift when $\mathbf{T}_\pm$ are symmetric. We will need a slightly more specialized version of this statement in order to give a general definition of spectral flow.

In the following, we let

$$\mathrm{Fred}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{D},\mathcal{H}) := \mathrm{Fred}^{0}_{\mathbb{F}}(\mathcal{D},\mathcal{H}) \cap \mathscr{L}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{D},\mathcal{H})$$

denote the space of symmetric Fredholm operators with index 0, and for $k\in\mathbb{N}$,

$$\mathrm{Fred}^{\mathrm{sym},k}_{\mathbb{F}}(\mathcal{D},\mathcal{H}) := \mathrm{Fred}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{D},\mathcal{H}) \cap \mathrm{Fred}^{0,k}_{\mathbb{F}}(\mathcal{D},\mathcal{H}).$$

Given $\mathbf{T}_{\mathrm{ref}} \in \mathrm{Fred}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{D},\mathcal{H})$, consider the space

$$\mathrm{Fred}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}}) := \left\{ \mathbf{T}_{\mathrm{ref}} + \mathbf{K} : \mathcal{D}\to\mathcal{H} \;\middle|\; \mathbf{K}\in\mathscr{L}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{H}) \right\}.$$

Note that the restriction of each $\mathbf{K}\in\mathscr{L}_{\mathbb{F}}(\mathcal{H})$ to $\mathcal{D}$ is a compact operator $\mathcal{D}\to\mathcal{H}$, thus $\mathrm{Fred}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$ has a natural continuous inclusion into $\mathrm{Fred}^{\mathrm{sym}}_{\mathbb{R}}(\mathcal{D},\mathcal{H})$. It is also an affine space over $\mathscr{L}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{H})$ and can thus be regarded naturally as a smooth Banach manifold locally modeled on $\mathscr{L}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{H})$; in particular, its tangent spaces are

$$T_{\mathbf{T}}\left(\mathrm{Fred}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})\right) = \mathscr{L}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{H}).$$

A remark about the case $\mathbb{F}=\mathbb{C}$ is in order: $\mathscr{L}^{\mathrm{sym}}_{\mathbb{C}}(\mathcal{D},\mathcal{H})$ is a *real*-linear and not a complex subspace of $\mathscr{L}_{\mathbb{C}}(\mathcal{D},\mathcal{H})$, thus $\mathrm{Fred}^{\mathrm{sym}}_{\mathbb{C}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$ is a real Banach manifold but does not carry a natural complex structure.

LEMMA 3.8. *For any* $\mathbf{T}\in\mathscr{L}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{D},\mathcal{H})$ *that is Fredholm with index* 0, $\ker\mathbf{T}$ *is the orthogonal complement of* $\operatorname{im}\mathbf{T}$ *in* $\mathcal{H}$, *hence there exist splittings into closed linear subspaces*

$$\mathcal{D} = V\oplus K, \qquad \mathcal{H} = W\oplus C$$

where $K = C = \ker\mathbf{T}$, $W = \operatorname{im}\mathbf{T}$ *and* $V = W\cap\mathcal{D}$.

PROOF. If $x\in K := \ker\mathbf{T}$, then symmetry implies $\langle x,\mathbf{T}y\rangle_{\mathcal{H}} = \langle\mathbf{T}x,y\rangle_{\mathcal{H}} = 0$ for all $y\in\mathcal{D}$, hence $K\subset W^\perp$, where $W := \operatorname{im}\mathbf{T}$. But since $\operatorname{ind}\mathbf{T} = 0$, the dimension of $\ker\mathbf{T}$ equals the codimension of $\operatorname{im}\mathbf{T}$, implying that K already has the largest possible dimension for a subspace that intersects W trivially, and therefore $W\oplus K = \mathcal{H}$. Since K is also a subspace of $\mathcal{D}$ and the latter is a subspace of $\mathcal{H}$, any $x\in\mathcal{D}$ can be written uniquely as $x = v + k$ where $k\in K$ and $v\in W\cap\mathcal{D} =: V$. The continuous inclusion of $\mathcal{D}$ into $\mathcal{H}$ and the fact that W is closed in $\mathcal{H}$ imply that V is a closed subspace of $\mathcal{D}$. □

We now have the following modification of Prop. 3.5.

PROPOSITION 3.9. *For each integer $k \geq 0$, the subset*

$$\mathrm{Fred}^{\mathrm{sym},k}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}}) := \left\{ \mathbf{T} \in \mathrm{Fred}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}}) \;\middle|\; \dim_{\mathbb{F}} \ker \mathbf{T} = k \right\}$$

is a smooth finite-codimensional Banach submanifold of $\mathrm{Fred}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$, *with*

$$\operatorname{codim}_{\mathbb{R}} \mathrm{Fred}^{\mathrm{sym},k}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}}) = \begin{cases} k(k+1)/2 & \text{if } \mathbb{F} = \mathbb{R}, \\ k^2 & \text{if } \mathbb{F} = \mathbb{C}. \end{cases}$$

In particular, $\mathrm{Fred}^{\mathrm{sym},1}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$ *is a submanifold of* $\mathrm{Fred}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$ *with codimension* 1, *and moreover, it carries a canonical co-orientation.*

PROOF. Given $\mathbf{T}_0 \in \mathrm{Fred}^{\mathrm{sym},k}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$, fix the splittings $\mathcal{D} = V \oplus K$ and $\mathcal{H} = W \oplus K$ as in Lemma 3.8. Using these in the construction of the map Φ from (3.5) produces a neighborhood $\mathcal{O} \subset \mathrm{Fred}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$ of $\mathbf{T}_0$ such that, by Lemma 3.4, $\{\mathbf{T} \in \mathcal{O} \mid \dim_{\mathbb{F}} \ker \mathbf{T} = k\} = \Phi^{-1}(0)$, where

$$\Phi : \mathcal{O} \to \mathrm{End}_{\mathbb{F}}(K) : \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \mapsto \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}.$$

Since the splittings are orthogonal, an element $\mathbf{T} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \in \mathcal{O}$ is symmetric if and only if

$$\begin{aligned} \langle x, \mathbf{A}y\rangle_{\mathcal{H}} &= \langle \mathbf{A}x, y\rangle_{\mathcal{H}} && \text{for all } x, y \in V, \\ \langle x, \mathbf{D}y\rangle_{\mathcal{H}} &= \langle \mathbf{D}x, y\rangle_{\mathcal{H}} && \text{for all } x, y \in K, \\ \langle x, \mathbf{B}y\rangle_{\mathcal{H}} &= \langle \mathbf{C}x, y\rangle_{\mathcal{H}} && \text{for all } x \in V,\ y \in K, \\ \langle x, \mathbf{C}y\rangle_{\mathcal{H}} &= \langle \mathbf{B}x, y\rangle_{\mathcal{H}} && \text{for all } x \in K,\ y \in V, \end{aligned}$$

and it follows then that $\Phi(\mathbf{T}) \in \mathrm{End}^{\mathrm{sym}}_{\mathbb{F}}(K)$, where $\mathrm{End}^{\mathrm{sym}}_{\mathbb{F}}(K) \subset \mathrm{End}_{\mathbb{F}}(K)$ is the real vector space of symmetric (or Hermitian when $\mathbb{F} = \mathbb{C}$) linear maps on $(K, \langle\ ,\ \rangle_{\mathcal{H}})$. We thus have $\mathcal{O} \cap \mathrm{Fred}^{\mathrm{sym},k}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}}) = \Phi^{-1}(0)$ with Φ regarded as a smooth map $\mathcal{O} \cap \mathrm{Fred}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}}) \to \mathrm{End}^{\mathrm{sym}}_{\mathbb{F}}(K)$. The derivative at $\mathbf{T}_0$ again takes the form

$$d\Phi(\mathbf{T}_0) : \mathscr{L}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{H}) \to \mathrm{End}^{\mathrm{sym}}_{\mathbb{F}}(K) : \begin{pmatrix} \mathbf{A}' & \mathbf{B}' \\ \mathbf{C}' & \mathbf{D}' \end{pmatrix} \mapsto \mathbf{D}',$$

where now the block matrix represents an element of $\mathscr{L}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{H})$ with respect to the splitting $\mathcal{H} = W \oplus K$. This operator is evidently surjective, hence by the implicit function theorem, $\Phi^{-1}(0)$ is a smooth Banach submanifold with codimension equal to $\dim_{\mathbb{R}} \mathrm{End}^{\mathrm{sym}}_{\mathbb{F}}(K)$.

Finally, we observe that in the case $k = 1$, the above identifies $\mathrm{Fred}^{\mathrm{sym},1}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$ locally with the zero set of a submersion to $\mathrm{End}^{\mathrm{sym}}_{\mathbb{F}}(K)$, which is a real 1-dimensional vector space since K is a 1-dimensional vector space over $\mathbb{F}$. The canonical isomorphism

$$\mathbb{R} \to \mathrm{End}^{\mathrm{sym}}_{\mathbb{F}}(K) : a \mapsto a\mathbb{1}$$

thus determines a co-orientation on $\mathrm{Fred}^{\mathrm{sym},1}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$. $\square$

The canonical co-orientation of $\mathrm{Fred}^{\mathrm{sym},1}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$ makes it natural to define signed intersection numbers between $\mathrm{Fred}^{\mathrm{sym},1}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$ and smooth paths in the ambient space $\mathrm{Fred}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$. The codimensions of $\mathrm{Fred}^{\mathrm{sym},k}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$ for each $k \ge 2$ are still at least 3, hence large enough to ensure that generic paths or homotopies of paths will never intersect them. The following notion is therefore independent of choices.

DEFINITION 3.10. Suppose $\mathbf{T}_+ : \mathbf{T}_- \in \mathrm{Fred}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$ are both Banach space isomorphisms $\mathcal{D} \to \mathcal{H}$. The **spectral flow**

$$\mu^{\mathrm{spec}}(\mathbf{T}_-,\mathbf{T}_+) \in \mathbb{Z}$$

from $\mathbf{T}_-$ to $\mathbf{T}_+$ is then defined as the signed count of intersections of $\mathbf{T} : [-1,1] \to \mathrm{Fred}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$ with $\mathrm{Fred}^{\mathrm{sym},1}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$, where the latter is assumed to carry the co-orientation given by Prop. 3.9, and $\mathbf{T} : [-1,1] \to \mathrm{Fred}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$ is any smooth path that is transverse to $\mathrm{Fred}^{\mathrm{sym},k}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$ for every $k \ge 1$ and satisfies $\mathbf{T}(\pm 1) = \mathbf{T}_\pm$.

3.2.3. Perturbation of eigenvalues. Continuing in the setting of the previous subsection, we shall now regard each $\mathbf{T} \in \mathrm{Fred}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$ as an unbounded operator on $\mathcal{H}$ with domain $\mathcal{D}$, see e.g. [**RS80**, Chapter VIII]. Notice that for each scalar $\lambda \in \mathbb{F}$, the operator $\mathbf{T}-\lambda$ also belongs to $\mathrm{Fred}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$. The **spectrum**

$$\sigma(\mathbf{T}) \subset \mathbb{F}$$

of $\mathbf{T}$ is defined as the set of all $\lambda \in \mathbb{F}$ for which $\mathbf{T} - \lambda : \mathcal{D} \to \mathcal{H}$ does not admit a bounded inverse. In particular, $\lambda \in \sigma(\mathbf{T})$ is an **eigenvalue** of $\mathbf{T}$ whenever $\mathbf{T} - \lambda : \mathcal{D} \to \mathcal{H}$ has nontrivial kernel, and the dimension of this kernel is called the **multiplicity** of the eigenvalue. We call λ a **simple eigenvalue** if it has multiplicity 1. By a standard argument familiar to both mathematicians and physicists, the eigenvalues of a symmetric complex-linear operator are always real.

REMARK 3.11. If $\mathcal{D} \subset \mathcal{H}$ is dense, then the **adjoint** of $\mathbf{T}$ is defined as an unbounded operator $\mathbf{T}^*$ with domain $\mathcal{D}^*$ satisfying

$$\langle x, \mathbf{T}y\rangle_{\mathcal{H}} = \langle \mathbf{T}^* x, y\rangle_{\mathcal{H}} \quad \text{for all } x \in \mathcal{D}^*,\ y \in \mathcal{D},$$

where $\mathcal{D}^*$ is the set of all $x \in \mathcal{H}$ such that there exists $z \in \mathcal{H}$ satisfying $\langle x, \mathbf{T}y\rangle_{\mathcal{H}} = \langle z, y\rangle_{\mathcal{H}}$ for all $y \in \mathcal{D}$. One says that $\mathbf{T}$ is **self-adjoint** if $\mathbf{T} = \mathbf{T}^*$, which means both that $\mathbf{T}$ is symmetric and $\mathcal{D} = \mathcal{D}^*$. In many applications (e.g. in Exercise 3.29), the latter amounts to a condition on "regularity of weak solutions". This condition implies that the inclusion $\ker \mathbf{T} \hookrightarrow (\operatorname{im} \mathbf{T})^\perp$—valid for all symmetric operators—is also surjective, so if $\mathbf{T} : \mathcal{D} \to \mathcal{H}$ is Fredholm, it is then automatic that $\mathrm{ind}(\mathbf{T}) = 0$.

PROPOSITION 3.12. *Assume* $\mathbf{T}_0 \in \mathrm{Fred}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$. *Then:*

(1) *Every* $\lambda \in \sigma(\mathbf{T}_0)$ *is an eigenvalue with finite multiplicity.*
(2) *The spectrum* $\sigma(\mathbf{T}_0)$ *is a discrete subset of* $\mathbb{R}$.
(3) *Suppose* $\lambda_0 \in \sigma(\mathbf{T}_0)$ *is an eigenvalue with multiplicity* $m \in \mathbb{N}$ *and* $\epsilon > 0$ *is chosen such that no other eigenvalues lie in* $[\lambda_0 - \epsilon, \lambda_0 + \epsilon]$. *Then* $\mathbf{T}_0$ *has a*

neighorhood $\mathcal{O} \subset \mathrm{Fred}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{D}, \mathcal{H}, \mathbf{T}_{\mathrm{ref}})$ *such that for all* $\mathbf{T} \in \mathcal{O}$*,*

$$\sum_{\lambda \in \sigma(\mathbf{T}) \cap [\lambda_0 - \epsilon, \lambda_0 + \epsilon]} m(\lambda) = m,$$

where $m(\lambda) \in \mathbb{N}$ *denotes the multiplicity of* $\lambda \in \sigma(\mathbf{T})$*.*

PROOF. For every $\lambda \in \mathbb{F}$, $\mathbf{T}_0 - \lambda$ is a Fredholm operator with index 0, so it is a Banach space isomorphism $\mathcal{D} \to \mathcal{H}$ and thus has a bounded inverse if and only if its kernel is trivial. The Fredholm property also implies that the kernel is finite dimensional whenever it is nontrivial, so this proves (1).

For (2) and (3), let us assume $\mathbb{F} = \mathbb{C}$, as the case $\mathbb{F} = \mathbb{R}$ will follow by taking complexifications of real vector spaces. We claim therefore that $\sigma(\mathbf{T}_0)$ is a discrete subset of $\mathbb{C}$. To see this, suppose $\lambda_0 \in \mathbb{R}$ is an eigenvalue of $\mathbf{T}_0$ with multiplicity m, so

$$\mathbf{T}_0 - \lambda_0 \in \mathrm{Fred}^{\mathrm{sym},m}_{\mathbb{C}}(\mathcal{D}, \mathcal{H}).$$

By Lemma 3.8, there are splittings $\mathcal{D} = V \oplus K$ and $\mathcal{H} = W \oplus K$ with $K = \ker(\mathbf{T}_0 - \lambda_0)$, $W = \mathrm{im}(\mathbf{T}_0 - \lambda_0)$ and $V = W \cap \mathcal{D}$. Any scalar $\lambda \in \mathbb{C}$ appears in block-diagonal form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ with respect to these splittings, and the block form for $\mathbf{T}_0$ is thus

$$\mathbf{T}_0 = \begin{pmatrix} \mathbf{A}_0 + \lambda_0 & 0 \\ 0 & \lambda_0 \end{pmatrix}$$

for some Banach space isomorphism $\mathbf{A}_0 : V \to W$. Writing nearby operators $\mathbf{T} \in \mathrm{Fred}_{\mathbb{C}}(\mathcal{D}, \mathcal{H})$ as $\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}$, we can imitate the construction in (3.5) to produce neighborhoods $\mathcal{O}(\mathbf{T}_0) \subset \mathrm{Fred}_{\mathbb{C}}(\mathcal{D}, \mathcal{H})$ of $\mathbf{T}_0$ and $\mathbb{D}_\epsilon(\lambda_0) \subset \mathbb{C}$ of λ_0, admitting a holomorphic map

$$\Phi : \mathcal{O}(\mathbf{T}_0) \times \mathbb{D}_\epsilon(\lambda_0) \to \mathrm{End}_{\mathbb{C}}(K) : (\mathbf{T}, \lambda) \mapsto (\mathbf{D} - \lambda) - \mathbf{C}\,(\mathbf{A} - \lambda)^{-1}\,\mathbf{B}$$

such that $\ker(\mathbf{T} - \lambda) \cong \ker \Phi(\mathbf{T}, \lambda)$. The set of eigenvalues of $\mathbf{T}_0$ near λ_0 is then the zero set of the holomorphic function

$$\mathbb{D}_\epsilon(\lambda_0) \to \mathbb{C} : \lambda \mapsto \det \Phi(\mathbf{T}_0, \lambda). \tag{3.6}$$

This function cannot be identically zero since there are no eigenvalues outside of $\mathbb{R}$, thus the zero at λ_0 is isolated, proving (2).

To prove (3), note finally that if the neighborhood $\mathcal{O}(\mathbf{T}_0) \subset \mathrm{Fred}_{\mathbb{C}}(\mathcal{D}, \mathcal{H})$ of $\mathbf{T}_0$ is sufficiently small, then for every $\mathbf{T} \in \mathcal{O}(\mathbf{T}_0)$, the holomorphic function

$$f_{\mathbf{T}} : \mathbb{D}_\epsilon(\lambda_0) \to \mathbb{C} : \lambda \mapsto \det \Phi(\mathbf{T}, \lambda)$$

has the same algebraic count of zeroes in $\mathbb{D}_\epsilon(\lambda_0)$, all of which lie in $[\lambda_0 - \epsilon, \lambda_0 + \epsilon]$ if $\mathbf{T}$ is symmetric. Observe moreover that since

$$\partial_\lambda \Phi(\mathbf{T}_0, \lambda_0) = -\mathbb{1} \in \mathrm{End}_{\mathbb{C}}(K),$$

we are free to assume after possibly shrinking ϵ and $\mathcal{O}(\mathbf{T}_0)$ that $\partial_\lambda \Phi(\mathbf{T}, \lambda)$ is always a nonsingular transformation in $\mathrm{End}_{\mathbb{C}}(K)$. Since $\Phi(\mathbf{T}, \lambda)$ is in $\mathrm{End}^{\mathrm{sym}}_{\mathbb{C}}(K)$ and thus diagonalizable whenever $\mathbf{T}$ is symmetric and $\lambda \in \mathbb{R}$, it follows via Exercise 3.13

below that the order of any zero $f_{\mathbf{T}}(\lambda) = 0$ is precisely the multiplicity of λ as an eigenvalue of $\mathbf{T}$.

$\square$

EXERCISE 3.13. Suppose $\mathcal{U} \subset \mathbb{C}$ is an open subset, $A : \mathcal{U} \to \mathbb{C}^{n\times n}$ is a holomorphic map and $z_0 \in \mathcal{U}$ is a point at which $A(z_0)$ is noninvertible but diagonalizable, and $A'(z_0) \in \mathrm{GL}(n, \mathbb{C})$. Show that $\dim_{\mathbb{C}} \ker A(z_0)$ is the order of the zero of the holomorphic function $\det A : \mathcal{U} \to \mathbb{C}$ at z_0.

The next result implies that for a generic path of symmetric index 0 operators as appears in our definition of $\mu^{\mathrm{spec}}(\mathbf{T}_-, \mathbf{T}_+)$, the spectral flow is indeed a signed count of eigenvalues crossing 0.

PROPOSITION 3.14. *Suppose* $\{\mathbf{T}_t \in \mathrm{Fred}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{D}, \mathcal{H}, \mathbf{T}_{\mathrm{ref}})\}_{t\in(-1,1)}$ *is a smooth path and* $\lambda_0 \in \mathbb{R}$ *is a simple eigenvalue of* $\mathbf{T}_0$*. Then:*

(1) For sufficiently small $\epsilon > 0$*, there exists a unique smooth function* $\lambda : (-\epsilon, \epsilon) \to \mathbb{R}$ *such that* $\lambda(0) = \lambda_0$ *and* $\lambda(t)$ *is a simple eigenvalue of* $\mathbf{T}_t$ *for each* $t \in (-\epsilon, \epsilon)$*.*

(2) The derivative $\lambda'(0)$ *is nonzero if and only if the intersection of the path* $\{\mathbf{T}_t - \lambda_0 \in \mathrm{Fred}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{D}, \mathcal{H}, \mathbf{T}_{\mathrm{ref}})\}_{t\in(-1,1)}$ *with* $\mathrm{Fred}^{\mathrm{sym},1}_{\mathbb{F}}(\mathcal{D}, \mathcal{H}, \mathbf{T}_{\mathrm{ref}})$ *at* $t = 0$ *is transverse, and the sign of* $\lambda'(0)$ *is then the sign of the intersection.*

PROOF. Using the same construction as in the proof of Proposition 3.12, we can find small numbers $\epsilon > 0$ and $\delta > 0$ such that

$$\big\{(t, \lambda) \in (-\epsilon, \epsilon) \times (\lambda_0 - \delta, \lambda_0 + \delta) \bigm| \lambda \in \sigma(\mathbf{T}_t)\big\} = \Phi^{-1}(0),$$

where

$$\Phi : (-\epsilon, \epsilon) \times (\lambda_0 - \delta, \lambda_0 + \delta) \to \mathrm{End}^{\mathrm{sym}}_{\mathbb{F}}(K) : (t, \lambda) \mapsto (\mathbf{D}_t - \lambda) - \mathbf{C}_t \left(\mathbf{A}_t - \lambda\right)^{-1} \mathbf{B}_t,$$

and we write $\mathbf{T}_t = \begin{pmatrix} \mathbf{A}_t & \mathbf{B}_t \\ \mathbf{C}_t & \mathbf{D}_t \end{pmatrix}$ with respect to splittings $\mathcal{D} = V \oplus K$ and $\mathcal{H} = W \oplus K$ with $K = \ker(\mathbf{T}_0 - \lambda_0)$, $W = \mathrm{im}(\mathbf{T}_0 - \lambda_0)$ and $V = W \cap \mathcal{D}$. In saying this, we've implicitly used the assumption that λ_0 is a simple eigenvalue, as it follows that $\dim_{\mathbb{F}} \ker(\mathbf{T} - \lambda)$ cannot be larger than 1 for any $\mathbf{T}$ near $\mathbf{T}_0$ and λ near λ_0, so that $\Phi^{-1}(0)$ catches all nearby eigenvalues. Simplicity also means that $\mathrm{End}^{\mathrm{sym}}_{\mathbb{F}}(K)$ is real 1-dimensional, and we have

$$\partial_t \Phi(0, \lambda_0) = \partial_t \mathbf{D}_t|_{t=0}, \qquad \partial_\lambda \Phi(0, \lambda_0) = -1.$$

The implicit function theorem thus gives $\Phi^{-1}(0)$ near $(0, \lambda_0)$ the structure of a smooth 1-manifold with tangent space at $(0, \lambda_0)$ spanned by the vector

$$\partial_t + \left(\partial_t \mathbf{D}_t|_{t=0}\right) \partial_\lambda,$$

where we are identifying $\partial_t \mathbf{D}_t|_{t=0} \in \mathrm{End}^{\mathrm{sym}}_{\mathbb{F}}(K)$ with a real number via the natural isomorphism $\mathrm{End}^{\mathrm{sym}}_{\mathbb{F}}(K) = \mathbb{R}$. Therefore $\Phi^{-1}(0)$ can be written as the graph of a uniquely determined smooth function λ, whose derivative at zero is a multiple of $\partial_t \mathbf{D}_t|_{t=0}$. This proves both statements in the proposition, since by the proof of Proposition 3.9, the intersection of $\{\mathbf{T}_t\}_{t\in(-1,1)}$ with $\mathrm{Fred}^{\mathrm{sym},1}_{\mathbb{F}}(\mathcal{D}, \mathcal{H}, \mathbf{T}_{\mathrm{ref}})$ is transverse if and only if $\partial_t \mathbf{D}_t|_{t=0} \neq 0$, and its sign is then the sign of $\partial_t \mathbf{D}_t|_{t=0}$. $\square$

The purpose of the next lemma is to prevent eigenvalues from escaping to $\pm\infty$ under smooth families of operators in $\mathrm{Fred}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$.

LEMMA 3.15. *Suppose $\{\mathbf{K}_t \in \mathscr{L}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{H})\}_{t\in(a,b)}$ is a smooth path of symmetric bounded linear operators, and $\lambda : (a,b) \to \mathbb{R}$ is a smooth function such that for every $t \in (a,b)$, $\lambda(t)$ is a simple eigenvalue of $\mathbf{T}_t := \mathbf{T}_{\mathrm{ref}} + \mathbf{K}_t \in \mathrm{Fred}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$. Then*

$$|\dot{\lambda}(t)| \le \|\partial_t \mathbf{K}_t\|_{\mathscr{L}(\mathcal{H})} \qquad \textit{for all } t \in (a,b).$$

PROOF. Since $\{\mathbf{T}_t - \lambda(t) \in \mathrm{Fred}^{\mathrm{sym}}_{\mathbb{F}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})\}_{t\in(a,b)}$ is a smooth family of operators in $\mathrm{Fred}_{\mathbb{F}}(\mathcal{D},\mathcal{H})$ with 1-dimensional kernel, one can use the local families of isomorphisms $\Psi(\mathbf{T}_t - \lambda(t)) \in \mathscr{L}_{\mathbb{F}}(\mathcal{D})$ from Lemma 3.4 to find a smooth family of eigenvectors $x(t) \in \ker(\mathbf{T}_t - \lambda(t))$ for $t \in (a,b)$. Normalize these so that $\|x(t)\|_{\mathcal{H}} = 1$ for all t. Then $0 = \partial_t\langle x(t), x(t)\rangle_{\mathcal{H}} = \langle \dot{x}(t), x(t)\rangle_{\mathcal{H}} + \langle x(t), \dot{x}(t)\rangle_{\mathcal{H}}$ and $\lambda(t) = \langle x(t), \mathbf{T}_t x(t)\rangle_{\mathcal{H}}$, so writing $\dot{\mathbf{K}}_t := \partial_t \mathbf{K}_t = \partial_t \mathbf{T}_t$, we have

$$\begin{aligned}\dot{\lambda}(t) &= \partial_t\langle x(t), \mathbf{T}_t x(t)\rangle_{\mathcal{H}} = \langle x(t), \dot{\mathbf{K}}_t x(t)\rangle_{\mathcal{H}} + \langle \dot{x}(t), \mathbf{T}_t x(t)\rangle_{\mathcal{H}} + \langle x(t), \mathbf{T}_t \dot{x}(t)\rangle_{\mathcal{H}} \\ &= \langle x(t), \dot{\mathbf{K}}_t x(t)\rangle_{\mathcal{H}},\end{aligned}$$

as the last two terms in the first line become $\lambda(t)\left[\langle \dot{x}(t), x(t)\rangle_{\mathcal{H}} + \langle x(t), \dot{x}(t)\rangle_{\mathcal{H}}\right] = 0$ since $\mathbf{T}_t$ is symmetric and $\mathbf{T}_t x(t) = \lambda(t)x(t)$. We obtain

$$|\dot{\lambda}(t)| \le \|x(t)\|_{\mathcal{H}} \|\dot{\mathbf{K}}_t\|_{\mathscr{L}(\mathcal{H})} \|x(t)\|_{\mathcal{H}} = \|\dot{\mathbf{K}}_t\|_{\mathscr{L}(\mathcal{H})}.$$

□

3.2.4. Homotopies of eigenvalues. Specializing further, we now set $\mathcal{H}$ and $\mathcal{D}$ equal to the specific real Hilbert spaces

$$\mathcal{H} := L^2(S^1, \mathbb{R}^{2n}), \qquad \mathcal{D} := H^1(S^1, \mathbb{R}^{2n}),$$

and set $\mathbf{T}_{\mathrm{ref}} := -J_0\,\partial_t$, where J_0 denotes the standard complex structure on $\mathbb{R}^{2n} = \mathbb{C}^n$. Observe that any bounded linear operator on L^2 determines a compact operator $H^1 \to L^2$ via composition with the compact inclusion. In particular, we shall consider compact perturbations of $-J_0\,\partial_t$ in the form

$$\mathbf{A} = -J_0\,\partial_t - S(t) \tag{3.7}$$

with $S : S^1 \to \mathrm{End}^{\mathrm{sym}}_{\mathbb{R}}(\mathbb{R}^{2n})$ smooth. It is straightforward to check that this operator is symmetric with respect to the L^2-product since $S(t)$ is symmetric for every t. The following then implies that $\mathbf{A} \in \mathrm{Fred}^{\mathrm{sym}}_{\mathbb{R}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$.

LEMMA 3.16. *The operator $-J_0\,\partial_t : H^1(S^1,\mathbb{R}^{2n}) \to L^2(S^1,\mathbb{R}^{2n})$ is Fredholm with index 0.*

PROOF. Since J_0 defines an isomorphism, it suffices actually to show that the ordinary differential operator

$$\partial_t : H^1(S^1,\mathbb{R}^{2n}) \to L^2(S^1,\mathbb{R}^{2n})$$

is Fredholm with index 0. The kernel of this operator is the space of constant functions $S^1 \to \mathbb{R}^{2n}$, which has dimension $2n$. To compute the dimension of the cokernel, we observe that if $f = \partial_t F$ lies in the image of this operator, we have $\int_{S^1} f(t)\,dt = 0$ since F is periodic in t. Conversely, if $\int_{S^1} f(t)\,dt = 0$ with $f \in$

$L^2(S^1, \mathbb{R}^{2n})$, then the function $F(s) = \int_0^s f(t)\,dt$ is periodic in s and defines an element of $H^1(S^1, \mathbb{R}^{2n})$ satisfying $\partial_t F = f$. Hence the image of ∂_t is exactly the set

$$\operatorname{im}(\partial_t) = \left\{ f \in L^2(S^1, \mathbb{R}^{2n}) \;\middle|\; \int_{S^1} f(t)\,dt = 0 \right\},$$

which has codimension $2n$. □

The proof of Theorem 3.3 requires only one more technical ingredient, whose proof is given in Appendix C and should probably be skipped on first reading unless you have already read Lecture 7 or seen similar applications of the Sard-Smale theorem. You might however find the result plausible in accordance with the notion that maps from 2-dimensional domains, such as a map of the form

$$(-1, 1) \times \mathbb{R} \to \operatorname{Fred}^{\mathrm{sym}}_{\mathbb{R}}(\mathcal{D}, \mathcal{H}, \mathbf{T}_{\mathrm{ref}}) : (t, \lambda) \mapsto \mathbf{T}_t - \lambda$$

should *generically* not intersect submanifolds that have codimension 3 or more, such as $\operatorname{Fred}^{\mathrm{sym},k}_{\mathbb{R}}(\mathcal{D}, \mathcal{H}, \mathbf{T}_{\mathrm{ref}})$ when $k \ge 2$.

LEMMA 3.17. *Fix a smooth map* $S : [-1, 1] \times S^1 \to \operatorname{End}^{\mathrm{sym}}_{\mathbb{R}}(\mathbb{R}^{2n})$ *and consider the* 1*-parameter family of unbounded linear operators*

$$\mathbf{A}_s := -J_0\,\partial_t - S(s, \cdot) : L^2(S^1, \mathbb{R}^{2n}) \supset H^1(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n})$$

for $s \in [-1, 1]$*. One can arrange after a* C^∞*-small perturbation of* S *fixed at* $s = \pm 1$ *that the following conditions hold:*

(1) For each $s \in (-1, 1)$*, all eigenvalues of* $\mathbf{A}_s$ *are simple.*
(2) All intersections of the path

$$(-1, 1) \to \operatorname{Fred}^{\mathrm{sym}}_{\mathbb{R}}(\mathcal{D}, \mathcal{H}, \mathbf{T}_{\mathrm{ref}}) : s \mapsto \mathbf{A}_s$$

with $\operatorname{Fred}^{\mathrm{sym},1}_{\mathbb{R}}(\mathcal{D}, \mathcal{H}, \mathbf{T}_{\mathrm{ref}})$ *are transverse.*

□

PROOF OF THEOREM 3.3. Given a smooth family $\{\mathbf{A}_s\}_{s\in[-1,1]}$ as stated in the theorem, use Lemma 3.17 to obtain a C^∞-small perturbation for which the eigenvalues are simple for $s \in (-1, 1)$ and all intersections with $\operatorname{Fred}^{\mathrm{sym},1}_{\mathbb{R}}(\mathcal{D}, \mathcal{H})$ are transverse. Proposition 3.14 then implies that the eigenvalues depend smoothly on s, and Lemma 3.15 imposes a uniform bound on their derivatives with respect to s so that each one varies only in a bounded subset of $\mathbb{R}$ for $s \in (-1, 1)$. The smooth families of eigenvalues for $s \in (-1, 1)$ therefore extend to continuous families for $s \in [-1, 1]$ since the space of noninvertible Fredholm operators with index 0 is closed. Proposition 3.12 ensures moreover that these continuous families hit every eigenvalue with the correct multiplicity at $s = \pm 1$, and by Proposition 3.14, the formula for $\mu^{\mathrm{spec}}(\mathbf{A}_-, \mathbf{A}_+)$ stated in the theorem is correct for the perturbed family with simple eigenvalues and transverse crossings. To obtain the same result for the original family, suppose we have a sequence of perturbations $\{\mathbf{A}^\nu_s\}_{s\in[-1,1]}$ converging in C^∞ as $\nu \to \infty$ to $\{\mathbf{A}_s\}_{s\in[-1,1]}$. Lemma 3.15 then provides a uniform C^1-bound for each sequence of smooth families of eigenvalues, so they have C^0-convergent subsequences as $\nu \to \infty$, giving rise to the continuous families in the statement of the theorem. □

REMARK 3.18. It is important to understand that the definition of spectral flow depends on the particular co-orientation of $\text{Fred}_{\mathbb{F}}^{\text{sym},1}(\mathcal{D},\mathcal{H},\mathbf{T}_{\text{ref}})$ that arose in the proof of Prop. 3.9; we saw in Prop. 3.14 that this is indeed the *right* co-orientation to use if we want to interpret signed intersections with $\text{Fred}_{\mathbb{F}}^{\text{sym},1}(\mathcal{D},\mathcal{H},\mathbf{T}_{\text{ref}})$ as signed crossing numbers of eigenvalues. In the non-symmetric setting of §3.2.1, one can show that $\text{Fred}_{\mathbb{R}}^{0,1}(X,Y)$ is also co-orientable; this is obvious in the finite-dimensional case since $\text{Fred}_{\mathbb{R}}^{0,1}(\mathbb{R}^n,\mathbb{R}^n)$ is then a regular level set of the determinant function. Moreover, $\text{Fred}_{\mathbb{R}}^{0,1}(\mathbb{R}^n,\mathbb{R}^n)$ is connected (see Exercise 3.19 below), so the co-orientation is unique up to a sign. One can therefore lift the $\mathbb{Z}_2$-valued spectral flow of §3.2.1 to $\mathbb{Z}$, but as in Exercise 3.7, the result will be a different and much less interesting invariant than $\mu^{\text{spec}}(A_-,A_+)$, as its value will always be either 0 (if $\det A_-$ and $\det A_+$ have the same sign) or ± 1 (if they don't). The reason for the discrepancy is that the canonical co-orientation of $\text{Fred}_{\mathbb{R}}^{\text{sym},1}(\mathcal{D},\mathcal{H},\mathbf{T}_{\text{ref}})$ must generally differ on some connected components from any possible co-orientation of the larger hypersurface $\text{Fred}_{\mathbb{R}}^{0,1}(\mathcal{D},\mathcal{H}) \subset \text{Fred}_{\mathbb{R}}^{0}(\mathcal{D},\mathcal{H})$.

EXERCISE 3.19. Show that the space $\text{Fred}_{\mathbb{R}}^{0,1}(\mathbb{R}^2,\mathbb{R}^2)$ of rank 1 matrices in $\mathbb{R}^{2\times 2}$ is connected, but the space $\text{Fred}_{\mathbb{R}}^{\text{sym},1}(\mathbb{R}^2,\mathbb{R}^2)$ of *symmetric* rank 1 matrices is not, and that the canonical co-orientation of $\text{Fred}_{\mathbb{R}}^{\text{sym},1}(\mathbb{R}^2,\mathbb{R}^2)$ coming from Prop. 3.9 differs on some components from any possible co-orientation of $\text{Fred}_{\mathbb{R}}^{0,1}(\mathbb{R}^2,\mathbb{R}^2) \subset \mathbb{R}^{2\times 2}$. *Hint: A non-symmetric* 2*-by-*2 *matrix may have rank* 1 *even if both of its eigenvalues are* 0*. For symmetric matrices this cannot happen.*

EXERCISE 3.20. Find a smooth path $A : [-1,1] \to \mathbb{R}^{2\times 2}$ of symmetric matrices such that $A_\pm := A(\pm 1)$ are both invertible and $\mu^{\text{spec}}(A_-,A_+) = 2$, but A_+ and A_- can also be connected by a smooth path of (not necessarily symmetric) invertible matrices in $\mathbb{R}^{2\times 2}$.

3.3. The Hessian of the contact action functional

Before returning to contact geometry, let's quickly revisit the Floer homology for a time-dependent Hamiltonian $\{H_t : M \to \mathbb{R}\}_{t\in S^1}$ on a symplectic manifold (M,ω). In Lecture 1, we introduced the symplectic action functional $\mathcal{A}_H : C^\infty_{\text{contr}}(S^1,M) \to \mathbb{R}$ and wrote down the formula

$$\nabla \mathcal{A}_H(\gamma) = J_t(\gamma)\,(\dot\gamma - X_t(\gamma)) \in \Gamma(\gamma^*TM) =: T_\gamma C^\infty_{\text{contr}}(S^1,M)$$

for the "unregularized" gradient of $\mathcal{A}_H$ at a contractible loop $\gamma \in C^\infty_{\text{contr}}(S^1,M)$. Here X_t denotes the Hamiltonian vector field and J_t is a time-dependent family of compatible almost complex structures, which determines the L^2-product

$$\langle \eta_1,\eta_2\rangle_{L^2} = \int_{S^1} \omega(\eta_1(t), J_t\eta_2(t))\,dt.$$

The critical points of $\mathcal{A}_H$ are the loops γ such that $\nabla\mathcal{A}_H(\gamma) = 0$. Formally, the Hessian of $\mathcal{A}_H$ at $\gamma \in \text{Crit}(\mathcal{A}_H)$ is the "linearization of $\mathcal{A}_H$ at γ," which gives a linear operator

$$\mathbf{A}_\gamma := \nabla^2\mathcal{A}_H(\gamma) : \Gamma(\gamma^*TM) \to \Gamma(\gamma^*TM).$$

To write it down, one can choose any connection ∇ on M, and choose for $\eta \in \Gamma(\gamma^*TM)$ a smooth family $\{\gamma_\rho : S^1 \to M\}_{\rho \in (-\epsilon,\epsilon)}$ with $\gamma_0 = \gamma$ and $\partial_\rho \gamma_\rho|_{\rho=0} = \eta$, and then compute

$$\mathbf{A}_\gamma \eta := \nabla_\rho \left[\nabla \mathcal{A}_H(\gamma_\rho)\right]\big|_{\rho=0}.$$

The result is independent of the choice of connection since $\nabla \mathcal{A}_H(\gamma) = 0$.

EXERCISE 3.21. Show that if the connection ∇ on M is chosen to be symmetric, then $\mathbf{A}_\gamma \eta = J_t(\nabla_t \eta - \nabla_\eta X_t)$.

We now introduce the class of symmetric operators that appear in asymptotic formulas in SFT. Fix a $(2n-1)$-dimensional contact manifold (M, ξ) with contact form α, induced Reeb vector field R_α, and a complex structure $J : \xi \to \xi$ compatible with the symplectic structure $d\alpha|_\xi$. Let

$$\pi_\xi : TM \to \xi$$

denote the projection along R_α. The **contact action functional** is defined by

$$\mathcal{A}_\alpha : C^\infty(S^1, M) \to \mathbb{R} : \gamma \mapsto \int_{S^1} \gamma^*\alpha.$$

The first variation of this functional for $\gamma \in C^\infty(S^1, M)$ and $\eta \in \Gamma(\gamma^*TM)$ is

$$d\mathcal{A}_\alpha(\gamma)\eta = \int_{S^1} d\alpha(\eta, \dot{\gamma})\, dt = -\int_{S^1} d\alpha(\pi_\xi \dot{\gamma}, \eta)\, dt.$$

The functional has a built-in degeneracy since it is parametrization-invariant; in particular, $d\mathcal{A}_\alpha(\gamma)\eta = 0$ whenever η points in the direction of the Reeb vector field, a symptom of the fact that closed Reeb orbits always come in families related to each other by reparametrization. A loop $\gamma : S^1 \to M$ is critical for $\mathcal{A}_\alpha$ if and only if $\dot{\gamma}$ is everywhere tangent to R_α, allowing for an infinite-dimensional family of distinct perturbations—however, there exist preferred parametrizations, namely those for which $\dot{\gamma}$ is a *constant* multiple of R_α, meaning

$$\dot{\gamma} = T \cdot R_\alpha(\gamma), \qquad T := \mathcal{A}_\alpha(\gamma). \tag{3.8}$$

Such a loop corresponds to a T-periodic solution $x : \mathbb{R} \to M$ to $\dot{x} = R_\alpha(x)$, where $\gamma(t) = x(Tt)$.

The discussion above indicates that we cannot derive a "Hessian" of $\mathcal{A}_\alpha$ in the same straightforward way as in Floer homology, as the resulting operator will always have nontrivial kernel due to the degeneracy in the R_α direction. To avoid this, we shall consider only preferred parametrizations $\gamma : S^1 \to M$ of the form (3.8), and perturbations in directions tangent to ξ, which is transverse to every Reeb orbit. For $\eta \in \Gamma(\gamma^*\xi)$, we then have

$$d\mathcal{A}_\alpha(\gamma)\eta = \int_{S^1} d\alpha(-J\pi_\xi \dot{\gamma}, J\eta)\, dt = \langle -J\pi_\xi \dot{\gamma}, \eta \rangle_{L^2},$$

where we define an L^2-product for sections of $\gamma^*\xi$ by

$$\langle \eta, \eta' \rangle_{L^2} := \int_{S^1} d\alpha(\eta, J\eta')\, dt. \tag{3.9}$$

It therefore seems sensible to write

$$\nabla \mathcal{A}_\alpha(\gamma) := -J\pi_\xi \dot{\gamma} \in \Gamma(\gamma^*\xi),$$

and we shall define the Hessian at a critical point γ as the linearization in ξ directions, i.e.

$$\nabla^2 \mathcal{A}_\alpha(\gamma) : \Gamma(\gamma^*\xi) \to \Gamma(\gamma^*\xi).$$

Given $\eta \in \Gamma(\gamma^*\xi)$, choose a smooth family $\{\gamma_\rho : S^1 \to M\}_{\rho \in (-\epsilon,\epsilon)}$ with $\gamma_0 = \gamma$ and $\partial_\rho \gamma_\rho|_{\rho=0} = \eta$, and fix a symmetric connection ∇ on M. Since $\pi_\xi \dot{\gamma} = 0$, the covariant derivative of $\nabla \mathcal{A}_\alpha(\gamma_\rho)$ at $\rho = 0$ is then

$$\begin{aligned}\nabla_\rho \left(-J\pi_\xi \dot{\gamma}_\rho\right)\big|_{\rho=0} &= -J\, \nabla_\rho \left(\pi_\xi \dot{\gamma}_\rho\right)\big|_{\rho=0} = -J\, \nabla_\rho \left[\dot{\gamma}_\rho - \alpha(\dot{\gamma}_\rho) R_\alpha(\gamma_\rho)\right]\big|_{\rho=0} \\ &= -J \left(\nabla_t \eta - T\nabla_\eta R_\alpha - \partial_\rho \left[\alpha(\dot{\gamma}_\rho)\right]|_{\rho=0} \cdot R_\alpha(\gamma)\right).\end{aligned}$$

In the last term, we can write $\partial_\rho \left[\alpha(\dot{\gamma}_\rho)\right]|_{\rho=0} = d\alpha(\eta, \dot{\gamma}) + \partial_t \left[\alpha(\eta)\right] = 0$ since $\dot{\gamma} = TR_\alpha(\gamma)$ and $\alpha(\eta) = 0$ for $\eta \in \Gamma(\gamma^*\xi)$. One can now check that the remaining terms define a section of $\gamma^*\xi$, thus we are led to the following definition.

Definition 3.22. Given a loop $\gamma : S^1 \to M$ parametrizing a closed Reeb orbit in $(M, \xi = \ker \alpha)$ with period $T \equiv \alpha(\dot{\gamma})$, the **asymptotic operator** associated to γ is the first-order differential operator on $\gamma^*\xi$ defined by

$$\mathbf{A}_\gamma : \Gamma(\gamma^*\xi) \to \Gamma(\gamma^*\xi) : \eta \mapsto -J(\nabla_t \eta - T\nabla_\eta R_\alpha)$$

Exercise 3.23. Show that $\mathbf{A}_\gamma$ is symmetric with respect to the L^2 inner product (3.9) on $\Gamma(\gamma^*\xi)$. Moreover, γ is nondegenerate (see §1.3) if and only if $\ker \mathbf{A}_\gamma$ is trivial. *Hint for nondegeneracy: Consider the pullback of $\gamma^*\xi$ via the cover $\mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$, and show that solutions to $\nabla_t \eta - T\nabla_\eta R_\alpha = 0$ on the pullback are given by operating on $\xi_{\gamma(0)}$ with the linearized Reeb flow. To see this, try differentiating families of solutions to the equation $\dot{x} = TR_\alpha(x)$.*

Remark 3.24. Another way of phrasing the hint in the the above exercise is as follows: $\mathbf{A}_\gamma$ can also be written as $-J\widehat{\nabla}_t$, where $\widehat{\nabla}_t$ is the unique *symplectic connection* on $(\gamma^*\xi, d\alpha)$ for which parallel transport is given by the linearized Reeb flow.

You might be slightly concerned about the sign difference between the two formulas we've derived for asymptotic operators in contact geometry and in Floer homology. I also find this troubling, but the discrepancy seems to originate from the fact that our account of Floer homology has referred always to the *negative* gradient flow of $\mathcal{A}_H$, while SFT is actually defined via the *positive* gradient flow of $\mathcal{A}_\alpha$. The words "gradient flow" in SFT must in any case be interpreted very loosely. If

$$u : [0, \infty) \times S^1 \to \mathbb{R} \times M$$

is the cylindrical end of a finite-energy J-holomorphic curve for some $J \in \mathcal{J}(\alpha)$ as we described in Lecture 1, then $u(s,t)$ does not satisfy anything so straightforward as $\partial_s - \nabla \mathcal{A}_\alpha(u(s,\cdot)) = 0$, but it does satisfy

$$\pi_\xi \partial_s u + J\pi_\xi \partial_t u = 0,$$

which can be interpreted as the projection of a positive gradient flow equation to the contact bundle. This observation is a local symptom of a more important global fact that follows from Stokes' theorem: any asymptotically cylindrical J-holomorphic curve $u : \dot{\Sigma} \to \mathbb{R} \times M$ with positive and negative punctures $\Gamma^{\pm}$ asymptotic to orbits $\{\gamma_z\}_{z \in \Gamma^{\pm}}$ satisfies

$$\sum_{z \in \Gamma^+} \mathcal{A}_\alpha(\gamma) - \sum_{z \in \Gamma^-} \mathcal{A}_\alpha(\gamma) = \int_{\dot{\Sigma}} u^* d\alpha \geq 0.$$

This generalizes the basic fact in Floer homology that flow lines decrease action and, conversely, have their energy controlled by the action.

We would now like to develop some of the general properties of asymptotic operators. Recall that on any symplectic vector bundle (E, ω), a compatible complex structure J determines a Hermitian inner product

$$\langle v, w \rangle = \omega(v, Jw) + i\omega(v, w),$$

and conversely, any Hermitian inner product on a complex vector bundle determines a symplectic structure via the same relation. For this reason, we shall refer to any vector bundle E with a compatible pair (J, ω) as a **Hermitian vector bundle**. A **unitary trivialization** of such a bundle is a trivialization that identifies fibers with $\mathbb{R}^{2n}$ such that J and ω become the standard complex structure J_0 and symplectic structure ω_0 respectively.

DEFINITION 3.25. Fix a Hermitian vector bundle (E, J, ω) over S^1. An **asymptotic operator** on (E, J, ω) is any real-linear differential operator $\mathbf{A} : \Gamma(E) \to \Gamma(E)$ that takes the form

$$\mathbf{A} : C^\infty(S^1, \mathbb{R}^{2n}) \to C^\infty(S^1, \mathbb{R}^{2n}) : \eta \mapsto -J_0 \partial_t \eta - S(t)\eta \tag{3.10}$$

in unitary trivializations, where $S : S^1 \to \mathrm{End}(\mathbb{R}^{2n})$ is a smooth loop of symmetric matrices.

Equivalently, an asymptotic operator on (E, J, ω) is any operator of the form $-J\nabla$ where ∇ is a symplectic connection on E.

EXERCISE 3.26. Show that any asymptotic operator on a Hermitian vector bundle (E, J, ω) over S^1 is symmetric with respect to the real L^2 bundle metric

$$\langle \eta_1, \eta_2 \rangle_{L^2} := \int_{S^1} \omega(\eta_1(t), J\eta_2(t))\, dt.$$

EXERCISE 3.27. Show that the asymptotic operator $\mathbf{A}_\gamma$ for a closed Reeb orbit γ is also an asymptotic operator on $(\gamma^*\xi, J, d\alpha)$ in the sense of Definition 3.25.

For functional analytic purposes, we shall regard asymptotic operators on Hermitian bundles (E, J, ω) as bounded real-linear operators

$$\mathbf{A} : H^1(E) \to L^2(E).$$

By Lemma 3.16, all asymptotic operators are then Fredholm with index 0, and any two such operators on the same bundle are compact perturbations of each other. Regarding them alternatively as unbounded symmetric operators on $L^2(E)$, the spectral flow

$$\mu^{\mathrm{spec}}(\mathbf{A}_-, \mathbf{A}_+) \in \mathbb{Z}$$

between two such operators $\mathbf{A}_\pm$ with trivial kernel is defined by choosing any unitary trivialization to write both in the form $-J_0\,\partial_t - S(t)$, and it is independent of this choice. The following is what we mean when we say that critical points of the action functional have "infinite Morse index" and "infinite Morse co-index":

PROPOSITION 3.28. *Every asymptotic operator has infinitely many eigenvalues of both signs.*

PROOF. It is easy to verify that this is true for $\mathbf{A}_0 := -J_0\partial_t : H^1(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n})$; see the proof of theorem 3.35 below. It is therefore also true for $\mathbf{A}_0 + \epsilon$ for any $\epsilon \in \mathbb{R}$, and this operator has trivial kernel whenever $\epsilon \notin 2\pi\mathbb{Z}$. For any other trivialized asymptotic operator $\mathbf{A}$ with $0 \notin \sigma(\mathbf{A})$, the result then follows from Theorem 3.3 since $\mu^{\text{spec}}(\mathbf{A}_0 + \epsilon, \mathbf{A})$ is finite, and this is precisely the signed count of eigenvalues which change sign. The condition $0 \notin \sigma(\mathbf{A})$ can then be lifted by replacing $\mathbf{A}$ with $\mathbf{A} + \epsilon$. ☐

EXERCISE 3.29. Show that asymptotic operators are self-adjoint (as unbounded operators on L^2 with domain H^1) in the sense of Remark 3.11.

3.4. The Conley-Zehnder index

We are now in a position to define a suitable replacement for the Morse index in the context of SFT. We shall say that an asymptotic operator $\mathbf{A}$ is **nondegenerate** whenever $0 \notin \sigma(\mathbf{A})$. We will begin by defining the Conley-Zehnder index as an integer-valued invariant of homotopy classes of nondegenerate asymptotic operators on the trivial Hermitian bundle $S^1 \times \mathbb{R}^{2n}$; the definition on arbitrary Hermitian bundles will then depend on a choice of trivialization.

It is customary elsewhere in the literature (see e.g. [**SZ92**]) to adopt a somewhat different perspective on the Conley-Zehnder index, in which it defines an integer-valued invariant of connected components of the space of "nondegenerate symplectic arcs"

$$\left\{ \Psi \in C^0([0,1], \mathrm{Sp}(2n)) \;\middle|\; \Psi(0) = \mathbb{1} \text{ and } 1 \notin \sigma(\Psi(1)) \right\}.$$

These are two different perspectives on the same notion. A dictionary from ours to the other perspective is provided by associating to any trivialized nondegenerate asymptotic operator $\mathbf{A} = -J_0\partial_t - S(t)$ the symplectic arc Ψ defined by the initial value problem

$$(-J_0\partial_t - S(t))\Psi(t) = 0, \qquad \Psi(0) = \mathbb{1}.$$

Conversely, any smooth symplectic arc determines via this same formula a smooth path of symmetric matrices $S : [0,1] \to \mathrm{End}(\mathbb{R}^{2n})$, producing a mild generalization of our notion of an asymptotic operator.[4]

[4]If $S(t)$ is not continuous on S^1 but is continuous on $[0,1]$, then $-J_0\partial_t - S(t)$ cannot be regarded as a linear operator on $C^\infty(S^1, \mathbb{R}^{2n})$ but is still a very well-behaved symmetric Fredholm operator from $H^1(S^1)$ to $L^2(S^1)$. All of the important functional analytic results in this lecture can thus be generalized to allow this.

DEFINITION 3.30. The **Conley-Zehnder index** associates to every trivialized nondegenerate asymptotic operator $\mathbf{A} : H^1(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n})$ as in (3.10) an integer

$$\mu_{\mathrm{CZ}}(\mathbf{A}) \in \mathbb{Z}$$

determined uniquely by the following properties:

(1) Set $\mu_{\mathrm{CZ}}(\mathbf{A}) := 0$ for the operator $\mathbf{A} = -J_0\partial_t - \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$.

(2) For any two nondegenerate operators $\mathbf{A}_\pm$, set

$$\mu_{\mathrm{CZ}}(\mathbf{A}_-) - \mu_{\mathrm{CZ}}(\mathbf{A}_+) := \mu^{\mathrm{spec}}(\mathbf{A}_-, \mathbf{A}_+).$$

DEFINITION 3.31. Given a nondegenerate asymptotic operator $\mathbf{A}$ on a Hermitian bundle (E, J, ω) over S^1 and a choice of complex trivialization τ for (E, J), the **Conley-Zehnder index** of $\mathbf{A}$ with respect to τ is the integer

$$\mu^\tau_{\mathrm{CZ}}(\mathbf{A}) \in \mathbb{Z}$$

defined by choosing any unitary trivialization homotopic to τ to write $\mathbf{A}$ as an operator $H^1(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n})$ and then plugging in Definition 3.30.

If γ is a nondegenerate Reeb orbit γ in a $(2n-1)$-dimensional contact manifold $(M, \xi = \ker\alpha)$, then for any complex trivialization τ of $\gamma^*\xi \to S^1$, the **Conley-Zehnder index** of γ relative to τ is defined as

$$\mu^\tau_{\mathrm{CZ}}(\gamma) := \mu^\tau_{\mathrm{CZ}}(\mathbf{A}_\gamma).$$

REMARK 3.32. From the perspective of [SZ92], $\mu^\tau_{\mathrm{CZ}}(\gamma)$ is the Conley-Zehnder index of the linearized Reeb flow along γ restricted to ξ, expressed via a choice of unitary trivialization as a nondegenerate arc in $\mathrm{Sp}(2n-2)$.

EXERCISE 3.33. Show that if $\mathbf{A}_1$ and $\mathbf{A}_2$ are nondegenerate asymptotic operators on Hermitian bundles E_1 and E_2 respectively, then $\mathbf{A}_1 \oplus \mathbf{A}_2$ defines a nondegenerate asymptotic operator on $E_1 \oplus E_2$, and given trivializations τ_j for $j = 1, 2$,

$$\mu^{\tau_1 \oplus \tau_2}_{\mathrm{CZ}}(\mathbf{A}_1 \oplus \mathbf{A}_2) = \mu^{\tau_1}_{\mathrm{CZ}}(\mathbf{A}_1) + \mu^{\tau_2}_{\mathrm{CZ}}(\mathbf{A}_2).$$

The following is a functional-analytic version of the well-known fact that the Conley-Zehnder index classifies homotopy classes of nondegenerate symplectic arcs.

THEOREM 3.34. *On any Hermitian bundle $(E, J, \omega) \to S^1$ with complex trivialization τ, two nondegenerate asymptotic operators $\mathbf{A}_\pm$ lie in the same connected component of the space of nondegenerate asymptotic operators if and only if $\mu^\tau_{CZ}(\mathbf{A}_+) = \mu^\tau_{CZ}(\mathbf{A}_-)$.*

PROOF. Trivializing the bundle, we need to show that if $\mathbf{A}_\pm = -J_0\partial_t - S_\pm(t)$ satisfy $\mu^{\mathrm{spec}}(\mathbf{A}_-, \mathbf{A}_+) = 0$, then there exists a path of asymptotic operators between them for which no eigenvalues cross 0. To see this, we can first choose any path $\{\mathbf{A}_t\}_{t\in[-1,1]}$ of asymptotic operators with $\mathbf{A}_{\pm 1} = \mathbf{A}_\pm$, and then use Lemma 3.17 to add generic compact perturbations producing a family

$$\left\{\mathbf{A}'_t \in \mathrm{Fred}^{\mathrm{sym}}_{\mathbb{R}}(H^1, L^2, \mathbf{A}_+)\right\}_{t\in[-1,1]}$$

whose intersections with $\mathrm{Fred}^{\mathrm{sym},k}_{\mathbb{R}}(H^1, L^2, \mathbf{A}_+)$ are transverse for every $k \geq 1$, hence only simple eigenvalues cross 0 and they cross transversely. Any neighboring pair of crossings with opposite signs can then be eliminated by changing $\{\mathbf{A}'_t\}_{t\in[-1,1]}$ to $\{\mathbf{A}'_t + c(t)\}_{t\in[-1,1]}$ for a suitable choice of smooth function $c : [-1,1] \to \mathbb{R}$. Since the spectral flow is zero, one can repeat this modification until one obtains a path of perturbed operators with no crossings, and it is a small perturbation of the path of asymptotic operators $\{\mathbf{A}_t + c(t)\}_{t\in[-1,1]}$. Since $\mathbf{A}_\pm$ are both nondegenerate, one can assume moreover that all eigenvalues of $\mathbf{A}_t + c(t)$ stay a fixed distance $\delta > 0$ away from 0, where δ is independent of the perturbation. One can therefore "turn off the perturbation" as in the proof of Theorem 3.3, i.e. there exists a sequence of perturbed paths $\{\mathbf{A}^\nu_t\}_{t\in[-1,1]}$ converging to $\{\mathbf{A}_t + c(t)\}$ whose eigenvalues stay a fixed distance away from 0, and the same is therefore true for the continuous families of eigenvalues of $\mathbf{A}_t + c(t)$ obtained as $\nu \to \infty$. □

To compute Conley-Zehnder indices, Exercise 3.33 shows that it suffices if we know how to compute them for operators on Hermitian line bundles. The next two theorems provide a tool for handling the latter.

THEOREM 3.35. *Let* $\mathbf{A} = -J_0\partial_t - S(t) : H^1(S^1, \mathbb{R}^2) \to L^2(S^1, \mathbb{R}^2)$, *where* $S(t)$ *is a smooth loop of symmetric 2-by-2 matrices. For each* $\lambda \in \sigma(\mathbf{A})$, *denote the corresponding eigenspace by* $E_\lambda \subset H^1(S^1, \mathbb{R}^2)$.

(1) *Every nontrivial eigenfunction* $e_\lambda \in E_\lambda$ *is nowhere zero and thus has a well-defined winding number* $\mathrm{wind}(e_\lambda) \in \mathbb{Z}$.
(2) *Any two nontrivial eigenfunctions in the same eigenspace* E_λ *have the same winding number.*
(3) *If* $\lambda, \mu \in \sigma(\mathbf{A})$ *satisfy* $\lambda < \mu$, *then any two nontrivial eigenfunctions* $e_\lambda \in E_\lambda$ *and* $e_\mu \in E_\mu$ *satisfy* $\mathrm{wind}(e_\lambda) \leq \mathrm{wind}(e_\mu)$.
(4) *For every* $k \in \mathbb{Z}$, $\mathbf{A}$ *has exactly two eigenvalues (counting multiplicity) for which the corresponding eigenfunctions have winding number equal to* k.

PROOF. We follow the proof given in [**HWZ95**].

Observe first that (1) follows from the fact that nontrivial eigenfunctions are solutions to an ODE, for which classical existence and uniqueness results are available. Since the trivial map is a solution, every eigenfunction which vanishes at a point must be itself trivial, by uniqueness.

To prove (2), let ν_0 and ν_1 be nontrivial eigenfunctions for the same eigenvalue λ. If their winding numbers are different, then there exists $t_0 \in S^1$ at which $\nu_1(t_0)$ is a nonzero real multiple of $\nu_0(t_0)$, so after rescaling, we can assume $\nu_0(t_0) = \nu_1(t_0)$. But ν_0 and ν_1 are both solutions to the same linear ODE, so this implies $\nu_0(t) = \nu_1(t)$ for all t and thus contradicts the assumption on the winding numbers.

We first prove the rest for the case $S = 0$ and the operator $\mathbf{A}_0 = -J_0\partial_t$. Given $\nu \in H^1(S^1, \mathbb{R}^2)$, written as $\nu(t) = (x(t), y(t))$, we have that ν is an element of E_λ for the operator $\mathbf{A}_0$ if and only $(\dot{y}, -\dot{x}) = \lambda(x, y)$. This has solutions of the form

$$\begin{cases} x(t) = A\cos(\lambda t) - B\sin(\lambda t) \\ y(t) = B\cos(\lambda t) + A\sin(\lambda t) \end{cases},$$

for some constants $A, B \in \mathbb{R}$, which are defined on S^1 as long as $\lambda \in 2\pi\mathbb{Z}$. In other words, the spectrum of this operator is $\sigma(\mathbf{A}_0) = 2\pi\mathbb{Z}$. Hence $\nu(t) = \nu(0)e^{i\lambda t}$, which has winding number

$$\operatorname{wind}(\nu) = \frac{\lambda}{2\pi}$$

Statements (2) and (3) are now obvious, and (4) follows from the observation that E_λ is two-dimensional, so in this case each eigenvalue is to be counted with multiplicity two.

For the general case, consider the path of asymptotic operators given by

$$\{\mathbf{A}_\tau = -J_0\partial_t - \tau S(t)\}_{\tau\in[0,1]} .$$

Theorem 3.3 gives continuous families $\{\lambda_j : [0,1] \to \mathbb{R}\}_{j\in\mathbb{Z}}$ and $\{\nu_j : [0,1] \to H^1(S^1, \mathbb{R}^2)\}_{j\in\mathbb{Z}}$ such that for every $\tau \in [0,1]$, $\nu_j(\tau)$ is an eigenfunction for the operator $\mathbf{A}_\tau$ with eigenvalue $\lambda_j(\tau)$, whose multiplicity is given by the number of $i \in \mathbb{Z}$ for which $\lambda_i(\tau) = \lambda_j(\tau)$, and such that $\lambda_{2n+k}(0) = 2\pi n$, for $k = 0, 1$ (this eigenvalue has multiplicity 2). Now, since the winding number is a homotopy invariant (hence invariant under deformations), we have

$$\operatorname{wind}(\nu_{2n+k}(\tau)) = \operatorname{wind}(\nu_{2n+k}(0)) = n,$$

for $k = 0, 1$. Moreover, since the winding only depends on the eigenvalue, the only paths that can possibly meet are λ_{2n} and λ_{2n+1}, which implies that the multiplicity of every eigenvalue $\lambda_i(\tau)$ is at most two, with equality where these two "branches" meet. Hence (3) and (4) follow, where equality in (3) holds if and only if the two branches of paths of eigenvalues with same winding number end up at different points. □

The theorem implies the existence of a well-defined and nondecreasing function

$$\sigma(\mathbf{A}) \to \mathbb{Z} : \lambda \mapsto \operatorname{wind}(\lambda),$$

where $\operatorname{wind}(\lambda)$ is defined as $\operatorname{wind}(e_\lambda)$ for any nontrivial $e_\lambda \in E_\lambda$, and this function attains every value exactly twice (counting multiplicity of eigenvalues). Since eigenvalues of $\mathbf{A}$ are isolated, we can therefore associate to any nondegenerate asymptotic operator $\mathbf{A}$ on the trivial Hermitian line bundle its **extremal winding numbers** and its **parity**,

$$\begin{aligned} \alpha_+(\mathbf{A}) &= \min_{\lambda\in\sigma(\mathbf{A})\cap(0,\infty)} \operatorname{wind}(\lambda) \in \mathbb{Z}, \\ \alpha_-(\mathbf{A}) &= \max_{\lambda\in\sigma(\mathbf{A})\cap(-\infty,0)} \operatorname{wind}(\lambda) \in \mathbb{Z}, \\ p(\mathbf{A}) &= \alpha_+(\mathbf{A}) - \alpha_-(\mathbf{A}) \in \{0, 1\}. \end{aligned} \tag{3.11}$$

THEOREM 3.36. *If $\mathbf{A}$ is a nondegenerate asymptotic operator on the trivial Hermitian line bundle $S^1 \times \mathbb{R}^2 \to S^1$, then*

$$\mu_{CZ}(\mathbf{A}) = 2\alpha_-(\mathbf{A}) + p(\mathbf{A}) = 2\alpha_+(\mathbf{A}) - p(\mathbf{A}).$$

PROOF. The operator $\mathbf{A}_0 = -J_0\partial_t - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ satisfies $\mu_{\mathrm{CZ}}(\mathbf{A}_0) = 0$ by definition, and it has two constant eigenfunctions with eigenvalues of opposite signs, hence

$$\alpha_-(\mathbf{A}_0) = \alpha_+(\mathbf{A}_0) = 0,$$

consistent with the stated formula. The general case then follows by computing the spectral flow from $\mathbf{A}_0$ to any other nondegenerate operator $\mathbf{A}$, and observing that the winding number associated to any continuous family of eigenvalues (as in Theorem 3.3) for a path $\{\mathbf{A}_t\}_{t\in[-1,1]}$ of asymptotic operators cannot change. □

For any Hermitian line bundle (E, J, ω) over S^1 with a nondegenerate asymptotic operator $\mathbf{A}$, we can similarly choose a complex trivialization τ to define the winding numbers $\alpha^\tau_\pm(\mathbf{A}) \in \mathbb{Z}$ and parity $p(\mathbf{A}) = \alpha^\tau_+(\mathbf{A}) - \alpha^\tau_-(\mathbf{A}) \in \{0, 1\}$; note that the dependence on τ cancels out in the last formula, so that $p(\mathbf{A})$ is independent of choices. We then can associate to any nondegenerate Reeb orbit γ in a contact 3-manifold $(M, \xi = \ker\alpha)$ with a trivialization τ of $\gamma^*\xi$ the integers $\alpha^\tau_\pm(\gamma)$ and $p(\gamma)$, such that

$$\mu^\tau_{\mathrm{CZ}}(\gamma) = 2\alpha^\tau_-(\gamma) + p(\gamma) = 2\alpha^\tau_+(\gamma) - p(\gamma)$$

holds.

EXERCISE 3.37. Given a Hermitian vector bundle $(E, J, \omega) \to S^1$ with two complex trivializations $\tau_j : E \to S^1 \times \mathbb{R}^{2n}$ for $j = 1, 2$, denote by

$$\deg(\tau_1 \circ \tau_2^{-1}) \in \mathbb{Z}$$

the winding number of $\det g : S^1 \to \mathbb{C} \setminus \{0\}$, where $g : S^1 \to \mathrm{GL}(n, \mathbb{C})$ is the transition map appearing in the formula $\tau_1 \circ \tau_2^{-1}(t, v) = (t, g(t)v)$. Show that for any asymptotic operator $\mathbf{A}$ on (E, J, ω),

$$\mu^{\tau_2}_{\mathrm{CZ}}(\mathbf{A}) = \mu^{\tau_1}_{\mathrm{CZ}}(\mathbf{A}) + 2\deg(\tau_2 \circ \tau_1^{-1}).$$

Exercise 3.37 provides the useful formula

$$\mu^{\tau_2}_{\mathrm{CZ}}(\gamma) = \mu^{\tau_1}_{\mathrm{CZ}}(\gamma) + 2\deg(\tau_2 \circ \tau_1^{-1})$$

for any two trivializations τ_1, τ_2 of ξ along a nondegenerate Reeb orbit γ. In particular, this shows that the **parity**

$$\mu^{\mathbb{Z}_2}_{\mathrm{CZ}}(\gamma) := [\mu^\tau_{\mathrm{CZ}}(\gamma)] \in \mathbb{Z}_2$$

of the orbit does not depend on a choice of trivialization. We sometimes refer to **even orbits** and **odd orbits** accordingly.

EXERCISE 3.38. Show that if a Reeb orbit $\gamma : S^1 \to M$ in a contact 3-manifold $(M, \xi = \ker\alpha)$ is nondegenerate and has even parity, then the same is true for all of its multiple covers

$$\gamma^k : S^1 \to M : t \mapsto \gamma(kt), \qquad k \in \mathbb{N}.$$

LECTURE 4

Fredholm theory with cylindrical ends

Contents

In this lecture we will study the class of linear Cauchy-Riemann type operators that arise by linearizing the nonlinear equation for moduli spaces in SFT. We saw in the previous lecture that linearizing PDEs over domains with cylindrical ends naturally leads one to consider certain symmetric *asymptotic operators* (e.g. the Hessian of a Morse function at its critical points), which have trivial kernel if and only if a nondegeneracy (i.e. Morse) condition is satisfied. Our goal in this lecture is to write down the SFT version of this story and show that the linear Cauchy-Riemann type operators are Fredholm if their asymptotic operators are nondegenerate.

4.1. Cauchy-Riemann operators with punctures

The setup throughout this lecture will be as follows.

Assume (Σ, j) is a closed connected Riemann surface of genus $g \geq 0$, $\Gamma \subset \Sigma$ is a finite set partitioned into two subsets

$$\Gamma = \Gamma^+ \cup \Gamma^-,$$

and $\dot{\Sigma} := \Sigma \setminus \Gamma$ denotes the resulting punctured Riemann surface. We shall fix a choice of **holomorphic cylindrical coordinate** near each puncture $z \in \Gamma^\pm$, meaning the following. Given $R \geq 0$, let $(Z_\pm^R, i)$ denote the half-cylinders

$$Z_+^R := [R, \infty) \times S^1, \qquad Z_-^R := (-\infty, -R] \times S^1, \qquad Z_\pm := Z_\pm^0,$$

with complex structure $i\partial_s = \partial_t$, $i\partial_t = -\partial_s$ in coordinates $(s,t) \in \mathbb{R} \times S^1$. The standard half-cylinders $Z_\pm$ are each biholomorphically equivalent to the punctured disk $\dot{\mathbb{D}} := \mathbb{D} \setminus \{0\}$ via the maps

$$\psi_\pm : Z_\pm \to \dot{\mathbb{D}} : (s,t) \mapsto e^{\mp 2\pi(s+it)}.$$

For $z \in \Gamma^\pm$, we choose a closed neighborhood $\mathcal{U}_z \subset \Sigma$ of z with a biholomorphic map

$$\varphi_z : (\dot{\mathcal{U}}_z, j) \to (Z_\pm, i),$$

where $\dot{\mathcal{U}}_z := \mathcal{U}_z \setminus \{z\}$, such that $\psi_\pm \circ \varphi_z : \dot{\mathcal{U}}_z \to \dot{\mathbb{D}}$ extends holomorphically to $\mathcal{U}_z \to \mathbb{D}$ with $z \mapsto 0$. One can always find such coordinates by choosing holomorphic coordinates near z. We can thus view the punctured neighborhoods $\dot{\mathcal{U}}_z \subset \dot{\Sigma}$ as **cylindrical ends** $Z_\pm$.

Suppose (E, J) is a smooth complex vector bundle of rank m over $(\dot{\Sigma}, j)$. An **asymptotically Hermitian structure** on (E, J) is a choice of Hermitian vector bundles (E_z, J_z, ω_z) of rank m associated to each puncture $z \in \Gamma^\pm$, together with choices of complex bundle isomorphisms

$$E|_{\dot{\mathcal{U}}_z} \to \mathrm{pr}_2^* E_z$$

covering $\varphi_z : \dot{\mathcal{U}}_z \to Z_\pm$, where $\mathrm{pr}_2 : Z_\pm \to S^1$ denotes the natural projection to the S^1 factor. This isomorphism induces from any unitary trivialization τ of (E_z, J_z, ω_z) a complex trivialization

$$\tau : E|_{\dot{\mathcal{U}}_z} \to Z_\pm \times \mathbb{R}^{2m} \tag{4.1}$$

over the cylindrical end, which we will call an **asymptotic trivialization** near z. The bundle (E_z, J_z, ω_z) will be referred to as the **asymptotic bundle** associated to (E, J) near z.

Fixing asymptotic trivializations near every puncture, we can now define Sobolev spaces of sections of E by

$$W^{k,p}(E) := \left\{ \eta \in W^{k,p}_{\mathrm{loc}}(E) \;\middle|\; \eta_z \in W^{k,p}(\mathring{Z}_\pm, \mathbb{R}^{2m}) \text{ for every } z \in \Gamma^\pm \right\},$$

where $\eta_z : Z_\pm \to \mathbb{R}^{2m}$ denotes the expression of $\eta|_{\dot{\mathcal{U}}_z}$ in terms of the asymptotic trivialization, and we use the standard area form $ds \wedge dt$ on $Z_\pm$ to define the norm. Since S^1 is compact, different choices of asymptotic trivialization give rise to equivalent norms, however:

EXERCISE 4.1. Convince yourself that different choices of asymptotically Hermitian structure on $E \to \dot{\Sigma}$ can give rise to *inequivalent* $W^{k,p}$-norms.

Any linear Cauchy-Riemann type operator on E has as its target the complex vector bundle

$$F := \overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, E),$$

so sections of F are the same thing as E-valued $(0,1)$-forms. An asymptotic trivialization τ as in (4.1) then also induces a complex trivialization

$$F|_{\dot{\mathcal{U}}_z} \to Z_\pm \times \mathbb{R}^{2m} : \lambda \mapsto \tau(\lambda(\partial_s)),$$

where ∂_s is the vector field on $\dot{\mathcal{U}}_z$ arising from its identification with $Z_\pm$. This trivialization yields a corresponding definition for the Sobolev spaces $W^{k,p}(F)$, which depend on the asymptotically Hermitian structure of E but not on the choices of asymptotic trivializations. Having made these choices, a Cauchy-Riemann type operator $\mathbf{D} : \Gamma(E) \to \Gamma(F)$ always appears over $\dot{\mathcal{U}}_z$ as a linear map on $C^\infty(Z_\pm, \mathbb{R}^{2m})$ of the form

$$\mathbf{D}\eta(s,t) = \bar{\partial}\eta(s,t) + S(s,t)\eta(s,t), \tag{4.2}$$

where $\bar{\partial} := \partial_s + J_0 \partial_t$ and $S \in C^\infty(Z_\pm, \mathrm{End}(\mathbb{R}^{2m}))$.

DEFINITION 4.2. Suppose $\mathbf{A}_z$ is an asymptotic operator on (E_z, J_z, ω_z) and $\mathbf{D}$ is a linear Cauchy-Riemann type operator on (E, J). We say that $\mathbf{D}$ is **asymptotic to** $\mathbf{A}_z$ at z if $\mathbf{D}$ appears in the form (4.2) with respect to an asymptotic trivialization near z, with

$$\|S - S_\infty\|_{C^k(Z^R_\pm)} \to 0 \quad \text{as} \quad R \to \infty$$

for all $k \in \mathbb{N}$, where $S_\infty(s,t) := S_\infty(t)$ is a smooth loop of symmetric matrices such that $\mathbf{A}_z$ appears in the corresponding unitary trivialization of (E_z, J_z, ω_z) as $-J_0\partial_t - S_\infty$.

Recall that an asymptotic operator is called **nondegenerate** if 0 is not in its spectrum, which means it defines an isomorphism $H^1 \to L^2$. The objective of this lecture will be to prove the following:

THEOREM 4.3. *Suppose (E, J) is an asymptotically Hermitian vector bundle over $(\dot{\Sigma}, j)$, $\mathbf{A}_z$ is a nondegenerate asymptotic operator on the associated asymptotic bundle (E_z, J_z, ω_z) for each $z \in \Gamma$, and $\mathbf{D}$ is a linear Cauchy-Riemann type operator asymptotic to $\mathbf{A}_z$ at each puncture z. Then for every $k \in \mathbb{N}$ and $1 < p < \infty$,*

$$\mathbf{D} : W^{k,p}(E) \to W^{k-1,p}(F)$$

is Fredholm. Moreover, $\operatorname{ind} \mathbf{D}$ *and* $\ker \mathbf{D}$ *are each independent of k and p, the latter being a space of smooth sections whose derivatives of all orders decay to 0 at infinity.*

REMARK 4.4. The asymptotic decay conditions on $S(s,t)$ in Definition 4.2 can be relaxed at the cost of limiting the range of $k \in \mathbb{N}$ for which Theorem 4.3 is valid. To prove that $\mathbf{D} : W^{1,p} \to L^p$ is Fredholm, it suffices to assume $S(s,\cdot) \to S_\infty$ uniformly as $|s| \to \infty$.

The index of $\mathbf{D}$ is determined by a generalization of the Riemann-Roch formula involving the Conley-Zehnder indices $\mu^\tau_{\mathrm{CZ}}(\mathbf{A}_z)$ that were introduced in the previous lecture. We will postpone serious discussion of the index formula until the next lecture, but here is the statement:

THEOREM 4.5. *In the setting of Theorem 4.3,*

$$\operatorname{ind} \mathbf{D} = m\chi(\dot{\Sigma}) + 2c_1^\tau(E) + \sum_{z\in\Gamma^+} \mu^\tau_{CZ}(\mathbf{A}_z) - \sum_{z\in\Gamma^-} \mu^\tau_{CZ}(\mathbf{A}_z),$$

where τ is an arbitrary choice of asymptotic trivializations, $c_1^\tau(E) \in \mathbb{Z}$ is the relative first Chern number *of E with respect to τ, and the sum is independent of this choice.*

For the rest of this lecture, we maintain as standing assumptions that $k \in \mathbb{N}$, $1 < p < \infty$, and $\mathbf{D}$ is a linear Cauchy-Riemann type operator on E asymptotic at the punctures to a fixed set of asymptotic operators $\{\mathbf{A}_z\}_{z\in\Gamma}$. We will not always need to assume that the $\mathbf{A}_z$ are nondegenerate, so this condition will be specified whenever it is relevant. For subdomains $\Sigma_0 \subset \dot{\Sigma}$, we will sometimes denote the $W^{k,p}$-norm on sections of E restricted to Σ_0 by

$$\|\eta\|_{W^{k,p}(\Sigma_0)} := \|\eta\|_{W^{k,p}(E|_{\Sigma_0})},$$

and we will use the same notation for sections of other bundles such as $F = \overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, E)$ over this domain when there is no danger of confusion. The space

$$W_0^{k,p}(\Sigma_0) \subset W^{k,p}(E)$$

is defined in this case as the $W^{k,p}$-closure of the space of smooth sections of E with compact support in $\Sigma_0 \setminus \partial\Sigma_0$. For some background discussion on Sobolev spaces of sections of vector bundles, see Appendix A.

4.2. A global weak regularity result

In Lecture 2 we proved that for $1 < p < \infty$, weak solutions of class L^p_{loc} to linear Cauchy-Riemann type equations are always smooth. Here is a global version of that result.

PROPOSITION 4.6. *Suppose $1 < p < \infty$ and $k \in \mathbb{N}$. If $\eta \in L^p(E)$ weakly satisfies*

$$\mathbf{D}\eta \in W^{k-1,p}(F),$$

then $\eta \in W^{k,p}(E)$.

PROOF. By induction, it suffices to show that if $\eta \in W^{k-1,p}$ and $\mathbf{D}\eta \in W^{k-1,p}$ then $\eta \in W^{k,p}$. We already know that this is true locally, so the task is to bound the $W^{k,p}$-norm of η on the cylindrical ends. Pick an asymptotic trivialization and write $\mathbf{D}$ on one of the ends $Z_\pm \cong \dot{\mathcal{U}}_z$ as $\bar{\partial} + S(s,t)$. Let us assume for concreteness that the puncture is a positive one, and now consider the $W^{k,p}$-norm of η on $(N, N+1) \times S^1 \subset \dot{\mathcal{U}}_z$ for $N \in \mathbb{N}$. Choosing a smooth bump function $\beta : \mathbb{R} \times S^1 \to [0,1]$ supported in $(N-1, N+2) \times S^1$ with $\beta = 1$ on $[N, N+1] \times S^1$, we can use the usual elliptic estimate to write

$$\begin{aligned}
\|\eta\|_{W^{k,p}((N,N+1)\times S^1)} &\le \|\beta\eta\|_{W^{k,p}((N-1,N+2)\times S^1)} \le c\|\bar{\partial}(\beta\eta)\|_{W^{k-1,p}((N-1,N+2)\times S^1)} \\
&\le c\|\eta\|_{W^{k-1,p}((N-1,N+2)\times S^1)} + c\|\bar{\partial}\eta\|_{W^{k-1,p}((N-1,N+2)\times S^1)} \\
&= c\|\eta\|_{W^{k-1,p}((N-1,N+2)\times S^1)} + c\|\mathbf{D}\eta - S\eta\|_{W^{k-1,p}((N-1,N+2)\times S^1)} \\
&\le c'\|\eta\|_{W^{k-1,p}((N-1,N+2)\times S^1)} + c'\|\mathbf{D}\eta\|_{W^{k-1,p}((N-1,N+2)\times S^1)}.
\end{aligned}$$

An important detail here is that the constants in these estimates can be assumed independent of N: indeed, one can use shifts of the same cutoff function for any N, and the C^{k-1}-norm of S on $[N-1, N+2] \times S^1$ is also bounded uniformly in N since $S(s,t)$ converges asymptotically to some $S_\infty(t)$. We can therefore take the sum of this estimate for all $N \in \mathbb{N}$, producing

$$\|\eta\|_{W^{k,p}(\mathring{Z}^1_+)} \le c\|\eta\|_{W^{k-1,p}(\mathring{Z}_+)} + c\|\mathbf{D}\eta\|_{W^{k-1,p}(\mathring{Z}_+)}.$$

□

COROLLARY 4.7. *For $1 < p < \infty$, any weak solution $\eta \in L^p(E)$ of $\mathbf{D}\eta = 0$ is smooth, with derivatives of all orders decaying to 0 at infinity.*

PROOF. Proposition 4.6 implies $\eta \in W^{k,p}(E)$ for every $k \in \mathbb{N}$, so smoothness follows from the Sobolev embedding theorem. Moreover, suppose k and p are large enough to have a continuous inclusion $W^{k,p} \hookrightarrow C^m$ for some $m \in \mathbb{N}$. Then the finiteness of the $W^{k,p}$-norm also implies that for each end $\dot{\mathcal{U}}_z = Z_\pm$,

$$\|\eta\|_{C^m(Z_\pm^R)} \le c\|\eta\|_{W^{k,p}(\mathring{Z}_\pm^R)} \to 0 \quad \text{as} \quad R \to \infty.$$

□

4.3. Elliptic estimates on cylindrical ends

The local elliptic estimates for $\bar{\partial} = \partial_s + J_0\partial_t$ in Lecture 2 applied to functions on $\mathring{\mathbb{D}} \subset \mathbb{C}$ with compact support. Using a finite open covering with a subordinate partition of unity, it is a straightforward matter to turn these local estimates into the following global result (cf. [Wend, Lemma 3.3.2]):

PROPOSITION 4.8. *If $\Sigma_0 \subset \dot{\Sigma}$ is a compact 2-dimensional submanifold with boundary, then there exists a constant $c > 0$ such that*

$$\|\eta\|_{W^{k,p}(\Sigma_0)} \le c\|\mathbf{D}\eta\|_{W^{k-1,p}(\Sigma_0)} + c\|\eta\|_{W^{k-1,p}(\Sigma_0)}$$

for all $\eta \in W_0^{k,p}(\Sigma_0)$. □

This unfortunately is unsufficient for the global problem under consideration, since one has to chop off the cylindrical ends of $\dot{\Sigma}$ in order to obtain a compact domain. We therefore supplement the previous local estimates with an asymptotic estimate.

PROPOSITION 4.9. *Suppose $z \in \Gamma^\pm$ is a puncture such that the asymptotic operator $\mathbf{A}_z$ is nondegenerate. Then on $Z_\pm^R \subset \dot{\mathcal{U}}_z$ for sufficiently large $R \ge 0$, there exists a constant $c > 0$ such that*

$$\|\eta\|_{W^{k,p}(\mathring{Z}_\pm^R)} \le c\|\mathbf{D}\eta\|_{W^{k-1,p}(\mathring{Z}_\pm^R)} \qquad \textit{for all} \qquad \eta \in W_0^{k,p}(\mathring{Z}_\pm^R).$$

REMARK 4.10. Recall that $W_0^{k,p}(\mathring{Z}_\pm^R)$ denotes the $W^{k,p}$-closure of $C_0^\infty(\mathring{Z}_\pm^R)$, so such functions remain in $W^{k,p}$ if they are extended as zero to larger domains containing $\mathring{Z}_\pm^R$. Note that functions of class $W_0^{k,p}$ on $\mathring{Z}_\pm^R$ need not actually have compact support; in fact C_0^∞ is dense in $W^{k,p}(\mathbb{R} \times S^1)$, see §A.4.

The proof of this requires a basic result about translation-invariant Cauchy-Riemann type operators on the cylinder. Other than the elliptic estimates we discussed in Lecture 2, this is the main analytical ingredient that makes all Floer-type theories in symplectic geometry work.

THEOREM 4.11. *Suppose $k \in \mathbb{N}$, $1 < p < \infty$, and $\mathbf{A} = -J_0\partial_t - S(t)$ is a nondegenerate asymptotic operator on the trivial Hermitian vector bundle $S^1 \times \mathbb{R}^{2n} \to S^1$. Then the operator*

$$\partial_s - \mathbf{A} = \partial_s + J_0\partial_t + S(t) : W^{k,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to W^{k-1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

is an isomorphism. □

A detailed proof of this result for $k = 1$ can be found in [Sal99, Lemma 2.4], and the general result follows easily from this using regularity (Proposition 4.6). I will not attempt to reproduce the proof in Salamon's notes here since it is somewhat involved, but let us informally sketch the first step, which is the interesting part. The goal is to prove that $\mathbf{D}_0 := \partial_s - \mathbf{A}$ is an invertible operator from $H^1(\mathbb{R} \times S^1)$ to $L^2(\mathbb{R} \times S^1)$. To gain some intuition on this, consider the special case where the asymptotic operator is of the form $\mathbf{A} = -i\partial_t - C$ for some constant $C \in \mathbb{R}$. One can then write down an inverse of $\mathbf{D}_0$ explicitly by combining a Fourier transform in the s variable with a Fourier series in the t variable. That is, sufficiently nice functions u on $\mathbb{R} \times S^1$ can be expressed as

$$u(s,t) = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \hat{u}_k(\sigma) e^{2\pi i\sigma s} e^{2\pi i k t}\, d\sigma,$$

where the hybrid Fourier transform/series $\hat{u}$ depends on a continuous variable $\sigma \in \mathbb{R}$ and a discrete variable $k \in \mathbb{Z}$. One can then obtain $\hat{u}$ from u by

$$\hat{u}_k(\sigma) = \int_{\mathbb{R}\times S^1} u(s,t) e^{-2\pi i\sigma s} e^{-2\pi i k t}\, ds\, dt,$$

and we have the usual derivative formulas $\widehat{\partial_s u}_k(\sigma) = 2\pi i\sigma \hat{u}_k(\sigma)$ and $\widehat{\partial_t u}_k(\sigma) = 2\pi i k \hat{u}_k(\sigma)$. The relation $(\partial_s + i\partial_t + C)u = f$ therefore produces an inversion formula of the form

$$\hat{u}_k(\sigma) = \frac{\hat{f}_k(\sigma)}{2\pi i\sigma - 2\pi k + C}.$$

This is a nice formula and produces from any $f \in L^2$ an element $u \in H^1$ unless $C \in 2\pi\mathbb{Z}$, in which case the denominator has a singularity. This condition means C must not be an eigenvalue of $-i\partial_t$, or in other words, $\mathbf{A} = -i\partial_t - C$ is nondegenerate. One can perhaps imagine carrying out a similar argument in the general case using an orthonormal set of eigenfunctions[1] for $\mathbf{A}$ in place of the functions $e^{2\pi i k t}$; this is presumably part of the idea behind the actual proof in [Sal99], which uses strongly continuous semigroups generated by the self-adjoint operator $\mathbf{A}$.

Proof of Proposition 4.9. Write $\mathbf{D} = \partial_s + J_0\partial_t + S(s,t)$ and $\mathbf{D}_0 = \partial_s + J_0\partial_t + S_\infty(t)$ in an asymptotic trivialization on $\dot{\mathcal{U}}_z = Z_\pm$, where the nondegenerate asymptotic operator is $\mathbf{A} = -J_0\partial_t - S_\infty(t)$ and we assume

$$\|S - S_\infty\|_{C^{k-1}(Z^R_\pm)} \to 0 \quad \text{as} \quad R \to \infty.$$

For $\eta \in W^{k,p}_0(\mathring{Z}^R_\pm)$, there is a canonical extension $\eta \in W^{k,p}(\mathbb{R} \times S^1)$ that equals zero outside $Z^R_\pm$, so by Theorem 4.11 we have

$$\|\eta\|_{W^{k,p}(\mathring{Z}^R_\pm)} = \|\eta\|_{W^{k,p}(\mathbb{R}\times S^1)} \le c\|\mathbf{D}_0\eta\|_{W^{k-1,p}(\mathbb{R}\times S^1)} = c\|\mathbf{D}_0\eta\|_{W^{k-1,p}(\mathbb{R}\times S^1)}.$$

Rewriting this in terms of $\mathbf{D}$ gives

$$\|\eta\|_{W^{k,p}(\mathring{Z}^R_\pm)} \le c\|\mathbf{D}\eta\|_{W^{k-1,p}(Z^R_\pm)} + c\|(S_\infty - S)\eta\|_{W^{k-1,p}(Z^R_\pm)},$$

[1] Recall from Lecture 3 that the spectrum $\sigma(\mathbf{A})$ of an arbitrary asymptotic operator $\mathbf{A}$ always consists only of isolated real eigenvalues, thus one can find $\lambda \in \mathbb{R}$ for which $\lambda - \mathbf{A} : H^1(S^1) \to L^2(S^1)$ is invertible. Its inverse, also known as the **resolvent**, then defines a *compact* self-adjoint operator $(\lambda - \mathbf{A})^{-1} : L^2(S^1) \to L^2(S^1)$ due to the compact inclusion $H^1(S^1) \hookrightarrow L^2(S^1)$. The spectral theorem for compact self-adjoint operators now provides an orthonormal basis of $L^2(S^1)$ consisting of eigenfunctions of $(\lambda - \mathbf{A})^{-1}$, which are also eigenfunctions of $\mathbf{A}$.

where the constants $c > 0$ do not depend on R. For this reason, we are free to make $R \geq 0$ large enough to make the C^{k-1}-norm of $S_\infty - S$ on $Z^R_\pm$ less than an arbitrarily small number $\delta > 0$, in which case the above gives

$$\|\eta\|_{W^{k,p}(\mathring{Z}^R_\pm)} \leq c\|\mathbf{D}\eta\|_{W^{k-1,p}(Z^R_\pm)} + c\delta\|\eta\|_{W^{k-1,p}(Z^R_\pm)},$$

and thus by the inclusion $W^{k-1,p} \hookrightarrow W^{k,p}$,

$$\|\eta\|_{W^{k,p}(\mathring{Z}^R_\pm)} \leq \frac{c}{1-c\delta}\|\mathbf{D}\eta\|_{W^{k-1,p}(Z^R_\pm)}.$$

□

4.4. The semi-Fredholm property

The standard approach for proving that elliptic operators are Fredholm begins by proving that they are **semi-Fredholm**, meaning $\dim\ker\mathbf{D} < \infty$ and $\operatorname{im}\mathbf{D}$ is closed. In most settings, it is not hard to show that local elliptic estimates give rise to global estimates of the form $\|\eta\|_{W^{k,p}} \leq c\|\mathbf{D}\eta\|_{W^{k-1,p}} + \|\eta\|_{W^{k-1,p}}$. The step from these estimates to the semi-Fredholm property is then provided by the following lemma.

LEMMA 4.12. *Suppose X, Y and Z are Banach spaces, $\mathbf{T} \in \mathscr{L}(X,Y)$, $\mathbf{K} \in \mathscr{L}(X,Z)$ is compact, and there is a constant $c > 0$ such that for all $x \in X$,*

$$\|x\|_X \leq c\|\mathbf{T}x\|_Y + c\|\mathbf{K}x\|_Z. \tag{4.3}$$

Then $\ker\mathbf{T}$ is finite dimensional and $\operatorname{im}\mathbf{T}$ is closed.

PROOF. A vector space is finite dimensional if and only if the unit ball in that space is a compact set, so we begin by proving the latter holds for $\ker\mathbf{T}$. Suppose $x_k \in \ker\mathbf{T}$ is a bounded sequence. Then since $\mathbf{K}$ is a compact operator, $\mathbf{K}x_k$ has a convergent subsequence in Z, which is therefore Cauchy. But (4.3) then implies that the corresponding subsequence of x_k in X is also Cauchy, and thus converges.

Since we now know $\ker\mathbf{T}$ is finite dimensional, we also know there is a closed complement $V \subset X$ with $\ker\mathbf{T} \oplus V = X$. Then the restriction $\mathbf{T}|_V$ has the same image as $\mathbf{T}$, thus if $y \in \overline{\operatorname{im}\mathbf{T}}$, there is a sequence $x_k \in V$ such that $\mathbf{T}x_k \to y$. We claim that x_k is bounded. If not, then $\mathbf{T}(x_k/\|x_k\|_X) \to 0$ and $\mathbf{K}(x_k/\|x_k\|_X)$ has a convergent subsequence, so (4.3) implies that a subsequence of $x_k/\|x_k\|_X$ also converges to some $x_\infty \in V$ with $\|x_\infty\| = 1$ and $\mathbf{T}x_\infty = 0$, a contradiction. But now since x_k is bounded, $\mathbf{K}x_k$ also has a convergent subsequence and $\mathbf{T}x_k$ converges by assumption, thus (4.3) yields also a convergent subsequence of x_k, whose limit x satisfies $\mathbf{T}x = y$. This completes the proof that $\operatorname{im}\mathbf{T}$ is closed. □

In the analysis of closed J-holomorphic curves, one makes use of the above lemma by placing the inclusion $W^{k-1,p} \hookrightarrow W^{k,p}$ in the role of the compact operator $\mathbf{K}$. Unfortunately, $W^{k-1,p} \hookrightarrow W^{k,p}$ is not compact when the domain $\dot{\Sigma}$ has cylindrical ends; in contrast to the case of a compact domain, there is no way to write the norm on the ends as a finite sum of norms for functions on domains of finite measure. To circumvent this problem, let

$$\Sigma^R \subset \dot{\Sigma}$$

denote the compact complement of the ends $\mathring{Z}_\pm^R \subset \dot{\mathcal{U}}_z$ for all $z \in \Gamma$.

LEMMA 4.13. *Fix $k \in \mathbb{N}$ and $1 < p < \infty$, and assume all the $\mathbf{A}_z$ are nondegenerate. Then for sufficiently large $R > 0$, there exists a constant $c > 0$ such that*

$$\|\eta\|_{W^{k,p}(\dot{\Sigma})} \le c\|\mathbf{D}\eta\|_{W^{k-1,p}(\dot{\Sigma})} + c\|\eta\|_{W^{k-1,p}(\Sigma^R)}$$

for all $\eta \in W^{k,p}(E)$.

PROOF. Fix a smooth cutoff function $\beta \in C_0^\infty(\Sigma^R)$ such that $\beta|_{\Sigma^{R-1}} \equiv 1$, and write

$$\dot{\mathcal{U}}_\Gamma^R \subset \dot{\Sigma}$$

for the union of all the ends $\mathring{Z}_\pm^R \subset \dot{\mathcal{U}}_z$ for $z \in \Gamma^+ \cup \Gamma^-$. Then we can write any $\eta \in W^{k,p}(E)$ as $\eta = \beta\eta + (1-\beta)\eta$ so that $\beta\eta \in W_0^{k,p}(\Sigma^R)$ and $(1-\beta)\eta \in W_0^{k,p}(\dot{\mathcal{U}}_\Gamma^{R-1})$. Choosing R large enough to make Proposition 4.9 valid, we can apply this together with Proposition 4.8 to show

$$\begin{aligned}\|\eta\|_{W^{k,p}(\dot{\Sigma})} &\le \|\beta\eta\|_{W^{k,p}(\Sigma^R)} + \|(1-\beta)\eta\|_{W^{k,p}(\dot{\mathcal{U}}_\Gamma^{R-1})} \\ &\le c\|\mathbf{D}(\beta\eta)\|_{W^{k-1,p}(\Sigma^R)} + c\|\beta\eta\|_{W^{k-1,p}(\Sigma^R)} + \|\mathbf{D}\left[(1-\beta)\eta\right]\|_{W^{k-1,p}(\dot{\mathcal{U}}_\Gamma^{R-1})}.\end{aligned}$$

After applying the Leipbniz rule and absorbing the norms of β and $\bar{\partial}\beta$ into the constants, this produces the stated inequality since the term involving the $W^{k-1,p}$-norm of η on the cylindrical ends includes $\bar{\partial}(1-\beta)$, which vanishes outside of Σ^R. □

Lemma 4.12 is now applicable since the operator

$$W^{k,p}(\dot{\Sigma}) \to W^{k-1,p}(\Sigma^R) : \eta \mapsto \eta|_{\Sigma^R}$$

involves the compact inclusion $W^{k,p}(\Sigma^R) \hookrightarrow W^{k-1,p}(\Sigma^R)$ and is thus compact.

COROLLARY 4.14. *If all the $\mathbf{A}_z$ are nondegenerate, then*

$$\mathbf{D} : W^{k,p}(E) \to W^{k-1,p}(F)$$

is semi-Fredholm. □

4.5. Formal adjoints and proof of the Fredholm property

In order to show that coker $\mathbf{D}$ is also finite dimensional, we will apply the above arguments to the formal adjoint of $\mathbf{D}$, an operator whose kernel is naturally isomorphic to the cokernel of $\mathbf{D}$. Let us choose Hermitian bundle metrics $\langle\ ,\ \rangle_E$ on E and $\langle\ ,\ \rangle_F$ on F, and fix an area form $d\,\mathrm{vol}$ on $\dot{\Sigma}$ that takes the form $d\,\mathrm{vol} = ds \wedge dt$ on the cylindrical ends. The **formal adjoint** of $\mathbf{D}$ is then defined as the unique first-order linear differential operator

$$\mathbf{D}^* : \Gamma(F) \to \Gamma(E)$$

that satisfies the relation

$$\langle \lambda, \mathbf{D}\eta \rangle_{L^2(F)} = \langle \mathbf{D}^*\lambda, \eta \rangle_{L^2(E)} \quad \text{for all} \quad \eta \in C_0^\infty(E),\ \lambda \in C_0^\infty(F),$$

where we use the real-valued L^2-pairings

$$\langle \eta, \xi \rangle_{L^2(E)} := \mathrm{Re} \int_{\dot\Sigma} \langle \eta, \xi \rangle_E \, d\,\mathrm{vol}, \qquad \text{for} \qquad \eta, \xi \in \Gamma(E),$$
$$\langle \alpha, \lambda \rangle_{L^2(F)} := \mathrm{Re} \int_{\dot\Sigma} \langle \alpha, \lambda \rangle_F \, d\,\mathrm{vol}, \qquad \text{for} \qquad \alpha, \lambda \in \Gamma(F).$$

The word "formal" refers to the fact that we are not viewing $\mathbf{D}^*$ as the adjoint of an unbounded operator on a Hilbert space (cf. [**RS80**]); that would be a stronger condition.

EXERCISE 4.15. Show that $\mathbf{D}^*$ is well defined and, for suitable choices of complex local trivializations of E and F and holomorphic coordinates on open subsets $\mathcal{U} \subset \dot\Sigma$, can be written locally as

$$\mathbf{D}^* = -\partial + A : C^\infty(\mathcal{U}, \mathbb{R}^{2n}) \to C^\infty(\mathcal{U}, \mathbb{R}^{2n})$$

for some $A \in C^\infty(\mathcal{U}, \mathrm{End}(\mathbb{R}^{2n}))$, where $\partial := \partial_s - J_0 \partial_t$.

The formula in the above exercise reveals that $\mathbf{D}^*$ is also an elliptic operator[2] and thus has the same local properties as $\mathbf{D}$; indeed, $-\partial + A$ can be transformed into $\bar\partial + B$ for some zeroth-order term B if we conjugate it by a suitable complex-antilinear change of trivialization. In particular, our local estimates for $\mathbf{D}$ and their consequences, notably Proposition 4.8, are all equally valid for $\mathbf{D}^*$.

To obtain suitable asymptotic estimates for $\mathbf{D}^*$, let us fix asymptotic trivializations τ of E, use the corresponding trivializations of F over the ends as described in §4.1, and choose the bundle metrics such that both appear standard in these trivializations over the ends. We will say that the bundle metrics are **compatible with the asymptotically Hermitian structure** of E whenever they are chosen in this way outside of a compact subset of $\dot\Sigma$. We can then express $\mathbf{D}$ as $\bar\partial + S(s,t)$ on $\dot{\mathcal{U}}_z = Z_\pm$, and integrate by parts to obtain

$$\mathbf{D}^* = -\partial + S(s,t)^{\mathrm{T}}.$$

To identify this expression with a Cauchy-Riemann type operator, let $C := \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$ denote the $\mathbb{R}$-linear transformation on $\mathbb{R}^{2n} = \mathbb{C}^n$ representing complex conjugation. Then since C anticommutes with J_0, we have

$$\begin{aligned} (C^{-1}\mathbf{D}^*C)\eta &= -C\partial_s(C\eta) + CJ_0\partial_t(C\eta) + CS(s,t)^{\mathrm{T}}C\eta \\ &= -\partial_s\eta - J_0\partial_t\eta + CS(s,t)^{\mathrm{T}}C\eta = -(\bar\partial\eta - CS(s,t)^{\mathrm{T}}C\eta) \\ &=: -(\bar\partial + \bar{S}(s,t))\eta, \end{aligned}$$

where we've defined $\bar{S}(s,t) := -CS(s,t)^{\mathrm{T}}C$. Now if the asymptotic operator $\mathbf{A}_z$ at $z \in \Gamma^\pm$ is written in the chosen trivialization as $\mathbf{A} := -J_0\partial_s - S_\infty(t)$, the asymptotic

[2]Technically, this property of the formal adjoint is part of the definition of ellipticity: we call a differential operator elliptic whenever (1) it has the properties necessary for proving fundamental estimates using Fourier transforms as we did with $\bar\partial$ in §2.3, and (2) its formal adjoint also has this property. The former requires the principal symbol of the operator to be everywhere injective, and the latter requires it to be surjective.

convergence of $S(s,t)$ implies that similarly

$$\|\bar{S} - \bar{S}_\infty\|_{C^k(Z^R_\pm)} \to 0 \quad \text{as} \quad R \to \infty$$

for all $k \in \mathbb{N}$, where

$$\bar{S}_\infty(t) := -CS_\infty(t)C.$$

This defines a trivialized asymptotic operator $\overline{\mathbf{A}} = -J_0\partial_t - \bar{S}_\infty(t)$ to which $-\mathbf{D}^*$ is (after a suitable change of trivialization) asymptotic at the puncture z; in particular, our proof of the global regularity result, Proposition 4.6, now also works for $\mathbf{D}^*$. Finally, notice that $\mathbf{A}$ and $-\overline{\mathbf{A}}$ are conjugate: indeed,

$$(C^{-1}\overline{\mathbf{A}}C)\eta = -CJ_0\partial_t(C\eta) + CCS_\infty(t)C(C\eta) = J_0\partial_t\eta + S_\infty(t)\eta = -\mathbf{A}\eta.$$

This implies that $\mathbf{A}$ is nondegenerate if and only if $\overline{\mathbf{A}}$ is; applying this assumption for all of the $\mathbf{A}_z$, the proofs of Proposition 4.9 and Lemma 4.13 now also go through for $\mathbf{D}^*$.

We've proved:

PROPOSITION 4.16. *Suppose $\mathbf{D}^*$ is defined with respect Hermitian bundle metrics on E and $F = \overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, E)$ that are compatible with the asymptotically Hermitian structure of E. If additionally all the asymptotic operators $\mathbf{A}_z$ are nondegenerate, then*

$$\mathbf{D}^* : W^{k,p}(F) \to W^{k-1,p}(E)$$

is semi-Fredholm, and its kernel is a space of smooth sections contained in $W^{m,q}(F)$ for all $m \in \mathbb{N}$ and $q \in (1,\infty)$. □

Since $\ker \mathbf{D}^*$ is now known to be finite dimensional, the next result completes the proof of the Fredholm property for $\mathbf{D}$ by showing that its image has finite codimension:

LEMMA 4.17. *Under the same assumptions as in Proposition 4.16,*

$$W^{k-1,p}(F) = \mathrm{im}\,\mathbf{D} + \ker \mathbf{D}^*.$$

PROOF. Consider first the case $k = 1$. Since $\mathbf{D} : W^{1,p}(E) \to L^p(F)$ is semi-Fredholm, its image is closed, hence $\mathrm{im}\,\mathbf{D} + \ker \mathbf{D}^*$ is a closed subspace of $L^p(F)$. Then if $\mathrm{im}\,\mathbf{D} + \ker \mathbf{D}^* \neq L^p(F)$, the Hahn-Banach theorem[3] provides a nontrivial element $\alpha \in (L^p(F))^* \cong L^q(F)$ for $\frac{1}{p} + \frac{1}{q} = 1$ such that

$$\langle \mathbf{D}\eta + \lambda, \alpha \rangle_{L^2(F)} = 0 \quad \text{for all} \quad \eta \in W^{1,p}(E),\ \lambda \in \ker \mathbf{D}^*. \tag{4.4}$$

Choosing $\lambda = 0$, this implies in particular

$$\langle \mathbf{D}\eta, \alpha \rangle_{L^2(F)} = 0 \quad \text{for all} \quad \eta \in C_0^\infty(E),$$

which means that α is a weak solution of class L^q to the formal adjoint equation $\mathbf{D}^*\alpha = 0$. By Propositiion 4.6, α is therefore smooth and belongs to $\ker \mathbf{D}^*$. But this contradicts (4.4) if we plug in $\eta = 0$ and $\lambda = \alpha$, so this completes the proof for $k = 1$.

[3]In the case $p = 2$, one can forego the Hahn-Banach theorem and simply take an L^2-orthogonal complement.

For $k \geq 2$, suppose $\alpha \in W^{k-1,p}(F) \subset L^p(F)$ is given: then the case $k = 1$ provides elements $\eta \in W^{1,p}(E)$ and $\lambda \in \ker \mathbf{D}^*$ such that $\mathbf{D}\eta + \lambda = \alpha$. Since Proposition 4.6 implies $\lambda \in W^{m,q}(F)$ for all $m \in \mathbb{N}$ and $q \in (1, \infty)$, we have $\mathbf{D}\eta = \alpha - \lambda \in W^{k-1,p}(F)$ and thus, by Prop. 4.6 again, $\eta \in W^{k,p}(E)$, completing the proof for all $k \in \mathbb{N}$. □

The proof of Theorem 4.3 is now complete, but as long as we're talking about the formal adjoint, let us take note of a few more properties that will be useful in the future. Assume from now on that all the assumptions of Proposition 4.16 are satisfied. We can now strengthen Lemma 4.17 as follows.

PROPOSITION 4.18. $W^{k-1,p}(F) = \operatorname{im} \mathbf{D} \oplus \ker \mathbf{D}^*$ *and* $W^{k-1,p}(E) = \operatorname{im} \mathbf{D}^* \oplus \ker \mathbf{D}$. *In particular, the projections defined by these splittings give isomorphisms*

$$\operatorname{coker} \mathbf{D} \cong \ker \mathbf{D}^* \quad \textit{and} \quad \operatorname{coker} \mathbf{D}^* \cong \ker \mathbf{D},$$

thus $\mathbf{D}^* : W^{k,p}(F) \to W^{k-1,p}(E)$ *is a Fredholm operator with*

$$\operatorname{ind} \mathbf{D}^* = -\operatorname{ind} \mathbf{D}.$$

PROOF. By Lemma 4.17, the first splitting follows if we can show that $\operatorname{im} \mathbf{D} \cap \ker \mathbf{D}^* = \{0\}$. Recall first (see §A.4) that $C_0^\infty(\dot{\Sigma})$ is dense in $W^{k,p}(\dot{\Sigma})$ for every $k \geq 0$ and $p \in [1, \infty)$, so the definition of the formal adjoint implies via density and Hölder's inequality that if $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$,

$$\langle \lambda, \mathbf{D}\eta \rangle_{L^2(F)} = \langle \mathbf{D}^*\lambda, \eta \rangle_{L^2(E)} \quad \text{for all} \quad \eta \in W^{1,p}(E),\ \lambda \in W^{1,q}(F). \tag{4.5}$$

Now suppose $\lambda \in \operatorname{im} \mathbf{D} \cap \ker \mathbf{D}^*$ and write $\lambda = \mathbf{D}\eta$, assuming $\eta \in W^{k,p}(E)$. Regularity implies that since $\mathbf{D}^*\lambda = 0$, $\lambda \in W^{1,q}(F)$, where q can be chosen to satisfy $\frac{1}{p} + \frac{1}{q} = 1$. We can therefore apply (4.5) and obtain

$$\langle \lambda, \lambda \rangle_{L^2(F)} = \langle \lambda, \mathbf{D}\eta \rangle_{L^2(F)} = \langle \mathbf{D}^*\lambda, \eta \rangle_{L^2(E)} = 0,$$

hence $\lambda = 0$.

The proof that $W^{k-1,p}(E) = \operatorname{im} \mathbf{D}^* \oplus \ker \mathbf{D}$ is analogous. □

This result hints at the fact that $\mathbf{D}^*$ is in fact—under some natural extra assumptions—globally equivalent to another Cauchy-Riemann type operator. To see this, let us impose a further constraint on the relation between the Hermitian bundle metrics $\langle\ ,\ \rangle_E$ and $\langle\ ,\ \rangle_F$. Note that since the area form $d\,\mathrm{vol}$ is necessarily j-invariant, it induces a Hermitian structure on $T\dot{\Sigma}$, namely

$$\langle X, Y \rangle_\Sigma := d\,\mathrm{vol}(X, jY) + i\, d\,\mathrm{vol}(X, Y),$$

which matches the standard bundle metric in the trivializations over the ends defined via the cylindrical coordinates. This induces real-linear isomorphisms from $T\dot{\Sigma}$ to the complex-linear and -antilinear parts of the complexified cotangent bundle,

$$T\dot{\Sigma} \to \Lambda^{1,0}T^*\dot{\Sigma} : X \mapsto X^{1,0} := \langle X, \cdot \rangle_\Sigma,$$
$$T\dot{\Sigma} \to \Lambda^{0,1}T^*\dot{\Sigma} : X \mapsto X^{0,1} := \langle \cdot, X \rangle_\Sigma,$$

where the first isomorphism is complex antilinear and the second is complex linear. We use these to define Hermitian bundle metrics on $\Lambda^{1,0}T^*\dot{\Sigma}$ and $\Lambda^{0,1}T^*\dot{\Sigma}$ in terms of the metric on $T\dot{\Sigma}$; note that this is a straightforward definition for $\Lambda^{0,1}T^*\dot{\Sigma}$, but since the isomorphism to $\Lambda^{1,0}T^*\dot{\Sigma}$ is complex *antilinear*, we really mean

$$\langle X^{1,0}, Y^{1,0}\rangle_\Sigma := \langle Y, X\rangle_\Sigma \quad \text{for} \quad X, Y \in T\dot{\Sigma}.$$

Now observe that as a vector bundle with complex structure $\lambda \mapsto J \circ \lambda$, $F = \overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, E)$ is naturally isomorphic to the complex tensor product

$$F = \Lambda^{0,1}T^*\Sigma \otimes E.$$

We can therefore make a natural choice for $\langle\ ,\ \rangle_F$ as the tensor product metric determined by $\langle\ ,\ \rangle_\Sigma$ and $\langle\ ,\ \rangle_E$. It is easy to check that this choice is compatible with the asymptotically Hermitian structure of E.

Next, we notice that the area form $d\,\mathrm{vol}$ also induces a natural complex bundle isomorphism

$$E \to \mathrm{Hom}_{\mathbb{C}}(T\dot{\Sigma}, F).$$

Indeed, the right hand side is canonically isomorphic to the complex tensor product

$$\mathrm{Hom}_{\mathbb{C}}(T\dot{\Sigma}, F) = \Lambda^{1,0}T^*\dot{\Sigma} \otimes F = \Lambda^{1,0}T^*\dot{\Sigma} \otimes \Lambda^{0,1}T^*\dot{\Sigma} \otimes E,$$

and $\Lambda^{1,0}T^*\dot{\Sigma}\otimes\Lambda^{0,1}T^*\dot{\Sigma}$ is isomorphic to the trivial complex line bundle $\epsilon^1 := \dot{\Sigma}\times\mathbb{C} \to \dot{\Sigma}$ via

$$\Lambda^{1,0}T^*\dot{\Sigma} \otimes \Lambda^{0,1}T^*\dot{\Sigma} \to \epsilon^1 : X^{1,0} \otimes Y^{0,1} \mapsto X^{1,0}(Y) = \langle X, Y\rangle_\Sigma.$$

EXERCISE 4.19. Assuming $\langle\ ,\ \rangle_F$ is chosen as the tensor product metric described above, show that under the natural identification of E with $\mathrm{Hom}_{\mathbb{C}}(T\dot{\Sigma}, F)$,

$$-\mathbf{D}^* : \Gamma(F) \to \Omega^{1,0}(\dot{\Sigma}, F)$$

satisfies the Leibniz rule

$$-\mathbf{D}^*(f\lambda) = (\partial f)\lambda + f(-\mathbf{D}^*\lambda)$$

for all $f \in C^\infty(\dot{\Sigma}, \mathbb{R})$, where $\partial f \in \Omega^{1,0}(\dot{\Sigma})$ denotes the complex-valued $(1,0)$-form $df - i\, df \circ j$.

We might summarize this exercise by saying that $-\mathbf{D}^*$ is an "anti-Cauchy-Riemann type" operator on F. But such an object is easily transformed into an honest Cauchy-Riemann type operator: let $\bar{F}$ denote the **conjugate bundle** to F, which we define as the same real vector bundle F but with the sign of its complex structure reversed, so $\lambda \mapsto -J \circ \lambda$. Now there is a canonical isomorphism

$$\mathrm{Hom}_{\mathbb{C}}(T\dot{\Sigma}, F) = \overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, \bar{F}),$$

and the same operator defines a real-linear map

$$-\mathbf{D}^* : \Gamma(\bar{F}) \to \Omega^{0,1}(\dot{\Sigma}, \bar{F})$$

which satisfies our usual Leibniz rule for Cauchy-Riemann type operators.

Its asymptotic behavior also fits into the scheme we've been describing: we have already seen this by computing $\mathbf{D}^*$ on the ends with respect to asymptotic trivializations. To express this in trivialization-invariant language, observe that each of the Hermitian bundles (E_z, J_z, ω_z) over S^1 for $z \in \Gamma$ has a conjugate bundle $\bar{E}_z$ with complex structure $-J_z$ and symplectic structure $-\omega_z$; its natural Hermitian inner product is then the complex conjugate of the one on E_z. The asymptotic operator $\mathbf{A}_z$ on E_z can be expressed as $-J_z\widehat{\nabla}_t$, where $\widehat{\nabla}_t$ is a symplectic connection on (E_z, ω_z). Then $\widehat{\nabla}_t$ is also a symplectic connection on $(\bar{E}_z, -\omega_z)$, so we naturally obtain an asymptotic operator on $\bar{E}_z$ in the form

$$\overline{\mathbf{A}}_z := -\mathbf{A}_z : \Gamma(\bar{E}_z) \to \Gamma(\bar{E}_z), \tag{4.6}$$

where the sign reversal arises from the reversal of the complex structure. One can check that if we choose a unitary trivialization of E_z and the conjugate trivialization of $\bar{E}_z$, this relationship between $\mathbf{A}_z$ and $\overline{\mathbf{A}}_z$ produces precisely the relationship between $\mathbf{A} = -J_0\partial_t - S_\infty(t)$ and $\overline{\mathbf{A}} = -J_0\partial_t - \bar{S}_\infty(t)$ that we saw previously, with $\bar{S}_\infty(t) = -CS_\infty(t)C$. Let us summarize all this with a theorem.

Theorem 4.20. *Assume $\langle\ ,\ \rangle_F$ is chosen to be the tensor product metric on $F = \Lambda^{0,1}T^*\Sigma \otimes E$ induced by $\langle\ ,\ \rangle_E$ and the area form $d\,\mathrm{vol}$. Then under the isomorphism induced by $d\,\mathrm{vol}$ from E to $\mathrm{Hom}_{\mathbb{C}}(T\dot{\Sigma}, F)$ and the natural identification of the latter with its conjugate $\overline{\mathrm{Hom}_{\mathbb{C}}}(T\dot{\Sigma}, \bar{F})$, the operator $-\mathbf{D}^* : \Gamma(F) \to \Gamma(E)$ defines a linear Cauchy-Riemann type operator on the conjugate bundle $\bar{F}$,*

$$-\mathbf{D}^* : \Gamma(\bar{F}) \to \Omega^{0,1}(\dot{\Sigma}, \bar{F}),$$

and it is asymptotic at each puncture $z \in \Gamma$ to the conjugate asymptotic operator (4.6). □

LECTURE 5

The index formula

Contents

5.1. Riemann-Roch with punctures

As in the previous lecture, let $\mathbf{D}$ denote a linear Cauchy-Riemann type operator on an asymptotically Hermitian vector bundle E of complex rank m over a punctured Riemann surface $(\dot{\Sigma} = \Sigma \setminus (\Gamma^+ \cup \Gamma^-), j)$, and assume that $\mathbf{D}$ is asymptotic at each puncture $z \in \Gamma$ to a nondegenerate asymptotic operator $\mathbf{A}_z$ on the asymptotic bundle (E_z, J_z, ω_z) over S^1. Writing

$$F := \overline{\operatorname{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, E)$$

for the bundle of complex-antilinear homomorphisms $T\dot{\Sigma} \to E$, the main result of the previous lecture was that

$$\mathbf{D} : W^{k,p}(E) \to W^{k-1,p}(F)$$

is Fredholm for any $k \in \mathbb{N}$ and $p \in (1, \infty)$, and its kernel and index do not depend on k or p. The main goal of this lecture is to compute $\operatorname{ind}(\mathbf{D}) \in \mathbb{Z}$.

The index will depend on the Conley-Zehnder indices $\mu_{\mathrm{CZ}}^{\tau}(\mathbf{A}_z) \in \mathbb{Z}$ introduced in Lecture 3, but since these depend on arbitrary choices of unitary trivializations τ, we need a way of selecting preferred trivializations. The most natural condition is to require that every (E_z, J_z, ω_z) be endowed with a unitary trivialization such that the corresponding asymptotic trivializations of (E, J) extend to a global trivialization[1]; if there is only one puncture z, for instance, then this condition determines $\mu_{\mathrm{CZ}}^{\tau}(\mathbf{A}_z)$ uniquely. This convention has been used to state the formula for $\operatorname{ind}(\mathbf{D})$ in several of the standard references, e.g. in [HWZ99]. We would prefer however to state a formula which is also valid when $\Gamma = \emptyset$ and $E \to \Sigma$ is nontrivial. One way to do

[1] Note that (E, J) is always globally trivializable unless $\Gamma = \emptyset$, as a punctured surface can be retracted to its 1-skeleton.

this is by allowing completely arbitrary asymptotic trivializations, but introducing a topological invariant to measure their failure to extend globally over E.

DEFINITION 5.1. Fix a compact oriented surface S with boundary. The **relative first Chern number** associates to every complex vector bundle (E, J) over S and trivialization τ of $E|_{\partial S}$ an integer

$$c_1^\tau(E) \in \mathbb{Z}$$

satisfying the following properties:

(1) If $(E, J) \to S$ is a line bundle, then $c_1^\tau(E)$ is the signed count of zeroes for a generic smooth section $\eta \in \Gamma(E)$ that appears as a nonzero constant at ∂S with respect to τ.
(2) For any two bundles (E_1, J_1) and (E_2, J_2) with trivializations τ_1 and τ_2 respectively over ∂S,

$$c_1^{\tau_1 \oplus \tau_2}(E_1 \oplus E_2) = c_1^{\tau_1}(E_1) + c_1^{\tau_2}(E_2).$$

These two conditions uniquely determine $c_1^\tau(E)$ for all complex vector bundles since bundles of higher rank can always be split into direct sums of line bundles. The definition clearly matches the usual first Chern number $c_1(E)$ when $\partial S = \emptyset$, and it extends in an obvious way to the category of asymptotically Hermitian vector bundles with asymptotic trivializations.

EXERCISE 5.2. Given two distinct choices of asymptotic trivializations τ_1 and τ_2 for an asymptotically Hermitian bundle E of rank m, show that

$$c_1^{\tau_2}(E) = c_1^{\tau_1}(E) - \deg(\tau_2 \circ \tau_1^{-1}),$$

where $\deg(\tau_2 \circ \tau_1^{-1}) \in \mathbb{Z}$ denotes the sum over all punctures of the winding numbers of the determinants of the transition maps $S^1 \to \mathrm{U}(m)$.[2]

EXERCISE 5.3. Combining Exercise 5.2 above with Exercise 3.37, show that for our asymptotically Hermitian vector bundle E with Cauchy-Riemann type operator $\mathbf{D}$ and asymptotic operators $\mathbf{A}_z$, the number

$$2c_1^\tau(E) + \sum_{z \in \Gamma^+} \mu_{\mathrm{CZ}}^\tau(\mathbf{A}_z) - \sum_{z \in \Gamma^-} \mu_{\mathrm{CZ}}^\tau(\mathbf{A}_z)$$

is independent of the choice of asymptotic trivializations τ.

The above exercise shows that the right hand side of the following index formula is independent of all choices.

THEOREM 5.4. *The Fredholm index of* $\mathbf{D}$ *is given by*

$$\operatorname{ind} \mathbf{D} = m\chi(\dot{\Sigma}) + 2c_1^\tau(E) + \sum_{z \in \Gamma^+} \mu_{CZ}^\tau(\mathbf{A}_z) - \sum_{z \in \Gamma^-} \mu_{CZ}^\tau(\mathbf{A}_z),$$

where $m = \operatorname{rank}_{\mathbb{C}} E$ *and* τ *is an arbitrary choice of asymptotic trivializations.*

[2] Caution: to compute this winding number at a negative puncture using cylindrical coordinates $(s, t) \in (-\infty, 0] \times S^1$, one must traverse $\{-s\} \times S^1$ for $s \gg 1$ in the *wrong direction*, as this is consistent with the orientation induced on $\{-s\} \times S^1$ as a boundary component of a large compact subdomain of $\dot{\Sigma}$.

NOTATION. Throughout this lecture, we shall denote the integer on the right hand side in Theorem 5.4 by

$$I(\mathbf{D}) := m\chi(\dot{\Sigma}) + 2c_1^\tau(E) + \sum_{z\in\Gamma^+} \mu_{\mathrm{CZ}}^\tau(\mathbf{A}_z) - \sum_{z\in\Gamma^-} \mu_{\mathrm{CZ}}^\tau(\mathbf{A}_z) \in \mathbb{Z}.$$

Our goal is thus to prove that $\mathrm{ind}(\mathbf{D}) = I(\mathbf{D})$.

When $\Gamma = \emptyset$, Theorem 5.4 is equivalent to the classical Riemann-Roch formula, which is more often stated for *holomorphic* vector bundles over a closed Riemann surface (Σ, j) with genus g as

$$\mathrm{ind}_{\mathbb{C}}(\mathbf{D}_0) = m(1-g) + c_1(E). \tag{5.1}$$

This formula assumes that the Cauchy-Riemann type operator $\mathbf{D}_0$ is complex linear, but an arbitrary real-linear Cauchy-Riemann operator is then of the form $\mathbf{D} = \mathbf{D}_0 + B$, where the zeroth-order term $B \in \Gamma(\mathrm{Hom}_{\mathbb{R}}(E,F))$ defines a compact perturbation since the inclusion $W^{k,p}(\Sigma) \hookrightarrow W^{k-1,p}(\Sigma)$ is compact. It follows that $\mathbf{D}$ has the same *real* Fredholm index as $\mathbf{D}_0$, namely twice the complex index shown on the right hand side of (5.1), which matches what we see in Theorem 5.4.

REMARK 5.5. Now seems a good moment to clarify explicitly that all dimensions (and therefore also Fredholm indices) in this lecture are *real* dimensions, not complex dimensions, unless otherwise stated.

Reduction to the complex-linear case does not work in general if there are punctures: it remains true that arbitrary Cauchy-Riemann type operators can be written as $\mathbf{D} = \mathbf{D}_0 + B$ where $\mathbf{D}_0$ is complex linear, but the perturbation introduced by the zeroth-order term B is not compact since $W^{k,p}(\dot{\Sigma}) \hookrightarrow W^{k-1,p}(\dot{\Sigma})$ is not compact when $\Gamma \neq \emptyset$. Another indication that this idea cannot work is the fact that while the formula in Theorem 5.4 always gives an *even* integer when $\Gamma = \emptyset$, it can be odd when there are punctures, in which case $\mathbf{D}$ clearly cannot have the same index is any complex-linear operator. Our proof will therefore have to deal with more than just the complex category.

The punctured version of Theorem 5.4 was first proved by Schwarz in his thesis [**Sch95**], its main purpose at the time being to help define algebraic operations (notably the *pair-of-pants product*) in Hamiltonian Floer homology. Schwarz's proof used a "linear gluing" construction that gives a relation between indices of operators on bundles over surfaces obtained by gluing together constituent surfaces along matching cylindrical ends. Since any surface with ends can be "capped off" to form a closed surface, one obtains the general index formula if one already knows how to compute it for closed surfaces and for planes (i.e. caps). For the latter, it is simple enough to write down model Cauchy-Riemann operators on planes and compute their kernels and cokernels explicitly, so in this way the general case is reduced to the classical Riemann-Roch formula. An analogous linear gluing argument for compact surfaces with boundary is used in [**MS04**, Appendix C] to reduce the general Riemann-Roch formula to an explicit computation for Cauchy-Riemann operators on the disk with a totally real boundary condition.

In this lecture, we will follow a different path and use an argument that was first sketched by Taubes for the closed case in [**Tau96a**, §7], with an additional argument for the punctured case suggested by Chris Gerig [**Ger**]. The argument is (in my opinion) analytically somewhat easier than the more standard approaches, and in addition to proving the formula we need for punctured surfaces, it produces a new proof in the closed case without assuming the classical Riemann-Roch formula. It also provides a gentle preview of two analytical phenomena that will later assume prominent roles in our discussion of SFT: *bubbling* and *gluing*.

To see the idea behind Taubes's argument, we can start by noticing an apparent numerical coincidence in the closed case. Assume (E, J) is a complex line bundle over a closed Riemann surface (Σ, j), and $\mathbf{D} : \Gamma(E) \to \Gamma(F) = \Omega^{0,1}(\Sigma, E)$ is a Cauchy-Riemann type operator. We know that $\operatorname{ind}(\mathbf{D}) = \operatorname{ind}(\mathbf{D} + B)$ for any zeroth-order term $B \in \Gamma(\operatorname{Hom}_{\mathbb{R}}(E, F))$. But E and F are both complex vector bundles, so B can always be split uniquely into its complex-linear and complex-antilinear parts, i.e. there is a natural splitting of $\operatorname{Hom}_{\mathbb{R}}(E, F)$ into a direct sum of complex line bundles[3]

$$\operatorname{Hom}_{\mathbb{R}}(E, F) = \operatorname{Hom}_{\mathbb{C}}(E, F) \oplus \overline{\operatorname{Hom}}_{\mathbb{C}}(E, F).$$

Out of curiosity, let's compute the first Chern number of the second factor; this will be the signed count of zeroes of a generic complex-*antilinear* zeroth-order perturbation. To start with, note that

$$\overline{\operatorname{Hom}}_{\mathbb{C}}(E, F) = \overline{\operatorname{Hom}}_{\mathbb{C}}(E, \mathbb{C}) \otimes F,$$

and then observe that $\overline{\operatorname{Hom}}_{\mathbb{C}}(E, \mathbb{C})$ and E are isomorphic: indeed, any Hermitian bundle metric $\langle\ ,\ \rangle_E$ on E gives rise to a bundle isomorphism[4]

$$E \to \overline{\operatorname{Hom}}_{\mathbb{C}}(E, \mathbb{C}) : \eta \mapsto \langle \cdot, \eta \rangle_E.$$

We thus have $\overline{\operatorname{Hom}}_{\mathbb{C}}(E, F) \cong E \otimes F$, so $c_1(\overline{\operatorname{Hom}}_{\mathbb{C}}(E, F)) = c_1(E) + c_1(F)$. We can compute $c_1(F)$ by the same trick since

$$F = \overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, E) = \overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, \mathbb{C}) \otimes E \cong T\Sigma \otimes E,$$

so $c_1(F) = c_1(T\Sigma) + c_1(E) = \chi(\Sigma) + c_1(E)$, and thus

$$c_1(\overline{\operatorname{Hom}}_{\mathbb{C}}(E, F)) = \chi(\Sigma) + 2c_1(E).$$

Since we're looking at a line bundle over a surface without punctures, this number is the same as $I(\mathbf{D})$. This coincidence is too improbable to ignore, and indeed, it turns out not to be coincidental. Here is an informal statement of a result that we will later prove a more precise version of in order to deduce Theorem 5.4.

"THEOREM". *Given a Cauchy-Riemann type operator $\mathbf{D} : H^1(E) \to L^2(F)$ on a line bundle (E, J) over a closed Riemann surface (Σ, j), choose a complex-antilinear zeroth-order perturbation $B \in \Gamma(\overline{\operatorname{Hom}}_{\mathbb{C}}(E, F))$ whose zeroes are all nondegenerate.*

[3] Here the complex structure on $\operatorname{Hom}_{\mathbb{R}}(E, F)$ and its subbundles is defined in terms of the complex structure of F, i.e. it sends $B \in \operatorname{Hom}_{\mathbb{R}}(E, F)$ to $J \circ B \in \operatorname{Hom}_{\mathbb{R}}(E, F)$.

[4] We are assuming as usual that Hermitian inner products are complex antilinear in the first argument and linear in the second.

Then for sufficiently large $\sigma > 0$, $\ker(\mathbf{D} + \sigma B)$ *is approximately spanned by* 1-*dimensional spaces of sections with support localized near the positive zeroes of* B. *In particular,* $\dim \ker(\mathbf{D} + \sigma B)$ *equals the number of positive zeroes of* B.

To deduce $\operatorname{ind}(\mathbf{D}) = I(\mathbf{D})$ from this, we need to apply the same trick to the formal adjoint $\mathbf{D}^*$. As we will review in §5.2, $-\mathbf{D}^*$ can be regarded under certain natural assumptions as a Cauchy-Riemann type operator on the bundle $\bar{F}$ conjugate to F, and the formal adjoint of $\mathbf{D} + \sigma B$ then gives rise to a Cauchy-Riemann type operator of the form

$$-\mathbf{D}^* + \sigma B' : \Gamma(\bar{F}) \to \Gamma(\bar{E}) = \Omega^{0,1}(\Sigma, \bar{F}),$$

where $B' : \bar{F} \to \bar{E}$ is also complex antilinear and has the same zeroes as B, but with opposite signs. Applying the above "theorem" to $-\mathbf{D}^*$ thus identifies $\ker(\mathbf{D} + \sigma B)^*$ for sufficiently large $\sigma > 0$ with a space whose dimension equals the number of *negative* zeroes of B. This gives

$$\begin{aligned} \operatorname{ind}(\mathbf{D}) = \operatorname{ind}(\mathbf{D} + \sigma B) &= \dim \ker(\mathbf{D} + \sigma B) - \dim \ker(\mathbf{D} + \sigma B)^* \\ &= c_1(\overline{\operatorname{Hom}}_{\mathbb{C}}(E, F)) = I(\mathbf{D}). \end{aligned}$$

It's worth mentioning that the "large perturbation" argument we've just sketched is only one simple example of an idea with a long and illustrious history: another simple example is the observation by Witten [**Wit82**] that after choosing a Morse function on a Riemannian manifold, certain large deformations of the de Rham complex lead to an approximation of the Morse complex, with generators of the de Rham complex having support concentrated near the critical points of the Morse function—this yields a somewhat novel proof of de Rham's theorem. A much deeper example is Taubes's isomorphism [**Tau96b**] between the Seiberg-Witten invariants of symplectic 4-manifolds and certain holomorphic curve invariants: here also, the idea is to consider a large compact perturbation of the Seiberg-Witten equations and show that, in the limit where the perturbation becomes infinitely large, solutions of the Seiberg-Witten equations localize near J-holomorphic curves. For a more recent exploration of this idea in the context of Dirac operators, see [**Mar**].

Before proceeding with the details, let us fix two simplifying assumptions that can be imposed without loss of generality:

ASSUMPTION 5.6. (E, J) *has complex rank* 1.

Indeed, an asymptotically Hermitian bundle E of complex rank $m \in \mathbb{N}$ always admits a decomposition into asymptotically Hermitian line bundles $E = E_1 \oplus \ldots \oplus E_m$, producing a corresponding splitting of the target bundle $F = F_1 \oplus \ldots \oplus F_m$. The operator $\mathbf{D}$ need not respect these splittings, but it is always *homotopic through Fredholm operators* to one that does: we saw in Theorem 3.34 that the asymptotic operators $\mathbf{A}_z$ are homotopic through nondegenerate asymptotic operators to any other operators $\mathbf{A}'_z$ that have the same Conley-Zehnder indices, so one can choose $\mathbf{A}'_z$ to respect the splitting. Any homotopy of Cauchy-Riemann operators following such a homotopy of nondegenerate asymptotic operators then produces a continuous family of Fredholm operators by the main result of Lecture 4, implying that their indices do not change. The general index formula then follows from the line bundle case since any two Cauchy-Riemann type Fredholm operators $\mathbf{D}_1$ and $\mathbf{D}_2$ over the same Riemann surface satisfy

$$\operatorname{ind}(\mathbf{D}_1 \oplus \mathbf{D}_2) = \operatorname{ind}(\mathbf{D}_1) + \operatorname{ind}(\mathbf{D}_2) \quad \text{and} \quad I(\mathbf{D}_1 \oplus \mathbf{D}_2) = I(\mathbf{D}_1) + I(\mathbf{D}_2).$$

ASSUMPTION 5.7. *$k = 1$ and $p = 2$.*

This means we will concretely be considering the operator

$$\mathbf{D} : H^1(E) \to L^2(F),$$

where H^1 as usual is an abbreviation for $W^{1,2}$. This assumption is clearly harmless since we know that $\operatorname{ind} \mathbf{D}$ does not depend on the choice of k and p.

5.2. Some remarks on the formal adjoint

For the beginning of this section we can drop the assumption that (E, J) is a line bundle and assume $\operatorname{rank}_{\mathbb{C}} E = m \in \mathbb{N}$, though later we will again set $m = 1$.

Recall from the end of Lecture 4 that if we fix global Hermitian structures $\langle\ ,\ \rangle_E$ and $\langle\ ,\ \rangle_F$ on (E, J) and (F, J) respectively and an area form $d\,\mathrm{vol}$ on $\dot{\Sigma}$ that matches $ds \wedge dt$ on the cylindrical ends, then $\mathbf{D}$ has a *formal adjoint*

$$\mathbf{D}^* : \Gamma(F) \to \Gamma(E)$$

satisfying

$$\langle \lambda, \mathbf{D}\eta \rangle_{L^2(F)} = \langle \mathbf{D}^*\lambda, \eta \rangle_{L^2(E)} \quad \text{for all} \quad \eta \in H^1(E),\ \lambda \in H^1(F).$$

Here the real-valued L^2 pairings are defined by

$$\langle \eta, \xi \rangle_{L^2(E)} := \operatorname{Re} \int_{\dot{\Sigma}} \langle \eta, \xi \rangle_E \, d\,\mathrm{vol} \quad \text{for} \quad \eta, \xi \in \Gamma(E),$$

and similarly for sections of F. The essential features of the formal adjoint are that $\ker \mathbf{D}^* \cong \operatorname{coker} \mathbf{D}$ and $\operatorname{coker} \mathbf{D}^* \cong \ker \mathbf{D}$, hence $\operatorname{ind}(\mathbf{D}^*) = -\operatorname{ind}(\mathbf{D})$. Recall moreover that $d\,\mathrm{vol}$ induces a natural Hermitian bundle metric on $\dot{\Sigma}$ by

$$\langle \cdot, \cdot \rangle_{\Sigma} = d\,\mathrm{vol}(\cdot, j\cdot) + i\, d\,\mathrm{vol}(\cdot, \cdot),$$

which determines a bundle isomorphism

$$T\dot{\Sigma} \to \Lambda^{0,1}T^*\dot{\Sigma} : X \mapsto X^{0,1} := \langle \cdot, X \rangle_{\Sigma},$$

as well as a complex-*antilinear* isomorphism

$$T\dot{\Sigma} \to \Lambda^{1,0}T^*\dot{\Sigma} : X \mapsto X^{1,0} := \langle X, \cdot \rangle_{\Sigma}.$$

If $\langle\ ,\ \rangle_F$ is then chosen to be the tensor product metric determined via the natural isomorphism

$$F = \overline{\operatorname{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, E) = \Lambda^{0,1}T^*\dot{\Sigma} \otimes E = T\dot{\Sigma} \otimes E,$$

then E admits a natural isomorphism to $\Lambda^{1,0}T^*\dot{\Sigma} \otimes F$ such that

$$-\mathbf{D}^* : \Gamma(F) \to \Gamma(E) = \Omega^{1,0}(\dot{\Sigma}, F)$$

becomes an *anti-Cauchy-Riemann* type operator, i.e. it satisfies the Leibniz rule

$$-\mathbf{D}^*(f\lambda) = (\partial f)\lambda + f(-\mathbf{D}^*\lambda)$$

for all $f \in C^\infty(\dot\Sigma, \mathbb{R})$, with $\partial f := df - i\, df \circ j \in \Omega^{1,0}(\dot\Sigma)$. Equivalently, $-\mathbf{D}^*$ defines a Cauchy-Riemann type operator on the **conjugate** bundle $\bar F \to \dot\Sigma$, defined as the real bundle $F \to \dot\Sigma$ but with the sign of its complex structure reversed; we shall distinguish this Cauchy-Riemann operator from $-\mathbf{D}^*$ by writing it as

$$-\overline{\mathbf{D}}^* : \Gamma(\bar F) \to \Omega^{0,1}(\dot\Sigma, \bar F),$$

though it is technically the same operator. Recall that the identity map defines a natural complex-antilinear isomorphism between any complex vector bundle and its conjugate bundle; we shall denote this isomorphism generally by

$$E \to \bar E : v \mapsto \bar v,$$

so in particular it satisfies $\overline{cv} = \bar c \bar v$ for all scalars $c \in \mathbb{C}$, and similarly

$$\overline{\mathbf{D}}^* \bar\lambda = \overline{\mathbf{D}^* \lambda}$$

for $\lambda \in \Gamma(F)$. The asymptotic operators for $-\overline{\mathbf{D}}^*$ are

$$\overline{\mathbf{A}}_z = -\mathbf{A}_z : \Gamma(\bar E_z) \to \Gamma(\bar E_z).$$

LEMMA 5.8. *If τ is a choice of asymptotic trivialization on E and $\bar\tau$ denotes the* conjugate *asymptotic trivialization*[5]*, then*

$$c_1^{\bar\tau}(\bar E) = -c_1^\tau(E), \qquad \textit{and} \qquad \mu_{CZ}^{\bar\tau}(\overline{\mathbf{A}}_z) = -\mu_{CZ}^\tau(\mathbf{A}_z) \textit{ for all } z \in \Gamma.$$

PROOF. Assuming E is a line bundle, suppose η is a generic section of E that matches a nonzero constant with respect to τ on the cylindrical ends, so $c_1^\tau(E)$ is the signed count of zeroes of η. Then $\bar\eta \in \Gamma(\bar E)$ is similarly a nonzero constant on the ends with respect to $\bar\tau$, but the signs of its zeroes are opposite those of η because they are defined as winding numbers with respect to *conjugate* local trivializations. This proves $c_1^{\bar\tau}(\bar E) = -c_1^\tau(E)$.

The Conley-Zehnder indices can be computed from the formula

$$\mu_{\mathrm{CZ}}^\tau(\mathbf{A}_z) = \alpha_+^\tau(\mathbf{A}_z) + \alpha_-^\tau(\mathbf{A}_z),$$

see Theorem 3.36. Here $\alpha_-^\tau(\mathbf{A}_z)$ is the largest possible winding number relative to τ of an eigenfunction for $\mathbf{A}_z$ with negative eigenvalue, and $\alpha_+^\tau(\mathbf{A}_z)$ is the smallest possible winding number with positive eigenvalue. The eigenfunctions of $\overline{\mathbf{A}}_z = -\mathbf{A}_z$ are the same, but the signs of their eigenvalues are reversed, and the signs of their winding numbers are also reversed because they must be measured relative to the conjugate trivialization, thus

$$\alpha_\pm^{\bar\tau}(\overline{\mathbf{A}}_z) = -\alpha_\mp^\tau(\mathbf{A}_z),$$

implying

$$\mu_{\mathrm{CZ}}^{\bar\tau}(\overline{\mathbf{A}}_z) = \alpha_+^{\bar\tau}(\overline{\mathbf{A}}_z) + \alpha_-^{\bar\tau}(\overline{\mathbf{A}}_z) = -\alpha_-^\tau(\mathbf{A}_z) - \alpha_+^\tau(\mathbf{A}_z) = -\mu_{\mathrm{CZ}}^\tau(\mathbf{A}_z).$$

The above calculations are all valid for line bundles, but the general case follows by taking direct sums. □

[5]If $\tau : E|_{\mathcal{U}} \to \mathcal{U} \times \mathbb{C}^m$ is a local trivialization of E with $\tau(v) = (z, w)$, the conjugate trivialization $\bar\tau : \bar E|_{\mathcal{U}} \to \mathcal{U} \times \mathbb{C}^m$ is defined by $\bar\tau(\bar v) = (z, \bar w)$.

We are now able to show that Theorem 5.4 is consistent with what we already know about the formal adjoint.

PROPOSITION 5.9. $I(-\overline{\mathbf{D}}^*) = -I(\mathbf{D})$.

PROOF. Under the isomorphism $F = \Lambda^{0,1}T^*\dot{\Sigma} \otimes E = T\dot{\Sigma} \otimes E$, an asymptotic trivialization τ on E induces an asymptotic trivialization $\partial_s \otimes \tau$ on F, where ∂_s denotes the asymptotic trivialization of $T\dot{\Sigma}$ defined via an outward pointing vector field on the cylindrical ends. Counting zeroes of vector fields then proves $c_1^{\partial_s}(T\dot{\Sigma}) = \chi(\dot{\Sigma})$, so

$$c_1^{\partial_s\otimes\tau}(F) = c_1^{\partial_s\otimes\tau}(T\dot{\Sigma}\otimes E) = mc_1^{\partial_s}(T\dot{\Sigma}) + c_1^\tau(E) = m\chi(\dot{\Sigma}) + c_1^\tau(E).$$

Applying Lemma 5.8 to the conjugate bundle then gives

$$c_1^{\overline{\partial_s\otimes\tau}}(\bar{F}) = -m\chi(\dot{\Sigma}) - c_1^\tau(E).$$

The unitary trivializations of the asymptotic bundles $\bar{E}_z$ corresponding to $\overline{\partial_s \otimes \tau}$ are simply $\bar{\tau}$, thus using Lemma 5.8 again for the Conley-Zehnder terms,

$$\begin{aligned} I(-\overline{\mathbf{D}}^*) &= m\chi(\dot{\Sigma}) + 2c_1^{\overline{\partial_s\otimes\tau}}(\bar{F}) + \sum_{z\in\Gamma^+}\mu_{\mathrm{CZ}}^{\bar{\tau}}(\overline{\mathbf{A}}_z) - \sum_{z\in\Gamma^-}\mu_{\mathrm{CZ}}^{\bar{\tau}}(\overline{\mathbf{A}}_z) \\ &= -m\chi(\dot{\Sigma}) - 2c_1^\tau(E) - \sum_{z\in\Gamma^+}\mu_{\mathrm{CZ}}^{\tau}(\mathbf{A}_z) + \sum_{z\in\Gamma^-}\mu_{\mathrm{CZ}}^{\tau}(\mathbf{A}_z) \\ &= -I(\mathbf{D}). \end{aligned}$$

□

We next consider the effect of an antilinear zeroth-order perturbation on the formal adjoint. By "antilinear zeroth-order perturbation," we generally mean a smooth section

$$B \in \Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(E, F)).$$

It is perhaps easier to understand B in terms of the conjugate bundle $\bar{E}$: indeed, there exists a unique

$$\beta \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(\bar{E}, F))$$

such that

$$B\eta = \beta\bar{\eta},$$

and this correspondence defines a bundle isomorphism $\overline{\mathrm{Hom}}_{\mathbb{C}}(E, F) = \mathrm{Hom}_{\mathbb{C}}(\bar{E}, F)$.

EXERCISE 5.10. Assume X and Y are complex vector bundles over the same base.

(a) Show that $\bar{X} \otimes \bar{Y}$ is canonically isomorphic to the conjugate bundle of $X \otimes Y$.
(b) Show that $\mathrm{Hom}_{\mathbb{C}}(\bar{X}, \bar{Y})$ is canonically isomorphic to the conjugate bundle of $\mathrm{Hom}_{\mathbb{C}}(X, Y)$, and $\overline{\mathrm{Hom}}_{\mathbb{C}}(\bar{X}, \bar{Y})$ is canonically isomorphic to the conjugate bundle of $\overline{\mathrm{Hom}}_{\mathbb{C}}(X, Y)$.
(c) Show that $\Lambda^{0,1}X := \overline{\mathrm{Hom}}_{\mathbb{C}}(X, \mathbb{C})$ is canonically isomorphic to the conjugate bundle of $\Lambda^{1,0}X := \mathrm{Hom}_{\mathbb{C}}(X, \mathbb{C})$.

Define the Cauchy-Riemann type operator

$$\mathbf{D}_B := \mathbf{D} + B : \Gamma(E) \to \Gamma(F) = \Omega^{0,1}(\dot{\Sigma}, E),$$

so $\mathbf{D}_B\eta = \mathbf{D}\eta + \beta\bar{\eta}$. To write down $\mathbf{D}_B^*$, observe that since $\beta : \bar{E} \to F$ is a complex-linear bundle map between Hermitian bundles, it has a complex-linear adjoint

$$\beta^\dagger : F \to \bar{E} \quad \text{such that} \quad \langle \beta^\dagger \lambda, \bar{\eta} \rangle_{\bar{E}} = \langle \lambda, \beta\bar{\eta} \rangle_F \text{ for } \lambda \in F,\, \bar{\eta} \in \bar{E}.$$

Here the bundle metric on $\bar{E}$ is defined by $\langle \bar{\eta}, \bar{\xi} \rangle_{\bar{E}} := \langle \xi, \eta \rangle_E$. We then have

$$\begin{aligned} \operatorname{Re}\langle \lambda, B\eta \rangle_F &= \operatorname{Re}\langle \lambda, \beta\bar{\eta} \rangle_F = \operatorname{Re}\langle \beta^\dagger\lambda, \bar{\eta} \rangle_{\bar{E}} = \operatorname{Re}\langle \eta, \overline{\beta^\dagger\lambda} \rangle_E = \operatorname{Re}\langle \overline{\beta^\dagger\lambda}, \eta \rangle_E \\ &= \operatorname{Re}\langle \overline{\beta^\dagger}\bar{\lambda}, \eta \rangle_E, \end{aligned}$$

where $\overline{\beta^\dagger} \in \Gamma(\operatorname{Hom}_{\mathbb{C}}(\bar{F}, E))$ denotes the image of $\beta^\dagger \in \Gamma(\operatorname{Hom}_{\mathbb{C}}(F, \bar{E}))$ under the complex-antilinear identity map from $\operatorname{Hom}_{\mathbb{C}}(F, \bar{E})$ to its conjugate bundle (see Exercise 5.10). The formal adjoint of $\mathbf{D}_B$ is thus

$$\mathbf{D}_B^* = \mathbf{D}^* + B^* : \Gamma(F) \to \Gamma(E),$$

where $B^* : F \to E$ is defined by

$$B^*\lambda := \overline{\beta^\dagger}\bar{\lambda}.$$

To write down the resulting Cauchy-Riemann type operator on $\bar{F}$, we replace $B^* : F \to E$ with $\overline{B}^* : \bar{F} \to \bar{E}$, defined by

$$\overline{B}^*\bar{\lambda} := \overline{B^*\lambda} = \beta^\dagger\lambda,$$

giving a Cauchy-Riemann operator

$$-\overline{\mathbf{D}}_B^* = -\overline{\mathbf{D}}^* + (-\overline{B}^*) : \Gamma(\bar{F}) \to \Gamma(\bar{E}) = \Omega^{0,1}(\dot{\Sigma}, \bar{F}).$$

The point of writing down this formula is to make the following observations:

LEMMA 5.11. *The zeroth-order perturbation $-\overline{B}^* : \bar{F} \to \bar{E}$ appearing in $-\overline{\mathbf{D}}_B^*$ has the following properties:*

(1) $-\overline{B}^* : \bar{F} \to \bar{E}$ is complex antilinear;
(2) There is a natural complex bundle isomorphism $\overline{\operatorname{Hom}}_{\mathbb{C}}(\bar{F}, \bar{E}) = \operatorname{Hom}_{\mathbb{C}}(F, \bar{E})$ that identifies $-\overline{B}^*$ with $-\beta^\dagger$;
(3) If $m = 1$ and $B \in \Gamma(\overline{\operatorname{Hom}}_{\mathbb{C}}(E, F))$ has only nondegenerate zeroes, then $-\overline{B}^* \in \Gamma(\overline{\operatorname{Hom}}_{\mathbb{C}}(\bar{F}, \bar{E}))$ has the same zeroes but with opposite signs.

PROOF. The first two statements follow immediately from the fact that $-\overline{B}^*$ is the composition of the canonical conjugation map $\bar{F} \to F$ with the complex-linear bundle map $-\beta^\dagger : F \to \bar{E}$. For the third, it suffices to compare what $\beta \in \Gamma(\operatorname{Hom}_{\mathbb{C}}(\bar{E}, F))$ and $-\beta^\dagger : \Gamma(\operatorname{Hom}_{\mathbb{C}}(F, \bar{E}))$ look like in local trivializations near a zero: one is minus the complex conjugate of the other, hence their zeroes count with opposite signs. □

5.3. The index zero case on a torus

As a warmup for the general case, we now fill in the details of Taubes's proof of Theorem 5.4 in the case

$$\dot{\Sigma} = \mathbb{T}^2 := \mathbb{C} \setminus (\mathbb{Z} \oplus i\mathbb{Z})$$

and $E = \mathbb{T}^2 \times \mathbb{C}$, i.e. a trivial line bundle. In this case $I(\mathbf{D}) = \chi(\mathbb{T}^2) + 2c_1(E) = 0$, so our aim is to prove $\operatorname{ind}(\mathbf{D}) = 0$. What we will show in fact is that $\mathbf{D}$ is homotopic through a continuous family of Fredholm operators to one that is an isomorphism. Since E and F are now both trivial, it will suffice to consider the operator

$$\mathbf{D} := \bar{\partial} = \partial_s + i\partial_t : H^1(\mathbb{T}^2, \mathbb{C}) \to L^2(\mathbb{T}^2, \mathbb{C}),$$

whose formal adjoint is $\mathbf{D}^* := -\partial = -\partial_s + i\partial_t$. An antilinear zeroth-order perturbation is then equivalent to a choice of function $\beta : \mathbb{T}^2 \to \mathbb{C}$, giving rise to a family of operators

$$\mathbf{D}_\sigma \eta := \bar{\partial}\eta + \sigma\beta\bar{\eta}$$

for $\sigma \in \mathbb{R}$, where $\bar{\eta} : \mathbb{T}^2 \to \mathbb{C}$ now denotes the straightforward complex conjugate of η. Let us assume that $\beta : \mathbb{T}^2 \to \mathbb{C}$ is nowhere zero; note that this would not be possible in more general situations, but is possible here because $\operatorname{Hom}_{\mathbb{C}}(\bar{E}, F)$ is a trivial bundle.

LEMMA 5.12. *$\mathbf{D}_\sigma$ is injective for all $\sigma > 0$ sufficiently large.*

PROOF. Elliptic regularity implies any $\eta \in \ker \mathbf{D}_\sigma$ is smooth, so we shall restrict our attention to smooth functions $\eta : \mathbb{T}^2 \to \mathbb{C}$. We start by comparing the two second-order differential operators

$$\mathbf{D}^*\mathbf{D} \text{ and } \mathbf{D}_\sigma^*\mathbf{D}_\sigma : C^\infty(\mathbb{T}^2, \mathbb{C}) \to C^\infty(\mathbb{T}^2, \mathbb{C}).$$

Both are nonnegative L^2-symmetric operators, and in fact the first is simply the Laplacian

$$\mathbf{D}^*\mathbf{D} = -\partial\bar{\partial} = (-\partial_s + i\partial_t)(\partial_s + i\partial_t) = -\partial_s^2 - \partial_t^2 = -\Delta.$$

The formal adjoint of $\mathbf{D}_\sigma$ takes the form

$$\mathbf{D}_\sigma^*\eta = \mathbf{D}^*\eta + \sigma B^*\eta = \mathbf{D}^*\eta + \sigma\beta\bar{\eta},$$

thus for any $\eta \in C^\infty(\mathbb{T}^2, \mathbb{C})$,

$$\begin{aligned} \mathbf{D}_\sigma^*\mathbf{D}_\sigma\eta &= (\mathbf{D}^* + \sigma B^*)(\mathbf{D} + \sigma B)\eta \\ &= \mathbf{D}^*\mathbf{D}\eta + \sigma\left(\beta\overline{\bar{\partial}\eta} - \partial(\beta\bar{\eta})\right) + \sigma^2 B^*B\eta \\ &= \mathbf{D}^*\mathbf{D}\eta + \sigma\left(\beta\partial\bar{\eta} - (\partial\beta)\bar{\eta} - \beta\partial\bar{\eta}\right) + \sigma^2 B^*B\eta \\ &= \mathbf{D}^*\mathbf{D}\eta + \sigma^2 B^*B\eta - \sigma(\partial\beta)\bar{\eta}. \end{aligned} \tag{5.2}$$

This is a *Weitzenböck formula*: its main message is that the Laplacian $\mathbf{D}^*\mathbf{D}$ and the related operator $\mathbf{D}_\sigma^*\mathbf{D}_\sigma$ differ from each other only by a zeroth-order term that will be positive definite if σ is sufficiently large. Indeed, since β is nowhere zero, we have $|B\eta| \geq c|\eta|$ for some constant $c > 0$, thus

$$\begin{aligned}\|\mathbf{D}_\sigma \eta\|_{L^2}^2 &= \langle \eta, \mathbf{D}_\sigma^* \mathbf{D}_\sigma \eta \rangle_{L^2} = \langle \eta, \mathbf{D}^* \mathbf{D} \eta \rangle_{L^2} + \sigma^2 \langle \eta, B^* B \eta \rangle_{L^2} - \sigma \langle \eta, (\partial \beta) \bar{\eta} \rangle_{L^2} \\ &= \|\mathbf{D}\eta\|_{L^2}^2 + \sigma^2 \|B\eta\|_{L^2}^2 - \sigma \langle \eta, (\partial\beta)\bar{\eta} \rangle_{L^2} \\ &\geq \left(\sigma^2 c^2 - \sigma \|\partial\beta\|_{C^0}\right) \|\eta\|_{L^2}^2.\end{aligned}$$

We conclude that as soon as $\sigma > 0$ is large enough to make the quantity in parentheses positive, $\mathbf{D}_\sigma \eta$ cannot vanish unless $\|\eta\|_{L^2} = 0$. □

PROOF OF THEOREM 5.4 FOR $E = \mathbb{T}^2 \times \mathbb{C}$. The lemma above shows that one can add a large antilinear perturbation to $\mathbf{D} = \bar{\partial}$ making the deformed operator $\mathbf{D}_\sigma$ injective. By Lemma 5.11, the same argument applies to the formal adjoint $\mathbf{D}^*$, implying that for sufficiently large $\sigma > 0$, $\mathbf{D}_\sigma^*$ is injective and thus $\mathbf{D}_\sigma$ is also surjective, and therefore an isomorphism. This proves $\operatorname{ind}(\mathbf{D}) = \operatorname{ind}(\mathbf{D}_\sigma) = 0$. □

Let's consider which particular details of the setup made the proof above possible.

First, the zeroth-order perturbation is complex antilinear. We used this, if only implicitly, in deriving the Weitzenböck formula (5.2): the key step is in the third line, where the two terms involving $\partial\bar{\eta}$ cancel each other out and leave nothing but zeroth-order terms remaining. This would not have happened if e.g. $B : E \to F$ had been complex linear—we would then have seen terms depending on the first derivative of η in $\mathbf{D}_\sigma^* \mathbf{D}_\sigma \eta - \mathbf{D}^* \mathbf{D} \eta$, and this would have killed the whole argument. The fact that this cancellation happens when the perturbation is antilinear probably looks like magic at this point, but there is a principle behind it; we will discuss it further in §5.4 below, see Remark 5.17.

The second crucial fact we used was that $\beta : \mathbb{T}^2 \to \mathbb{C}$ is nowhere zero, in order to obtain the lower bound on $\|B\eta\|_{L^2}$ in terms of $\|\eta\|_{L^2}$. This cannot always be achieved—it is possible in this special case only because E and F are both trivial bundles and thus so is $\operatorname{Hom}_{\mathbb{C}}(\bar{E}, F)$. On more general bundles, the best we could hope for would be to pick $\beta \in \Gamma(\operatorname{Hom}_{\mathbb{C}}(\bar{E}, F))$ with finitely many zeroes, all nondegenerate. In this case the above argument fails, but it still tells us something. Suppose $\Sigma_\epsilon \subset \mathbb{T}^2$ is a region disjoint from the isolated zeroes of β. Then there exists a constant $c_\epsilon > 0$, dependent on the region Σ_ϵ, such that

$$\|\beta\bar{\eta}\|_{L^2(\mathbb{T}^2)}^2 \geq \|\beta\bar{\eta}\|_{L^2(\Sigma_\epsilon)}^2 \geq c_\epsilon \|\eta\|_{L^2(\Sigma_\epsilon)}^2,$$

so instead of the estimate at the end of the proof above implying $\mathbf{D}_\sigma$ is injective, we obtain one of the form

$$\|\mathbf{D}_\sigma \eta\|_{L^2(\mathbb{T}^2)}^2 \geq c_\epsilon \sigma^2 \|\eta\|_{L^2(\Sigma_\epsilon)} - c\sigma \|\eta\|_{L^2(\mathbb{T}^2)}^2.$$

To see what this means, imagine we have sequences $\sigma_\nu \to \infty$ and $\eta_\nu \in \ker \mathbf{D}_{\sigma_\nu}$, normalized so that $\|\eta_\nu\|_{L^2} = 1$ for all ν. The estimate above then implies

$$\|\eta_\nu\|_{L^2(\Sigma_\epsilon)}^2 \leq \frac{c}{c_\epsilon \sigma_\nu} \to 0 \quad \text{as} \quad \nu \to \infty,$$

so while all sections η_ν have the same amount of "energy" (as measured via their L^2-norms), the energy is escaping from Σ_ϵ as σ_ν increases. This is true for *any* domain Σ_ϵ disjoint from the zeroes, so we conclude that in the limit as $\sigma \to \infty$, sections in

$\ker \mathbf{D}_\sigma$ have their energy concentrated in infinitesimally small neighborhoods of the zeroes of β. We will see in the following how to extract useful information from this concentration of energy.

5.4. A Weitzenböck formula for Cauchy-Riemann operators

The Weitzenböck formula (5.2) can be generalized to a useful relation between any two Cauchy-Riemann type operators that differ by an *antilinear* zeroth-order term. To see this, we start with a short digression on holomorphic and antiholomorphic vector bundles.

A smooth function $f : \mathbb{C} \supset \mathcal{U} \to \mathbb{C}$ is called **antiholomorphic** if it satisfies $(\partial_s - i\partial_t) f = 0$, which means its differential anticommutes with the complex structure on $\mathbb{C}$. The class of antiholomorphic functions is not closed under composition, but it is closed under products, hence one can define an **antiholomorphic structure** on a complex vector bundle to be a system of local trivializations for which all transition maps are antiholomorphic. Given the standard correspondence between holomorphic structures and Cauchy-Riemann type operators, it is easy to establish a similar correspondence between aniholomorphic structures and (complex-linear) **anti-Cauchy-Riemann type** operators, i.e. those which satisfy

$$\mathbf{D}(f\eta) = (\partial f)\eta + f\mathbf{D}\eta$$

for all $f \in C^\infty(\dot{\Sigma}, \mathbb{C})$, where $\partial f := df - i\, df \circ j \in \Omega^{1,0}(\dot{\Sigma})$. We've seen one important example of such an operator already: if $\mathbf{D} : \Gamma(E) \to \Gamma(F)$ is complex linear, then $-\mathbf{D}^*$ is a complex-linear anti-Cauchy-Riemann operator on F and thus endows F with an antiholomorphic structure. Another natural example occurs naturally on conjugate bundles: if E has a holomorphic structure, then $\bar{E}$ inherits from this an antiholomorphic structure. This is immediate from the fact that $f : \mathbb{C} \supset \mathcal{U} \to \mathbb{C}$ is holomorphic if and only if $\bar{f} : \mathcal{U} \to \mathbb{C}$ is antiholomorphic. If $\mathbf{D} : \Gamma(E) \to \Gamma(F) = \Omega^{0,1}(\dot{\Sigma}, E)$ is the corresponding complex-linear Cauchy-Riemann type operator on E, we shall denote the resulting anti-Cauchy-Riemann operator by

$$\overline{\mathbf{D}} : \Gamma(\bar{E}) \to \Gamma(\bar{F}) = \Omega^{1,0}(\dot{\Sigma}, \bar{E}),$$

where by definition $\overline{\mathbf{D}}\bar{\eta} = \overline{\mathbf{D}\eta}$.

EXERCISE 5.13. Show that if X and Y are antiholomorphic vector bundles over the same base, then $X \otimes Y$ and $\mathrm{Hom}_{\mathbb{C}}(X, Y)$ both naturally inherit antiholomorphic bundle structures such that the obvious Leibniz rules are satisfied. *Remark: the proof of this is exactly the same as for holomorphic bundles, one only needs to change some signs.*

EXERCISE 5.14. Suppose X and Y are complex vector bundles over the same base, carrying real-linear anti-Cauchy-Riemann operators ∂_X and ∂_Y respectively. Show that $H := \mathrm{Hom}_{\mathbb{R}}(X, Y)$ then admits a real-linear anti-Cauchy-Riemann operator ∂_H such that for all $\Phi \in \Gamma(H)$ and $\eta \in \Gamma(X)$,

$$\partial_Y(\Phi\eta) = (\partial_H \Phi)\eta + \Phi(\partial_X \eta).$$

Hint: write ∂_X *and* ∂_Y *as complex-linear operators with real-linear zeroth-order perturbations, and apply Exercise 5.13. Show moreover that any* C^k*-bounds satisfied by the zeroth-order terms in* ∂_X *and* ∂_Y *are inherited by the zeroth-order term in* ∂_H*.*

The setup for the next result is as follows. We assume again $m = 1$, so E and F are line bundles. Fix $\beta \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(\bar{E}, F))$, define $B \in \Gamma(\overline{\mathrm{Hom}_{\mathbb{C}}}(E, F))$ by $B\eta := \beta\bar{\eta}$, and use this to define the perturbed Cauchy-Riemann type operator

$$\mathbf{D}_B := \mathbf{D} + B : \Gamma(E) \to \Gamma(F),$$

whose formal adjoint is $\mathbf{D}_B^* = \mathbf{D}^* + B^*$ with $B^*\lambda := \overline{\beta^\dagger}\bar{\lambda}$.

PROPOSITION 5.15. *The second-order differential operators* $\mathbf{D}^*\mathbf{D}$ *and* $\mathbf{D}_B^*\mathbf{D}_B$ *on* E *are related by*

$$\mathbf{D}_B^*\mathbf{D}_B\eta = \mathbf{D}^*\mathbf{D}^*\eta + B^*B\eta - (\partial_H\beta)\bar{\eta},$$

where ∂_H *is a real-linear anti-Cauchy-Riemann type operator on* $\mathrm{Hom}_{\mathbb{C}}(\bar{E}, F)$*. Moreover, if* β *is* C^1*-bounded on* $\dot{\Sigma}$*, then* $\partial_H\beta$ *is* C^0*-bounded.*

PROOF. We have real-linear anti-Cauchy-Riemann operators $\overline{\mathbf{D}}$ and $-\mathbf{D}^*$ on $\bar{E}$ and F respectively, so Exercise 5.14 produces an operator ∂_H on $\mathrm{Hom}_{\mathbb{C}}(\bar{E}, F)$ for which the Leibniz rule is satisfied. We can then write

$$\begin{aligned}
\mathbf{D}_B^*\mathbf{D}_B\eta &= (\mathbf{D}^* + B^*)(\mathbf{D} + B)\eta \\
&= \mathbf{D}^*\mathbf{D}\eta + \overline{\beta^\dagger}\overline{\mathbf{D}\eta} - (-\mathbf{D}^*)(\beta\bar{\eta}) + B^*B\eta \\
&= \mathbf{D}^*\mathbf{D}\eta + \overline{\beta^\dagger}\overline{\mathbf{D}}\bar{\eta} - (\partial_H\beta)\bar{\eta} - \beta\overline{\mathbf{D}}\bar{\eta} + B^*B\eta \\
&= \mathbf{D}^*\mathbf{D}\eta + B^*B\eta - (\partial_H\beta)\bar{\eta} + \left(\overline{\beta^\dagger} - \beta\right)\overline{\mathbf{D}}\bar{\eta}.
\end{aligned}$$

Here β and $\overline{\beta^\dagger}$ are both viewed as complex-linear bundle maps $\bar{F} \to E$, the latter in the obvious way, and the former acting as $\mathbb{1} \otimes \beta$ on $\bar{F} = \Lambda^{1,0}T^*\dot{\Sigma} \otimes \bar{E}$ with target $\Lambda^{1,0}T^*\dot{\Sigma} \otimes F = \Lambda^{1,0}T^*\dot{\Sigma} \otimes \Lambda^{0,1}T^*\dot{\Sigma} \otimes E = E$. Choosing unitary local trivializations, β and $\overline{\beta^\dagger}$ are represented by the same complex-valued function: indeed, the latter is the transpose of the former as m-by-m complex matrices, but since $m = 1$, this means they are identical.

Finally, we observe that the asymptotic convergence conditions satisfied by $\mathbf{D}$ on the cylindrical ends imply similar conditions for all other Cauchy-Riemann and anti-Cauchy-Riemann operators in this picture, yielding an estimate of the form $\|\partial_H\beta\|_{C^0} \le c\|\beta\|_{C^1}$ globally on $\dot{\Sigma}$. □

REMARK 5.16. The above proof used the assumption $m = 1$ in order to conclude $\overline{\beta^\dagger} - \beta \equiv 0$. For higher rank bundles, this imposes a nontrivial condition that must be satisfied in order for the Weitzenböck formula to hold, cf. [**GW**].

REMARK 5.17. We can now pick out a geometric reason for the miraculous cancellation in the Weitzenböck formula: the perturbation B is described by a complex bundle map $\bar{E} \to F$, where $\bar{E}$ and F both have natural antiholomorphic bundle structures defined via the complex-linear parts of $\overline{\mathbf{D}}$ and $-\mathbf{D}^*$ respectively. A complex-linear perturbation $B : E \to F$ would not work because E is holomorphic rather than antiholomorphic: while $\overline{\mathbf{D}}$ can be fit into the same Leibniz rule with $-\mathbf{D}^*$, the same is not true of $\mathbf{D}$.

5.5. Large antilinear perturbations and energy concentration

We continue in the setting of Proposition 5.15 and set

$$\mathbf{D}_\sigma := \mathbf{D} + \sigma B : \Gamma(E) \to \Gamma(F)$$

for $\sigma > 0$. After a compact perturbation of $\mathbf{D}$, we can without loss of generality also impose the following assumptions on $\mathbf{D}$, $\beta \in \Gamma(\operatorname{Hom}_{\mathbb{C}}(\bar{E}, F))$ and the area form $d\,\mathrm{vol}$:

(i) All zeroes of β are nondegenerate.
(ii) Both $|\beta|$ and $1/|\beta|$ are bounded outside of a compact subset of $\dot{\Sigma}$.
(iii) Near each point $\zeta \in \dot{\Sigma}$ with $\beta(\zeta) = 0$, there exists a neighborhood $\mathcal{D}(\zeta) \subset \dot{\Sigma}$ of ζ, a holomorphic coordinate chart identifying $(\mathcal{D}(\zeta), j, \zeta)$ with the unit disk $(\mathbb{D}, i, 0)$, and a local trivialization of E over $\mathcal{D}(\zeta)$ that identifies $\mathbf{D}$ with $\bar{\partial} = \partial_s + i\partial_t : C^\infty(\mathbb{D}, \mathbb{C}) \to C^\infty(\mathbb{D}, \mathbb{C})$ and β with one of the functions

$$\beta(z) = z \quad \text{or} \quad \beta(z) = \bar{z},$$

the former if ζ is a positive zero and the latter if it is negative.
(iv) In the holomorphic coordinate on $\mathcal{D}(\zeta)$ described above, $d\,\mathrm{vol}$ is the standard Lebesgue measure.

As in the torus case discussed in §5.3, we will see that the Weitzenböck formula implies a concentration of energy near the zeroes of β for sections $\eta \in \ker \mathbf{D}_\sigma$ as $\sigma \to \infty$. To understand what really happens in this limit, we will use a rescaling trick. Denote the zero set of β by

$$Z(\beta) = Z^+(\beta) \cup Z^-(\beta) \subset \dot{\Sigma},$$

partitioned into the positive and negative zeroes. For any $\eta \in \Gamma(E)$, $\zeta \in Z^\pm(\beta)$ and $\sigma > 0$, we then define a rescaled function

$$\eta^{(\zeta,\sigma)} : \mathbb{D}_{\sqrt{\sigma}} \to \mathbb{C} : z \mapsto \frac{1}{\sqrt{\sigma}} \eta(z/\sqrt{\sigma}),$$

where the right hand side denotes the local representation of η on $\mathcal{D}(\zeta)$ in the chosen coordinate and trivialization. Notice that the equation $\mathbf{D}_\sigma \eta = 0$ appears in this local representation as either $\bar{\partial}\eta + \sigma z \bar{\eta} = 0$ or $\bar{\partial}\eta + \sigma \bar{z}\bar{\eta} = 0$ depending on the sign of ζ, and the function $f := \eta^{(\zeta,\sigma)}$ then satisfies

$$\bar{\partial} f + z\bar{f} = 0 \quad \text{or} \quad \bar{\partial} f + \bar{z}\bar{f} = 0 \qquad \text{on } \mathbb{D}_{\sqrt{\sigma}}.$$

We will take a closer look at these two PDEs in §5.6 below. But first, observe that by change of variables,

$$\left\| \eta^{(\zeta,\sigma)} \right\|_{L^2(\mathbb{D}_{\sqrt{\sigma}})} = \|\eta\|_{L^2(\mathcal{D}(\zeta))}.$$

LEMMA 5.18. *Assume $\sigma_\nu \to \infty$, and $\eta_\nu \in \ker \mathbf{D}_{\sigma_\nu}$ is a sequence satisfying a uniform L^2-bound. Then after passing to a subsequence, the rescaled functions $\eta_\nu^\zeta := \eta_\nu^{(\zeta,\sigma_\nu)} : \mathbb{D}_{\sqrt{\sigma_\nu}} \to \mathbb{C}$ for each $\zeta \in Z^\pm(\beta)$ converge in $C^\infty_{\mathrm{loc}}(\mathbb{C})$ to smooth functions*

$\eta^\zeta_\infty \in L^2(\mathbb{C})$ satisfying

$$\bar\partial \eta^\zeta_\infty + z\overline{\eta^\zeta_\infty} = 0 \qquad \text{if } \zeta \in Z^+(\beta),$$
$$\bar\partial \eta^\zeta_\infty + \bar z\overline{\eta^\zeta_\infty} = 0 \qquad \text{if } \zeta \in Z^-(\beta).$$

Moreover, if $\xi_\nu \in \ker \mathbf{D}_{\sigma_\nu}$ is another sequence with these same properties and convergence $\xi^\zeta_\nu \to \xi^\zeta_\infty$, then

$$\lim_{\nu\to\infty} \langle \eta_\nu, \xi_\nu \rangle_{L^2(E)} = \sum_{\zeta \in Z(\beta)} \langle \eta^\zeta_\infty, \xi^\zeta_\infty \rangle_{L^2(\mathbb{C})}.$$

Proof. The uniform L^2-bound implies uniform bounds on $\|\eta^\zeta_\nu\|_{L^2(\mathbb{D}_R)}$ for every $R > 0$, where ν here is assumed sufficiently large so that $R < \sqrt{\sigma_\nu}$. Since η^ζ_ν satisfies a Cauchy-Riemann type equation on $\mathbb{D}_R$, the usual elliptic estimates (see Lecture 2) then imply uniform H^k-bounds for every $k \in \mathbb{N}$ on every compact subset in the interior of $\mathbb{D}_R$, hence η^ζ_ν has a C^∞_{loc}-convergent subsequence on $\mathbb{C}$, and the limit η^ζ_∞ clearly satisfies the stated PDE. The uniform L^2-bound also implies a uniform bound on $\|\eta^\zeta_\nu\|_{L^2(\mathbb{D}_{\sqrt{\sigma_\nu}})}$ and thus an R-independent uniform bound on $\|\eta^\zeta_\nu\|_{L^2(\mathbb{D}_R)}$ as $\nu \to \infty$, implying that η^ζ_∞ is in $L^2(\mathbb{C})$.

The limit of $\langle \eta_\nu, \xi_\nu \rangle_{L^2(E)}$ is now proved using the Weitzenböck formula. Let

$$\dot\Sigma_\epsilon := \dot\Sigma \setminus \bigcup_{\zeta \in Z(\beta)} \mathcal{D}(\zeta),$$

so there exists a constant $c > 0$ such that β satisfies $|\beta(z)\bar v| \ge c|v|$ for all $v \in E_z$, $z \in \dot\Sigma_\epsilon$. (Note that this depends on the assumption of $1/|\beta|$ being bounded outside of a compact subset.) Now by Proposition 5.15,

$$\begin{aligned}
0 = \|\mathbf{D}_{\sigma_\nu}\eta_\nu\|^2_{L^2(\dot\Sigma)} &= \langle \eta_\nu, \mathbf{D}^*_{\sigma_\nu}\mathbf{D}_{\sigma_\nu}\eta_\nu \rangle_{L^2(\dot\Sigma)} \\
&= \langle \eta_\nu, \mathbf{D}^*\mathbf{D}\eta_\nu \rangle_{L^2(\dot\Sigma)} + \sigma_\nu^2 \langle \eta_\nu, B^*B\eta_\nu \rangle_{L^2(\dot\Sigma)} - \sigma_\nu \langle \eta_\nu, (\partial_H \beta)\bar\eta_\nu \rangle_{L^2(\dot\Sigma)} \\
&\ge \|\mathbf{D}\eta_\nu\|^2_{L^2(\dot\Sigma)} + \sigma_\nu^2 c^2 \|\eta_\nu\|^2_{L^2(\dot\Sigma_\epsilon)} - \sigma_\nu c' \|\eta_\nu\|^2_{L^2(\dot\Sigma)} \\
&\ge \sigma_\nu^2 c^2 \|\eta_\nu\|^2_{L^2(\dot\Sigma_\epsilon)} - \sigma_\nu c' \|\eta_\nu\|^2_{L^2(\dot\Sigma)}
\end{aligned}$$

for some constant $c' > 0$ independent of ν. This implies

$$\|\eta_\nu\|^2_{L^2(\dot\Sigma_\epsilon)} \le \frac{c'}{c^2\sigma_\nu} \|\eta_\nu\|^2_{L^2(\dot\Sigma)} \to 0 \quad \text{as} \quad \nu \to \infty$$

since $\|\eta_\nu\|_{L^2(\dot\Sigma)}$ is uniformly bounded. The same estimate applies to ξ_ν, so that $\langle \eta_\nu, \xi_\nu \rangle_{L^2(\dot\Sigma_\epsilon)} \to 0$ and thus by change of variables,

$$\begin{aligned}
\lim_{\nu\to\infty} \langle \eta_\nu, \xi_\nu \rangle_{L^2(\dot\Sigma)} &= \lim_{\nu\to\infty} \sum_{\zeta \in Z(\beta)} \langle \eta_\nu, \xi_\nu \rangle_{L^2(\mathcal{D}(\zeta))} = \lim_{\nu\to\infty} \sum_{\zeta \in Z(\beta)} \langle \eta^\zeta_\nu, \xi^\zeta_\nu \rangle_{L^2(\mathbb{D}_{\sqrt{\sigma_\nu}})} \\
&= \sum_{\zeta \in Z(\beta)} \langle \eta^\zeta_\infty, \xi^\zeta_\infty \rangle_{L^2(\mathbb{C})}.
\end{aligned}$$

$\square$

5.6. Two Cauchy-Riemann type problems on the plane

The rescaling trick in the previous section produced smooth solutions $f : \mathbb{C} \to \mathbb{C}$ of class $L^2(\mathbb{C})$ to the two equations

$$\bar{\partial} f + z\bar{f} = 0, \qquad \bar{\partial} f + \bar{z}\bar{f} = 0.$$

It turns out that we can say precisely what all such solutions are. Write $\mathbf{D}_+ f := \bar{\partial} f + z\bar{f}$ and $\mathbf{D}_- f := \bar{\partial} f + \bar{z}\bar{f}$. Both operators differ from $\bar{\partial}$ by antilinear perturbations, so they satisfy Weitzenböck formulas relating $\mathbf{D}_\pm^* \mathbf{D}_\pm$ to the Laplacian $-\Delta = \bar{\partial}^* \bar{\partial} = -\partial_s^2 - \partial_t^2$. Indeed, repeating Proposition 5.15 in these special cases gives

$$\mathbf{D}_+^* \mathbf{D}_+ f = -\Delta f + |z|^2 f - 2\bar{f} \qquad \text{and} \qquad \mathbf{D}_-^* \mathbf{D}_- f = -\Delta f + |z|^2 f.$$

To make use of this, recall that a smooth function $u : \mathcal{U} \to \mathbb{R}$ on an open subset $\mathcal{U} \subset \mathbb{C}$ is called **subharmonic** if it satisfies

$$-\Delta u \le 0.$$

Subharmonic functions satisfy a **mean value property**:

$$-\Delta u \le 0 \text{ on } \mathcal{U} \qquad \Rightarrow \qquad u(z_0) \le \frac{1}{\pi r^2} \int_{\mathbb{D}_r(z_0)} u(z)\, d\mu(z) \quad \text{for all} \quad \mathbb{D}_r(z_0) \subset \mathcal{U},$$

where $\mathbb{D}_r(z_0) \subset \mathbb{C}$ denotes the disk of radius $r > 0$ about a point $z_0 \in \mathcal{U}$, and $d\mu(z)$ is the Lebesgue measure on $\mathbb{C}$; see e.g. [**Eva98**, p. 85].

EXERCISE 5.19. Show that for any smooth complex-valued function f on an open subset of $\mathbb{C}$,

$$\Delta |f|^2 = 2 \operatorname{Re}\langle f, \Delta f \rangle + 2|\nabla f|^2,$$

where $\langle\ ,\ \rangle$ denotes the standard Hermitian inner product on $\mathbb{C}$ and $|\nabla f|^2 := |\partial_s f|^2 + |\partial_t f|^2$.

PROPOSITION 5.20. *The equation $\bar{\partial} f + \bar{z}\bar{f} = 0$ does not admit any nontrivial smooth solutions $f \in L^2(\mathbb{C}, \mathbb{C})$.*

PROOF. If $f : \mathbb{C} \to \mathbb{C}$ is smooth with $\mathbf{D}_- f = 0$, then the Weitzenböck formula for $\mathbf{D}_-$ implies $\Delta f = |z|^2 f$. Then by Exercise 5.19,

$$\Delta |f|^2 = 2 \operatorname{Re}\langle f, |z|^2 f \rangle + 2|\nabla f|^2 = 2|z|^2 |f|^2 + 2|\nabla f|^2,$$

implying that $|f|^2 : \mathbb{C} \to \mathbb{R}$ is subharmonic. Now if $f(z_0) \ne 0$ for some $z_0 \in \mathbb{C}$, the mean value property implies

$$\int_{\mathbb{D}_r(z_0)} |f(z)|^2\, d\mu(z) \ge \pi r^2 |f(z_0)|^2 \to \infty \quad \text{as} \quad r \to \infty,$$

so $f \notin L^2(\mathbb{C})$. □

PROPOSITION 5.21. *Every smooth solution $f \in L^2(\mathbb{C}, \mathbb{C})$ to the equation $\bar{\partial} f + z\bar{f} = 0$ is a constant real multiple of $f_0(z) := e^{-\frac{1}{2}|z|^2}$.*

PROOF. We claim first that every smooth solution in $L^2(\mathbb{C}, \mathbb{C})$ of $\mathbf{D}_+ f = 0$ is purely real valued. The Weitzenböck formula for this case gives $\Delta f = |z|^2 f - 2\bar{f}$, and taking the difference between this equation and its complex conjugate then implies that $u := \operatorname{Im} f : \mathbb{C} \to \mathbb{R}$ satisfies

$$\Delta u = (|z|^2 + 2)u.$$

Now by Exercise 5.19,

$$\Delta(u^2) = 2|\nabla u|^2 + 2(|z|^2 + 2)u^2 \geq 0,$$

so $u^2 : \mathbb{C} \to \mathbb{R}$ is subharmonic, and the mean value property implies as in the proof of Prop. 5.20 that $u \notin L^2(\mathbb{C})$ and hence $f \notin L^2(\mathbb{C})$ unless $u \equiv 0$. This proves the claim.

It is easy to check however that f_0 is a solution and is in $L^2(\mathbb{C})$. Since it is also nowhere zero, every other solution f must then take the form $f(z) = v(z) f_0(z)$ for some *real-valued* function $v : \mathbb{C} \to \mathbb{R}$. Since $\mathbf{D}_+$ is a Cauchy-Riemann type operator, the Leibniz rule then implies $\bar{\partial} v \equiv 0$. But the only globally holomorphic functions with trivial imaginary parts are constant. □

5.7. A linear gluing argument

Now we're getting somewhere.

LEMMA 5.22. *Suppose the assumptions of §5.5 hold and $\beta \in \Gamma(\operatorname{Hom}_{\mathbb{C}}(\bar{E}, F))$ has $I_+ \geq 0$ positive and $I_- \geq 0$ negative zeroes. Then for all $\sigma > 0$ sufficiently large,*

$$\dim \ker \mathbf{D}_\sigma \leq I_+ \quad \text{and} \quad \dim \operatorname{coker} \mathbf{D}_\sigma \leq I_-.$$

In particular, for sufficiently large σ, $\mathbf{D}_\sigma$ is injective if all zeroes of β are negative and surjective if all zeroes are positive.

PROOF. Arguing by contradiction, suppose there exists a sequence $\sigma_\nu \to \infty$ such that $\dim \ker \mathbf{D}_{\sigma_\nu} > I_+$, and pick $(I_+ + 1)$ sequences of sections $\eta_\nu^1, \ldots, \eta_\nu^{I_+ + 1} \in \ker \mathbf{D}_{\sigma_\nu}$ which form L^2-orthonormal sets for each ν. By Lemma 5.18, we can then extract a subsequence such that rescaling near the zeroes of β produces C^∞_{loc}-convergent sequences whose limits form an $(I_+ + 1)$-dimensional orthonormal set in

$$\bigoplus_{\zeta \in Z(\beta)} L^2(\mathbb{C}, \mathbb{C}),$$

where the component functions $f \in L^2(\mathbb{C}, \mathbb{C})$ for $\zeta \in Z^+(\zeta)$ satisfy $\bar{\partial} f + z\bar{f} = 0$, while those for $\zeta \in Z^-(\zeta)$ satisfy $\bar{\partial} f + \bar{z}\bar{f} = 0$. Proposition 5.20 now implies that the component functions for $\zeta \in Z^-(\zeta)$ are all trivial, and by Proposition 5.21, the components for $\zeta \in Z^+(\zeta)$ belong to 1-dimensional subspaces $\ker \mathbf{D}_+ \subset L^2(\mathbb{C})$ generated by the function $e^{-\frac{1}{2}|z|^2}$. We conclude that the limiting orthonormal set lives in a precisely I_+-dimensional subspace

$$\bigoplus_{\zeta \in Z^+(\beta)} \ker \mathbf{D}_+ \subset \bigoplus_{\zeta \in Z(\beta)} L^2(\mathbb{C}, \mathbb{C}),$$

and this is a contradiction since there are $I_+ + 1$ elements in the set.

Applying the same argument to the formal adjoint implies similarly $\dim \ker \mathbf{D}_\sigma^* \leq I_-$ for σ sufficiently large. $\square$

We would next like to turn the two inequalities in the above lemma into equalities, which means showing that the I_+-dimensional subspace of $\bigoplus_{\zeta \in Z^+(\beta)} L^2(\mathbb{C}, \mathbb{C})$ generated by solutions of $\bar{\partial} f + z\bar{f} = 0$ is isomorphic to $\ker \mathbf{D}_\sigma$ for σ sufficiently large. This requires a simple example of a *linear gluing* argument, the point of which is to reverse the "convergence after rescaling" process that we saw in Lemma 5.18. The first step is a **pregluing** construction which turns elements of $\bigoplus_{\zeta \in Z^+(\beta)} \ker \mathbf{D}_+$ into *approximate* solutions to $\mathbf{D}_\sigma \eta = 0$ for large σ. To this end, fix a smooth bump function

$$\rho \in C_0^\infty(\mathring{\mathbb{D}}, [0,1]), \qquad \rho|_{\mathbb{D}_{1/2}} \equiv 1$$

and define for each $\zeta \in Z^+(\beta)$ and $\sigma > 0$ a linear map

$$\Phi_\sigma^\zeta : \ker \mathbf{D}_+ \to \Gamma(E)$$

such that $\Phi_\sigma^\zeta(f)$ is a section with support in $\mathcal{D}(\zeta)$ whose expression in our fixed coordinate and trivialization on that neighborhood is the function

$$f_\sigma^\zeta(z) = \rho(z)\sqrt{\sigma} f(\sqrt{\sigma} z).$$

Adding up the Φ_σ^ζ for all $\zeta \in Z^+(\beta)$ then produces a linear map

$$\Phi_\sigma : \bigoplus_{\zeta \in Z^+(\beta)} \ker \mathbf{D}_+ \to \Gamma(E)$$

whose image consists of sections supported near $Z^+(\beta)$, each a linear combination of cut-off Gaussians with energy concentrated in smaller neighborhoods of $Z^+(\beta)$ for larger σ. These sections are manifestly not in $\ker \mathbf{D}_\sigma$ since they vanish on open subsets and thus violate unique continuation, but they are close, in a quantitative sense:

LEMMA 5.23. *For each $\sigma > 0$, there exists a constant $c_\sigma > 0$ such that*

$$\|\mathbf{D}_\sigma \Phi_\sigma(f)\|_{L^2} \leq c_\sigma \|f\|_{L^2} \quad \text{for all} \quad f \in \bigoplus_{\zeta \in Z^+(\beta)} \ker \mathbf{D}_+,$$

and $c_\sigma \to 0$ as $\sigma \to \infty$. Moreover, for every pair $f, g \in \bigoplus_{\zeta \in Z^+(\beta)} \ker \mathbf{D}_+$,

$$\langle \Phi_\sigma(f), \Phi_\sigma(g) \rangle_{L^2} \to \langle f, g \rangle_{L^2}$$

as $\sigma \to \infty$.

PROOF. First, observe that any $f \in \bigoplus_{\zeta \in Z^+(\beta)} \ker \mathbf{D}_+$ is described by a collection of functions $\{f_\zeta \in L^2(\mathbb{C})\}_{\zeta \in \beta^+(Z)}$ which take the form

$$f_\zeta(z) = K_\zeta e^{-\frac{1}{2}|z|^2},$$

for some constants $K_\zeta \in \mathbb{R}$. Since each f_ζ is in $\ker \mathbf{D}_+$, we find

$$\begin{aligned} \mathbf{D}_\sigma \left(\Phi_\sigma(f)|_{\mathcal{D}(\zeta)}\right)(z) &= \overline{\partial}\rho(z)\sqrt{\sigma} f_\zeta(\sqrt{\sigma}z) + \rho(z)\sigma\overline{\partial} f_\zeta(\sqrt{\sigma}z) \\ &\quad + \sigma z \rho(z)\sqrt{\sigma} f_\zeta(\sqrt{\sigma}z) \\ &= \overline{\partial}\rho(z)\sqrt{\sigma} f_\zeta(\sqrt{\sigma}z) + \rho(z)\sigma(\mathbf{D}_+ f_\zeta)(\sqrt{\sigma}z) \\ &= \overline{\partial}\rho(z)\sqrt{\sigma} K_\zeta e^{-\frac{1}{2}\sigma|z|^2}. \end{aligned} \tag{5.3}$$

Now since $\overline{\partial}\rho = 0$ in $\mathbb{D}_{1/2}$, we obtain

$$\begin{aligned} \|\mathbf{D}_\sigma \Phi_\sigma(f)\|_{L^2}^2 &= \sum_{\zeta \in Z^+(\beta)} \int_{\mathcal{D}(\zeta)} |\mathbf{D}_\sigma \Phi_\sigma(f)(z)|^2 \, d\mu(z) \\ &= \sum_{\zeta \in Z^+(\beta)} \int_{\mathbb{D}\setminus\mathbb{D}_{1/2}} |\overline{\partial}\rho(z)|^2 \sigma K_\zeta^2 e^{-\sigma|z|^2} \, d\mu(z) \\ &\le I\sigma e^{-\sigma/4} \sum_{\zeta \in Z^+(\beta)} K_\zeta^2, \end{aligned}$$

where we abbreviate $I := \int_{\mathbb{D}\setminus\mathbb{D}_{1/2}} \left|\bar{\partial}\rho(z)\right|^2 \, d\mu(z)$. The norm of f is given by

$$\|f\|_{L^2}^2 = \sum_{\zeta \in Z^+(\beta)} \int_{\mathbb{C}} K_\zeta^2 e^{-|z|^2} \, d\mu(z) = \left(\int_{\mathbb{C}} e^{-|z|^2} \, d\mu(z)\right) \sum_{\zeta \in Z^+(\beta)} K_\zeta^2.$$

We conclude that there is a bound of the form

$$\|\mathbf{D}_\sigma \Phi_\sigma(f)\|_{L^2} \le C\sqrt{\sigma} e^{-\sigma/2} \|f\|_{L^2},$$

which proves the first statement since $\sqrt{\sigma}e^{-\sigma/2} \to 0$ as $\sigma \to \infty$.

The second statement follows by a change of variable, since

$$\begin{aligned} \langle \Phi_\sigma(f), \Phi_\sigma(g)\rangle_{L^2} &= \sum_{\zeta \in Z^+(\beta)} \langle \Phi_\sigma(f)|_{\mathcal{D}(\zeta)}, \Phi_\sigma(g)|_{\mathcal{D}(\zeta)}\rangle_{L^2(\mathcal{D}(\zeta))} \\ &= \sum_{\zeta \in Z^+(\beta)} \int_{\mathbb{D}} \rho^2(z)\sigma f_\zeta(\sqrt{\sigma}z) g_\zeta(\sqrt{\sigma}z) \, d\mu(z) \\ &= \sum_{\zeta \in Z^+(\beta)} \int_{\mathbb{D}_{\sqrt{\sigma}}} \rho^2\left(\frac{z}{\sqrt{\sigma}}\right) f_\zeta(z) g_\zeta(z) \, d\mu(z) \end{aligned}$$

The functions f_ζ and g_ζ are both real multiples of $e^{-\frac{1}{2}|z|^2}$, so this last integral for each $\zeta \in Z^+(\beta)$ is bounded between $\int_{\mathbb{D}_{\sqrt{\sigma}/2}} f_\zeta(z) g_\zeta(z) \, d\mu(z)$ and $\int_{\mathbb{D}_{\sqrt{\sigma}}} f_\zeta(z) g_\zeta(z) \, d\mu(z)$, both of which converge to $\int_{\mathbb{C}} f_\zeta(z) g_\zeta(z) \, d\mu(z)$ as $\sigma \to \infty$, thus

$$\lim_{\sigma\to\infty} \langle \Phi_\sigma(f), \Phi_\sigma(g)\rangle_{L^2} = \langle f, g\rangle_{L^2}.$$

□

To turn approximate solutions into actual solutions, let

$$\Pi_\sigma : L^2(E) \to \ker \mathbf{D}_\sigma$$

denote the orthogonal projection. We will prove:

PROPOSITION 5.24. *If all zeroes of β are positive, then the linear map*

$$\Pi_\sigma \circ \Phi_\sigma : \bigoplus_{\zeta \in Z^+(\beta)} \ker \mathbf{D}_+ \to \ker \mathbf{D}_\sigma$$

is injective for all $\sigma > 0$ sufficiently large.

This statement says in effect that whenever $\sigma > 0$ is large enough and $\eta := \Phi_\sigma(f) \in \Gamma(E)$ is in the image of the pregluing map, with f normalized by $\|f\|_{L^2} = 1$, we can find a "correction" $\xi \in (\ker \mathbf{D}_\sigma)^\perp$ such that

$$\eta + \xi \neq 0 \quad \text{but} \quad \mathbf{D}_\sigma(\eta + \xi) = 0.$$

An element $\xi \in (\ker \mathbf{D}_\sigma)^\perp$ with the second property certainly exists, and in fact it's unique: indeed, the assumption $Z^-(\beta) = \emptyset$ implies via Lemma 5.22 that $\mathbf{D}_\sigma$ is surjective and thus restricts to an isomorphism from $(\ker \mathbf{D})^\perp \cap H^1(E)$ to $L^2(F)$, with a bounded right inverse

$$\mathbf{Q}_\sigma : L^2(F) \to H^1(E) \cap (\ker \mathbf{D})^\perp,$$

hence $\xi := -\mathbf{Q}_\sigma(\mathbf{D}_\sigma \eta)$. We know moreover from Lemma 5.23 that $\|\eta\|_{L^2}$ is close to $\|f\|_{L^2} = 1$, so to prove $\eta + \xi \neq 0$, it would suffice to show $\|\xi\|_{L^2}$ is small, which sounds likely since we also know $\|\mathbf{D}_\sigma \eta\|_{L^2}$ is small and $\mathbf{Q}_\sigma$ is a bounded operator. To make this reasoning precise, we just need to have some control over $\|\mathbf{Q}_\sigma\|$ as $\sigma \to \infty$, or equivalently, a quantitative measure of the injectivity of $\mathbf{D}_\sigma|_{(\ker \mathbf{D}_\sigma)^\perp \cap H^1(E)}$. This requires one last appeal to the Weitzenböck formula.

LEMMA 5.25. *Assume all zeroes of β are positive. Then there exist constants $c > 0$ and σ_0 such that for all $\sigma > \sigma_0$,*

$$\|\eta\|_{L^2} \le c \|\mathbf{D}_\sigma \eta\|_{L^2} \quad \textit{for all} \quad \eta \in H^1(E) \cap (\ker \mathbf{D}_\sigma)^\perp.$$

PROOF. Let us instead prove that if zeroes of β are all *negative*, then the same bound holds for all $\eta \in H^1(E)$. The stated result follows from this by considering the formal adjoint and using Exercise 5.26 below. Note that by density, it suffices to prove the estimate holds for all $\eta \in C_0^\infty(E)$.

Assume therefore that $Z^+(\beta) = \emptyset$ and, arguing by contradiction, suppose there exist sequences $\sigma_\nu \to \infty$ and $\eta_\nu \in C_0^\infty(E)$ with $\|\eta_\nu\|_{L^2} = 1$ and

$$\|\mathbf{D}_{\sigma_\nu} \eta_\nu\|_{L^2} \to 0.$$

The usual rescaling trick and application of the Weitzenböck formula then produces for each $\zeta \in Z^-(\beta)$ a sequence of functions $\eta_\nu^\zeta := \eta_\nu^{(\zeta,\sigma_\nu)} : \mathbb{D}_{\sqrt{\sigma_\nu}} \to \mathbb{C}$ which satisfy

$$\sum_{\zeta \in Z^-(\beta)} \|\eta_\nu^\zeta\|^2_{L^2(\mathbb{D}_{\sqrt{\sigma_\nu}})} \to 1 \quad \text{and} \quad \|\mathbf{D}_- \eta_\nu^\zeta\|_{L^2(\mathbb{D}_{\sqrt{\sigma_\nu}})} \to 0$$

as $\nu \to \infty$. Indeed, defining $\dot{\Sigma}_\epsilon$ as in the proof of Lemma 5.18, a similar application of the Weitzenböck formula yields

$$\|\mathbf{D}_{\sigma_\nu} \eta_\nu\|^2_{L^2(\dot{\Sigma})} \ge \sigma_\nu^2 c^2 \|\eta_\nu\|^2_{L^2(\dot{\Sigma}_\epsilon)} - \sigma_\nu c' \|\eta_\nu\|^2_{L^2(\dot{\Sigma})} = \sigma_\nu^2 c^2 \|\eta_\nu\|^2_{L^2(\dot{\Sigma}_\epsilon)} - \sigma_\nu c',$$

for some $c' > 0$. Thus we obtain

$$\|\eta_\nu\|^2_{L^2(\dot\Sigma_\epsilon)} \le \frac{\|\mathbf{D}_{\sigma_\nu}\eta_\nu\|^2_{L^2(\dot\Sigma)}}{c^2\sigma_\nu^2} + \frac{c'}{\sigma_\nu c^2} \to 0 \quad \text{as} \quad \nu \to \infty,$$

so there is again concentration of energy near the zeroes of the antilinear perturbation: in particular,

$$\begin{aligned} 1 &= \lim_{\nu\to\infty} \|\eta_\nu\|^2_{L^2(\dot\Sigma)} \\ &= \lim_{\nu\to\infty} \|\eta_\nu\|^2_{L^2(\dot\Sigma_\epsilon)} + \lim_{\nu\to\infty} \sum_{\zeta\in Z^-(\beta)} \|\eta_\nu\|^2_{L^2(\mathcal{D}(\zeta))} \\ &= \lim_{\nu\to\infty} \sum_{\zeta\in Z^-(\beta)} \|\eta_\nu^\zeta\|^2_{L^2(\mathbb{D}_{\sqrt{\sigma_\nu}})}. \end{aligned}$$

Moreover, we have

$$\mathbf{D}_-\eta_\nu^\zeta(z) = \frac{1}{\sigma_\nu}\bar\partial\eta_\nu\left(\frac{z}{\sqrt{\sigma_\nu}}\right) + \frac{\bar z}{\sqrt{\sigma_\nu}}\bar\eta_\nu\left(\frac{z}{\sqrt{\sigma_\nu}}\right) = \frac{1}{\sigma_\nu}\mathbf{D}_{\sigma_\nu}\eta_\nu\left(\frac{z}{\sqrt{\sigma_\nu}}\right).$$

Taking the square of the norms on each side, we may integrate and use change of variables to obtain

$$\|\mathbf{D}_-\eta_\nu^\zeta\|_{L^2(\mathbb{D}_{\sqrt{\sigma_\nu}})} = \frac{1}{\sqrt{\sigma_\nu}}\|\mathbf{D}_{\sigma_\nu}\eta_\nu\|_{L^2(\mathcal{D}(\zeta))} \to 0 \quad \text{as} \quad \nu \to \infty.$$

The elliptic estimates from Lecture 2 now provide uniform H^k-bounds for each η_ν^ζ on compact subsets of $\mathbb{C}$ for every $k \in \mathbb{N}$, so that a subsequence converges in $C^\infty_{\text{loc}}(\mathbb{C})$ to a smooth map $\eta_\infty^\zeta \in L^2(\mathbb{C}, \mathbb{C})$ satisfying $\mathbf{D}_-\eta_\infty^\zeta = 0$. But $\sum_{\zeta\in Z^-(\beta)} \|\eta_\infty^\zeta\|^2_{L^2(\mathbb{C})} = 1$, so at least one of these solutions is nontrivial and thus contradicts Proposition 5.20. □

EXERCISE 5.26. Show that for any Fredholm Cauchy-Riemann type operator $\mathbf{D}$ on E, the following two estimates are equivalent, with the same constant $c > 0$ in both:

(i) $\|\eta\|_{L^2(E)} \le c\|\mathbf{D}\eta\|_{L^2(F)}$ for all $\eta \in H^1(E) \cap (\ker \mathbf{D})^\perp$;
(ii) $\|\lambda\|_{L^2(F)} \le c\|\mathbf{D}^*\lambda\|_{L^2(E)}$ for all $\lambda \in H^1(F) \cap (\ker \mathbf{D}^*)^\perp$.

Hint: elliptic regularity implies that for $\mathbf{D}$ and $\mathbf{D}^*$ as bounded linear operators $H^1 \to L^2$, $(\ker \mathbf{D})^\perp = \operatorname{im} \mathbf{D}^*$ and $(\ker \mathbf{D}^*)^\perp = \operatorname{im} \mathbf{D}$.

PROOF OF PROPOSITION 5.24. If the statement is not true, then there exist sequences $\sigma_\nu \to \infty$ and

$$f_\nu \in \bigoplus_{\zeta\in Z^+(\beta)} \ker \mathbf{D}_+$$

such that $\|f_\nu\|_{L^2} = 1$ and $\eta_\nu := \Phi_{\sigma_\nu}(f_\nu) \in (\ker \mathbf{D}_{\sigma_\nu})^\perp$ for all ν. Lemmas 5.23 and 5.25 then provide estimates of the form

- $\|\eta_\nu\|_{L^2} \to 1$,
- $\|\mathbf{D}_{\sigma_\nu}\eta_\nu\|_{L^2} \to 0$, and
- $\|\eta_\nu\|_{L^2} \le c\|\mathbf{D}_{\sigma_\nu}\eta_\nu\|_{L^2}$

as $\nu \to \infty$, with $c > 0$ independent of ν. These imply:

$$1 = \lim_{\nu\to\infty} \|\eta_\nu\|_{L^2} \le \lim_{\nu\to\infty} c\|\mathbf{D}_{\sigma_\nu}\eta_\nu\|_{L^2} = 0.$$

□

We've proved:

PROPOSITION 5.27. *Suppose the assumptions of §5.5 hold and that the section $\beta \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(\bar{E}, F))$ has $I_+ \ge 0$ positive and $I_- \ge 0$ negative zeroes. If $I_- = 0$, then $\mathbf{D}_\sigma$ is surjective with* $\dim\ker \mathbf{D}_\sigma = I_+$ *for all $\sigma > 0$ sufficiently large. If $I_+ = 0$, then $\mathbf{D}_\sigma$ is injective with* $\dim\operatorname{coker}\mathbf{D}_\sigma = I_-$ *for all $\sigma > 0$ sufficiently large. In either case,*

$$\mathrm{ind}(\mathbf{D}_\sigma) = I_+ - I_-$$

for all $\sigma > 0$ sufficiently large. □

5.8. Antilinear deformations of asymptotic operators

Proposition 5.27 suffices to prove the index formula in the closed case, but there is an additional snag if $\Gamma \neq \emptyset$: since $H^1(\dot{\Sigma}) \hookrightarrow L^2(\dot{\Sigma})$ is not a compact inclusion, we have no guarantee that $\mathbf{D}$ and $\mathbf{D}_\sigma := \mathbf{D} + \sigma B$ will have the same index, and generally they will not. A solution to this problem has been pointed out by Chris Gerig [**Ger**], using a special class of asymptotic operators that also originate in the work of Taubes (see [**Tau10**, Lemma 2.3]).

In general, the only obvious way to guarantee $\mathrm{ind}(\mathbf{D}) = \mathrm{ind}(\mathbf{D}_\sigma)$ for large $\sigma > 0$ is if we can arrange for *every* operator in the family $\{\mathbf{D}_\sigma\}_{\sigma\ge 0}$ to be Fredholm, which is not automatic since the zeroth-order perturbation $B : E \to F$ is required to be bounded away from zero near ∞ and must therefore change the asymptotic operators at the punctures. We are therefore led to ask:

QUESTION. *For what nondegenerate asymptotic operators $\mathbf{A} : H^1(E) \to L^2(E)$ on a Hermitian line bundle $(E, J, \omega) \to S^1$ can one find complex-antilinear bundle maps $B : E \to E$ such that*

$$\mathbf{A}_\sigma := \mathbf{A} - \sigma B : H^1(E) \to L^2(E)$$

is an isomorphism for every $\sigma \ge 0$?

It turns out that it will suffice to find, for each unitary trivialization σ and every $k \in \mathbb{Z}$, a particular pair $(\mathbf{A}_k, B_k)$ such that $\mathbf{A}_k - \sigma B_k$ is nondegenerate for all $\sigma \ge 0$ and $\mu^\tau_{\mathrm{CZ}}(\mathbf{A}_k) = k$. To see why, let us proceed under the assumption that such pairs can be found, and use them to compute the index:

LEMMA 5.28. *Given $\mathbf{D}$ as in Theorem 5.4, fix asymptotic trivializations τ and suppose that for each puncture $z \in \Gamma$ there exists an asymptotic operator $\mathbf{A}'_z$ on (E_z, J_z, ω_z) with $\mu^\tau_{CZ}(\mathbf{A}'_z) = \mu^\tau_{CZ}(\mathbf{A}_z)$, such that if $\mathbf{A}'_z$ is written with respect to τ as $-J_0\partial_t - S_z(t)$, then the deformed asymptotic operator*

$$C^\infty(S^1, \mathbb{R}^2) \to C^\infty(S^1, \mathbb{R}^2) : \eta \mapsto -J_0\partial_t\eta - S_z(t)\eta - \sigma\beta_z(t)\bar{\eta} \tag{5.4}$$

is nondegenerate for some loop $\beta_z : S^1 \to \mathbb{C} \setminus \{0\}$ and every $\sigma \geq 0$. Then

$$\operatorname{ind}(\mathbf{D}) = \chi(\dot{\Sigma}) + 2c_1^\tau(E) + \sum_{z \in \Gamma^+} \operatorname{wind}(\beta_z) - \sum_{z \in \Gamma^-} \operatorname{wind}(\beta_z).$$

PROOF. Since $\mu_{\mathrm{CZ}}^\tau(\mathbf{A}_z) = \mu_{\mathrm{CZ}}^\tau(\mathbf{A}'_z)$, we can deform $\mathbf{A}_z$ to $\mathbf{A}'_z$ continuously through a family of nondegenerate asymptotic operators. It follows that we can deform $\mathbf{D}$ through a continuous family of Fredholm Cauchy-Riemann type operators to a new operator $\mathbf{D}'$ whose asymptotic operators are $\mathbf{A}'_z$ for $z \in \Gamma$, and $\operatorname{ind}(\mathbf{D}') = \operatorname{ind}(\mathbf{D})$. We are free to assume in fact that $\mathbf{D}'$ is written with respect to the trivialization τ on the cylindrical end near $z \in \Gamma^\pm$ as

$$\partial_s + J_0 \partial_t + S_z(t).$$

Now choose $\beta \in \Gamma(\operatorname{Hom}_{\mathbb{C}}(\bar{E}, F))$ with nondegenerate zeroes such that the deformed operators $\mathbf{D}_\sigma \eta := \mathbf{D}'\eta + \sigma\beta\bar{\eta}$ appear in trivialized form on the cylindrical end near $z \in \Gamma^\pm$ as

$$\mathbf{D}_\sigma \eta = \partial_s \eta + J_0 \partial_t \eta + S_z(t)\eta + \sigma\beta_z(t)\bar{\eta}.$$

This means $\mathbf{D}_\sigma$ is asymptotic at z to (5.4), which is nondegenerate for every $\sigma \geq 0$, implying $\mathbf{D}_\sigma$ is Fredholm for every $\sigma \geq 0$ and thus

$$\operatorname{ind}(\mathbf{D}) = \operatorname{ind}(\mathbf{D}_\sigma).$$

The trivializations τ induce trivializations over the cylindrical ends for $\bar{E}$ and $F = \Lambda^{0,1}T^*\dot{\Sigma} \otimes E$, and the expression for β in the resulting asymptotic trivialization of $\operatorname{Hom}_{\mathbb{C}}(\bar{E}, F)$ near $z \in \Gamma$ is $\beta_z(t)$. It follows that the signed count of zeroes of β is

$$\begin{aligned} i(\mathbf{D}) &:= c_1^\tau(\operatorname{Hom}_{\mathbb{C}}(\bar{E}, F)) + \sum_{z \in \Gamma^+} \operatorname{wind}(\beta_z) - \sum_{z \in \Gamma^-} \operatorname{wind}(\beta_z) \\ &= \chi(\dot{\Sigma}) + 2c_1^\tau(E) + \sum_{z \in \Gamma^+} \operatorname{wind}(\beta_z) - \sum_{z \in \Gamma^-} \operatorname{wind}(\beta_z), \end{aligned}$$

where the computation $c_1^\tau(\operatorname{Hom}_{\mathbb{C}}(\bar{E}, F)) = \chi(\dot{\Sigma}) + 2c_1^\tau(E)$ follows from the natural isomorphism

$$\begin{aligned} \operatorname{Hom}_{\mathbb{C}}(\bar{E}, F) &= \bar{E}^* \otimes F = E \otimes F = E \otimes \Lambda^{0,1}T^*\dot{\Sigma} \otimes E = \Lambda^{0,1}T^*\dot{\Sigma} \otimes E \otimes E \\ &= T\dot{\Sigma} \otimes E \otimes E. \end{aligned}$$

We are free to assume that all zeroes of β are either positive or negative, depending on the sign of $i(\mathbf{D})$. Proposition 5.27 then implies $\operatorname{ind}(\mathbf{D}_\sigma) = i(\mathbf{D})$ for large σ. ☐

Notice that instead of nondegenerate families $\mathbf{A} - \sigma B$ parametrized by $\sigma \in [0, \infty)$, it is just as well to find such families which are nondegenerate and have the right Conley-Zehnder index for all $\sigma > 0$, as the $\sigma \geq 1$ portion of this family can be rewritten as $(\mathbf{A} - B) - \sigma B$ for $\sigma \geq 0$. The following lemma thus completes the proof of Theorem 5.4.

LEMMA 5.29. *For every $k \in \mathbb{Z}$, the trivial Hermitian line bundle over S^1 admits an asymptotic operator $\mathbf{A}_k$ and a loop $\beta_k : S^1 \to \mathbb{C} \setminus \{0\}$ such that the deformed asymptotic operators*

$$\mathbf{A}_{k,\sigma}\eta := \mathbf{A}_k \eta - \sigma\beta_k\bar{\eta}$$

are nondegenerate for every $\sigma > 0$ and satisfy

$$\mu_{CZ}(\mathbf{A}_{k,\sigma}) = \text{wind}(\beta_k) = k.$$

PROOF. We claim that the choices

$$\mathbf{A}_k\eta := -J_0\partial_t\eta - \pi k\eta \quad \text{and} \quad \beta_k(t) := e^{2\pi ikt}$$

do the trick. We prove this in three steps.

Step 1: $k = 0$. The above formula gives $\mathbf{A}_{0,\sigma} = -J_0\partial_t\eta - \sigma\bar{\eta}$, in which the $\sigma = 1$ case is precisely the operator that we used in Lecture 3 to normalize the Conley-Zehnder index, hence $\mu_{\text{CZ}}(\mathbf{A}_{0,1}) = 0$ by definition. More generally, all of these operators can be expressed in the form $\mathbf{A} := -J_0\partial_t - S$ where $S \in \text{End}_{\mathbb{R}}(\mathbb{R}^2)$ is a constant nonsingular 2-by-2 symmetric matrix that anticommutes with J_0. We claim that *all* asymptotic operators of this form are nondegenerate. Indeed, the conditions $S^{\text{T}} = S$ and $SJ_0 = -J_0S$ for $J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ imply that S takes the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ with $\det S = -a^2 - b^2 \neq 0$, and moreover S is of this form if and only if J_0S also is. In particular, J_0S is traceless, symmetric, and nonsingular. Solutions of $\mathbf{A}\eta = 0$ then satisfy $\dot{\eta} = J_0S\eta$, which has no periodic solutions since J_0S has one positive and one negative eigenvalue, hence $\ker \mathbf{A} = \{0\}$.

Step 2: even k. There is a cheap trick to deduce the case $k = 2m$ for any $m \in \mathbb{N}$ from the $k = 0$ case. Recall that by Exercise 3.37 in Lecture 3, conjugating $\mathbf{A}_{0,\sigma}$ by a change of trivialization changes its Conley-Zehnder index by twice the degree of that change. In particular, the operator

$$\tilde{\mathbf{A}}_{0,\sigma}\eta := e^{2\pi imt}\mathbf{A}_{0,\sigma}(e^{-2\pi imt}\eta)$$

is also a nondegenerate asymptotic operator, but with $\mu_{\text{CZ}}(\tilde{\mathbf{A}}_{0,\sigma}) = \mu_{\text{CZ}}(\mathbf{A}_{0,\sigma}) + 2m = k$. Explicitly, we compute

$$\tilde{\mathbf{A}}_{0,\sigma}\eta = -J_0\partial_t\eta - \pi k\eta - \sigma ke^{2\pi ikt}\bar{\eta},$$

so $\mathbf{A}_{k,\sigma} = \tilde{\mathbf{A}}_{0,\sigma/k}$ is also nondegenerate for every $\sigma > 0$.

Step 3: odd k. Another cheap trick relates each $\mathbf{A}_{k,\sigma}$ to $\mathbf{A}_{2k,\sigma}$ after an adjustment in σ. Given an arbitrary asymptotic operator $\mathbf{A} = -J_0\partial_t - S(t)$ and $m \in \mathbb{N}$, define

$$\mathbf{A}^m := -J_0\partial_t - mS(mt).$$

Geometrically, if $\mathbf{A}$ is a trivialized representation for the asymptotic operator of a Reeb orbit $\gamma : S^1 \to M$, then $\mathbf{A}^m$ is the operator for the m-fold covered orbit $\gamma^m : S^1 \to M : t \mapsto \gamma(mt)$. It is easy to check in particular that if we define $\eta^m(t) := \eta(mt)$ for any given loop $\eta : S^1 \to \mathbb{R}^2$, then

$$\mathbf{A}^m\eta^m = m(\mathbf{A}\eta)^m,$$

so this gives an embedding of $\ker \mathbf{A}$ into $\ker \mathbf{A}^m$, implying that whenever $\mathbf{A}^m$ is nondegenerate for some $m \in \mathbb{N}$, so is $\mathbf{A}$. To make use of this, observe that

$$\mathbf{A}^2_{k,\sigma}\eta = -J_0\partial_t\eta - \pi 2k\eta - 2\sigma e^{4\pi ikt}\bar{\eta} = \mathbf{A}_{2k,2\sigma}\eta,$$

so $\mathbf{A}^2_{k,\sigma}$ is nondegenerate for all $\sigma > 0$ by Step 2, and therefore so is $\mathbf{A}_{k,\sigma}$. □

The proof of Theorem 5.4 is now complete.

EXERCISE 5.30. Derive a Weitzenböck formula for asymptotic operators and use it to show that for any asymptotic operator $\mathbf{A}$ on the trivial Hermitian line bundle and any smooth $\beta : S^1 \to \mathbb{C} \setminus \{0\}$, the deformed operators $\mathbf{A}_\sigma \eta := \mathbf{A}\eta - \sigma\beta\bar{\eta}$ are all nondegenerate for $\sigma > 0$ sufficiently large. Deduce from this that $\mu_{\mathrm{CZ}}(\mathbf{A}_\sigma) = \mathrm{wind}(\beta)$ for large $\sigma > 0$.

LECTURE 6

Symplectic cobordisms and moduli spaces

Contents

In this lecture we introduce the moduli spaces of holomorphic curves that are used to define SFT.

6.1. Stable Hamiltonian structures and their symplectizations

In Lecture 1, we motivated the notion of a contact manifold by considering hypersurfaces M in a symplectic manifold (W, ω) that satisfy a *convexity* (also known as "contact type") condition. The point of that condition was that it presents M as one member of a smooth 1-parameter family of hypersurfaces that all have the same Hamiltonian dynamics; that 1-parameter family furnishes the basic model of what we call the *symplectization* of M with its induced contact structure. A useful generalization of this notion was introduced in [HZ94] and was later recognized to be the most natural geometric setting for punctured holomorphic curves. It has the advantage of allowing us to view seemingly distinct theories such as Hamiltonian Floer homology as special cases of SFT—and even if we are only interested in contact manifolds, the generalization sometimes makes computations easier than they might be in a purely contact setting.

Recall that every smooth hypersurface M in a $2n$-dimensional symplectic manifold (W, ω) has a **characteristic line field**

$$\ker(\omega|_{TM}) \subset TM,$$

whose integral curves are the orbits on M of any Hamiltonian vector field generated by a function $H : W \to \mathbb{R}$ that has M as a regular level set. We say that $M \subset (W, \omega)$ is **stable** if a neighborhood of M admits a **stabilizing vector field** V: this means that V is transverse to M and the 1-parameter family of hypersurfaces

$$M_t := \varphi_V^t(M), \qquad -\epsilon < t < \epsilon$$

generated by the flow φ_V^t of V has the property that each of the diffeomorphisms $M \to M_t$ defined by flowing along V preserves characteristic line fields.

EXERCISE 6.1. Show that if V is a stabilizing vector field for $M \subset (W, \omega)$, then the 2-form and 1-form pair (Ω, Λ) defined on M by

$$\Omega := \omega|_{TM}, \qquad \Lambda := \iota_V \omega|_{TM}$$

has the following properties:

(i) $\Omega|_{\ker \Lambda}$ is nondegenerate;
(ii) $\ker \Omega \subset \ker d\Lambda$.

Show moreover that if M is assigned the orientation for which V is *positively* transverse to M and $\xi := \ker \Lambda \subset TM$ is assigned the natural co-orientation determined by Λ, then the induced orientation of ξ matches the orientation determined by the symplectic vector bundle structure $\Omega|_\xi$, hence condition (i) can equivalently be written as

(iii) $\Lambda \wedge \Omega^{n-1} > 0$

where $\dim W = 2n$.

A **stable Hamiltonian structure** (or "SHS" for short) on an arbitrary oriented $(2n-1)$-dimensional manifold M is a pair (Ω, Λ) consisting of a closed 2-form Ω and 1-form Λ such that properties (ii) and (iii) in Exercise 6.1 are satisfied.

EXERCISE 6.2. Show that if (Ω, Λ) is a stable Hamiltonian structure, then

$$\omega := d(r\Lambda) + \Omega$$

is a symplectic form on $(-\epsilon, \epsilon) \times M$ for $\epsilon > 0$ sufficiently small, where r denotes the coordinate on $(-\epsilon, \epsilon)$; moreover, $\{0\} \times M$ is a stable hypersurface in $((-\epsilon, \epsilon) \times M, \omega)$.

EXAMPLE 6.3. If $M \subset (W, \omega)$ is a contact type hypersurface, then a Liouville vector field V transverse to M is a stabilizing vector field, and the induced stable Hamiltonian structure is $(d\alpha, \alpha)$, where $\alpha := \lambda|_{TM}$ with $\lambda := \omega(V, \cdot)$. We will refer to this example henceforward as the **contact case**.

PROPOSITION 6.4. *Suppose $M \subset (W, \omega)$ is a closed stable hypersurface with stabilizing vector field V and induced stable Hamiltonian structure (Ω, Λ) where $\Omega = \omega|_{TM}$ and $\Lambda = \iota_V \omega|_{TM}$. Then a neighborhood of M in (W, ω) admits a symplectomorphism to $((-\epsilon, \epsilon) \times M, d(r\Lambda) + \Omega)$ for some $\epsilon > 0$, identifying $M \subset W$ with $\{0\} \times M \subset (-\epsilon, \epsilon) \times M$.*

PROOF. By the smooth tubular neighbourhood theorem and the preceeding exercise, we can view $\omega_0 = d(r\Lambda) + \Omega$ as a symplectic form in some neighbourhood $\mathcal{U}_0 \cong ((-\epsilon, \epsilon) \times M)$ of M. In this neighbourhood,

$$(\omega_0 - \omega)|_M = 0$$

by definition of ω_0 and thus

$$\omega_0 - \omega = d\mu$$

for some 1-form μ such that $\mu|_M = 0$. Now define

$$\omega_t = \omega + t\, d\mu$$

and observe that it is a closed 2-form which can be assumed to be non-degenerate for a small enough choice of $\mathcal{U}_0$. Solving the Moser equation

$$\iota_{v_t}\omega_t = -\mu$$

yields a well-defined, time-dependent vector field v_t with the property that $v_t|_M = 0$. Working back we produce an isotopy as follows:

$$d\iota_{v_t}\omega_t = -d\mu \Rightarrow$$

$$\mathcal{L}_{v_t}\omega_t = d\iota_{v_t}\omega_t + \iota_{v_t}d\omega_t = d\iota_{v_t}\omega_t = -d\mu = -\frac{d\omega_t}{dt} \Rightarrow$$

$$\frac{d}{dt}(\rho_t^*\omega_t) = \mathcal{L}_{v_t}\omega_t + \frac{d\omega_t}{dt} = 0$$

where ρ_t^* is the flow of v_t. Then

$$\rho_t^*\omega_t = \rho_0^*\omega = \omega$$

since ρ_0 is the identity. The required symplectomorphism is then

$$\rho_1 : \rho_1^{-1}\mathcal{U}_0 \to \mathcal{U}_0$$

and the fact that M is fixed under the isotopy follows from $v_t|_M = 0$. □

EXAMPLE 6.5. In the contact case $(\Omega, \Lambda) = (d\alpha, \alpha)$, the symplectic form on the collar neighborhood in Proposition 6.4 can be rewritten as $d(e^t\alpha)$ by defining the coordinate $t := \ln(r + 1)$. The proposition is easier to prove in this case: one can construct the collar neighborhood simply by flowing along V, with no need for the Moser isotopy trick.

A stable Hamiltonian structure $\mathcal{H} = (\Omega, \Lambda)$ gives rise to two important additional objects: a co-oriented hyperplane distribution

$$\xi := \ker \Lambda,$$

and a positively transverse vector field R determined by the conditions

$$\Omega(R, \cdot) \equiv 0 \quad \text{and} \quad \Lambda(R) \equiv 1.$$

By analogy with the contact case, we will refer to R as the **Reeb vector field** of $\mathcal{H}$. The condition $\ker \Omega \subset \ker d\Lambda$ implies that it reduces to the usual contact notion of the Reeb vector field for Λ whenever the latter happens also to be a contact form.

The **symplectization of** $(M, \mathcal{H})$ for any stable Hamiltonian structure $\mathcal{H} = (\Omega, \Lambda)$ can be defined by choosing suitable diffeomorphisms of $(-\epsilon, \epsilon) \times M$ with $\mathbb{R} \times M$: equivalently, this means we consider $\mathbb{R} \times M$ with the family of symplectic forms ω_φ defined by

$$\omega_\varphi := d\left(\varphi(r)\Lambda\right) + \Omega \tag{6.1}$$

where φ is chosen arbitrarily from the set

$$\mathcal{T} := \left\{\varphi \in C^\infty(\mathbb{R}, (-\epsilon, \epsilon)) \;\middle|\; \varphi' > 0\right\}. \tag{6.2}$$

EXAMPLE 6.6. The following stable Hamiltonian structure places Hamiltonian Floer homology into the setting of SFT. Suppose (W, ω) is a closed symplectic manifold and $H : S^1 \times W \to \mathbb{R}$ is a smooth function, and denote $H_t := H(t, \cdot) : W \to \mathbb{R}$. The time-dependent Hamiltonian vector field X_t defined by $dH_t = -\omega(X_t, \cdot)$ can then be viewed as defining a *symplectic connection* on the trivial symplectic fiber bundle

$$M := S^1 \times W \xrightarrow{t} S^1,$$

i.e. the flow of $R(t, x) := \partial_t + X_t(x)$ defines symplectic parallel transport maps between fibers. The horizontal subbundle for this connection is the "symplectic complement" of the vertical subbundle with respect to the closed 2-form

$$\Omega = \omega + dt \wedge dH.$$

In other words, Ω restricts to the fibers of $M \to S^1$ as ω and the subbundle $\{X \in TM \mid \omega(X, \cdot)|_{T(\{\text{const}\} \times W)}\}$ is generated by R, so Ω is the **connection 2-form** defining the connection, cf. [MS98]. Setting $\Lambda := dt$ then makes $\mathcal{H} := (\Omega, \Lambda)$ a stable Hamiltonian structure with Reeb vector field R, and its closed orbits in homotopy classes that project to S^1 with degree one are in 1-to-1 correspondence with the 1-periodic Hamiltonian orbits on W. Notice that this is very different from the contact case: $\xi = \ker dt$ is as far as possible from being a contact structure, it is instead an integrable distribution whose integral submanifolds are the fibers of $M \to S^1$.

EXERCISE 6.7. Show that for any stable Hamiltonian structure $\mathcal{H} = (\Omega, \Lambda)$, the flow of R preserves $\xi = \ker \Lambda$ along with its symplectic bundle structure $\Omega|_\xi$.

DEFINITION 6.8. A T-periodic orbit $x : \mathbb{R} \to M$ of R is called **nondegenerate** if 1 is not an eigenvalue of $d\varphi^T|_{\xi_{x(0)}} : \xi_{x(0)} \to \xi_{x(0)}$, where φ^t denotes the flow of R.

EXERCISE 6.9. Show that in Example 6.6, the notions of nondegeneracy for closed Reeb orbits on M and for 1-periodic Hamiltonian orbits on W (see Lecture 1) coincide.

If $\gamma : S^1 \to M$ parametrizes a T-periodic orbit of R with $\dot{\gamma} = T \cdot R(\gamma)$, then the formula of Lecture 3 for the **asymptotic operator**

$$\mathbf{A}_\gamma \eta = -J(\nabla_t \eta - T \nabla_\eta R)$$

still makes sense in this more general context, and it defines an L^2-symmetric operator on the Hermitian vector bundle $(\gamma^*\xi, J, \Omega)$ over S^1. It can also be interpreted as a Hessian at a critical point, though for an action functional that is only locally defined: indeed, while Ω need not be globally exact, it is necessarily exact on a neighborhood of $\gamma_0(S^1)$ for any given loop $\gamma_0 : S^1 \to M$, so one can pick any primitive λ of Ω on this neighborhood and, for a sufficiently small neighborhood $\mathcal{U}(\gamma_0) \subset C^\infty(S^1, M)$ of γ_0, consider the action functional

$$\mathcal{A}_\mathcal{H} : \mathcal{U}(\gamma_0) \to \mathbb{R} : \gamma \mapsto \int_{S^1} \gamma^*\lambda. \tag{6.3}$$

Its first variation at $\gamma \in \mathcal{U}(\gamma_0)$ in the direction $\eta \in \Gamma(\gamma^*\xi)$ is then

$$d\mathcal{A}_{\mathcal{H}}(\gamma)\eta = -\int_{S^1} \Omega(\dot{\gamma}, \eta)\, dt = \langle -J\pi_\xi\dot{\gamma}, \eta\rangle_{L^2},$$

where $\pi_\xi : TM \to \xi$ denotes the projection along R and the L^2-pairing on $\gamma^*\xi$ is defined via the bundle metric $\Omega(\cdot, J\cdot)|_\xi$. This leads us to interpret $-J\pi_\xi\dot{\gamma}$ as a "gradient" $\nabla\mathcal{A}_{\mathcal{H}}(\gamma)$, and if $\dot{\gamma} = T \cdot R(\gamma)$, then differentiating this gradient in the direction of $\eta \in \Gamma(\gamma^*\xi)$ gives $\mathbf{A}_\gamma\eta$. As one would expect, nondegeneracy of γ is then equivalent to the condition $\ker \mathbf{A}_\gamma = \{0\}$, and one can in this case define the Conley-Zehnder index $\mu^\tau_{\mathrm{CZ}}(\gamma) \in \mathbb{Z}$ as in Lecture 3, relative to a choice of unitary trivialization τ for (ξ, J, Ω).

EXERCISE 6.10. In the setting of Example 6.6, work out the relationship between $\mathcal{A}_{\mathcal{H}}$ and the symplectic action functional for Hamiltonian systems that we discussed in Lecture 1. (Try not to worry too much about signs.)

DEFINITION 6.11. Given a stable Hamiltonian structure $\mathcal{H} = (\Omega, \Lambda)$, denote by

$$\mathcal{J}(\mathcal{H}) \subset \mathcal{J}(\mathbb{R} \times M)$$

the space of smooth almost complex structures J on $\mathbb{R} \times M$ with the following properties:

- J is invariant under the $\mathbb{R}$-action on $\mathbb{R}\times M$ by translation of the first factor;
- $J\partial_r = R$ and $JR = -\partial_r$, where r denotes the natural coordinate on the first factor;
- $J(\xi) = \xi$ and $J|_\xi$ is compatible with the symplectic vector bundle structure $\Omega|_\xi$.

Notice that if $\mathcal{H} = (d\alpha, \alpha)$ for a contact form α, then $\mathcal{J}(\mathcal{H})$ matches the space $\mathcal{J}(\alpha)$ defined in Lecture 1.

EXERCISE 6.12. Show that every $J \in \mathcal{J}(\mathcal{H})$ is tamed by all of the symplectic structures ω_φ as defined in (6.1) for $\varphi \in \mathcal{T}$.

Given $J \in \mathcal{J}(\mathcal{H})$, we define the **energy** of a J-holomorphic curve $u : (\Sigma, j) \to (\mathbb{R} \times M, J)$ by

$$E(u) := \sup_{\varphi\in\mathcal{T}} \int_\Sigma u^*\omega_\varphi.$$

Exercise 6.12 above implies that $E(u) \geq 0$, with equality if and only if u is constant. In the contact case, this notion of energy is not identical to the "Hofer energy" that we defined in Lecture 1, nor to Hofer's original definition from [**Hof93**], but all three are equivalent for our purposes since uniform bounds on any of them imply uniform bounds on the others.

Just as in the contact case, the simplest example of a finite-energy J-holomorphic curve is a **trivial cylinder**

$$u_\gamma : \mathbb{R} \times S^1 \to \mathbb{R} \times M : (s, t) \mapsto (Ts, \gamma(t)),$$

where $\gamma : S^1 \to M$ is a "constant velocity" parametrization of a T-periodic orbit of R, i.e. $\dot{\gamma} = T{\cdot}R(\gamma)$. More generally, given a punctured Riemann surface $(\dot{\Sigma} = \Sigma\backslash\Gamma, j)$

with $\Gamma = \Gamma^+ \cup \Gamma^-$, we consider **asymptotically cylindrical** J-holomorphic curves $u : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$, which are assumed to have the property that for each $z \in \Gamma^\pm$, there exist holomorphic cylindrical coordinates identifying a punctured neighborhood $\dot{\mathcal{U}}_z \subset \dot{\Sigma}$ of z with $Z_+ = [0, \infty) \times S^1$ or $Z_- = (-\infty, 0] \times S^1$ respectively, and a trivial cylinder $u_{\gamma_z} : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ such that

$$u(s,t) = \exp_{u_{\gamma_z}(s,t)} h_z(s,t) \qquad \text{for } |s| \text{ sufficiently large,}$$

where $h_z(s,t)$ is a vector field along u_{γ_z} satisfying $|h_z(s,\cdot)| \to 0$ uniformly as $s \to \pm\infty$. As usual, both the norm $|h_z(s,t)|$ and the exponential map here are assumed to be defined with respect to a translation-invariant choice of Riemannian metric on $\mathbb{R} \times M$. The vector fields h_z along u_{γ_z} for each $z \in \Gamma$ are sometimes called **asymptotic representatives** of u near z.

Asymptotic representatives satisfy a regularity estimate that will be important to know about, though its proof (given originally in [**HWZ96**]) would be too lengthy to present here. The methods behind the following statement involve a combination of nonlinear regularity arguments as in Lecture 2 with the asymptotic elliptic estimates from Lecture 4. To prepare for the statement, note that $\mathcal{H}$ induces a splitting of complex vector bundles

$$T(\mathbb{R} \times M) = \epsilon \oplus \xi, \tag{6.4}$$

where ϵ denotes the trivial complex line bundle generated by the vector field ∂_r, or equivalently, the Reeb vector field. It follows that if $\gamma : S^1 \to M$ is a Reeb orbit and $u_\gamma : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ is the corresponding trivial cylinder, then any unitary trivialization τ of the Hermitian bundle $(\gamma^*\xi, J, \Omega)$ naturally induces a trivialization of $u_\gamma^* T(\mathbb{R} \times M)$.

PROPOSITION 6.13 ([**HWZ96**]). *Assume $J \in \mathcal{J}(\mathcal{H})$, $u : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$ is J-holomorphic and asymptotically cylindrical, and its asymptotic orbit γ_z at $z \in \Gamma^\pm$ is nondegenerate. Let $h(s,t) \in \mathbb{C}^n$ denote the asymptotic representative of u near z expressed via the trivialization induced by a choice of unitary trivialization for $(\gamma_z^*\xi, J, \Omega)$. If $\delta > 0$ is small enough so that the asymptotic operator $\mathbf{A}_{\gamma_z}$ has no eigenvalues in the closed interval between 0 and $\mp\delta$, then*

$$h(s,t) = e^{\mp\delta s} g(s,t)$$

for some bounded function $g(s,t) \in \mathbb{C}^n$ whose derivatives of all orders are bounded as $s \to \pm\infty$.

REMARK 6.14. The range of $\delta > 0$ for which Prop. 6.13 holds is open, thus by adjusting δ slightly, one can equivalently say that $h(s,t) = e^{\mp\delta s} g(s,t)$ where the derivatives of all orders of $g(s,t)$ *decay to zero* as $s \to \pm\infty$.

EXERCISE 6.15. Convince yourself that the analogue of Proposition 6.13 in Morse theory is true. Namely, suppose (M, g) is a Riemannian manifold, $f : M \to \mathbb{R}$ is smooth and $u : \mathbb{R} \to M$ is a solution to $\dot{u} + \nabla f(u) = 0$ with $\lim_{s \to \pm\infty} u(s) = x_\pm \in \operatorname{Crit}(f)$, where $x_\pm$ are nondegenerate critical points. We can write $u(s)$ asymptotically as

$$u(s) = \exp_{x_\pm} h_\pm(s)$$

for some functions $h_\pm(s) \in T_{x_\pm} M$ that are defined for s close to $\pm\infty$ and satisfy $|h_\pm(s)| \to 0$ as $s \to \pm\infty$. Show that if $\delta > 0$ is small enough so that $\nabla^2 f(x_\pm)$ has no eigenvalue in the closed interval between 0 and $\pm\delta$, then

$$h_\pm(s) = e^{\mp\delta s} g_\pm(s)$$

for some functions $g_\pm(s)$ with bounded derivatives of all orders as $s \to \pm\infty$.[1] *Hint: fix local coordinates identifying $x_\pm$ with $0 \in \mathbb{R}^n$ and first consider the case where $\nabla f(x)$ in these coordinates depends linearly on x. Then try to compare $u(s)$ with solutions of this idealized equation.*

EXAMPLE 6.16. In the setting of Example 6.6, a choice of $J \in \mathcal{J}(\mathcal{H})$ is equivalent to a choice of smooth S^1-parametrized family of compatible almost complex structures $\{J_t\}_{t\in S^1}$ on (W, ω), and J-holomorphic curves $u : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$ can be written as

$$u = (f, v) : \dot{\Sigma} \to \left(\mathbb{R} \times S^1\right) \times W,$$

where $f : (\dot{\Sigma}, j) \to (\mathbb{R} \times S^1, i)$ is holomorphic. In particular, if $(\dot{\Sigma}, j) = (\mathbb{R} \times S^1, i)$ and f is taken to have an extension to $S^2 \to S^2$ of degree one, then u can be reparametrized so that f is the identity map, hence $u = (\mathrm{Id}, v) : \mathbb{R} \times S^1 \to (\mathbb{R} \times S^1) \times W$ is a section of the trivial fiber bundle $(\mathbb{R} \times S^1) \times W \to \mathbb{R} \times S^1$, and one can check that the equation satisfied by $v : \mathbb{R} \times S^1 \to W$ is precisely the Floer equation

$$\partial_s v + J_t(v)(\partial_t v - X_t(v)) = 0.$$

6.2. Symplectic cobordisms with stable boundary

We discussed symplectic cobordisms between contact manifolds in Lecture 1. Let us now generalize this notion in the context of stable Hamiltonian structures.

A **symplectic cobordism with stable boundary** is a compact symplectic manifold (W, ω) with boundary $\partial W = -M_- \sqcup M_+$, equipped with a stabilizing vector field V that points transversely inward at M_- and outward at M_+. This induces stable Hamiltonian structures $\mathcal{H}_\pm = (\omega_\pm, \lambda_\pm)$ on $M_\pm$, where

$$\omega_\pm := \omega|_{TM_\pm}, \qquad \lambda_\pm := (\iota_V \omega)|_{TM_\pm},$$

and observe that the orientation conventions for M_+ and M_- (with the latter carrying the opposite of the natural boundary orientation) have been chosen such that if $\dim W = 2n$,

$$\lambda_\pm \wedge \omega_\pm^{n-1} > 0 \quad \text{on } M_\pm.$$

We can now identify neighborhoods of $M_\pm$ in (W, ω) symplectically with collars of the form

$$([0, \epsilon) \times M_+, d(r\lambda_+) + \omega_+),$$
$$((-\epsilon, 0] \times M_-, d(r\lambda_-) + \omega_-),$$

see Figure 6.1.

[1]The apparent discrepancy in signs between this and Proposition 6.13 is due to the fact that $u(s)$ satisfies a *negative* gradient flow equation, whereas the nonlinear Cauchy-Riemann equation in symplectizations is interpreted loosely as a *positive* gradient flow equation.

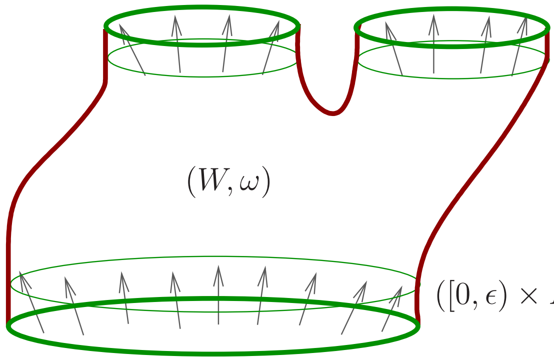

FIGURE 6.1. A symplectic cobordism with stable boundary components $\partial W = -M_- \sqcup M_+$ and symplectic collar neighborhoods induced by the stable Hamiltonian structures $\mathcal{H}_\pm = (\omega_\pm, \lambda_\pm)$ on $M_\pm$.

Modifying (6.2) by

$$\mathcal{T}_0 := \left\{ \varphi \in C^\infty(\mathbb{R}, (-\epsilon,\epsilon)) \;\middle|\; \varphi' > 0 \text{ and } \varphi(r) = r \text{ for } r \text{ near } 0 \right\}, \tag{6.5}$$

we can use any $\varphi \in \mathcal{T}_0$ to define a **symplectic completion** $(\widehat{W}, \omega_\varphi)$ of (W,ω) by

$$\widehat{W} := \big((-\infty,0]\times M_-\big) \cup_{M_-} W \cup_{M_+} \big([0,\infty)\times M_+\big),$$

where the above collar neighborhoods are used to glue the pieces together smoothly and the symplectic form is defined by

$$\omega_\varphi := \begin{cases} d\left(\varphi(r)\lambda_-\right) + \omega_- & \text{on } (-\infty,0]\times M_-, \\ \omega & \text{on } W, \\ d\left(\varphi(r)\lambda_+\right) + \omega_+ & \text{on } [0,\infty)\times M_+, \end{cases}$$

see Figure 6.2. For each $r_0 \ge 0$, we define the compact submanifold

$$W^{r_0} := ([-r_0,0]\times M_-) \cup_{M_-} W \cup_{M_+} ([0,r_0]\times M_+),$$

and observe that $(W^{r_0}, \omega_\varphi)$ is also a symplectic cobordism with stable boundary for every $\varphi \in \mathcal{T}_0$.

Since $\widehat{W}$ is noncompact, almost complex structures J on $\widehat{W}$ will need to satisfy conditions near infinity in order for moduli spaces of J-holomorphic curves to be well behaved, but we would like to preserve the freedom of choosing arbitrary compatible or tame almost complex structures in compact subsets.

DEFINITION 6.17. Given $\psi \in \mathcal{T}_0$ and $r_0 \ge 0$, let

$$\mathcal{J}_\tau(\omega_\psi, r_0, \mathcal{H}_+, \mathcal{H}_-) \subset \mathcal{J}(\widehat{W})$$

denote the space of smooth almost complex structures J on $\widehat{W}$ such that:

- J on $[r_0,\infty)\times M_+$ matches an element of $\mathcal{J}(\mathcal{H}_+)$;
- J on $(-\infty,-r_0]\times M_-$ matches an element of $\mathcal{J}(\mathcal{H}_-)$;
- J on W^{r_0} is tamed by ω_ψ.

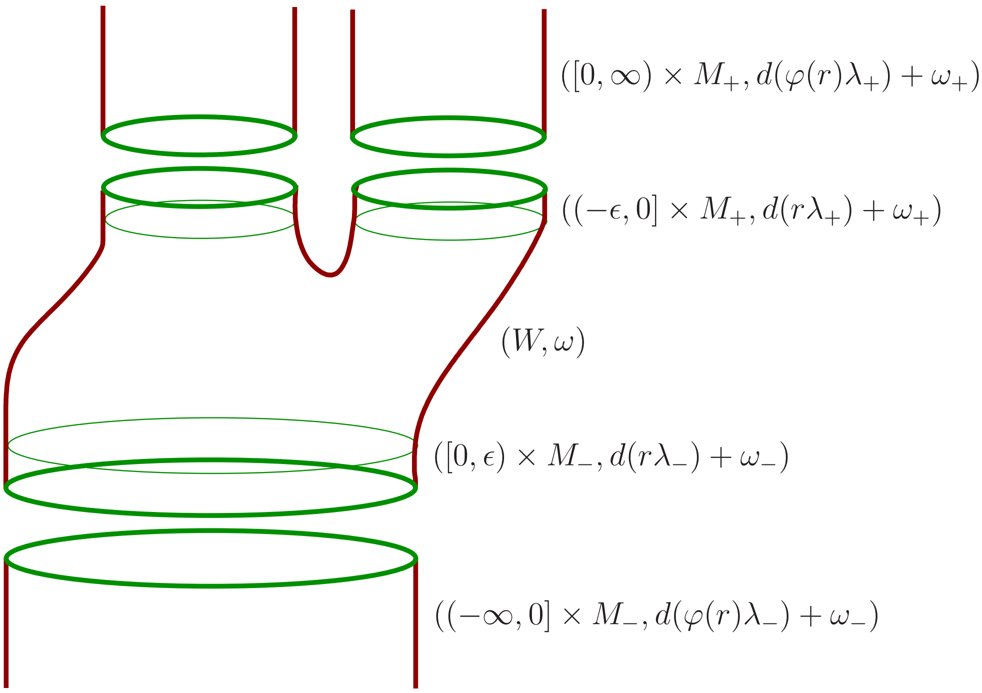

FIGURE 6.2. The completion $(\widehat{W},\omega_\varphi)$ of a symplectic cobordism with stable boundary.

Let

$$\mathcal{J}(\omega_\psi, r_0, \mathcal{H}_+, \mathcal{H}_-) \subset \mathcal{J}_\tau(\omega_\psi, r_0, \mathcal{H}_+, \mathcal{H}_-)$$

denote the subset for which J is additionally compatible with ω_ψ on W^{r_0}.

Setting

$$\mathcal{T}(\psi, r_0) := \left\{\varphi \in \mathcal{T}_0 \;\middle|\; \varphi \equiv \psi \text{ on } [-r_0, r_0]\right\}, \tag{6.6}$$

Exercise 6.12 implies that every $J \in \mathcal{J}(\omega_\psi, r_0, \mathcal{H}_+, \mathcal{H}_-)$ is tamed by ω_φ for every $\varphi \in \mathcal{T}(\psi, r_0)$. It is therefore sensible to define the energy of a J-holomorphic curve $u : (\Sigma, j) \to (\widehat{W}, J)$ by

$$E(u) := \sup_{\varphi\in\mathcal{T}(\psi,r_0)} \int_\Sigma u^*\omega_\varphi.$$

The notion of asymptotically cylindrical J-holomorphic curves extends in a straightforward way to the setting of $(\widehat{W}, J)$: such curves are proper maps whose positive/negative punctures are asymptotic to closed orbits of the Reeb vector field $R_\pm$ induced by $\mathcal{H}_\pm$ on $\{\pm\infty\} \times M_\pm$, see Figure 6.3. The exponential decay estimate in Proposition 6.13 is also immediately applicable in this more general setting since asymptotically cylindrical curves in $\widehat{W}$ are indistinguishable near their punctures from curves in the symplectizations $\mathbb{R} \times M_\pm$.

It is easy to check that asymptotically cylindrical J-holomorphic curves always have finite energy. We will prove in Lecture 8 that the converse is also true whenever the Reeb orbits are nondegenerate.

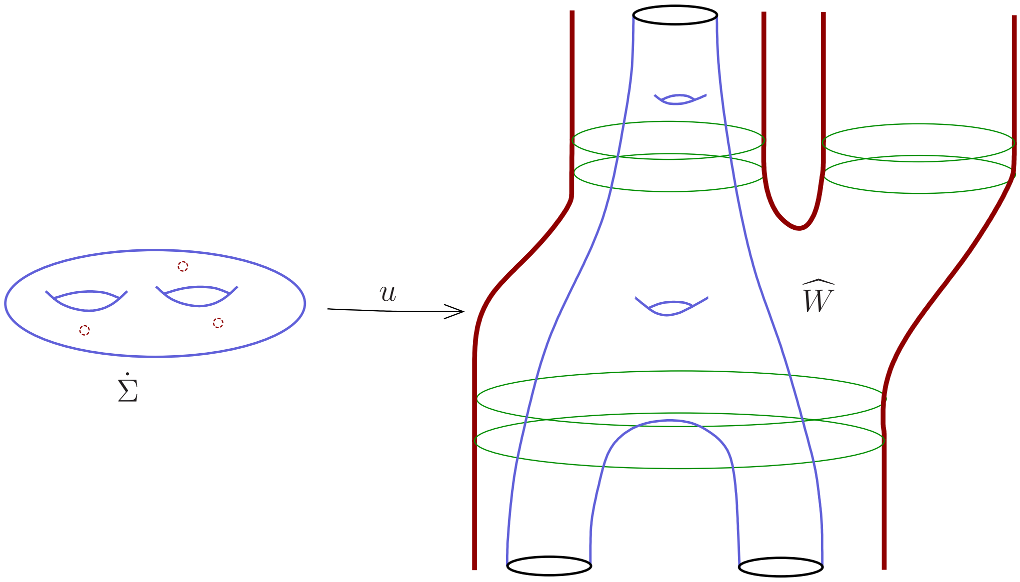

FIGURE 6.3. An asymptotically cylindrical holomorphic curve in $(\widehat{W}, J)$ with genus 2, one positive puncture and two negative punctures.

REMARK 6.18. Strictly speaking, the "trivial stable cobordism"

$$([0,1] \times M, d(\varphi(r)\Lambda, \Omega))$$

induces different stable Hamiltonian structures at $M_- := \{0\} \times M$ and $M_+ := \{1\} \times M$, thus one cannot technically regard $\mathcal{J}(\mathcal{H})$ as contained in any space of the form $\mathcal{J}(\omega_\psi, r_0, \mathcal{H}_+, \mathcal{H}_-)$ without inventing questionable new notions such as the "infinitesimal trivial cobordism" $[0,0] \times M$ (whose completion would be the symplectization of $(M, \mathcal{H})$). It is nonetheless true for fairly trivial reasons that most results about $\mathcal{J}(\omega, r_0, \mathcal{H}_+, \mathcal{H}_-)$ apply equally well to $\mathcal{J}(\mathcal{H})$, and we shall use this fact in the following without always mentioning it.

Every asymptotically cylindrical curve $u : \dot{\Sigma} \to \widehat{W}$ has a well-defined **relative homology class**, meaning the following. Denote the asymptotic orbits of u at its punctures $z \in \Gamma^\pm$ by γ_z, and let $\bar{\boldsymbol{\gamma}}^\pm \subset M_\pm$ denote the closed 1-dimensional submanifold defined as the union over $z \in \Gamma^\pm$ of the images of the orbits γ_z. Let $\overline{\Sigma}$ denote the compact oriented topological surface with boundary obtained from $\dot{\Sigma}$ by appending $\{\pm\infty\} \times S^1$ to each of its cylindrical ends, and let $\pi : \widehat{W} \to W$ denote the retraction defined as the identity on W and $\pi(r, x) = x \in M_\pm \subset \partial W$ for (r, x) in $[0, \infty) \times M_+$ or $(-\infty, 0] \times M_-$. Then $\pi \circ u : \dot{\Sigma} \to W$ has a natural continuous extension

$$\bar{u} : (\overline{\Sigma}, \partial\overline{\Sigma}) \to (W, \bar{\boldsymbol{\gamma}}^+ \cup \bar{\boldsymbol{\gamma}}^-)$$

and thus represents a relative homology class

$$[u] \in H_2(W, \bar{\boldsymbol{\gamma}}^+ \cup \bar{\boldsymbol{\gamma}}^-).$$

6.3. Moduli spaces of unparametrized holomorphic curves

We continue in the setting of a completed symplectic cobordism $\widehat{W}$ with fixed choices of $\psi \in \mathcal{T}_0$, $r_0 \geq 0$ and $J \in \mathcal{J}(\omega_\psi, r_0, \mathcal{H}_+, \mathcal{H}_-)$. We shall denote by $\xi_\pm$ and $R_\pm$ the hyperplane distribution and Reeb vector field respectively determined by the stable Hamiltonian structure $\mathcal{H}_\pm = (\omega_\pm, \lambda_\pm)$.

Fix integers $g, m, k_+, k_- \geq 0$ along with ordered sets of Reeb orbits

$$\boldsymbol{\gamma}^\pm = (\gamma_1^\pm, \dots, \gamma_{k_\pm}^\pm),$$

where each $\gamma_i^\pm$ is a closed orbit of $R_\pm$ in $M_\pm$. Denote the union of the images of the $\gamma_i^\pm$ by $\bar{\boldsymbol{\gamma}}^\pm \subset M_\pm$, and choose a relative homology class

$$A \in H_2(W, \bar{\boldsymbol{\gamma}}^+ \cup \bar{\boldsymbol{\gamma}}^-)$$

whose image under the boundary map $H_2(W, \bar{\boldsymbol{\gamma}}^+ \cup \bar{\boldsymbol{\gamma}}^-) \xrightarrow{\partial} H_1(\bar{\boldsymbol{\gamma}}^+ \cup \bar{\boldsymbol{\gamma}}^-)$ defined via the long exact sequence of the pair $(W, \bar{\boldsymbol{\gamma}}^+ \cup \bar{\boldsymbol{\gamma}}^-)$ is

$$\partial A = \sum_{i=1}^{k_+} [\gamma_i^+] - \sum_{i=1}^{k_-} [\gamma_i^-] \in H_1(\bar{\boldsymbol{\gamma}}^+ \cup \bar{\boldsymbol{\gamma}}^-).$$

The **moduli space** of **unparametrized** J-holomorphic curves of **genus** g with m **marked points**, **homologous to** A and **asymptotic to** $(\boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$ is then defined as a set of equivalence classes of tuples

$$\mathcal{M}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-) = \big\{(\Sigma, j, \Gamma^+, \Gamma^-, \Theta, u)\big\} \,/ \sim,$$

where:

(1) (Σ, j) is a closed connected Riemann surface of genus g;
(2) $\Gamma^+ = (z_1^+, \dots, z_{k_+}^+)$, $\Gamma^- = (z_1^-, \dots, z_{k_-}^-)$ and $\Theta = (\zeta_1, \dots, \zeta_m)$ are disjoint ordered sets of distinct points in Σ;
(3) $u : (\dot{\Sigma} := \Sigma \setminus (\Gamma^+ \cup \Gamma^-), j) \to (\widehat{W}, J)$ is an asymptotically cylindrical J-holomorphic map with $[u] = A$, asymptotic at $z_i^\pm \in \Gamma^\pm$ to $\gamma_i^\pm$ for $i = 1, \dots, k_\pm$;
(4) Equivalence

$$(\Sigma_0, j_0, \Gamma_0^+, \Gamma_0^-, \Theta_0, u_0) \sim (\Sigma_1, j_1, \Gamma_1^+, \Gamma_1^-, \Theta_1, u_1)$$

means the existence of a biholomorphic map $\psi : (\Sigma_0, j_0) \to (\Sigma_1, j_1)$, taking $\Gamma_0^\pm$ to $\Gamma_1^\pm$ and Θ_0 to Θ_1 with the ordering preserved, such that

$$u_1 \circ \psi = u_0.$$

We shall usually abuse notation by abbreviating elements $[(\Sigma, j, \Gamma^+, \Gamma^-, \Theta, u)]$ in this moduli space by

$$u \in \mathcal{M}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-).$$

The **automorphism group**

$$\mathrm{Aut}(u) = \mathrm{Aut}(\Sigma, j, \Gamma^+, \Gamma^-, \Theta, u)$$

of u is defined as the group of biholomorphic maps $\psi : (\Sigma, j) \to (\Sigma, j)$ which act as the identity on $\Gamma^+ \cup \Gamma^- \cup \Theta$ and satisfy $u = u \circ \psi$. Clearly the isomorphism class of this group depends only on the equivalence class $[(\Sigma, j, \Gamma^+, \Gamma^-, \Theta, u)] \in$

$\mathcal{M}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$, and we will see in §6.4 below that it is always finite unless $u : \dot{\Sigma} \to \widehat{W}$ is constant. The significance of the marked points is that they determine an **evaluation map**

$$\mathrm{ev} : \mathcal{M}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-) \to \widehat{W}^m : [(\Sigma, j, \Gamma^+, \Gamma^-, \Theta, u)] \mapsto (u(\zeta_1), \dots, u(\zeta_m))$$

where $\Theta = (\zeta_1, \dots, \zeta_m)$. For most of our applications we will be free to assume $m = 0$, as marked points are not needed for defining the most basic versions of SFT; the evaluation map does play a prominent role however in more algebraically elaborate versions of the theory, and especially in the Gromov-Witten invariants (the "closed case" of SFT).

We will assign a topology to $\mathcal{M}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$ in the next lecture by locally identifying it with subsets of certain manifolds of maps $\dot{\Sigma} \to \widehat{W}$ with Sobolev-type regularity and exponential decay conditions at the ends. In reality, this topology admits a simpler description: one can define convergence of a sequence

$$[(\Sigma_\nu, j_\nu, \Gamma^+_\nu, \Gamma^-_\nu, \Theta_\nu, u_\nu)] \to [(\Sigma, j, \Gamma^+, \Gamma^-, \Theta, u)]$$

to mean that for sufficiently large ν, the equivalence classes in the sequence admit representatives of the form $(\Sigma, j'_\nu, \Gamma^+, \Gamma^-, \Theta, u'_\nu)$ such that

(1) $j'_\nu \to j$ in C^∞;
(2) $u'_\nu \to u$ in $C^\infty_{\mathrm{loc}}(\dot{\Sigma}, \widehat{W})$;
(3) $\bar{u}'_\nu \to \bar{u}$ in $C^0(\overline{\Sigma}, W)$.

The proof that this topology matches what we will define in the next lecture in terms of weighted Sobolev spaces requires asymptotic elliptic regularity arguments along the lines of Proposition 6.13.

6.4. Simple curves and multiple covers

In Lecture 2, we proved that closed J-holomorphic curves are all either embedded in the complement of a finite set or are multiple covers of curves with this property. The same thing holds in the punctured case:

THEOREM 6.19. *Assume $u : (\dot{\Sigma}, j) \to (\widehat{W}, J)$ is a nonconstant asymptotically cylindrical J-holomorphic curve whose asymptotic orbits are all nondegenerate, where $\dot{\Sigma} = \Sigma \setminus \Gamma$ for some closed Riemann surface (Σ, j) and finite subset $\Gamma \subset \Sigma$. Then there exists a factorization $u = v \circ \varphi$, where*

- *$\varphi : (\Sigma, j) \to (\Sigma', j')$ is a holomorphic map of positive degree to another closed and connected Riemann surface (Σ', j');*
- *$v : (\dot{\Sigma}', j') \to (\widehat{W}, J)$ is an asymptotically cylindrical J-holomorphic curve which is embedded except at a finite set of critical points and self-intersections, where $\dot{\Sigma}' := \Sigma' \setminus \Gamma'$ with $\Gamma' := \varphi(\Gamma)$ and $\Gamma = \varphi^{-1}(\Gamma')$.*

As in the closed case, we call u a **simple** curve if the holomorphic map $\varphi : (\Sigma, j) \to (\Sigma', j')$ is a diffeomorphism, and u is otherwise a **k-fold multiple cover** of v with $k := \deg(\varphi) \geq 2$.

The proof of this theorem is an almost verbatim repeat of the proof of Theorem 2.29 in Lecture 2, but with one new ingredient added. Recall that in the closed case, our proof required two lemmas which described the local picture of a J-holomorphic curve $u : \dot{\Sigma} \to \widehat{W}$ near either a double point $u(z_0) = u(z_1)$ for $z_0 \neq z_1$ or a critical point $du(z_0) = 0$. Both statements were completely local and thus equally valid for non-closed curves, but we now need similar statements to describe what kinds of singularities can appear in the neighborhood of a puncture. The following lemma is due to Siefring [**Sie08**] and follows from a "relative asymptotic formula" analogous to Proposition 6.13.

LEMMA 6.20 (Asymptotics). *Assume $u : (\dot{\Sigma} = \Sigma \setminus \Gamma, j) \to (\widehat{W}, J)$ is asymptotically cylindrical and is asymptotic at $z_0 \in \Gamma$ to a nondegenerate Reeb orbit. Then a punctured neighborhood $\dot{\mathcal{U}}_{z_0} \subset \dot{\Sigma}$ of z_0 can be identified biholomorphically with the punctured disk $\dot{\mathbb{D}} = \mathbb{D} \setminus \{0\}$ such that*

$$u(z) = v(z^k) \quad \text{for} \quad z \in \dot{\mathbb{D}} = \dot{\mathcal{U}}_{z_0},$$

where $k \in \mathbb{N}$ and $v : (\dot{\mathbb{D}}, i) \to (\widehat{W}, J)$ is an embedded and asymptotically cylindrical J-holomorphic curve. Moreover, if $u' : (\dot{\Sigma}' = \Sigma' \setminus \Gamma', j') \to (\widehat{W}, J)$ is another asymptotically cylindrical curve with a puncture $z_0' \in \Gamma'$, then the images of u near z_0 and u' near z_0' are either identical or disjoint. □

EXERCISE 6.21. With Lemma 6.20 in hand, adapt the proof of Theorem 2.29 in Lecture 2 to prove Theorem 6.19. If you get stuck, see [**Nel15**, §3.2].

PROPOSITION 6.22. *If $[(\Sigma, j, \Gamma^+, \Gamma^-, \Theta, u)] \in \mathcal{M}_{g,m}(J, A, \gamma^+, \gamma^-)$ is represented by a simple curve, then $\mathrm{Aut}(u)$ is trivial. If it is represented by a k-fold cover of a simple curve, then $|\mathrm{Aut}(u)| \leq k$. In particular, $\mathrm{Aut}(u)$ is always finite unless u is constant.*

PROOF. If u is simple, then it is a diffeomorphism onto its image in a small neighbourhood of some point, and any map φ satisfying $u = u \circ \varphi$ would be the identity on such a neighbourhood. By unique continuation, we conclude that $\mathrm{Aut}(u)$ is trivial. In general if $u = v \circ \varphi$ for some simple

$$v : \Sigma' \to W$$

and

$$\varphi : \Sigma \to \Sigma'$$

a k-fold branched cover, we have

$$\mathrm{Aut}(u) = \{f : \Sigma \to \Sigma \mid v \circ \varphi \circ f = v \circ \varphi\}.$$

By a similar argument as in the previous case, knowing that v is simple implies we only need to look at solutions to

$$\varphi \circ f = \varphi.$$

Remove the set of branch points B from Σ' together with the set $\varphi^{-1}(B)$ from Σ, so that φ becomes an honest covering map. Any $\varphi \in \mathrm{Aut}(u)$ then defines a biholomorphic deck transformation of the cover, so it remains to argue that there are at most k of them. In fact, there is at most one transformation that takes w_1 to w_2 for any two given points $w_1, w_2 \in \varphi^{-1}(x)$. If there were two such transformations

f and g, then $f \circ g^{-1}$ would be the identity on an open neighbourhood and would thus be globally the identity by unique continuation. □

6.5. A local structure result

The following statement, which we will prove in the next lecture, is the main goal of most of the analysis we have discussed recently. It is essentially an application of the implicit function theorem for a smooth nonlinear Fredholm section of a Banach space bundle. The implicit function theorem (see [**Lan93**]) implies in particular that if F is a smooth map between Banach spaces such that $F(x_0) = 0$ and $dF(x_0)$ is a surjective Fredholm operator, then $F^{-1}(0)$ is a smooth manifold near x_0 with its dimension equal to the Fredholm index of $dF(x_0)$. Surjectivity is an extra hypothesis, referred to in the statement below as "Fredholm regularity," a notion that we will define precisely in the next lecture. The dimension formula should look familiar, but is only an *indirect* consequence of the index formula for Cauchy-Riemann type operators that we proved in Lecture 5; one also needs to account for the fact that in defining our moduli space $\mathcal{M}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$, we did not fix the complex structures on our domain curves, hence they are free to move about in the moduli space of Riemann surfaces, whose dimension therefore plays a role in determining the dimension of $\mathcal{M}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$.

THEOREM 6.23. *The set of* Fredholm regular *curves forms an open subset*

$$\mathcal{M}^{\mathrm{reg}}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-) \subset \mathcal{M}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$$

which naturally admits the structure of a smooth finite-dimensional orbifold of dimension

$$\dim \mathcal{M}^{\mathrm{reg}}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-) = (n-3)(2 - 2g - k_+ - k_-) + 2c_1^\tau(A) + \sum_{i=1}^{k_+} \mu^\tau_{CZ}(\gamma_i^+) - \sum_{i=1}^{k_-} \mu^\tau_{CZ}(\gamma_i^-) + 2m,$$

where $\dim W = 2n$*,* τ *is a choice of unitary trivialization for* $(\xi_\pm, J, \omega_\pm)$ *along each of the asymptotic orbits* $\gamma_i^\pm$*, and* $c_1^\tau(A)$ *denotes the normal first Chern number of the complex vector bundle* $(u^*T\widehat{W}, J) \to \dot{\Sigma}$ *with respect to the asymptotic trivialization determined by* τ *and the splitting* $T(\mathbb{R} \times M_\pm) = \epsilon \oplus \xi_\pm$ *(cf. (6.4)). The local isotropy group of* $\mathcal{M}^{\mathrm{reg}}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$ *at* u *is* $\mathrm{Aut}(u)$*, hence the moduli space is a manifold near any regular element with trivial automorphism group.*

EXERCISE 6.24. Verify that the number in the above index formula is independent of the choice of trivializations τ, and that $c_1^\tau(u^*T\widehat{W})$ depends only on the relative homology class A.

LECTURE 7

Smoothness of the moduli space

Contents

In this lecture, we continue the study of the moduli space

$$\mathcal{M}(J) := \mathcal{M}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-).$$

We assume as before that (W, ω) is a $2n$-dimensional symplectic cobordism with stable boundary $\partial W = -M_- \sqcup M_+$ inheriting stable Hamiltonian structures $\mathcal{H}_\pm = (\omega_\pm, \lambda_\pm)$ with induced Reeb vector fields $R_\pm$ and hyperplane distributions $\xi_\pm = \ker \lambda_\pm$, $g, m, k_+, k_- \geq 0$ are integers, $\boldsymbol{\gamma}^\pm = (\gamma_1^\pm, \ldots, \gamma_{k_\pm}^\pm)$ are ordered sets of periodic $R_\pm$-orbits in $M_\pm$, and $A \in H_2(W, \bar{\boldsymbol{\gamma}}^+ \cup \bar{\boldsymbol{\gamma}}^-)$ is a relative homology class with $\partial A = \sum_i [\gamma_i^+] - \sum_i [\gamma_i^-] \in H_1(W, \bar{\boldsymbol{\gamma}}^+ \cup \bar{\boldsymbol{\gamma}}^-)$. The noncompact completion of (W, ω) is denoted by $(\widehat{W}, \omega_\psi)$ for some fixed function $\psi : \mathbb{R} \to (-\epsilon, \epsilon)$ that scales the symplectic form on the cylindrical ends, and $r_0 \geq 0$ is a fixed constant which determines the size of the ends $[r_0, \infty) \times M_+$ and $(-\infty, -r_0] \times M_-$ on which we require our almost complex structures $J \in \mathcal{J}(\omega_\psi, r_0, \mathcal{H}_+, \mathcal{H}_-)$ to be $\mathbb{R}$-invariant. The complement of these ends has closure

$$W^{r_0} := ([-r_0, 0] \times M_-) \cup_{M_-} W \cup_{M_+} ([0, r_0] \times M_+) .$$

We will often make use of the fact that since J matches translation-invariant almost complex structures in $\mathcal{J}(\mathcal{H}_\pm)$ outside of W^{r_0}, there are natural complex vector bundle splittings

$$T(\mathbb{R} \times M_\pm) = \epsilon \oplus \xi_\pm,$$

where ϵ denotes the canonically trivial line bundle spanned by ∂_r and the Reeb vector field.

7.1. Transversality theorems in cobordisms

We concluded the previous lecture with the statement of the following theorem.

THEOREM 7.1. *If the orbits $\gamma_i^\pm$ are all nondegenerate and $J \in \mathcal{J}(\omega_\psi, r_0, \mathcal{H}_+, \mathcal{H}_-)$, then the moduli space $\mathcal{M}(J)$ contains an open subset*

$$\mathcal{M}^{\mathrm{reg}}(J) \subset \mathcal{M}(J)$$

consisting of so-called Fredholm regular *curves, which naturally admits the structure of a smooth finite-dimensional orbifold of dimension*

$$\dim \mathcal{M}^{\mathrm{reg}}(J) = (n-3)(2-2g-k_+-k_-) + 2c_1^\tau(A) + \sum_{i=1}^{k_+} \mu_{CZ}^\tau(\gamma_i^+) - \sum_{i=1}^{k_-} \mu_{CZ}^\tau(\gamma_i^-) + 2m,$$

where $\dim W = 2n$, *τ is a choice of unitary trivialization for $(\xi_\pm, J, \omega_\pm)$ along each of the asymptotic orbits $\gamma_i^\pm$, and $c_1^\tau(A)$ denotes the normal first Chern number of the complex vector bundle $(u^*T\widehat{W}, J) \to \dot{\Sigma}$ with respect to the asymptotic trivialization determined by τ and the splitting $T(\mathbb{R} \times M_\pm) = \epsilon \oplus \xi_\pm$. The local isotropy group of $\mathcal{M}^{\mathrm{reg}}(J)$ at u is* $\mathrm{Aut}(u)$, *hence the moduli space is a manifold near any regular element with trivial automorphism group.*

The integer in the above dimension formula is often called the **virtual dimension** of $\mathcal{M}(J)$ and denoted by

$$\text{vir-dim}\, \mathcal{M}(J) := (n-3)(2-2g-k_+-k_-) + 2c_1^\tau(A) + \sum_{i=1}^{k_+} \mu_{\mathrm{CZ}}^\tau(\gamma_i^+) - \sum_{i=1}^{k_-} \mu_{\mathrm{CZ}}^\tau(\gamma_i^-) + 2m.$$

Ignoring the marked points, the virtual dimension of a space $\mathcal{M}_{g,0}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$ containing a curve $u : (\dot{\Sigma}, j) \to (\widehat{W}, J)$ with punctures $z \in \Gamma^\pm$ and nondegenerate asymptotic orbits $\{\gamma_z\}_{z\in\Gamma^\pm}$ is sometimes also called the **index** of u,

$$\mathrm{ind}(u) := (n-3)\chi(\dot{\Sigma}) + 2c_1^\tau(u^*T\widehat{W}) + \sum_{z\in\Gamma^+} \mu_{\mathrm{CZ}}^\tau(\gamma_z) - \sum_{z\in\Gamma^-} \mu_{\mathrm{CZ}}^\tau(\gamma_z) \in \mathbb{Z},$$

and we will see that it is in fact the Fredholm index of an operator closely related to the linearized Cauchy-Riemann operator $\mathbf{D}_u$ at u. The word "virtual" refers to the fact that in general, the regularity condition may fail and thus $\mathcal{M}(J)$ might not be smooth, or if it is, it might actually be of a different dimension (see Example 7.5 below), but in an ideal world where transversality is always satisfied, its dimension would be vir-dim $\mathcal{M}(J)$. This notion makes sense in finite-dimensional contexts as well: if $f : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth map, then we would say that $f^{-1}(0)$ has virtual dimension $n-m$, even though $f^{-1}(0)$ might in general be all sorts of strange things other than a smooth $(n-m)$-dimensional manifold. In particular, $n-m$ could be negative, in which case $f^{-1}(0)$ would be empty if transversality were satisfied, but in general this need not be the case. It is true however that f can always be *perturbed* to a map whose zero set is an $(n-m)$-dimensional manifold (or empty if $n-m<0$). The same is true in principle of the nonlinear Cauchy-Riemann equation, but in general it is a formidably difficult problem to find perturbations that respect all symmetries inherent in the setup as well as the extra structure provided by the *compatification* of $\mathcal{M}(J)$, which is usually crucial for meangingful applications. Such issues require more sophisticated methods than we will discuss here, but a good place to read about them is [**FFGW**].

The first goal of this lecture is to define the notion "Fredholm regular" and prove Theorem 7.1. In practice, however, Fredholm regularity is a technical condition that can rarely be directly checked. To remedy this, we will also prove a genericity result for *somewhere injective* J-holomorphic curves. A smooth map $u : \dot{\Sigma} \to \widehat{W}$ is said to have an **injective point** $z \in \dot{\Sigma}$ if

$$du(z) : T_z\dot{\Sigma} \to T_{u(z)}\widehat{W} \text{ is injective} \quad \text{ and } \quad u^{-1}(u(z)) = \{z\}.$$

If u is a proper map, then it is easy to see that the set of injective points is open in $\dot{\Sigma}$, though in general it could also be empty; this is the case e.g. for multiply covered J-holomorphic curves. We say u is **somewhere injective** if its set of injective points is nonempty; for asymptotically cylindrical J-holomorphic curves with nondegenerate asymptotic orbits, Theorem 6.19 implies that somewhere injectivity is equivalent to being *simple*, i.e. not multiply covered.

Recall that if X is a topological space, a subset $Y \subset X$ is called **comeager** if it contains a countable intersection of open and dense sets.[1] If X is complete, then the Baire category theorem implies that comeager subsets are always dense; moreover, any countable intersection of comeager subsets is also comeager and therefore dense. Comeager subsets often play the role in infinite dimensions that the term "almost everywhere" plays in finite dimensions. Informally, we often say that a given statement dependent on a choice of auxiliary data (living in a complete metric space) is true **generically**, or "for **generic** choices," if it is true whenever the data are chosen from some comeager subset of the space of all possible data.

THEOREM 7.2. *Fix the same data as in Theorem 7.1, an almost complex structure $J^{\mathrm{fix}} \in \mathcal{J}(\omega_\psi, r_0, \mathcal{H}_+, \mathcal{H}_-)$ and an open subset*

$$\mathcal{U} \subset W^{r_0}.$$

Then there exists a comeager subset

$$\mathcal{J}^{\mathrm{reg}}_{\mathcal{U}} \subset \left\{ J \in \mathcal{J}(\omega_\psi, r_0, \mathcal{H}_+, \mathcal{H}_-) \;\middle|\; J = J^{\mathrm{fix}} \text{ on } \widehat{W} \setminus \mathcal{U} \right\},$$

such that for every $J \in \mathcal{J}^{\mathrm{reg}}_{\mathcal{U}}$, every curve $u \in \mathcal{M}(J)$ that has an injective point mapped into $\mathcal{U}$ is Fredholm regular. In particular, the curves with this property define an open subset of $\mathcal{M}(J)$ that is a smooth manifold with dimension equal to its virtual dimension.

REMARK 7.3. Since $\mathcal{U} \subset \widehat{W}$ has compact closure, the set

$$\left\{ J \in \mathcal{J}(\omega_\psi, r_0, \mathcal{H}_+, \mathcal{H}_-) \;\middle|\; J = J^{\mathrm{fix}} \text{ on } \widehat{W} \setminus \mathcal{U} \right\}$$

[1] Elsewhere in the symplectic literature, comeager subsets are sometimes referred to as "sets of second category," which is unfortunately slightly at odds with the standard meaning of "second category," though it is accurate to say that the *complement* of a comeager subset (also known as a "meager" subset) is a set of first category. The term *Baire subset* is also sometimes used as a synonym for "comeager subset".

has a natural C^∞-topology that makes it a Fréchet manifold and thus a complete metric space, hence comeager subsets of it are dense.

REMARK 7.4. Both of the above theorems admit easy extensions to the study of moduli spaces dependent on finitely many parameters. Concretely, suppose P is a smooth finite-dimensional manifold and $\{J_s\}_{s\in P}$ is a smooth family of almost complex structures satisfying the usual conditions. One can then define a *parametric moduli space*

$$\mathcal{M}(\{J_s\}_{s\in P}) = \big\{(s,u) \bigm| s \in P,\ u \in \mathcal{M}(J_s)\big\}$$

and a notion of *parametric regularity* for pairs $(s,u) \in \mathcal{M}(\{J_s\})$, which is again an open condition, such that the space $\mathcal{M}^{\mathrm{reg}}(\{J_s\})$ of parametrically regular elements will be an orbifold of dimension

$$\dim \mathcal{M}^{\mathrm{reg}}(\{J_s\}) = \operatorname{vir-dim} \mathcal{M}(J) + \dim P.$$

Similarly, one can show that if the family $\{J_s\}_{s\in P}$ is allowed to vary on an open subset $\mathcal{U} \subset W^{r_0}$ for s lying in some precompact open subset $\mathcal{V} \subset P$, then all elements (s,u) for which $s \in \mathcal{V}$ and u has an injective point mapping to $\mathcal{U}$ will be parametrically regular. See [**Wend**, §4.5] for details in the closed case, which is not fundamentally different from the punctured case. The standard and most important example is $P = [0,1]$ with $\mathcal{V} = (0,1)$, so we consider *generic homotopies* of almost complex structures. Here it is important to observe that while regularity in the sense of Theorem 7.1 always implies parametric regularity, the converse is false: there can exist parametrically regular pairs $(s,u) \in \mathcal{M}(\{J_s\})$ for which u is *not* a Fredholm regular element of $\mathcal{M}(J_s)$, hence $\mathcal{M}(\{J_s\})$ may be smooth even if $\mathcal{M}(J_s)$ is not smooth for some $s \in P$. This can happen in particular whenever s is a critical value of the projection map

$$\mathcal{M}(\{J_s\}) \to P : (s,u) \mapsto s,$$

see Figure 7.1. In general these cannot be excluded by making generic choices of the homotopy, though it is possible in certain cases using "automatic" transversality results, which guarantee regularity for all J_s with no need for genericity (cf. [**Wen10**]).

EXAMPLE 7.5. It is not hard to imagine situations in which transversality *must* fail generically for multiply covered curves. Suppose for instance that (W,ω) is an 8-dimensional symplectic manifold with compatible almost complex structure J_0, and $u_0 : S^2 \to W$ is a simple J_0-holomorphic sphere with no punctures and $[u_0] = A \in H_2(W)$, where $c_1(A) = -1$. This means u_0 represents an element of a moduli space $\mathcal{M}_{0,0}(J_0, A)$ with

$$\operatorname{vir-dim} \mathcal{M}_{0,0}(J_0, A) = 2 - 2g + 2c_1(A) = 0.$$

In particular if u_0 is regular and $\{J_s \in \mathcal{J}(\omega)\}_{s\in\mathbb{R}^k}$ is a smooth k-parameter family of compatible almost complex structures including J_0, then Remark 7.4 implies that a neighborhood of $(0, u_0)$ in the parametric moduli space $\mathcal{M}(\{J_s\}) = \{(s,u) \mid s \in P,\ u \in \mathcal{M}_{0,0}(J_s, A)\}$ is a smooth k-dimensional manifold, and this will be true no matter how the family $\{J_s\}$ is chosen. But for each of the elements $(s,u) \in \mathcal{M}(\{J_s\})$

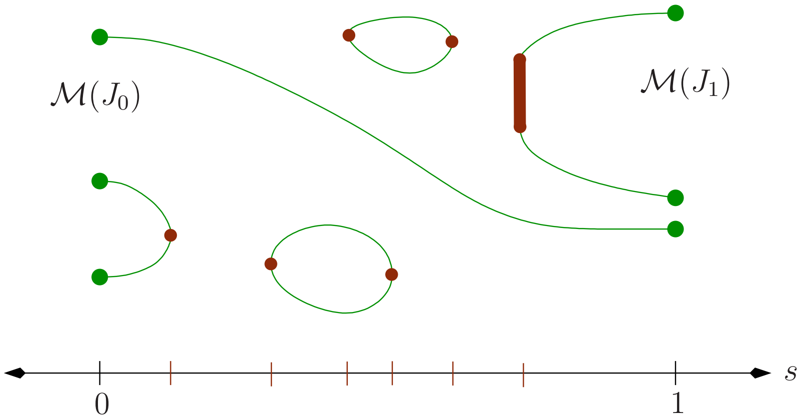

FIGURE 7.1. The picture shows a smooth parametric moduli space $\mathcal{M}(\{J_s\}_{s\in[0,1]})$ and its projection $\mathcal{M}(\{J_s\}) \to [0,1] : (s,u) \mapsto s$ in a case where vir-dim $\mathcal{M}(J_s) = 0$. The parametric moduli space is 1-dimensional and the spaces $\mathcal{M}(J_s)$ are regular and 0-dimensional for almost every $s \in [0,1]$, but this need not hold when s is a critical value of the projection; in the picture, one such space $\mathcal{M}(J_s)$ contains a 1-dimensional component consisting of non-regular curves, so its dimension differs from its virtual dimension.

parametrized by a J-holomorphic map $u : (S^2 = \mathbb{C} \cup \{\infty\}, i) \to (W, J_s)$, there is also a double cover

$$u' : S^2 \to W : z \mapsto u(z^2),$$

with $[u'] = 2A$, so $u' \in \mathcal{M}_{0,0}(J_s, 2A)$ and

$$\text{vir-dim}\, \mathcal{M}_{0,0}(J_s, 2A) = 2 - 2g + 2c_1(2A) = -2.$$

Negative virtual dimension means that $\mathcal{M}_{0,0}(J_0, 2A)$ should be empty whenever Fredholm regularity is achieved, but this is clearly impossible, even generically, since elements of $\mathcal{M}_{0,0}(J_s, A)$ always have double covers belonging to $\mathcal{M}_{0,0}(J_s, 2A)$.

REMARK 7.6. The most common way to apply Theorem 7.2 is by setting $\mathcal{U}$ equal to the interior of W^{r_0}, so generic perturbations of J are allowed everywhere except on the regions where it is required to be $\mathbb{R}$-invariant. The theorem then achieves transversality for all simple curves that are not confined to the $\mathbb{R}$-invariant regions. We will show in the next lecture that transversality for all curves of the latter type can also be achieved by generic perturbations within the spaces $\mathcal{J}(\mathcal{H}_\pm)$ of compatible $\mathbb{R}$-invariant almost complex structures on the symplectizations $\mathbb{R} \times M_\pm$, hence generic choices in $\mathcal{J}(\omega_\psi, r_0, \mathcal{H}_+, \mathcal{H}_-)$ do achieve transversality for all simple curves.

Our proofs of Theorems 7.1 and 7.2 will mostly follow the same line of argument that is carried out for the closed case in [**Wend**, Chapter 4], thus we will not discuss every detail but will instead emphasize aspects which are unique to the punctured case.

7.2. Functional analytic setup

Fix $k \in \mathbb{N}$ and $p \in (1, \infty)$ with $kp > 2$, a small number $\delta \geq 0$, and a Riemannian metric on $\widehat{W}$ that is translation-invariant in the cylindrical ends. Fix also a closed connected surface Σ of genus g, and disjoint finite ordered sets of distinct points

$$\Gamma^{\pm} = (z_1^{\pm}, \dots, z_{k_{\pm}}^{\pm}), \qquad \Theta = (\zeta_1, \dots, \zeta_m)$$

in Σ, together with disjoint neighborhoods

$$\mathcal{U}_j^{\pm} \subset \Sigma$$

of each $z_j^{\pm} \in \Gamma^{\pm}$ with complex structures j_{Γ} and biholomorphic identifications of $(\mathcal{U}_j^{\pm}, j_{\Gamma}, z_j)$ with $(\mathbb{D}, i, 0)$ for each $j = 1, \dots, k_{\pm}$. This determines holomorphic cylindrical coordinates identifying each of the punctured neighborhoods

$$\dot{\mathcal{U}}_j^{\pm} \subset \dot{\Sigma} := \Sigma \setminus (\Gamma^+ \cup \Gamma^-)$$

biholomorphically with the half-cylinder $Z_{\pm}$.

For reasons that will become clear when we study the linearized Cauchy-Riemann operator in the punctured setting, we will need to consider exponentially weighted Sobolev spaces. Suppose $E \to \dot{\Sigma}$ is an asymptotically Hermitian vector bundle: then the Banach space

$$W^{k,p,\delta}(E) \subset W^{k,p}_{\text{loc}}(E)$$

is defined to consist of sections $\eta \in W^{k,p}_{\text{loc}}(E)$ whose representatives $f : Z_{\pm} \to \mathbb{C}^m$ in cylindrical coordinates $(s,t) \in Z_{\pm}$ and asymptotic trivializations at the ends satisfy

$$\|e^{\pm \delta s} f\|_{W^{k,p}(Z_{\pm})} < \infty. \tag{7.1}$$

The norm of a section $\eta \in W^{k,p,\delta}(E)$ is defined by adding the $W^{k,p}$-norm of η over a large compact subdomain in $\dot{\Sigma}$ to the weighted norms (7.1) for each cylindrical end. If $\delta = 0$, this just produces the usual $W^{k,p}(E)$, but for $\delta > 0$, sections in $W^{k,p,\delta}(E)$ are guaranteed to have exponential decay at infinity.

REMARK 7.7. It is occasionally useful to observe that the definition of $W^{k,p,\delta}(E)$ also makes sense when $\delta < 0$. In this case, sections in $W^{k,p,\delta}(E)$ are of class $W^{k,p}_{\text{loc}}$ but need not be globally in $W^{k,p}(E)$, as they are also allowed to have exponential *growth* at infinity.

We now want to define a Banach manifold of maps $u : \dot{\Sigma} \to \widehat{W}$ that will contain all the asymptotically cylindrical J-holomorphic curves with our particular choice of asymptotic orbits. Recall that the asymptotically cylindrical condition means

$$u(s,t) = \exp_{(T_j^{\pm} s, \gamma_j^{\pm}(t))} h(s,t) \quad \text{for sufficiently large } |s| \tag{7.2}$$

in suitable cylindrical coordinates $(s,t) \in Z_{\pm}$ near each puncture $z_j^{\pm} \in \Gamma^{\pm}$, where $T_j^{\pm} > 0$ is the period of the orbit $\gamma_j^{\pm} : S^1 \to M_{\pm}$ and $h(s,t)$ is a vector field along the trivial cylinder that decays as $s \to \pm\infty$. The catch is that this definition was not formulated with respect to a *fixed* choice of the holomorphic cylindrical coordinates (s,t); in general the coordinates in which (7.2) is valid may depend on u, and different choices of coordinates might be required for different maps. One can show however that any two distinct choices of holomorphic cylindrical coordinates are related to each other by a transformation that converges asymptotically to a constant shift, which implies that for our *fixed* choice of coordinates (s,t), every asymptotically cylindrical map can be assumed to satisfy

$$u(s,t) = \exp_{(T_j^\pm s+a, \gamma_j^\pm(t+b))} h(s,t), \qquad \lim_{s\to\pm\infty} h(s,t) = 0$$

for some constants $a \in \mathbb{R}$ and $b \in S^1$. We therefore define the space

$$\mathcal{B}^{k,p,\delta} := W^{k,p,\delta}(\dot{\Sigma}, \widehat{W}\,; \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-) \subset C^0(\dot{\Sigma}, \widehat{W})$$

to consist of all continuous maps $u : \dot{\Sigma} \to \widehat{W}$ of the form

$$u = \exp_f h,$$

where:

- $f : \dot{\Sigma} \to \widehat{W}$ is smooth and, in our fixed cylindrical coordinates $(s,t) \in Z_\pm$ on neighborhoods of the punctures $z_j^\pm \in \Gamma^\pm$, takes the form
$$f(s,t) = (T_j^\pm s + a, \gamma_j^\pm(t+b)) \quad \text{for } |s| \text{ sufficiently large,}$$
where $a \in \mathbb{R}$ and $b \in S^1$ are arbitrary constants and $T_j^\pm > 0$ is the period of the Reeb orbit $\gamma_j^\pm : S^1 \to M_\pm$;
- $h \in W^{k,p,\delta}(f^*T\widehat{W})$.

Though it is not immediate since $\dot{\Sigma}$ is noncompact, one can generalize the ideas in [**Elĭ67**] to give $\mathcal{B}^{k,p,\delta}$ the structure of a smooth, separable and metrizable Banach manifold. The key point is the condition $kp > 2$, which guarantees the continuous inclusion $W^{k,p,\delta}(f^*T\widehat{W}) \hookrightarrow C^0(f^*T\widehat{W})$ as well as Banach algebra and C^k-continuity properties, cf. Propositions 2.4, 2.7 and 2.8 in Lecture 2. These properties are needed in order to show that the transition maps between pairs of charts of the form $\exp_f h \mapsto h$ are smooth.

The tangent space to $\mathcal{B}^{k,p,\delta}$ at $u \in \mathcal{B}^{k,p,\delta}$ can be written as

$$T_u\mathcal{B}^{k,p,\delta} = W^{k,p,\delta}(u^*T\widehat{W}) \oplus V_\Gamma,$$

where $V_\Gamma \subset \Gamma(u^*T\widehat{W})$ is a non-canonical choice of a $2(k_+ + k_-)$-dimensional vector space of smooth sections asymptotic at the punctures to constant linear combinations of the vector fields spanning the canonical trivialization of the first factor in $T(\mathbb{R} \times M_\pm) = \epsilon \oplus \xi_\pm$, i.e. they point in the $\mathbb{R}$- and $R_\pm$-directions. The space V_Γ appears due to the fact that two distinct elements of $\mathcal{B}^{k,p,\delta}$ are generally asymptotic to collections of trivial cylinders that differ from each other by $k_+ + k_-$ pairs of constant shifts $(a,b) \in \mathbb{R} \times S^1$.

Fix $J \in \mathcal{J}(\omega_\psi, r_0, \mathcal{H}_+, \mathcal{H}_-)$ and a smooth complex structure j on Σ that matches j_Γ in the neighborhoods $\mathcal{U}_j^\pm$ of the punctures. The nonlinear Cauchy-Riemann operator is then defined as a smooth section

$$\bar{\partial}_{j,J} : \mathcal{B}^{k,p,\delta} \to \mathcal{E}^{k-1,p,\delta} : u \mapsto Tu + J \circ Tu \circ j$$

of a Banach space bundle

$$\mathcal{E}^{k-1,p,\delta} \to \mathcal{B}^{k,p,\delta}$$

with fibers

$$\mathcal{E}_u^{k-1,p,\delta} = W^{k-1,p,\delta}(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, u^*T\widehat{W})).$$

The zero set of $\bar{\partial}_{j,J}$ is the set of all maps $u \in \mathcal{B}^{k,p,\delta}$ that are pseudoholomorphic from $(\dot{\Sigma}, j)$ to $(\widehat{W}, J)$. Note that the smoothness of $\bar{\partial}_{j,J}$ depends mainly on the fact that J is smooth. Indeed, in local coordinates $\bar{\partial}_{j,J}$ looks like $u \mapsto \partial_s u + (J \circ u)\partial_t u$, in which the most obviously nonlinear ingredient is $u \mapsto J \circ u$. If J were only of class C^k, then the C^k-continuity property would imply that the map $u \mapsto J \circ u$ sends maps of class $W^{k,p}$ continuously to maps of class $W^{k,p}$, and one can use an inductive argument to show that this map then becomes r-times differentiable if J is of class C^{k+r}, see [**Wend**, Lemma 2.12.5]. Moreover, the fact that $\bar{\partial}_{j,J} u$ satisfies the same exponential weighting condition as u at the cylindrical ends depends on the fact that J is $\mathbb{R}$-invariant near infinity.

For $u \in \bar{\partial}_{j,J}^{-1}(0)$, the linearization $D\bar{\partial}_{j,J}(u) : T_u\mathcal{B}^{k,p,\delta} \to \mathcal{E}_u^{k-1,p,\delta}$ defines a bounded linear operator

$$\mathbf{D}_u : W^{k,p,\delta}(u^*T\widehat{W}) \oplus V_\Gamma \to W^{k-1,p,\delta}(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, u^*T\widehat{W})).$$

We derived a formula for this operator in Lecture 2 and showed that it is of Cauchy-Riemann type. Since V_Γ is finite dimensional, $\mathbf{D}_u$ will be Fredholm if and only if its restriction to the first factor is Fredholm; denote this restriction by

$$\mathbf{D}_\delta : W^{k,p,\delta}(u^*T\widehat{W}) \to W^{k-1,p,\delta}(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, u^*T\widehat{W})),$$

where we've chosen the notation to emphasize the dependence of this operator on the choice of exponential weight $\delta \geq 0$ in the definition of our Banach space. We will see presently why it's important to pay attention to this detail.

To see whether $\mathbf{D}_\delta$ is Fredholm, consider first the special case where u is a trivial cylinder

$$u_\gamma : \mathbb{R} \times S^1 \to \mathbb{R} \times M : (s,t) \mapsto (Ts, \gamma(t))$$

over some Reeb orbit $\gamma : S^1 \to M$ with period $T > 0$ in M with stable Hamiltonian structure $\mathcal{H} = (\omega, \lambda)$ on M. In this case, there is a more convenient way to write down $\mathbf{D}_{u_\gamma}$ than the formula from Lecture 2. To start with, we use the splitting $T(\mathbb{R} \times M) = \epsilon \oplus \xi$ to decompose $u_\gamma^* T(\mathbb{R} \times M) = u_\gamma^*\epsilon \oplus u_\gamma^*\xi$ and thus write $\mathbf{D}_{u_\gamma}$ in block form

$$\mathbf{D}_{u_\gamma} = \begin{pmatrix} \mathbf{D}_{u_\gamma}^\epsilon & \mathbf{D}_{u_\gamma}^{\epsilon\xi} \\ \mathbf{D}_{u_\gamma}^{\xi\epsilon} & \mathbf{D}_{u_\gamma}^\xi \end{pmatrix}.$$

EXERCISE 7.8. Suppose $\mathbf{D} : \Gamma(E) \to \Omega^{0,1}(\dot{\Sigma}, E)$ is a linear Cauchy-Riemann type operator on a vector bundle E with a complex-linear splitting $E = E_1 \oplus E_2$, and

$$\mathbf{D} = \begin{pmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{D}_{21} & \mathbf{D}_{22} \end{pmatrix}$$

is the resulting block decomposition of $\mathbf{D}$. Use the Leibniz rule satisfied by $\mathbf{D}$ to show that $\mathbf{D}_{11}$ and $\mathbf{D}_{22}$ are also Cauchy-Riemann type operators on E_1 and E_2 respectively, while the off-diagonal terms are tensorial, i.e. they commute with multiplication by smooth real-valued functions and thus define bundle maps $\mathbf{D}_{12} : E_2 \to \Lambda^{0,1}T^*\dot{\Sigma} \otimes E_1$ and $\mathbf{D}_{21} : E_1 \to \Lambda^{0,1}T^*\dot{\Sigma} \otimes E_2$.

Now observe that if $u = (u_{\mathbb{R}}, u_M) : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ is another cylinder near u_γ, the nonlinear operator $(\bar{\partial}_{j,J} u)\partial_s = \partial_s u + J\,\partial_t u \in \Gamma(u^*T(\mathbb{R} \times M)) = \Gamma(u^*\epsilon \oplus u^*\xi)$ takes the form

$$(\bar{\partial}_{j,J} u)\partial_s = \begin{pmatrix} \partial_s u_{\mathbb{R}} - \lambda(\partial_t u_M) + i\,(\partial_t u_{\mathbb{R}} + \lambda(\partial_t u_M)) \\ \pi_\xi\,\partial_s u_M + J\pi_\xi\,\partial_t u_M \end{pmatrix},$$

where we are using the canonical trivialization of $u^*\epsilon$ via ∂_r and R to express the top block as a complex-valued function. As we observed in Lecture 3, the bottom block of this expression can be interpreted in terms of the gradient flow of an action functional $\mathcal{A}_{\mathcal{H}} : C^\infty(S^1) \to \mathbb{R}$, with $\nabla \mathcal{A}_{\mathcal{H}}(\gamma) = -J\pi_\xi\,\partial_t\gamma$. Linearizing in the direction of a section $\eta^\xi \in \Gamma(u_\gamma^*\xi)$ and taking the ξ component thus yields an expression involving the Hessian of $\mathcal{A}_{\mathcal{H}}$ at the critical point γ, namely

$$(\mathbf{D}^\xi_{u_\gamma}\eta^\xi)\partial_s = (\partial_s - \mathbf{A}_\gamma)\eta^\xi.$$

To compute the blocks $\mathbf{D}^\epsilon_{u_\gamma}$ and $\mathbf{D}^{\xi\epsilon}_{u_\gamma}$, notice that $\mathbf{D}_{u_\gamma}\eta^\epsilon = 0$ whenever η^ϵ is a constant linear combination of ∂_r and R, as η^ϵ is then the derivative of a smooth family of J-holomorphic reparametrizations of u_γ. This is enough to prove $\mathbf{D}^{\xi\epsilon}_{u_\gamma} = 0$ since the latter is tensorial by Exercise 7.8, and expressing arbitrary sections of $u_\gamma^*\epsilon$ as $f\partial_r + gR$, we can apply the Leibniz rule for $\mathbf{D}^\epsilon_{u_\gamma}$ and conclude

$$(\mathbf{D}^\epsilon_{u_\gamma}\eta^\epsilon)\partial_s = (\partial_s + i\,\partial_t)\eta^\epsilon$$

in the canonical trivialization. To compute the remaining off-diagonal term, one needs to compute $dr(\mathbf{D}_{u_\gamma}\eta^\xi)$ and $\lambda(\mathbf{D}_{u_\gamma}\eta^\xi)$ for an arbitrary section $\eta^\xi \in \Gamma(u_\gamma^*\xi)$, e.g. by picking a smooth family $u_\rho : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ with $\partial_\rho u_\rho|_{\rho=0} = \eta^\xi$ and a connection ∇ and computing

$$dr\big(\,\nabla_\rho(\bar{\partial}_{j,J} u_\rho)\big|_{\rho=0}\big) \quad \text{and} \quad \lambda\big(\,\nabla_\rho(\bar{\partial}_{j,J} u_\rho)\big|_{\rho=0}\big).$$

This calculation is straightforward but unenlightening, so I will leave it as an exercise for now—in the next lecture we'll derive a general formula (see Lemma 8.10), which implies that since $\pi_\xi\,\partial_s u_\gamma \equiv \pi_\xi\,\partial_t u_\gamma \equiv 0$ in the present setting, $\mathbf{D}^{\epsilon\xi}_{u_\gamma} = 0$. All this leads to the formula

$$(\mathbf{D}_{u_\gamma}\eta)\partial_s = \left(\partial_s - \begin{pmatrix} -i\partial_t & 0 \\ 0 & \mathbf{A}_\gamma \end{pmatrix}\right)\eta.$$

Here the upper left block is the "trivial" asymptotic operator acting on the trivial line bundle over S^1. Since every asymptotically cylindrical curve approximates a trivial cylinder near infinity, one can deduce from this calculuation the following:

Proposition 7.9. *The Cauchy-Riemann type operator $\mathbf{D}_u$ on $u^*T\widehat{W}$ is asymptotic at its punctures $z_j^\pm \in \Gamma^\pm$ for $j = 1, \dots, k_\pm$ to the asymptotic operators $(-i\partial_t) \oplus \mathbf{A}_{\gamma_j^\pm}$ on $(\gamma_j^\pm)^*(\epsilon \oplus \xi_\pm)$.*

Perhaps you can now see a problem: even if the orbits $\gamma_j^\pm$ are all nondegenerate, the asymptotic operators $(-i\partial_t) \oplus \mathbf{A}_\gamma$ are degenerate, as they have nontrivial kernel consisting of constant sections in the first (trivial) factor of $(\gamma_j^\pm)^*(\epsilon \oplus \xi_\pm)$. This implies in particular that

$$\mathbf{D}_0 : W^{k,p}(u^*T\widehat{W}) \to W^{k-1,p}(\overline{\operatorname{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, u^*T\widehat{W}))$$

is *not* Fredholm, except of course in the special case where there are no punctures.

The situation is saved by the exponential weight:

LEMMA 7.10. *For every $\delta > 0$ sufficiently small, the operator $\mathbf{D}_\delta$ is Fredholm and has index*

$$\operatorname{ind}(\mathbf{D}_\delta) = n\chi(\Sigma) - (n+1)\#\Gamma + 2c_1^\tau(u^*T\widehat{W}) + \sum_{j=1}^{k_+} \mu_{CZ}^\tau(\gamma_j^+) - \sum_{j=1}^{k_-} \mu_{CZ}^\tau(\gamma_j^-).$$

Moreover, every element of $\mathcal{M}(J)$ can be represented by a map $u \in \mathcal{B}^{k,p,\delta}$.

PROOF. The second claim follows from the exponential decay estimate of Hofer-Wysocki-Zehnder [**HWZ96**] mentioned in the previous lecture, see Proposition 6.13.

To see that $\mathbf{D}_\delta : W^{k,p,\delta} \to W^{k-1,p,\delta}$ is Fredholm and to compute its index, we can identify it with a Cauchy-Riemann type operator from $W^{k,p}$ to $W^{k-1,p}$. Indeed, pick any smooth function $f : \dot{\Sigma} \to \mathbb{R}$ with $f(s,t) = \mp\delta s$ on the cylindrical ends near $\Gamma^\pm$, define Banach space isomorphisms

$$\Phi_\delta : W^{k,p} \to W^{k,p,\delta} : \eta \mapsto e^f \eta,$$
$$\Psi_\delta : W^{k-1,p} \to W^{k-1,p,\delta} : \theta \mapsto e^f \theta,$$

and consider the bounded linear map

$$\mathbf{D}'_\delta := \Psi_\delta^{-1} \mathbf{D}_\delta \Phi_\delta : W^{k,p}(u^*T\widehat{W}) \to W^{k-1,p}(\overline{\operatorname{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, u^*T\widehat{W})).$$

Using the Leibniz rule for $\mathbf{D}_\delta$, it is straightforward to show that $\mathbf{D}'_\delta$ is also a linear Cauchy-Riemann type operator. Moreover, suppose $\mathbf{D}_\delta$ takes the form $\bar{\partial} + S(s,t)$ in coordinates and trivialization on the cylindrical end near $z_j^\pm$, where $S(s,t) \to S_\infty(t)$ as $s \to \pm\infty$ and $\mathbf{A}_{\gamma_j^\pm} = -i\partial_t - S_\infty(t)$. Then $\mathbf{D}'_\delta$ on this same end takes the form

$$\mathbf{D}'_\delta \eta = e^{\pm\delta s}(\bar{\partial} + S(s,t))(e^{\mp\delta s}\eta) = \bar{\partial}\eta + (S(s,t) \mp \delta)\eta$$

and is therefore asymptotic to the perturbed asymptotic operator

$$\tilde{\mathbf{A}}_j^\pm := \left((-i\partial_t) \oplus \mathbf{A}_{\gamma_j^\pm}\right) \pm \delta.$$

The latter is the direct sum of two asymptotic operators $-i\partial_t \pm \delta$ on the trivial line bundle and $\mathbf{A}_{\gamma_j^\pm} \pm \delta$ on $(\gamma_j^\pm)^*\xi_\pm$ respectively. Since $\gamma_j^\pm$ is nondegenerate by assumption and the spectrum of $\mathbf{A}_{\gamma_j^\pm}$ is discrete, we can assume $\ker(\mathbf{A}_{\gamma_j^\pm} \pm \delta)$ remains trivial if $\delta > 0$ is sufficiently small, and the Conley-Zehnder index of this perturbed operator will be the same as without the perturbation. On the other hand, the spectrum of $-i\partial_t$ consists of the integer multiples of 2π, thus $-i\partial_t \pm \delta$ also becomes nondegenerate for any $\delta > 0$ small. Its Conley-Zehnder index can be deduced from the winding numbers of its eigenfunctions using Theorem 3.36 in Lecture 3: $-i\partial_t$ has a 2-dimensional nullspace consisting of sections with winding number 0, and this becomes an eigenspace for the smallest positive eigenvalue if the puncture is positive or the largest negative eigenvalue if the puncture is negative. Theorem 3.36 thus gives

$$\mu_{\mathrm{CZ}}(-i\partial_t \pm \delta) = \mp 1,$$

and therefore,

$$\mu^\tau_{\mathrm{CZ}}(\tilde{\mathbf{A}}^\pm_j) = \mp 1 + \mu^\tau_{\mathrm{CZ}}(\gamma^\pm_j).$$

Plugging this into the general index formula from Lecture 5 then gives the stated result. □

Putting back the missing $2(\#\Gamma)$ dimensions in the domain of $\mathbf{D}_u$, we have:

Corollary 7.11. *For all $\delta > 0$ sufficiently small, the linearized Cauchy-Riemann operator $\mathbf{D}_u : T_u\mathcal{B}^{k,p,\delta} \to \mathcal{E}^{k-1,p,\delta}_u$ is Fredholm with index*

$$\operatorname{ind}(\mathbf{D}_u) = n\chi(\Sigma) - (n-1)\#\Gamma + 2c^\tau_1(u^*T\widehat{W}) + \sum_{j=1}^{k_+} \mu^\tau_{CZ}(\gamma^+_j) - \sum_{j=1}^{k_-} \mu^\tau_{CZ}(\gamma^-_j).$$

7.3. Teichmüller slices

Since the moduli space $\mathcal{M}(J)$ is not defined with reference to any fixed complex structure on the domains $\dot{\Sigma}$, we must build this freedom into the setup. For a more detailed version of the following discussion, see [Wend, §4.2.1].

For any integers $g, \ell \geq 0$, the **moduli space of Riemann surfaces** of genus g with ℓ marked points is a space of equivalence classes

$$\mathcal{M}_{g,\ell} = \{(\Sigma, j, \Theta)\} \,/ \sim$$

where (Σ, j) is a compact connected surface with genus g, $\Theta \subset \Sigma$ is an ordered set of ℓ points and equivalence is defined via biholomorphic maps that preserve the marked points with their ordering. This space has been studied extensively in algebraic geometry, though it can also be understood using the same global analytic methods that we have been applying for $\mathcal{M}(J)$. It is known in particular that $\mathcal{M}_{g,\ell}$ is always a smooth orbifold, and for any $[(\Sigma, j, \Theta)] \in \mathcal{M}_{g,\ell}$, it satisfies

$$\dim \operatorname{Aut}(\Sigma, j, \Theta) - \dim \mathcal{M}_{g,\ell} = 3\chi(\Sigma) - 2\ell, \tag{7.3}$$

where $\operatorname{Aut}(\Sigma, j, \Theta)$ is the group of biholomorphic transformations of (Σ, j) that fix the points in Θ. This group is finite whenever (Σ, j, Θ) is **stable**, meaning $\chi(\Sigma \setminus \Theta) < 0$, and in that case (7.3) turns into the well-known dimension formula

$$\dim \mathcal{M}_{g,\ell} = -3\chi(\Sigma) + 2\ell = 6g - 6 + 2\ell.$$

This is also the dimension of the **Teichmüller space**

$$\mathcal{T}(\Sigma, \Theta) := \mathcal{J}(\Sigma) / \operatorname{Diff}_0(\Sigma, \Theta),$$

where $\mathcal{J}(\Sigma)$ denotes the space of all smooth complex structures on Σ compatible with its orientation, and $\operatorname{Diff}_0(\Sigma, \Theta)$ is the identity component of the group of diffeomorphisms that fix Θ. It is a classical result that $\mathcal{T}(\Sigma, \Theta)$ is a smooth manifold of the same dimension as $\mathcal{M}_{g,\ell}$, and indeed, the latter can be presented as the quotient of the former by the discrete action of the mapping class group of (Σ, Θ).

Equation (7.3) is actually a formula for a Fredholm index. To see how this works, consider first the case $\ell = 0$. The right hand side is then $\chi(\Sigma) + 2c_1(T\Sigma)$, which is, according to Riemann-Roch, the index of the natural Cauchy-Riemann operator on $T\Sigma$ that defines its holomorphic structure. This operator can also be interpreted as the linearization at the identity map of the *nonlinear* Cauchy-Riemann operator for holomorphic maps $(\Sigma, j) \to (\Sigma, j)$, so its kernel is naturally isomorphic to $T_{\mathrm{Id}} \operatorname{Aut}(\Sigma, j)$. Similarly, one can show that the cokernel of this operator is naturally isomorphic to $T_{[j]}\mathcal{T}(\Sigma)$. This discussion remains valid if marked points are included: the main difference is then that the Cauchy-Riemann operator on $T\Sigma$ should be restricted to a space of vector fields that vanish at Θ, defining a 2ℓ-codimensional subspace as the domain and thus reducing the index by 2ℓ.

For a proof of the following, see [**Wend**, Chapter 4] and [**Wen10**, §3.1].

PROPOSITION 7.12. *Given a closed Riemann surface (Σ, j) with a finite ordered set $\Theta \subset \Sigma$, there exists a smooth finite-dimensional submanifold $\mathcal{T} \subset \mathcal{J}(\Sigma)$ with the following properties:*

(1) The map $\mathcal{T} \to \mathcal{T}(\Sigma, \Theta) : j' \mapsto [j']$ is bijective onto a neighborhood of $[j]$ in $\mathcal{T}(\Sigma, \Theta)$;
(2) The subspace $T_j\mathcal{T} \subset \Gamma(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma))$ is complementary in $W^{k-1,p}(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma))$ to the image of the standard Cauchy-Riemann operator of $T\Sigma$ acting on the domain $\{X \in W^{k,p}(T\Sigma) \mid X|_{\Theta} = 0\}$;
(3) Every $j' \in \mathcal{T}$ equals j near Θ and is invariant under the action of $\operatorname{Aut}(\Sigma, j, \Theta)$ by diffeomorphisms on Σ.

□

We will refer to the family $\mathcal{T} \subset \mathcal{J}(\Sigma)$ in this proposition as a **Teichmüller slice through** j.

7.4. Fredholm regularity and the implicit function theorem

We are now in a position to define the necessary regularity condition and prove that a neighborhood of any given regular element $[(\Sigma, j_0, \Gamma^+, \Gamma^-, \Theta, u_0)]$ in $\mathcal{M}(J)$ is an orbifold of the stated dimension. After reparametrizing, we can assume without loss of generality that Σ, $\Gamma^\pm$ and Θ are precisely the data that were fixed in §7.2, and $j_0 \in \mathcal{J}(\Sigma)$ matches j_Γ on our fixed coordinate neighborhoods of $\Gamma^\pm$. We can then choose a Teichmüller slice

$$\mathcal{T} \subset \mathcal{J}(\Sigma)$$

through j_0 as provided by Prop. 7.12, but with j in that statement replaced by j_0 and Θ replaced by $\Gamma^+ \cup \Gamma^- \cup \Theta$. In particular, $\mathcal{T}$ is invariant under the action of the group

$$G_0 := \operatorname{Aut}(\Sigma, j_0, \Gamma^+ \cup \Gamma^- \cup \Theta),$$

and (7.3) now becomes

$$\dim G_0 - \dim \mathcal{T} = 3\chi(\Sigma) - 2(k_+ + k_- + m). \tag{7.4}$$

There is a natural extension of the nonlinear operator $\bar\partial_{j,J}$ in §7.2 to a smooth section

$$\bar\partial_J : \mathcal{T} \times \mathcal{B}^{k,p,\delta} \to \mathcal{E}^{k-1,p,\delta} : (j, u) \mapsto Tu + J \circ Tu \circ j$$

of a Banach space bundle $\mathcal{E}^{k-1,p,\delta} \to \mathcal{T} \times \mathcal{B}^{k,p,\delta}$ with fibers

$$\mathcal{E}^{k-1,p,\delta}_{(j,u)} = W^{k-1,p,\delta}\big(\overline{\mathrm{Hom}}_{\mathbb{C}}((T\dot\Sigma, j), (u^*T\widehat{W}, J))\big).$$

The zero set $\bar{\partial}_J^{-1}(0) \subset \mathcal{T} \times \mathcal{B}^{k,p,\delta}$ consists of pairs (j, u) for which $u : (\dot{\Sigma}, j) \to (\widehat{W}, J)$ is pseudoholomorphic, and it contains (j_0, u_0) by construction. It also admits a natural action of the automorphism group G_0,

$$G_0 \times \bar{\partial}_J^{-1}(0) \to \bar{\partial}_J^{-1}(0) : (\varphi, (j, u)) \mapsto (\varphi^* j, u \circ \varphi),$$

whose stabilizer at (j_0, u_0) is $\mathrm{Aut}(u_0)$, a finite group whenever u_0 is not constant. Observe that any two elements in the same G_0-orbit of $\bar{\partial}_J^{-1}(0)$ define equivalent elements of the moduli space $\mathcal{M}(J)$, as they are related to each other by a biholomorphic reparametrization that fixes the punctures and marked points.

LEMMA 7.13. *The map*

$$\bar{\partial}_J^{-1}(0)/G_0 \to \mathcal{M}(J) : [(j, u)] \mapsto [(\Sigma, j, \Gamma^+, \Gamma^-, \Theta, u)]$$

is a homeomorphism between open neighborhoods of $[(j_0, u_0)]$ *and* $[(\Sigma, j_0, \Gamma^+, \Gamma^-, \Theta, u_0)]$.

PROOF. This depends fundamentally on the same fact underlying the smoothness of Teichmüller space: the action of $\mathrm{Diff}_0(\Sigma, \Gamma^+ \cup \Gamma^- \cup \Theta)$ on $\mathcal{J}(\Sigma)$ is free and proper.[2] See the proof of [**Wend**, Theorem 4.3.6]. □

DEFINITION 7.14. We say that $[(\Sigma, j_0, \Gamma^+, \Gamma^-, \Theta, u_0)]$ is **Fredholm regular** if there exists a choice of Teichmüller slice $\mathcal{T}$ through j_0 such that the linearization

$$D\bar{\partial}_J(j_0, u_0) : T_{j_0}\mathcal{T} \oplus T_{u_0}\mathcal{B}^{k,p,\delta} \to \mathcal{E}^{k-1,p,\delta}_{(j_0,u_0)}$$

is surjective.

One can show that the surjectivity condition in this definition does not actually depend on the choice of Teichmüller slice. This follows from the identification of $T_{j_0}\mathcal{T}$ with the cokernel of the natural Cauchy-Riemann operator on $T\dot{\Sigma}$; see [**Wend**, Lemma 4.3.2].

PROOF OF THEOREM 7.1. The fact that $\mathcal{M}(J)$ is an orbifold in a neighborhood of $[(\Sigma, j_0, \Gamma^+, \Gamma^-, \Theta, u_0)]$ with isotropy group $\mathrm{Aut}(u_0)$ follows from Lemma 7.13 and the implicit function theorem, which gives $\bar{\partial}_J^{-1}(0)$ the structure of a finite-dimensional manifold near (j_0, u_0) if Fredholm regularity is satisfied. There is a bit of work to be done in showing that transition maps relating any two overlapping charts that arise in this way from the implicit function theorem are smooth; for this, we refer again to the proof of Theorem 4.3.6 in [**Wend**] and merely comment that the key ingredient is elliptic regularity.

The dimension of $\mathcal{M}(J)$ is

$$\dim \mathcal{M}(J) = \dim \bar{\partial}_J^{-1}(0) - \dim G_0 = \operatorname{ind} D\bar{\partial}_J(j_0, u_0) - \dim G_0.$$

The restriction of $D\bar{\partial}_J(j_0, u_0)$ to $T_{u_0}\mathcal{B}^{k,p,\delta}$ is the operator $\mathbf{D}_{u_0}$ that we studied in §7.2, hence

$$\operatorname{ind} D\bar{\partial}_J(j_0, u_0) = \dim \mathcal{T} + \operatorname{ind} \mathbf{D}_{u_0}.$$

[2] This is true at least in the stable case, i.e. when $\chi(\dot{\Sigma} \setminus \Theta) < 0$. There are finitely many cases not satisfying this hypothesis, for which the lemma can be proved by more direct arguments since explicit descriptions of both Teichmüller space and the automorphism groups of Riemann surfaces are available; see [**Wen10**, §3.1 and §3.2] for more details.

Using (7.4) to replace $\dim \mathcal{T} - \dim G_0$ and combining this with Corollary 7.11 now gives the stated formula for $\dim \mathcal{M}(J)$. $\square$

7.5. A universal moduli space

The remainder of this lecture is devoted to the proof of Theorem 7.2. The main tool for this purpose is the Sard-Smale theorem [**Sma65**], an infinite-dimensional version of Sard's theorem stating that the regular values of a smooth nonlinear Fredholm map between separable Banach spaces (i.e. a smooth map whose derivative at every point is a Fredholm operator) form a comeager subset of the target space. In order to incorporate perturbations of the almost complex structure into our functional analytic setup, we need to choose a suitable Banach manifold of almost complex structures. All known ways of doing this are in some sense non-ideal, e.g. one could take almost complex structures of class C^k or $W^{k,p}$, but this necessarily introduces non-smooth almost complex structures into the picture, with the consequence that the nonlinear Cauchy-Riemann operator has only finitely many derivatives. That is not the end of the world, and indeed, this is the approach taken in [**MS04**], but I will instead present an approach that was introduced by Floer in [**Flo88b**], in terms of what is now called the "Floer C_ε space". The idea is to work with a Banach manifold that continuously embeds into the space of smooth almost complex structures, so that the nonlinear Cauchy-Riemann operator will always be smooth. It's a nice trick, but the catch is that we obtain a space that is strictly smaller than the actual space of smooth almost complex structures we're interested in, and has a much stronger topology. The C_ε space should be viewed as a useful tool but not a deeply meaningful object—you might notice that while some of the intermediate results stated below depend on its (somewhat ad hoc) definition, Theorem 7.2 does not. This is due to a general trick described in §7.7 below for turning results about C_ε into results about C^∞.

As in the statement of Theorem 7.2, assume $\mathcal{U} \subset W^{r_0}$ is open and $J^{\text{fix}} \in \mathcal{J}(\omega_\psi, r_0, \mathcal{H}_+, \mathcal{H}_-)$. Let

$$\mathcal{J}_{\mathcal{U}} := \left\{ J \in \mathcal{J}(\omega_\psi, r_0, \mathcal{H}_+, \mathcal{H}_-) \;\middle|\; J = J^{\text{fix}} \text{ on } \widehat{W} \setminus \mathcal{U} \right\},$$

and choose any almost complex structure

$$J^{\text{ref}} \in \mathcal{J}_{\mathcal{U}}.$$

We can regard $\mathcal{J}_{\mathcal{U}}$ as a smooth Fréchet manifold with tangent spaces

$$T_{J^{\text{ref}}}\mathcal{J}_{\mathcal{U}} = \left\{ Y \in \Gamma\left(\overline{\text{End}}_{\mathbb{C}}(T\widehat{W}, J^{\text{ref}})\right) \;\middle|\; Y|_{\widehat{W}\setminus\mathcal{U}} \equiv 0 \text{ and } \omega_\psi(\cdot, Y\cdot) + \omega_\psi(Y\cdot, \cdot) \equiv 0 \right\},$$

where the antilinearity of $Y \in T_{J^{\text{ref}}}\mathcal{J}_{\mathcal{U}}$ means that Y is tangent to the space almost complex structures, and the condition relating it to ω_ψ means that these structures are compatible with ω_ψ. One can check that the map

$$Y \mapsto J_Y := \left(\mathbb{1} + \frac{1}{2}J^{\text{ref}}Y\right) J^{\text{ref}} \left(\mathbb{1} + \frac{1}{2}J^{\text{ref}}Y\right)^{-1}$$

maps a neighborhood of $0 \in T_{J^{\mathrm{ref}}}\mathcal{J}_{\mathcal{U}}$ bijectively to a neighborhood of J^{ref} in $\mathcal{J}_{\mathcal{U}}$. We thus fix a sufficiently small constant $c > 0$ and define the space of "C_ε-small perturbations of J^{ref}" by

$$\mathcal{J}_{\mathcal{U}}^{\varepsilon} := \left\{ J_Y \in \mathcal{J}_{\mathcal{U}} \;\middle|\; Y \in T_{J^{\mathrm{ref}}}\mathcal{J}_{\mathcal{U}} \text{ with } \sum_{\ell=0}^{\infty} \varepsilon_\ell \|Y\|_{C^\ell(\overline{\mathcal{U}})} < c \right\},$$

where $\varepsilon := (\varepsilon_\ell)_{\ell=0}^{\infty}$ is a fixed sequence of positive numbers with $\varepsilon_\ell \to 0$ as $\ell \to \infty$. The sum

$$\|Y\|_{C_\varepsilon} := \sum_{\ell=0}^{\infty} \varepsilon_\ell \|Y\|_{C^\ell(\overline{\mathcal{U}})}$$

defines a norm, and the space of smooth sections $Y \in T_{J^{\mathrm{ref}}}\mathcal{J}_{\mathcal{U}}$ for which this norm is finite is then a separable Banach space; see Appendix B for a proof of this statement. This makes $\mathcal{J}_{\mathcal{U}}^{\varepsilon}$ a separable and metrizable Banach manifold, as the map $J_Y \mapsto Y$ can be viewed as a chart identifying it with an open subset of the aforementioned Banach space. Not every $J \in \mathcal{J}_{\mathcal{U}}$ near J^{ref} belongs to $\mathcal{J}_{\mathcal{U}}^{\varepsilon}$, but there is a continuous inclusion

$$\mathcal{J}_{\mathcal{U}}^{\varepsilon} \hookrightarrow \mathcal{J}_{\mathcal{U}},$$

where the latter carries its usual C^∞-topology and $\mathcal{J}_{\mathcal{U}}^{\varepsilon}$ carries the topology induced by the C_ε-norm. By a lemma due to Floer, choosing a sequence ε_ℓ that decays sufficiently fast makes $\mathcal{J}_{\mathcal{U}}^{\varepsilon}$ large enough to contain perturbations in arbitrary directions with arbitrarily small support near arbitrary points in $\mathcal{U}$; see Theorem B.6 in Appendix B for a precise version of this statement and its proof. We will assume from now on that a suitably fast decaying sequence has been fixed.

We now define a **universal moduli space**

$$\mathcal{M}^*(\mathcal{J}_{\mathcal{U}}^{\varepsilon}) := \big\{ (u, J) \;\big|\; J \in \mathcal{J}_{\mathcal{U}}^{\varepsilon},\ u \in \mathcal{M}(J) \text{ and } u \text{ has an injective point mapped into } \mathcal{U} \big\}.$$

The terminology is somewhat unfortunate, as $\mathcal{M}^*(\mathcal{J}_{\mathcal{U}}^{\varepsilon})$ depends on many auxiliary choices such as J^{ref} and $(\varepsilon_\ell)_{\ell=0}^{\infty}$ and thus should not really be thought of as a "universal" object. Nonetheless:

LEMMA 7.15. *The universal moduli space $\mathcal{M}^*(\mathcal{J}_{\mathcal{U}}^{\varepsilon})$ is a smooth separable Banach manifold, and the projection $\mathcal{M}^*(\mathcal{J}_{\mathcal{U}}^{\varepsilon}) \to \mathcal{J}_{\mathcal{U}}^{\varepsilon} : (u, J) \mapsto J$ is smooth.*

PROOF. As in the proof of Theorem 7.1, one can identify $\mathcal{M}^*(\mathcal{J}_{\mathcal{U}}^{\varepsilon})$ locally with the zero set of a smooth section of a Banach space bundle. Suppose $J_0 \in \mathcal{J}_{\mathcal{U}}^{\varepsilon}$ and $[(\Sigma, j_0, \Gamma^+, \Gamma^-, \Theta, u_0)] \in \mathcal{M}(J_0)$ where $u_0 : \dot{\Sigma} \to \widehat{W}$ has an injective point z_0 with $u_0(z_0) \in \mathcal{U}$. Choose a Teichmüller slice $\mathcal{T}$ through j_0 as in Proposition 7.12 and consider the smooth section

$$\bar{\partial} : \mathcal{T} \times \mathcal{B}^{k,p,\delta} \times \mathcal{J}_{\mathcal{U}}^{\varepsilon} \to \mathcal{E}^{k-1,p,\delta} : (j, u, J) \mapsto Tu + J \circ Tu \circ j,$$

where $\mathcal{E}^{k-1,p,\delta}$ is the obvious extension of our previous Banach space bundle to a bundle over $\mathcal{T} \times \mathcal{B}^{k,p,\delta} \times \mathcal{J}_{\mathcal{U}}^{\varepsilon}$. We're assuming as before that $k \in \mathbb{N}$, $1 < p < \infty$,

$kp > 2$, and $\delta > 0$ is small. A neighborhood of (u_0, J_0) in $\mathcal{M}^*(\mathcal{J}^\varepsilon_{\mathcal{U}})$ can then be identified with a neighborhood of $[(j_0, u_0, J_0)]$ in

$$\bar\partial^{-1}(0)/G_0,$$

where $G_0 := \mathrm{Aut}(\Sigma, j_0, \Gamma^+ \cup \Gamma^- \cup \Theta)$ acts on $\bar\partial^{-1}(0)$ by $\varphi \cdot (j, u, J) := (\varphi^* j, u \circ \varphi, J)$. Since u_0 has an injective point, $\mathrm{Aut}(u_0)$ is trivial and the G_0-action at (j_0, u_0, J_0) is therefore free; hence it suffices to show that $\bar\partial^{-1}(0)$ is a smooth Banach manifold near (j_0, u_0, J_0). This follows from the implicit function theorem if we can show that

$$D\bar\partial(j_0, u_0, J_0) : T_{j_0}\mathcal{T} \oplus T_{u_0}\mathcal{B}^{k,p,\delta} \oplus T_{J_0}\mathcal{J}^\varepsilon_{\mathcal{U}} \to \mathcal{E}^{k-1,p,\delta}_{(j_0,u_0,J_0)}$$

is surjective; indeed, the infinite-dimensional implicit function theorem (see [**Lan93**]) requires the additional hypothesis that $D\bar\partial(j_0, u_0, J_0)$ has a bounded right inverse, but this is immediate since the restriction of this operator to the factor $T_{u_0}\mathcal{B}^{k,p,\delta}$ is Fredholm (see Exercise 7.17 below). We claim in fact that

$$\begin{aligned} T_{u_0}\mathcal{B}^{k,p,\delta} \oplus T_{J_0}\mathcal{J}^\varepsilon_{\mathcal{U}} &\to \mathcal{E}^{k-1,p,\delta}_{(j_0,u_0,J_0)} \\ (\eta, Y) &\mapsto D\bar\partial(j_0, u_0, J_0)(0, \eta, Y) = \mathbf{D}_{u_0}\eta + Y \circ Tu_0 \circ j_0 \end{aligned}$$

is surjective. Consider first the case $k = 1$,[3] so we are looking at a bounded linear map

$$W^{1,p,\delta}(u_0^* T\widehat{W}) \oplus V_\Gamma \oplus T_{J_0}\mathcal{J}^\varepsilon_{\mathcal{U}} \to L^{p,\delta}(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot\Sigma, u_0^* T\widehat{W})).$$

Note that the dual of any space of sections of class $L^{p,\delta}$ can be identified with sections of class $L^{q,-\delta}$ for $\frac{1}{p} + \frac{1}{q} = 1$ (recall Remark 7.7). Indeed, choosing a suitable L^2-pairing defines a bounded bilinear map

$$\langle\ ,\ \rangle_{L^2} : L^{p,\delta} \times L^{q,-\delta} \to \mathbb{R}, \tag{7.5}$$

and one can use isomorphisms of the form $L^p \to L^{p,\delta} : \eta \mapsto e^f \eta$ as in the proof of Lemma 7.10 to prove $(L^{p,\delta})^* \cong L^{q,-\delta}$ as a corollary of the standard fact that $(L^p)^* \cong L^q$. With this understood, observe that since $\mathbf{D}_{u_0} : W^{1,p,\delta} \oplus V_\Gamma \to L^{p,\delta}$ is Fredholm, we know by Exercise 7.16 below that the map under consideration has closed range. Thus if it is not surjective, the Hahn-Banach theorem provides a nontrivial element $\theta \in L^{q,-\delta}(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot\Sigma, u_0^* T\widehat{W}))$ that annihilates its image under the pairing (7.5), which amounts to the two conditions

$$\begin{aligned} \langle \mathbf{D}_{u_0}\eta, \theta\rangle_{L^2} &= 0 \text{ for all } \eta \in W^{1,p,\delta}(u_0^* T\widehat{W}) \oplus V_\Gamma, \\ \langle Y \circ Tu_0 \circ j_0, \theta\rangle_{L^2} &= 0 \text{ for all } Y \in T_{J_0}\mathcal{J}^\varepsilon_{\mathcal{U}}. \end{aligned} \tag{7.6}$$

The first relation is valid in particular for all smooth sections η with compact support and thus means that θ is a weak solution to the formal adjoint equation $\mathbf{D}^*_{u_0}\theta = 0$; applying elliptic regularity and the similarity principle, θ is therefore smooth and has only isolated zeroes. We will see however that this contradicts the second relation as long as there exists an injective point $z_0 \in \dot\Sigma$ with $u_0(z_0) \in \mathcal{U}$. Indeed, since the set of injective points with this property is open and zeroes of θ are isolated, let us assume without loss of generality that $\theta(z_0) \neq 0$. Then by a standard lemma in symplectic linear algebra (see [**Wend**, Lemma 4.4.12]), one can find a smooth

[3] Since the present discussion is purely linear, it does not require the assumption $kp > 2$.

section $Y \in T_{J_0}\mathcal{J}_{\mathcal{U}}$ whose value at $u_0(z_0)$ is chosen such that $Y \circ Tu_0 \circ j_0 = \theta$ at z_0, so their pointwise inner product is positive in some neighborhood of z_0. But by Theorem B.6, one can multiply a small perturbation of Y by a bump function to produce a section (still denoted by Y) of class C_ε so that the pointwise inner product of $Y \circ Tu_0 \circ j_0$ with θ is positive near z_0 but vanishes everywhere else; note that this requires the assumption $u_0^{-1}(u_0(z_0)) = \{z_0\}$, so that the value of Y near $u_0(z_0)$ affects the value of $Y \circ Tu_0 \circ j_0$ near z_0 but nowhere else. This violates the second condition in (7.6) and thus completes the proof for $k = 1$. In the general case, suppose $\theta \in W^{k-1,p,\delta}(\overline{\operatorname{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, u_0^*T\widehat{W}))$. Then θ is also of class $L^{p,\delta}$, so surjectivity in the $k = 1$ case implies the existence of $\eta \in W^{1,p,\delta}$ and $Y \in T_{J_0}\mathcal{J}_{\mathcal{U}}^\varepsilon$ with $\mathbf{D}_{u_0}\eta + Y \circ Tu_0 \circ j_0 = \theta$. Since $Y \circ Tu_0 \circ j_0$ is smooth with compact support, one can then use elliptic regularity to show $\eta \in W^{k,p,\delta}$, and this proves surjectivity for arbitrary $k \in \mathbb{N}$ and $p \in (1, \infty)$.

The implicit function theorem now implies that whenever $kp > 2$ so that $\mathcal{B}^{k,p,\delta}$ is a well-defined Banach manifold, $\bar\partial^{-1}(0)$ is a smooth Banach submanifold of $\mathcal{T} \times \mathcal{B}^{k,p,\delta} \times \mathcal{J}_{\mathcal{U}}^\varepsilon$ in a neighborhood of (j_0, u_0, J_0). The projection map

$$\bar\partial^{-1}(0) \to \mathcal{J}_{\mathcal{U}}^\varepsilon : (j, u, J) \mapsto J$$

is also smooth since it is the restriction to a smooth submanifold of the obviously smooth projection map $\mathcal{T} \times \mathcal{B}^{k,p,\delta} \times \mathcal{J}_{\mathcal{U}}^\varepsilon \to \mathcal{J}_{\mathcal{U}}^\varepsilon$. Since G_0 acts freely and properly on $\bar\partial^{-1}(0)$, the quotient $\bar\partial^{-1}/G_0$ then inherits a smooth Banach manifold structure for which the projection is still smooth, and this quotient is identified locally with $\mathcal{M}^*(\mathcal{J}_{\mathcal{U}}^\varepsilon)$. Smoothness of transition maps is shown via the same regularity arguments as in the proof of Theorem 7.1. □

EXERCISE 7.16. Show that if X, Y and Z are Banach spaces, $\mathbf{T} : X \to Y$ is a Fredholm operator and $\mathbf{A} : Z \to Y$ is a bounded linear operator, then the linear map

$$\mathbf{L} : X \oplus Z \to Y : (x, z) \mapsto \mathbf{T}x + \mathbf{A}z$$

has closed range. *Hint: it might help to write* $X = V \oplus \ker \mathbf{T}$ *and* $Y = W \oplus \operatorname{coker} C$ *so that* $C \cong \operatorname{coker} \mathbf{T}$ *and* $V \xrightarrow{\mathbf{T}} W$ *is an isomorphism.*

EXERCISE 7.17. Under the same assumptions as in Exercise 7.16, show that if $\mathbf{T}$ is surjective, then $\mathbf{L}$ has a bounded right inverse.

7.6. Applying the Sard-Smale theorem

We claim now that the smooth map

$$\mathcal{M}^*(\mathcal{J}_{\mathcal{U}}^\varepsilon) \to \mathcal{J}_{\mathcal{U}}^\varepsilon : (u, J) \mapsto J \tag{7.7}$$

is a nonlinear Fredholm map, i.e. its derivative at every point is a Fredholm operator. Using the local identification of $\mathcal{M}^*(\mathcal{J}_{\mathcal{U}}^\varepsilon)$ with $\bar\partial^{-1}(0)/G_0$ as in the proof of Lemma 7.15 and lifting the projection to $\bar\partial^{-1}(0)$, the derivative of $\bar\partial^{-1}(0) \to \mathcal{J}_{\mathcal{U}}^\varepsilon$ at (j_0, u_0, J_0) takes the form

$$\ker D\bar\partial(j_0, u_0, J_0) \to T_{J_0}\mathcal{J}_{\mathcal{U}}^\varepsilon : (y, \eta, Y) \mapsto Y.$$

The Fredholm property for this projection is a consequence of the Fredholm property for $\mathbf{D}_{u_0}$ via the following general lemma, whose proof is a routine matter of linear algebra (cf. [**Wend**, Lemma 4.4.13]):

LEMMA 7.18. *Under the assumptions of Exercise 7.16, suppose* $\mathbf{L}$ *is surjective. Then the projection*

$$\mathbf{\Pi} : \ker \mathbf{L} \to Z : (x, z) \mapsto z$$

has kernel and cokernel isomorphic to the kernel and cokernel respectively of $\mathbf{T} : X \to Y$. □

By the Sard-Smale theorem, the set of regular values of the projection (7.7) is a comeager subset

$$\mathcal{J}_{\mathcal{U}}^{\varepsilon,\mathrm{reg}} \subset \mathcal{J}_{\mathcal{U}}^{\varepsilon},$$

and by Lemma 7.18, every $(u_0, J_0) \in \mathcal{M}^*(\mathcal{J}_{\mathcal{U}}^{\varepsilon})$ with $J \in \mathcal{J}_{\mathcal{U}}^{\varepsilon,\mathrm{reg}}$ then has the property that

$$D\bar{\partial}_{J_0}(j_0, u_0) : T_{j_0}\mathcal{T} \oplus T_{u_0}\mathcal{B}^{k,p,\delta} \to \mathcal{E}^{k-1,p,\delta}_{(j_0,u_0)}$$

is surjective, which means u_0 represents a Fredholm regular element of $\mathcal{M}(J_0)$.

7.7. From C_ε to C^∞

The arguments above would constitute a proof of Theorem 7.2 if we were allowed to replace the space of smooth almost complex structures $\mathcal{J}_{\mathcal{U}}$ with the space $\mathcal{J}_{\mathcal{U}}^{\varepsilon}$ of C_ε-small perturbations of J^{ref}. Let us *define*

$$\mathcal{J}_{\mathcal{U}}^{\mathrm{reg}} \subset \mathcal{J}_{\mathcal{U}}$$

to be the space of all $J \in \mathcal{J}_{\mathcal{U}}$ with the property that all curves in $\mathcal{M}(J)$ that have injective points mapping to $\mathcal{U}$ are Fredholm regular. The theorem claims that this set is comeager in $\mathcal{J}_{\mathcal{U}}$. We can already see at this point that it is dense: indeed, the Baire category theorem implies that $\mathcal{J}_{\mathcal{U}}^{\varepsilon,\mathrm{reg}}$ is dense in $\mathcal{J}_{\mathcal{U}}^{\varepsilon}$, so in particular there exists a sequence $J_\nu \in J_{\mathcal{U}}^{\varepsilon,\mathrm{reg}}$ that converges in to J^{ref} in the C_ε-topology and therefore also in the C^∞-topology. The choice of $J^{\mathrm{ref}} \in \mathcal{J}_{\mathcal{U}}$ in this discussion was arbitrary, so this proves density.

To prove that $\mathcal{J}_{\mathcal{U}}^{\mathrm{reg}}$ is not only dense but also contains a countable intersection of *open* and dense sets in $\mathcal{J}_{\mathcal{U}}$, we can adapt an argument originally due to Taubes. The idea is to present the sets of somewhere injective curves in $\mathcal{M}(J)$ as countable unions of compact subsets $\mathcal{M}_N^*(J)$ for $N \in \mathbb{N}$, and thus present $\mathcal{J}_{\mathcal{U}}^{\mathrm{reg}}$ as a corresponding countable intersection of spaces $\mathcal{J}_{\mathcal{U}}^{\mathrm{reg},N}$ that achieve regularity only for the elements in $\mathcal{M}_N^*(J)$. The compactness of $\mathcal{M}_N^*(J)$ will then permit us to prove that $\mathcal{J}_{\mathcal{U}}^{\mathrm{reg},N}$ is not only dense but also open.

The definition of $\mathcal{M}_N^*(J)$ is motivated in part by the knowledge that spaces of J-holomorphic curves have natural compactifications. We have not yet discussed the compactification $\overline{\mathcal{M}}(J)$ of $\mathcal{M}(J)$, but we have covered enough of the analytical techniques behind this construction to suffice for the present discussion. Recall first that the moduli space of Riemann surfaces $\mathcal{M}_{g,\ell}$ of genus g with ℓ marked points also has a natural compactification whenever $2g + \ell \geq 3$, known as the **Deligne-Mumford compactification**

$$\overline{\mathcal{M}}_{g,\ell} \supset \mathcal{M}_{g,\ell}.$$

The space $\overline{\mathcal{M}}_{g,\ell}$ consists of "nodal" Riemann surfaces, which can be understood as objects that arise from smooth Riemann surfaces with pair-of-pants decompositions in the limit where some of the lengths of the circles separating two pairs of pants from each other may degenerate to 0 (see e.g. [SS92]). We will discuss this in a bit more detail in Lecture 9; for now, all you really need to know is that $\overline{\mathcal{M}}_{g,\ell}$ is a compact and metrizable topological space that contains $\mathcal{M}_{g,\ell}$ as an open subset. Let us fix a metric on $\mathcal{M}_{g,\ell}$ and denote the distance function by dist(,).

Similarly, fix Riemannian metrics on $\widehat{W}$ and $\dot{\Sigma}$ with translation-invariance on the cylindrical ends and use dist(,) to denote the distance functions. For $N \in \mathbb{N}$ and $J \in \mathcal{J}_{\mathcal{U}}$, we define

$$\mathcal{M}^*_N(J) \subset \mathcal{M}(J)$$

to be the set of equivalence classes admitting representatives $(\Sigma, j, \Gamma^+, \Gamma^-, \Theta, u)$ with the following properties:

- The equivalence class in $\mathcal{M}_{g,k_++k_-+m}$ represented by $(\Sigma, j, \Gamma^+ \cup \Gamma^- \cup \Theta)$ lies at a distance of at most $1/N$ from $\overline{\mathcal{M}}_{g,k_++k_-+m} \setminus \mathcal{M}_{g,k_++k_-+m}$;[4]
- $\sup_{z\in\dot{\Sigma}} |du(z)| \leq N$;
- There exists $z_0 \in \dot{\Sigma}$ such that

$$\mathrm{dist}(u(z_0), \widehat{W} \setminus \mathcal{U}) \geq \frac{1}{N}, \qquad |du(z_0)| \geq \frac{1}{N},$$

and

$$\inf_{z\in\dot{\Sigma}\setminus\{z_0\}} \frac{\mathrm{dist}(u(z_0), u(z))}{\mathrm{dist}(z_0, z)} \geq \frac{1}{N}.$$

We observe that every element of $\mathcal{M}^*_N(J)$ has an injective point mapped into $\mathcal{U}$, and conversely, every asymptotically cylindrical J-holomorphic curve with that property belongs to $\mathcal{M}^*_N(J)$ for $N \in \mathbb{N}$ sufficiently large. It is crucial to observe that all three conditions in this definition are *closed* conditions: morally, we are defining $\mathcal{M}^*_N(J)$ to be a closed subset in the compactification of $\mathcal{M}(J)$, and it will therefore be compact.

Define

$$\mathcal{J}^{\mathrm{reg},N}_{\mathcal{U}} \subset \mathcal{J}_{\mathcal{U}}$$

as the set of all $J \in \mathcal{J}_{\mathcal{U}}$ for which every element of $\mathcal{M}^*_N(J)$ is Fredholm regular.

LEMMA 7.19. *For every $N \in \mathbb{N}$, $\mathcal{J}^{\mathrm{reg},N}_{\mathcal{U}}$ is open and dense.*

PROOF. Density is immediate, since we've seen already that every $J \in \mathcal{J}_{\mathcal{U}}$ admits a C^∞-small perturbation that achieves regularity for all curves in $\bigcup_{N\in\mathbb{N}} \mathcal{M}^*_N(J)$. For openness, suppose the contrary: then there exists $J_\infty \in \mathcal{J}^{\mathrm{reg},N}_{\mathcal{U}}$ and a sequence

[4] If the stability condition $2g + k_+ + k_- + m \geq 3$ is not satisfied, one should amend this by asking for the distance condition to hold for some tuple $(\Sigma, j, \Gamma^+ \cup \Gamma^-, \Theta')$, where Θ' is the union of Θ with enough extra marked points to achieve stability.

$J_\nu \in \mathcal{J}_\mathcal{U} \setminus \mathcal{J}_\mathcal{U}^{\mathrm{reg},N}$ with $J_\nu \to J_\infty$ in the C^∞-topology. There must also exist a sequence of curves $u_\nu \in \mathcal{M}_N^*(J_\nu)$ that are not Fredholm regular. By the definition of $\mathcal{M}_N^*(J_\nu)$, they have domains that are uniformly bounded away from the singular part of the Deligne-Mumford space of Riemann surfaces, so we can extract a subsequence for which these domains converge. Similarly, the first derivatives of u_ν are uniformly bounded, implying in particular a uniform $W^{1,p}$-bound locally for some $p > 2$, and elliptic regularity (Theorem 2.22 in Lecture 2) turns this into uniform C^∞-bound and thus a C^∞-convergent subsequence $u_\nu \to u_\infty \in \mathcal{M}_N^*(J_\infty)$. But u_∞ must then be Fredholm regular, which is an open condition, implying that u_ν is also regular for ν sufficiently large, and this is a contradiction. □

PROOF OF THEOREM 7.2. Since the space of all curves in $\mathcal{M}(J)$ with injective points mapped into $\mathcal{U}$ is the union of the spaces $\mathcal{M}_N^*(J)$ for $N \in \mathbb{N}$, we have

$$\mathcal{J}_\mathcal{U}^{\mathrm{reg}} = \bigcap_{N \in \mathbb{N}} \mathcal{J}_\mathcal{U}^{\mathrm{reg},N},$$

which is a countable intersection of open and dense sets. □

LECTURE 8

Transversality in symplectizations

Contents

This lecture is an addendum to the transversality discussion in Lecture 7: we need to prove that Fredholm regularity can also be achieved for generic *translation-invariant* almost complex structures on symplectizations.

8.1. Statement of the theorem and discussion

Theorem 7.2 in the previous lecture stated that generic perturbations of J in a precompact open subset $\mathcal{U}$ of a completed symplectic cobordism suffice to achieve regularity for all simple holomorphic curves that pass through that subset. In the more specialized setting of a symplectization $\mathbb{R} \times M$ with an $\mathbb{R}$-invariant almost complex structure $J \in \mathcal{J}(\mathcal{H})$, we need a more specialized transversality result, as the generic perturbation from Theorem 7.2 cannot be expected to stay in the space $\mathcal{J}(\mathcal{H})$, in particular it will usually not be $\mathbb{R}$-invariant. The following statement refers to a stable Hamiltonian structure $\mathcal{H} = (\omega, \lambda)$ with induced hyperplane distribution $\xi = \ker \lambda$ and Reeb vector field R, and we denote by

$$\pi_\xi : T(\mathbb{R} \times M) \to \xi$$

the projection along the trivial subbundle generated by ∂_r and R. We assume as usual that $\mathcal{M}(J)$ denotes a moduli space of asymptotically cylindrical J-holomorphic curves with a fixed genus and number of marked points, representing a fixed relative homology class and asymptotic to fixed sets of nondegenerate Reeb orbits at its positive and negative punctures.

THEOREM 8.1. *Suppose M is a closed $(2n-1)$-dimensional manifold carrying a stable Hamiltonian structure $\mathcal{H} = (\omega, \lambda)$, $J^{\text{fix}} \in \mathcal{J}(\mathcal{H})$, and*

$$\mathcal{U} \subset M$$

is an open subset. Then there exists a comeager subset

$$\mathcal{J}^{\text{reg}}_{\mathcal{U}} \subset \left\{ J \in \mathcal{J}(\mathcal{H}) \;\middle|\; J = J^{\text{fix}} \text{ on } \mathbb{R} \times (M \setminus \mathcal{U}) \right\}$$

such that for every $J \in \mathcal{J}^{\text{reg}}_{\mathcal{U}}$, every curve $u \in \mathcal{M}(J)$ with a representative $u : \dot{\Sigma} \to \mathbb{R} \times M$ that has an injective point $z \in \dot{\Sigma}$ satisfying

(i) $u(z) \in \mathbb{R} \times \mathcal{U}$,
(ii) $\pi_\xi \circ du(z) \neq 0$, *and*
(iii) $\operatorname{im}(\pi_\xi \circ du(z)) \cap \ker(d\lambda|_\xi) = \{0\}$

is Fredholm regular.

This result is applied most frequently with $\mathcal{U} = M$, in which case the condition $u(z) \in \mathbb{R} \times \mathcal{U}$ is vacuous. The second and third conditions on the injective point z can be rephrased by asking for the linear map

$$d\lambda(\pi_\xi Tu(X), \cdot)|_{\xi_{u(z)}} : \xi_{u(z)} \to \mathbb{R}$$

to be nontrivial for every nonzero $X \in T_z\dot{\Sigma}$. If λ is contact, then this is immediate whenever $\pi_\xi Tu(X) \neq 0$ since $d\lambda|_\xi$ is nondegenerate, and the condition $\pi_\xi Tu(X) \neq 0$ is also easy to achieve:

PROPOSITION 8.2. *If $J \in \mathcal{J}(\mathcal{H})$, then for any connected J-holomorphic curve $u : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$, the section*

$$\pi_\xi \circ du \in \Gamma(\operatorname{Hom}_{\mathbb{C}}(T\dot{\Sigma}, u^*\xi))$$

either is identically zero or has only isolated zeroes.

As you might guess, this result is a consequence of the similarity principle; see §8.2 for a proof. Notice that if $\pi_\xi \circ du \equiv 0$, then u is everywhere tangent to the vector fields ∂_r and R, so if it is asymptotically cylindrical, then it can only be a trivial cylinder or a cover thereof.

PROPOSITION 8.3. *All trivial cylinders over nondegenerate Reeb orbits have index 0 and are Fredholm regular.*

PROOF. Let $u_\gamma : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ denote the trivial cylinder over an orbit $\gamma : S^1 \to M$. The virtual dimension formula proved in Lecture 7 gives

$$\begin{aligned}\operatorname{ind}(u_\gamma) &= (n-3)\chi(\mathbb{R} \times S^1) + 2c_1^\tau(u_\gamma^* T(\mathbb{R} \times M)) + \mu_{\mathrm{CZ}}^\tau(\gamma) - \mu_{\mathrm{CZ}}^\tau(\gamma) \\ &= 2c_1^\tau(u_\gamma^* T(\mathbb{R} \times M)) = 0\end{aligned}$$

since the asymptotic trivialization τ has an obvious extension to a global trivialization of $u_\gamma^*\xi$, and $u_\gamma^* T(\mathbb{R} \times M)$ is globally the direct sum of the latter with the trivial line bundle spanned by ∂_r and R. Using this splitting, the linearized Cauchy-Riemann operator $\mathbf{D}_{u_\gamma}$ can be identified with $\bar{\partial} \oplus (\partial_s - \mathbf{A}_\gamma)$, where

$$\bar{\partial} = \partial_s + i\partial_t : W^{k,p,\delta}(\mathbb{R} \times S^1, \mathbb{C}) \oplus V_\Gamma \to W^{k-1,p,\delta}(\mathbb{R} \times S^1, \mathbb{C})$$

and

$$\partial_s - \mathbf{A}_\gamma : W^{k,p,\delta}(u_\gamma^*\xi) \to W^{k-1,p,\delta}(u_\gamma^*\xi).$$

Here we are assuming without loss of generality that V_Γ is a complex 2-dimensional space of smooth sections of the trivial line bundle spanned by ∂_r and R that are constant near infinity, and we are identifying this with a space of smooth complex-valued functions on $\mathbb{R} \times S^1$. Nondegeneracy implies that $\partial_s - \mathbf{A} : W^{k,p} \to W^{k-1,p}$ is an isomorphism, recall Theorem 4.11 in Lecture 4. Using weight functions as in the proof of Lemma 7.10 to define isomorphisms between $W^{k,p,\delta}$ and $W^{k,p}$, one can identify $\partial_s - \mathbf{A}_\gamma : W^{k,p,\delta} \to W^{k-1,p,\delta}$ with a small perturbation of the same operator $W^{k,p} \to W^{k-1,p}$, hence it is also an isomorphism for $\delta > 0$ sufficiently small. To see that $\bar{\partial} : W^{k,p,\delta} \oplus V_\Gamma \to W^{k-1,p,\delta}$ is also surjective, observe first that its index is 2; this follows from our calculation of $\operatorname{ind}(u_\gamma)$ and corresponds to the fact that $\dim \operatorname{Aut}(\mathbb{R} \times S^1, i) = 2$. The kernel of this operator consists of bounded holomorphic $\mathbb{C}$-valued functions on $\mathbb{R} \times S^1$, so it is precisely the real 2-dimensional space of constant functions, implying

$$\dim_{\mathbb{R}} \operatorname{coker}(\bar{\partial}) = \dim_{\mathbb{R}} \ker(\bar{\partial}) - \operatorname{ind}_{\mathbb{R}}(\bar{\partial}) = 2 - 2 = 0,$$

so $\mathbf{D}_{u_\gamma}$ is surjective. □

COROLLARY 8.4. *For any contact form α on a closed manifold M, there exists a comeager subset $\mathcal{J}^{\text{reg}}(\alpha) \subset \mathcal{J}(\alpha)$ such that for every $J \in \mathcal{J}^{\text{reg}}(\alpha)$, all somewhere injective asymptotically cylindrical J-holomorphic curves in $\mathbb{R} \times M$ are Fredholm regular.* □

Note that in the setting of Corollary 8.4, a curve that is not a cover of a trivial cylinder always belongs to a smooth 1-parameter family of curves related to each other by $\mathbb{R}$-translation, so that the kernel of the linearized Cauchy-Riemann operator automatically has kernel of dimension at least 1. This precludes Fredholm regularity for curves of index 0, thus:

COROLLARY 8.5. *If α is a contact form and $J \in \mathcal{J}^{\text{reg}}(\alpha)$, then all simple asymptotically cylindrical J-holomorphic curves $u : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$ other than trivial cylinders satisfy*

$$\operatorname{ind}(u) \geq 1.$$

□

The following example shows that the third condition on the injective point in Theorem 8.1 cannot be fully removed in general.

EXAMPLE 8.6 (cf. Examples 6.6 and 6.16 in Lecture 6). Assume (W, ω) is a closed symplectic manifold of dimension $2n-2$ with a periodic time-dependent Hamiltonian $H : S^1 \times W \to \mathbb{R}$, and $M := S^1 \times W$ is assigned the stable Hamiltonian structure $(\Omega, \Lambda) := (\omega + dt \wedge dH, dt)$. A choice of $J \in \mathcal{J}(\mathcal{H})$ is then equivalent to a choice of t-dependent family of ω-compatible almost complex structures $\{J_t \in \mathcal{J}(W, \omega)\}_{t \in S^1}$, and for any $t \in S^1$ and $s \in \mathbb{R}$, J_t-holomorphic curves $u : (\Sigma, j) \to (W, J_t)$ give rise to J-holomorphic curves

$$\bar{u} : (\Sigma, j) \to (\mathbb{R} \times M, J) : z \mapsto (s, t, u(z)).$$

In particular, when $n = 2$ one can consider the example where $W = \Sigma$ is a closed surface, so curves of this form exist for any choice of $J \in \mathcal{J}(\mathcal{H})$, no matter how generic (remember that the domain complex structure j is arbitrary, it is not fixed in advance). If Σ has genus g and the map $u : \Sigma \to \Sigma$ has degree 1, then since $\bar{u}$ has no punctures and satisfies $c_1([\bar{u}]) = c_1(\bar{u}^* T(\mathbb{R} \times S^1 \times \Sigma)) = c_1(T\Sigma) = \chi(\Sigma)$, the index of $\bar{u}$ is

$$\operatorname{ind}(\bar{u}) = (n-3)\chi(\Sigma) + 2\chi(\Sigma) = \chi(\Sigma) = 2 - 2g.$$

This shows that $\bar{u}$ cannot be Fredholm regular unless $g = 0$.

Theorem 8.1 appeared for the first time in the contact case in [**Dra04**], and alternative proofs have since appeared in the appendix of [**Bou06**] (for cylinders in the contact case) and in [**Wena**] (under slightly different assumptions in the stable Hamiltonian setting). What I will describe below is a generalization of Bourgeois's proof.

8.2. Injective points of the projected curve

One point of difficulty in proving transversality in $\mathbb{R} \times M$ is that in contrast to the setting of Theorem 7.2, generic perturbations within $\mathcal{J}(\mathcal{H})$ can never be truly local, i.e. if you perturb J near a point $(r, x) \in \mathbb{R} \times M$, then you are also perturbing it in a neighborhood of the entire line $\mathbb{R} \times \{x\}$. We therefore need to know that we can find a point $z \in \dot{\Sigma}$ that is the *only* point where $u : \dot{\Sigma} \to \mathbb{R} \times M$ passes through such a line; put another way, we need to know that not only $u = (u_{\mathbb{R}}, u_M) : \dot{\Sigma} \to \mathbb{R} \times M$ but also the projected map $u_M : \dot{\Sigma} \to M$ is somewhere injective. The first step in showing this is Proposition 8.2 above, as the zeroes of the section

$$\pi_\xi \circ du \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(T\dot{\Sigma}, u^*\xi))$$

are precisely the critical points of $u_M : \dot{\Sigma} \to M$; everywhere else, u_M is an immersion transverse to the Reeb vector field. To prove Proposition 8.2, we shall use the fact that the vector fields ∂_r and R generate an integrable J-invariant distribution on $\mathbb{R} \times M$. Indeed, the zeroes of $\pi_\xi \circ du$ are the points of tangency with this distribution, hence the result is an immediate consequence of the following statement:

LEMMA 8.7. *Suppose (W, J) is an almost complex manifold, $\Xi \subset TW$ is a smooth integrable J-invariant distribution and $u : (\Sigma, j) \to (W, J)$ is a connected pseudo-holomorphic curve whose image is not contained in a leaf of the foliation generated by Ξ. Then all points $z \in \Sigma$ with $\operatorname{im} du(z) \subset \Xi$ are isolated in Σ.*

PROOF. Statement is local, so assume $(\Sigma, j) = (\mathbb{D}, i)$ with coordinates $s + it$, $W = \mathbb{C}^n$, and $u(0) = 0$. Let $2m$ denote the real dimension of Ξ, and observe that since Ξ is integrable, we can change coordinates near 0 and assume without loss of generality that at every point $p \in \mathbb{C}^n$ near 0, $\Xi_p = \mathbb{C}^m \oplus \{0\} \subset \mathbb{C}^n = T_p\mathbb{C}^n$. The J-invariance of Ξ then implies that in coordinates $(w, \zeta) \in \mathbb{C}^m \times \mathbb{C}^{n-m}$, J takes the form

$$J(w, \zeta) = \begin{pmatrix} J_1(w, \zeta) & Y(w, \zeta) \\ 0 & J_2(w, \zeta) \end{pmatrix},$$

where J_1^2 and J_2^2 are both $-\mathbb{1}$, and $J_1 Y + Y J_2 = 0$. Writing $u(z) = (f(z), v(z)) \in \mathbb{C}^m \times \mathbb{C}^{n-m}$, the Cauchy-Riemann equation $\partial_s u + J(u)\partial_t u = 0$ is then equivalent to the two equations

$$\begin{aligned} \partial_s f + J_1(f, v)\, \partial_t f + Y(f, v)\, \partial_t v &= 0, \\ \partial_s v + J_2(f, v)\, \partial_t v &= 0. \end{aligned} \tag{8.1}$$

We have $\operatorname{im} du(z) \subset \Xi$ wherever $\partial_s v = \partial_t v = 0$; notice that it suffices to consider the condition $\partial_s v = 0$ since $\partial_t v = J_2(f, v)\, \partial_s v$. Differentiating the second equation in (8.1) with respect to s gives

$$\partial_s(\partial_s v) + J_2(f, v)\, \partial_t(\partial_s v) + \partial_s \left[J_2(f, v)\right] J_2(f, v)\, \partial_s v = 0,$$

where in the last term we've substituted $J_2(f,v)\,\partial_s v$ for $\partial_t v$. Setting $\bar{J}(z) := J_2(f(z),v(z))$ and $A(z) := \partial_s\,[J_2(f(z),f(z)]\,J_2(f(z),v(z))$, this becomes a linear Cauchy-Riemann type equation $\partial_s(\partial_s v) + \bar{J}\,\partial_t(\partial_s v) + A(\partial_s v) = 0$, so the similarity principle implies that zeroes of $\partial_s v$ are isolated unless it is identically zero. The latter would mean v is constant, so u is contained in a leaf of Ξ. □

LEMMA 8.8. *Suppose $J \in \mathcal{J}(\mathcal{H})$, $\gamma : S^1 \to M$ is a closed Reeb orbit, and $u = (u_{\mathbb{R}}, u_M) : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$ is an asymptotically cylindrical J-holomorphic curve that is not a cover of a trivial cylinder. Then all intersections of the map $u_M : \dot{\Sigma} \to M$ with the image of the orbit γ are isolated.*

PROOF. The trivial cylinder over γ is a J-holomorphic curve, so the statement follows from the fact that two asymptotically cylindrical J-holomorphic curves can only have isolated intersections unless both are covers of the same simple curve. □

We can now prove the statement we need about somewhere injectivity for $u_M : \dot{\Sigma} \to M$. This result first appeared in [**HWZ99**, Theorem 1.13].

PROPOSITION 8.9. *Suppose $J \in \mathcal{J}(\mathcal{H})$ and*

$$u = (u_{\mathbb{R}}, u_M) : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$$

is a simple asymptotically cylindrical J-holomorphic curve which is not a trivial cylinder and has only nondegenerate asymptotic orbits. Then the set of injective points $z \in \dot{\Sigma}$ of the map $u_M : \dot{\Sigma} \to M$ for which $u_M(z)$ is not contained in any of the asymptotic orbits of u is open and dense.

PROOF. Openness is clear, so our main task is to prove density. The idea is first to show via elementary topological arguments that if the set of injective points is not dense, then $\dot{\Sigma}$ contains two disjoint open sets on which u_M is an embedding with identical images. We will then conclude from this that if u is simple, it must be equivalent to one of its nontrivial $\mathbb{R}$-translations, and the latter is impossible for an asymptotically cylindrical curve.

Step 1: We begin by harmlessly removing some discrete sets of points in $\dot{\Sigma}$ that would make the subsequent arguments more complicated. Let

$$P \subset M$$

denote the union of the images of the asymptotic orbits of u, a finite disjoint union of circles. Lemma 8.8 implies that $u_M^{-1}(P)$ is a discrete subset of $\dot{\Sigma}$. By Proposition 8.2, there is also a discrete set $Z \subset \dot{\Sigma} \setminus u_M^{-1}(P)$ containing all points $z \notin u_M^{-1}(P)$ where $\pi_\xi \circ du(z) = 0$, and we claim that

$$Z' := u_M^{-1}(u_M(Z))$$

is a discrete subset of $\dot{\Sigma} \setminus u_M^{-1}(P)$. Indeed, $u_M(Z)$ is a discrete subset of $M \setminus P$ since the points in Z can only accumulate at infinity,[1] hence accumulation points of $u_M(Z) \subset M$ can occur only in P. For each individual point $p \in u_M(Z)$, the fact that $p \notin P$ implies $u_M^{-1}(p)$ is compact, and it consists of a discrete (and therefore finite)

[1]Actually the asymptotic formula of [**HWZ96**] implies that both Z and $u_M^{-1}(P)$ are always finite for curves that are not covers of trivial cylinders, but we do not need to use that here.

set of points with $\pi_\xi \circ du(z) = 0$, plus possibly some other points where $\pi_\xi \circ du(z) \neq 0$, but u_M is an embedding near each point of the latter type, so that these points of $u_M^{-1}(p)$ must always be isolated and are therefore also finite in number. This proves the claim, and we conclude that

$$\ddot{\Sigma} := \dot{\Sigma} \setminus \left(u_M^{-1}(P) \cup Z'\right)$$

an open and dense subset of $\dot{\Sigma}$, as it is obtained by removing a discrete subset from the open and dense subset $\dot{\Sigma} \setminus u_M^{-1}(P)$. To prove the proposition, it will now suffice to prove that the set of points $z \in \ddot{\Sigma}$ which are injective points of $u_M : \dot{\Sigma} \to M$ is dense in $\ddot{\Sigma}$. We shall argue by contradiction and assume from now on that density fails.

Step 2: We will find two open subsets $\mathcal{U}, \mathcal{V} \subset \dot{\Sigma}$ such that u_M restricts to an embedding on both, but

$$\mathcal{U} \cap \mathcal{V} = \emptyset \quad \text{and} \quad u_M(\mathcal{U}) = u_M(\mathcal{V}).$$

Indeed, assume the set of injective points of u_M lying in $\ddot{\Sigma}$ is not dense in $\ddot{\Sigma}$. Then there exists a point $z_0 \in \ddot{\Sigma}$ with a closed neighborhood $\mathcal{D}(z_0) \subset \ddot{\Sigma}$ such that no $z \in \mathcal{D}(z_0)$ is an injective point. Since $z \in \ddot{\Sigma}$ implies $\pi_\xi \circ du(z) \neq 0$, this means that for every $z \in \mathcal{D}(z_0)$, there exists $\zeta \in \dot{\Sigma} \setminus \{z\}$ with $u_M(z) = u_M(\zeta)$, and the definition of $\ddot{\Sigma}$ implies ζ is also in $\ddot{\Sigma}$, hence $\pi_\xi \circ du(\zeta) \neq 0$ and u_M is a local embedding near ζ. Since $u(z) \notin P$ and u_M maps $\dot{\Sigma} \setminus u_M^{-1}(P)$ properly to $M \setminus P$, we also conclude that $u_M^{-1}(u_M(z))$ is finite. Now suppose $u_M^{-1}(u_M(z_0)) = \{z_0, \zeta_1, \ldots, \zeta_m\}$, and let $\mathcal{D}(\zeta_j) \subset \ddot{\Sigma}$ for $j = 1, \ldots, m$ denote closed neighborhoods on which u_M is an embedding. We claim that after possibly shrinking $\mathcal{D}(z_0)$, we can assume

$$u_M(\mathcal{D}(z_0)) \subset \bigcup_{j=1}^m u_M(\mathcal{D}(\zeta_j)).$$

Let us first shrink $\mathcal{D}(z_0)$ so that u_M is an embedding on $\mathcal{D}(z_0)$, which is possible since $\pi_\xi \circ du(z_0) \neq 0$. Then if the claim is false, there exists a sequence $z_\nu \in \mathcal{D}(z_0)$ of noninjective points with $z_\nu \to z_0$, hence there is also a sequence $z'_\nu \in \ddot{\Sigma} \setminus \mathcal{D}(z_0)$ with $u_M(z_\nu) = u_M(z'_\nu)$ but z'_ν not converging to any of $\zeta_1, \ldots, \zeta_m$. But since $u_M(z'_\nu) \to u_M(z_0) \notin P$, the points z'_ν are confined to a compact subset of $\dot{\Sigma}$ and therefore have a subsequence $z'_\nu \to z'_\infty \in \dot{\Sigma}$ with $u_M(z'_\infty) = u_M(z_0)$. The limit cannot be z_0 itself since $z'_\nu \notin \mathcal{D}(z_0)$, thus z'_∞ must be one of the $\zeta_1, \ldots, \zeta_m$, and we have a contradiction. We claim next that at least one of the sets $u_M(\mathcal{D}(z_0)) \cap u_M(\mathcal{D}(\zeta_j))$ has nonempty interior. This is a simple exercise in metric space topology: it can be reduced to the fact that if X is a metric space with closed subsets $V, W \subset X$ that both have empty interior (meaning no open subset of X is contained in V or W), then $V \cup W$ also has empty interior. Since the subsets $u_M(\mathcal{D}(z_0)) \cap u_M(\mathcal{D}(\zeta_j)) \subset u_M(\mathcal{D}(z_0))$ for $j = 1, \ldots, m$ are all closed but their union is $u_M(\mathcal{D}(z_0))$, they cannot all have empty interior. This achieves the goal of Step 2.

Step 3: We show that u is biholomorphically equivalent to one of its $\mathbb{R}$-translations

$$\tau \cdot u := (u_{\mathbb{R}} + \tau, u_M) : \dot{\Sigma} \to \mathbb{R} \times M$$

for $\tau \in \mathbb{R} \setminus \{0\}$. To see this, note that for $J \in \mathcal{J}(\mathcal{H})$, the nonlinear Cauchy-Riemann equation $Tu \circ j = J(u) \circ Tu$ is equivalent to the two equations

$$\begin{aligned} du_{\mathbb{R}} &= u_M^* \lambda \circ j, \\ \pi_\xi \circ Tu_M \circ j &= J(u_M) \circ \pi_\xi \circ Tu_M. \end{aligned} \tag{8.2}$$

Since $\pi_\xi \circ Tu_M : \dot{\Sigma} \to u_M^* \xi$ is injective everywhere on the neighborhoods $\mathcal{U}$ and $\mathcal{V}$, the second equation determines j in terms of J on each of these regions; in particular, the identification of $u_M(\mathcal{U})$ with $u_M(\mathcal{V})$ provides a biholomorphic map of $\mathcal{V}$ to $\mathcal{U}$ so that $u|_{\mathcal{U}}$ and $u|_{\mathcal{V}}$ may be regarded as two J-holomorphic maps from the same Riemann surface which differ only in the $\mathbb{R}$-factor. But with j and u_M both fixed, the first equation in (8.2) determines $du_{\mathbb{R}}$ and thus determines $u_{\mathbb{R}}$ up to the addition of a constant $\tau \in \mathbb{R}$. If $\tau = 0$, this means u has two disjoint regions on which its images are identical, contradicting the assumption that u is simple. Thus $\tau \neq 0$, and since two distinct simple curves can only intersect each other at isolated points, we conclude $u = \tau \cdot u$ up to parametrization.

Step 4: We now derive a contradiction. The relation $u = \tau \cdot u$ implies that in fact $u = k\tau \cdot u$ for every $k \in \mathbb{Z}$, so we obtain a diverging sequence of $\mathbb{R}$-translations $\tau_k \to \infty$ such that u and $\tau_k \cdot u$ always have identical images in $\mathbb{R} \times M$. It follows that for some point $z \in \dot{\Sigma}$ with $u(z) = (r, x)$ where x is not contained in any of the asymptotic orbits of u, the points $(r - \tau_k, x)$ are all in the image of u as $\tau_k \to \infty$. But this contradicts the asymptotically cylindrical behavior of u. $\square$

8.3. Smoothness of the universal moduli space

The overall outline of the proof of Theorem 8.1 is the same as for Theorem 7.2: one needs to define a suitable space $\mathcal{J}_{\mathcal{U}}^{\varepsilon}$ of perturbed almost complex structures, giving rise to a universal moduli space $\mathcal{M}^*(\mathcal{J}_{\mathcal{U}}^{\varepsilon})$ that is a smooth Banach manifold, and then apply the Sard-Smale theorem to conclude that generic elements of $\mathcal{J}_{\mathcal{U}}^{\varepsilon}$ are regular values of the projection $\mathcal{M}^*(\mathcal{J}_{\mathcal{U}}^{\varepsilon}) \to \mathcal{J}_{\mathcal{U}}^{\varepsilon} : (u, J) \mapsto J$. If $\mathcal{J}_{\mathcal{U}}^{\varepsilon}$ is a space of C_ε-perturbed almost complex structures, then in the final step one can use the Taubes trick as in §7.7 to transform the genericity result in $\mathcal{J}_{\mathcal{U}}^{\varepsilon}$ into a genericity result within the space $\mathcal{J}(\mathcal{H})$ of smooth almost complex structures. The only step that differs meaningfully from what we've already discussed is the smoothness of the universal moduli space, so let us focus on this detail.

Assume $J^{\text{ref}} \in \mathcal{J}(\mathcal{H})$ with $J^{\text{ref}} = J^{\text{fix}}$ outside $\mathbb{R} \times \mathcal{U}$, and $\mathcal{J}_{\mathcal{U}}^{\varepsilon}$ is a Banach manifold of C_ε-small perturbations of J^{ref} in $\mathcal{J}(\mathcal{H})$ that are also fixed outside of $\mathbb{R} \times \mathcal{U}$. The relevant universal moduli space is then defined by

$$\begin{aligned} \mathcal{M}^*(\mathcal{J}_{\mathcal{U}}^{\varepsilon}) := \big\{ (u, J) \bigm| \; & J \in \mathcal{J}_{\mathcal{U}}^{\varepsilon},\ u \in \mathcal{M}(J) \text{ and} \\ & u : \dot{\Sigma} \to \mathbb{R} \times M \text{ has an injective point } z \in \dot{\Sigma} \text{ with} \\ & u(z) \in \mathbb{R} \times \mathcal{U} \text{ and } \operatorname{im}(\pi_\xi \circ du(z)) \cap \ker(d\lambda|_\xi) = \{0\} \big\}. \end{aligned}$$

Notice that both of the constraints satisfied by u at the injective point are open. The local structure of $\mathcal{M}^*(\mathcal{J}_{\mathcal{U}}^{\varepsilon})$ near an element (u_0, J_0) with representative $u_0 : (\dot{\Sigma}, j_0) \to (\mathbb{R} \times M, J_0)$ can again be described via the zero set of a smooth section

$$\bar{\partial} : \mathcal{T} \times \mathcal{B}^{k,p,\delta} \times \mathcal{J}_{\mathcal{U}}^{\varepsilon} \to \mathcal{E}^{k-1,p,\delta} : (j, u, J) \mapsto Tu \circ J \circ Tu \circ j,$$

where $\mathcal{T}$ is a Teichmüller slice through j_0, and it suffices to show that the linearization

$$\mathbf{L} : T_{u_0}\mathcal{B}^{k,p,\delta} \oplus T_{J_0}\mathcal{J}^\varepsilon_{\mathcal{U}} \to \mathcal{E}^{k-1,p,\delta}_{(j_0,u_0,J_0)} : (\eta, Y) \mapsto \mathbf{D}_{u_0}\eta + Y \circ Tu_0 \circ j_0$$

is always surjective. As usual, here we're assuming $k \in \mathbb{N}$, $1 < p < \infty$, and the exponential weight $\delta > 0$ is small but positive so that $\mathbf{D}_{u_0}$ is Fredholm. The image of $\mathbf{L}$ is then closed, and focusing on the $k = 1$ case, if $\mathbf{L}$ is not surjective then there exists a nontrivial element $\theta \in L^{q,-\delta}(\overline{\operatorname{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, u_0^* T(\mathbb{R} \times M)))$ such that

$$\begin{aligned} \langle \mathbf{D}_{u_0}\eta, \theta\rangle_{L^2} &= 0 \text{ for all } \eta \in W^{1,p,\delta}(u_0^*T(\mathbb{R} \times M)) \oplus V_\Gamma, \\ \langle Y \circ Tu_0 \circ j_0, \theta\rangle_{L^2} &= 0 \text{ for all } Y \in T_{J_0}\mathcal{J}^\varepsilon_{\mathcal{U}}. \end{aligned} \tag{8.3}$$

The first condition implies via elliptic regularity and the similarity principle that θ is smooth and has only isolated zeroes. So far this is all the same as in the proof of Theorem 7.2, but the next step is trickier: since perturbing J_0 within $\mathcal{J}(\mathcal{H})$ only changes the action of the almost complex structure on ξ but not on the trivial subbundle generated by ∂_r and R, it is not clear whether the range of values allowed for Y is large enough to force $\langle Y \circ Tu_0 \circ j_0, \theta\rangle_{L^2} > 0$.

To overcome this, let us decompose everything in this picture with respect to the natural splitting

$$T(\mathbb{R} \times M) = \epsilon \oplus \xi,$$

where ϵ denotes the trivial line bundle spanned by ∂_r and R. In particular, the domain and target bundles of the Cauchy-Riemann type operator $\mathbf{D}_{u_0}$ now split as

$$\begin{aligned} u_0^*T(\mathbb{R} \times M) &= u_0^*\epsilon \oplus u_0^*\xi, \\ \overline{\operatorname{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, u_0^*T(\mathbb{R} \times M)) &= \overline{\operatorname{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, u_0^*\epsilon) \oplus \overline{\operatorname{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, u_0^*\xi), \end{aligned}$$

and we shall write $\eta = (\eta^\epsilon, \eta^\xi)$ and $\theta = (\theta^\epsilon, \theta^\xi)$ accordingly. This gives a block decomposition of $\mathbf{D}_{u_0}$ as

$$\mathbf{D}_{u_0}\eta = \begin{pmatrix} (\mathbf{D}_{u_0}\eta)^\epsilon \\ (\mathbf{D}_{u_0}\eta)^\xi \end{pmatrix} = \begin{pmatrix} \mathbf{D}^\epsilon_{u_0} & \mathbf{D}^{\epsilon\xi}_{u_0} \\ \mathbf{D}^{\xi\epsilon}_{u_0} & \mathbf{D}^\xi_{u_0} \end{pmatrix} \begin{pmatrix} \eta^\epsilon \\ \eta^\xi \end{pmatrix}.$$

It is easy to verify that $\mathbf{D}^\epsilon_{u_0}$ and $\mathbf{D}^\xi_{u_0}$ each satisfy suitable Leibniz rules and are thus Cauchy-Riemann type operators on $u_0^*\epsilon$ and $u_0^*\xi$ respectively, while the off-diagonal terms are both tensorial, i.e. zeroth-order operators. Since perturbations of J_0 in $\mathcal{J}(\mathcal{H})$ only change its action on ξ, $Y \in T_{J_0}\mathcal{J}^\varepsilon_{\mathcal{U}}$ now takes the block form

$$Y = \begin{pmatrix} 0 & 0 \\ 0 & Y^\xi \end{pmatrix},$$

where Y^ξ is a C_ε-small section of the bundle $\overline{\operatorname{End}}_{\mathbb{C}}(\xi, J_0)$ over M. Assuming the L^2-pairings are defined so as to respect these splittings, the second condition in (8.3) now becomes

$$\langle Y^\xi \circ \pi_\xi \circ Tu_0 \circ j_0, \theta^\xi\rangle_{L^2} = 0,$$

and given any injective point $z_0 \in \dot{\Sigma}$ of $(u_0)_M : \dot{\Sigma} \to M$ satisfying $u_0(z_0) \in \mathbb{R} \times \mathcal{U}$, we have enough freedom to choose Y^ξ near $\mathbb{R} \times \{u_0(z_0)\}$ such that this pairing becomes positive unless

$$\theta^\xi = 0 \quad \text{near } z_0.$$

It remains to show that θ^ϵ also vanishes near z_0, which will contradict the fact that θ only has isolated zeroes. To this end, notice that the first condition in (8.3) implies via separate choices of the components η^ϵ and η^ξ with support near z_0 that

$$(8.4)\qquad \begin{aligned} \langle \mathbf{D}^\epsilon_{u_0}\eta^\epsilon, \theta^\epsilon\rangle_{L^2} &= 0 \text{ for all } \eta^\epsilon \text{ supported near } z_0,\\ \langle \mathbf{D}^{\epsilon\xi}_{u_0}\eta^\xi, \theta^\epsilon\rangle_{L^2} &= 0 \text{ for all } \eta^\xi \text{ supported near } z_0. \end{aligned}$$

The first of these two conditions gives no new information, since we already know that $\theta = (\theta^\epsilon, 0)$ solves an anti-Cauchy-Riemann equation. To get some information out of the second condition, we will need an explicit formula for $\mathbf{D}^{\epsilon\xi}_{u_0}$.

LEMMA 8.10. *The tensorial operator* $\mathbf{D}^{\epsilon\xi}_{u_0} : u_0^*\xi \to \overline{\operatorname{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, u_0^*\epsilon)$ *takes the form*

$$\mathbf{D}^{\epsilon\xi}_{u_0}\eta^\xi = \left[-d\lambda\big(\eta^\xi, J_0^\xi \circ \pi_\xi \circ Tu(\cdot)\big)\right]\partial_r + \left[d\lambda\big(\eta^\xi, \pi_\xi \circ Tu(\cdot)\big)\right]R.$$

PROOF. As a preliminary step, notice that $-dr \circ J = \lambda$ for any $J \in \mathcal{J}(\mathcal{H})$; indeed, the conditions $J(\xi) = \xi \subset \ker dr$ and $J\partial_r = R$ imply that these two 1-forms have matching values on ∂_r, R and ξ. As a consequence, $\lambda \circ J_0 = dr$, so in particular $\lambda \circ J_0$ is closed.

Choosing local holomorphic coordinates (s,t) in an arbitrary neighborhood in $\dot{\Sigma}$, we have

$$(\mathbf{D}^{\epsilon\xi}_{u_0}\eta^\xi)\partial_s = dr\big((\mathbf{D}_{u_0}\eta^\xi)\partial_s\big)\,\partial_r + \lambda\big((\mathbf{D}_{u_0}\eta^\xi)\partial_s\big)\,R.$$

Extend $u_0 : \dot{\Sigma} \to \mathbb{R} \times M$ to a smooth 1-parameter family of maps $\{u_\rho : \dot{\Sigma} \to \mathbb{R} \times M\}_{\rho\in\mathbb{R}}$ with $\partial_\rho u_\rho|_{\rho=0} = \eta^\xi \in \Gamma(u_0^*\xi)$. Then by the definition of the linearized Cauchy-Riemann operator,

$$(\mathbf{D}_{u_0}\eta^\xi)\partial_s = \nabla_\rho\left(\partial_s u_\rho + J_0(u_\rho)\partial_t u_\rho\right)|_{\rho=0},$$

for any choice of connection ∇ on $\mathbb{R} \times M$. Since $\partial_s u_0 + J_0(u_0)\partial_t u_0 = 0$, we find

$$\begin{aligned} \lambda\big((\mathbf{D}_{u_0}\eta^\xi)\partial_s\big) &= \lambda\big(\nabla_\rho\left(\partial_s u_\rho + J_0(u_\rho)\partial_t u_\rho\right)|_{\rho=0}\big) = \partial_\rho\left[\lambda(\partial_s u_\rho + J_0(u_\rho)\partial_t u_\rho)\right]|_{\rho=0}\\ &= \partial_\rho\left[\lambda(\partial_s u_\rho)\right]|_{\rho=0} + \partial_\rho\left[(\lambda \circ J_0)(\partial_t u_\rho)\right]|_{\rho=0}\\ &= d\lambda(\eta^\xi, \partial_s u) + d(\lambda \circ J_0)(\eta^\xi, \partial_t u)\\ &= d\lambda(\eta^\xi, \pi_\xi \partial_s u), \end{aligned}$$

where we've used the formula

$$d\lambda(X,Y) = \mathcal{L}_X\left[\lambda(Y)\right] - \mathcal{L}_Y\left[\lambda(X)\right] - \lambda([X,Y])$$

and eliminated several terms using the fact that $\lambda(\eta^\xi) = \lambda(J_0\eta^\xi) = 0$ since η^ξ is valued in ξ, plus $d(\lambda \circ J_0) = 0$. A similar computation gives

$$dr\big((\mathbf{D}_{u_0}\eta^\xi)\partial_s\big) = -d\lambda(\eta^\xi, \pi_\xi\partial_t u) = -d\lambda(\eta^\xi, J_0 \circ \pi_\xi \partial_s u),$$

so removing the local coordinates from the picture produces the stated formula. □

The following exercise in symplectic linear algebra shows that this bundle map $u_0^*\xi \to \overline{\operatorname{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, u_0^*\epsilon)$ is surjective on any fiber over a point z with $\pi_\xi \circ du_0(z) \neq 0$. (If you have no patience for the exercise, just convince yourself that it's true whenever $d\lambda|_\xi$ is nondegenerate and tames $J|_\xi$, i.e. the contact case.)

EXERCISE 8.11. Assume V is a finite-dimensional vector space, $X, Y \subset V$ are linearly independent vectors, and Ω is an alternating bilinear form on V. Show that the real-linear map

$$A : V \to \mathbb{C} : v \mapsto \Omega(v, X) + i\Omega(v, Y)$$

is surjective if and only if $\mathrm{Span}(X, Y) \cap \ker \Omega = \{0\}$.
Hint: Under the latter condition, one loses no generality by replacing V with a subspace that is complementary to $\ker \Omega$ *and contains* $\mathrm{Span}(X, Y)$*, in which case* (V, Ω) *becomes a symplectic vector space. Now consider the restriction of A to a* 2*-dimensional subspace transverse to the symplectic complement of* $\mathrm{Span}(X, Y)$*.*

The conclusion of this discussion is that unless θ^ϵ vanishes near z_0, η^ξ can be chosen with support near z_0 so that $\langle \mathbf{D}_{u_0}\eta^\xi, \theta^\epsilon \rangle_{L^2} > 0$, violating the second condition in (8.4). This proves that θ vanishes altogether near z_0 and thus, by unique continuation, $\theta \equiv 0$, a contradiction.

We've proved that the universal moduli space is smooth as claimed. Since the rest of the proof of Theorem 8.1 is the same as in the non-$\mathbb{R}$-invariant case, we leave those details to the reader.

REMARK 8.12. You may have noticed that in both this and the previous lecture, our proof that the universal moduli space is smooth relied on a surjectivity result that was actually stronger than needed: in both cases, we needed to prove that an operator of the form

$$T_{j_0}\mathcal{T} \oplus T_{u_0}\mathcal{B}^{k,p,\delta} \oplus T_{J_0}\mathcal{J}^\varepsilon_{\mathcal{U}} \xrightarrow{\mathbf{L}} \mathcal{E}^{k-1,p,\delta}_{(j_0,u_0,J_0)}$$

was surjective, but we ended up proving that its restriction to the smaller domain $T_{u_0}\mathcal{B}^{k,p,\delta} \oplus T_{J_0}\mathcal{J}^\varepsilon_{\mathcal{U}}$ is already surjective. This technical detail hints at a stronger result that can be proved using these methods: one can show that not only is $\mathcal{M}^*(\mathcal{J}^\varepsilon_{\mathcal{U}})$ smooth but also the **forgetful map**

$$\mathcal{M}^*(\mathcal{J}^\varepsilon_{\mathcal{U}}) \to \mathcal{M}_{g,k_++k_-+m}$$
$$([(\Sigma, j, \Gamma^+, \Gamma^-, \Theta, u)], J) \mapsto [(\Sigma, j, \Gamma^+ \cup \Gamma^- \cup \Theta)]$$

sending a J-holomorphic curve to its underlying domain in the moduli space of Riemann surfaces is a submersion, cf. the blog post [**Wenb**] and its sequel. One can use this to prove generic transversality results for spaces of J-holomorphic curves whose domains are constrained within the moduli space of Riemann surfaces, which can be used to define more elaborate algebraic structures on SFT, e.g. this idea plays a very prominent role in the study of Gromov-Witten invariants.

LECTURE 9

Asymptotics and compactness

Contents

Moduli spaces of pseudoholomorphic curves are generally not compact, but they have natural *compactifications*, obtained by allowing certain types of curves with singular behavior. For closed holomorphic curves, this fact is known as *Gromov's compactness theorem*, and our main goal in this lecture is to state its generalization to punctured curves, which is usually called the *SFT compactness theorem*. The theorem was first proved in [**BEH⁺03**] (see also [**CM05**] for an alternative approach), and we do not have space here to present a complete proof, but we can still describe the main geometric and analytical ideas behind it.

The overarching theme of this lecture is the notion of *bubbling*, of which we will see several examples. Bubbling arises in a natural way from elliptic regularity: recall that in Lecture 2, we proved that whenever $kp > 2$, any uniformly $W^{k,p}$-bounded sequence u_ν of holomorphic curves is also uniformly C^m_{loc}-bounded for every $m \ge \mathbb{N}$ (cf. Theorem 2.22). The Arzelà-Ascoli theorem implies that such sequences have C^∞_{loc}-convergent subsequences, and this is true in particular whenever u_ν is uniformly C^1-bounded, as a C^1-bound implies a $W^{1,p}$-bound with $p > 2$. Let us take note of this fact for future use:

Proposition 9.1. *If (W, J_ν) is a sequence of almost complex manifolds with $J_\nu \to J$ in C^∞, then any uniformly C^1-bounded sequence of J_ν-holomorphic maps $u_\nu : \mathbb{D} \to W$ has a subsequence convergent in C^∞_{loc} on $\mathring{\mathbb{D}}$.*

If one wants to prove compactness for a moduli space of J-holomorphic curves, it therefore suffices in general to establish a C^1-bound. The catch is, of course, that the first derivatives of u_ν might *not* be uniformly bounded, and this is when interesting things are seen to happen: while the sequence u_ν is not compact, it turns out that it becomes compact after removing finitely many points from its domain, and near those points one can take a sequence of reparametrizations to find additional nontrivial holomorphic curves in the limit, the so-called "bubbles". This is one of the ways that the "nodal" curves in Gromov's compactness theorem can arise, and we will see the same phenomenon at work in several other contexts as well.

9.1. Removal of singularities

As an important tool for use in the rest of this lecture, we begin with the following result from [**Gro85**]:

THEOREM 9.2 (Gromov's removable singularity theorem). *Assume (W, ω) is a symplectic manifold with a tame almost complex structure J, and $u : \mathbb{D} \setminus \{0\} \to W$ is a J-holomorphic curve that has its image contained in a compact subset of W and satisfies*

$$\int_{\mathbb{D}\setminus\{0\}} u^*\omega < \infty.$$

Then u admits a smooth extension to $\mathbb{D}$.

We will prove the slightly weaker statement that u has a *continuous* extension. If $\dim_{\mathbb{R}} W = 2$, then the smooth extension follows from this by classical complex analysis; in higher dimensions, one can instead apply results on local elliptic regularity, see e.g. [**MS04**]. We will use as a black box the following additional result from [**Gro85**], which is closely related to a standard result about minimal surfaces:

THEOREM (Gromov's monotonicity lemma [**Gro85**]). *Suppose (W, ω) is a compact symplectic manifold (possibly with boundary), J is an ω-tame almost complex structure, and $B_r(p) \subset W$ denotes the open ball of radius $r > 0$ about $p \in W$ with respect to the Riemannian metric $g(X, Y) := \frac{1}{2}\omega(X, JY) + \frac{1}{2}\omega(Y, JX)$. Then there exist constants $c, R > 0$ such that for all $r \in (0, R)$ and $p \in W$ with $B_r(p) \subset W$, every proper non-constant J-holomorphic curve $u : (\Sigma, j) \to (B_r(p), J)$ passing through p satisfies*

$$\int_\Sigma u^*\omega \ge cr^2.$$

In the above statement, (Σ, j) is assumed to be an arbitrary (generally noncompact) Riemann surface *without boundary*. In applications, one typically has a larger (e.g. closed or punctured) domain Σ' in the picture, and Σ is defined to be the connected component of $u^{-1}(B_r(p)) \subset \Sigma'$ containing some point $z \in u^{-1}(p)$. The main message of the theorem is that u must use up at least a certain amount of energy for every ball whose center it passes through, so e.g. the portion of the curve passing through $B_r(p)$ cannot become arbitrarily "thin" as in Figure 9.1.

Returning to the removable singularity theorem, we shall use the biholomorphic map

$$Z_+ := [0, \infty) \times S^1 \to \mathbb{D} \setminus \{0\} : (s, t) \mapsto e^{-2\pi(s+it)}$$

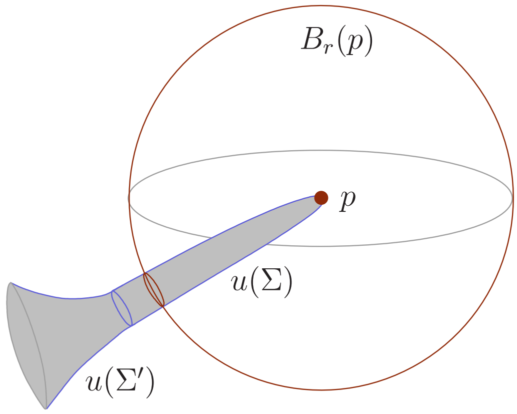

FIGURE 9.1. The intersection of a J-holomorphic curve u with an open ball $B_r(p)$ defines a proper map $\Sigma \to B_r(p)$. The monotonicity lemma prevents this map from having arbitrarily small area if it passes through p.

to transform J-holomorphic maps $\mathbb{D} \setminus \{0\} \to W$ into maps $Z_+ \to W$, and the goal will be to show that whenever such a map u has precompact image and satisfies $\int_{Z_+} u^*\omega < \infty$, there exists a point $p \in W$ such that

$$u(s, \cdot) \to p \quad \text{in} \quad C^\infty(S^1, W) \text{ as } s \to \infty. \tag{9.1}$$

Fix the obvious flat metric on Z_+ and any Riemannian metric on W in order to define norms such as $|du(s,t)|$ for $(s,t) \in Z_+$.

LEMMA 9.3. *There exists a constant $C > 0$ such that $|du(s,t)| \le C$ for all $(s,t) \in Z_+$.*

PROOF, PART 1. Arguing by contradiction, suppose there exists a sequence $z_k = (s_k, t_k) \in Z_+$ with $|du(z_k)| =: R_k \to \infty$. Choose a sequence of positive numbers $\epsilon_k > 0$ that converge to zero but not too fast, so that $\epsilon_k R_k \to \infty$. We then consider the sequence of reparametrized maps

$$v_k : \mathbb{D}_{\epsilon_k R_k} \to W : z \mapsto u(z_k + z/R_k).$$

These are also J-holomorphic since $z \mapsto z_k + z/R_k$ is holomorphic, and the values of v_k depend only on the values of u over the ϵ_k-disk about z_k. Notice that since $s_k \to \infty$ and $\epsilon_k \to 0$, we are free to assume that all of these ϵ_k-disks are disjoint; moreover, tameness of J implies $u^*\omega \ge 0$ and $v_k^*\omega \ge 0$, thus

$$\sum_k \int_{\mathbb{D}_{\epsilon_k R_k}} v_k^*\omega = \sum_k \int_{\mathbb{D}_{\epsilon_k}(z_k)} u^*\omega \le \int_{Z_+} u^*\omega < \infty,$$

implying

$$\int_{\mathbb{D}_{\epsilon_k R_k}} v_k^*\omega \to 0 \quad \text{as} \quad k \to \infty. \tag{9.2}$$

We would now like to say something about a limit of the maps v_k as $k \to \infty$, but this will require a brief pause in the proof, as we don't yet have quite enough information to do so. We know that the v_k are uniformly C^0-bounded since $u(Z_+)$ is contained in a compact subset. It would be ideal if we also had a uniform C^1-bound, as then elliptic regularity (Prop. 9.1) would give a C^∞_{loc} convergent subsequence on the union of all the domains $\mathbb{D}_{\epsilon_k R_k}$, i.e. on the entire plane. We have

$$dv_k(z) = \frac{1}{R_k} du(z_k + z/R_k),$$

hence $|dv_k(0)| = 1$, but we will need to know more about $|du|$ on the rest of $\mathbb{D}_{\epsilon_k}(z_k)$ in order to deduce a C^1-bound for v_k on all of $\mathbb{D}_{\epsilon_k R_k}$. We'll come back to this in a moment. PROOF TO BE CONTINUED...

Here is the auxiliary lemma that is needed to complete the above proof:

LEMMA 9.4 (Hofer). *Suppose (X, d) is a complete metric space, $g : X \to [0, \infty)$ is continuous, $x_0 \in X$ and $\epsilon_0 > 0$. Then there exist $x \in X$ and $\epsilon > 0$ such that,*

(a) $\epsilon \le \epsilon_0$,
(b) $g(x)\epsilon \ge g(x_0)\epsilon_0$,
(c) $d(x, x_0) \le 2\epsilon_0$, and
(d) $g(y) \le 2g(x)$ for all $y \in \overline{B_\epsilon(x)}$.

PROOF. If there is no $x_1 \in \overline{B_{\epsilon_0}(x_0)}$ such that $g(x_1) > 2g(x_0)$, then we can set $x = x_0$ and $\epsilon = \epsilon_0$ and are done. If such a point x_1 does exist, then we set $\epsilon_1 := \epsilon_0/2$ and repeat the above process for the pair (x_1, ϵ_1): that is, if there is no $x_2 \in \overline{B_{\epsilon_1}(x_1)}$ with $g(x_2) > 2g(x_1)$, we set $(x, \epsilon) = (x_1, \epsilon_1)$ and are finished, and otherwise define $\epsilon_2 = \epsilon_1/2$ and repeat for (x_2, ϵ_2). This process must eventually terminate, as otherwise we obtain a Cauchy sequence x_n with $g(x_n) \to \infty$, which is impossible if X is complete. □

PROOF OF LEMMA 9.3, PART 2. Applying Lemma 9.4 to $X = Z_+$ with $g(z) = |du(z)|$, we can replace the original sequences ϵ_k and z_k with new sequences for which all the previously stated properties still hold, but additionally,

$$|du(z)| \le 2|du(z_k)| \quad \text{for all} \quad z \in \mathbb{D}_{\epsilon_k}(z_k).$$

Our sequence of reparametrizations v_k then satisfies

$$|dv_k(z)| \le 2 \quad \text{for all} \quad z \in \mathbb{D}_{\epsilon_k R_k},$$

so by elliptic regularity, v_k has a subsequence convergent in $C^\infty_{\text{loc}}(\mathbb{C})$ to a J-holomorphic map

$$v_\infty : \mathbb{C} \to W$$

which is not constant since $|dv_\infty(0)| = \lim_{k\to\infty} |dv_k(0)| = 1$. Informally, we say that the blow-up of the derivatives at z_k has caused a plane to "bubble off". However, (9.2) implies that for every $R > 0$, one can write $\epsilon_k R_k \ge R$ for k sufficiently large and thus

$$\int_{\mathbb{D}_R} v_\infty^* \omega = \lim_{k\to\infty} \int_{\mathbb{D}_R} v_k^* \omega \le \lim_{k\to\infty} \int_{\mathbb{D}_{\epsilon_k R_k}} v_k^* \omega = 0,$$

implying $\int_{\mathbb{C}} v_\infty^* \omega = 0$. It follows that v_∞ must be constant, so we have a contradiction. □

To obtain the uniform limit of $u(s, \cdot)$ as $s \to \infty$, we now pick any sequence of nonnegative numbers $s_k \to \infty$ and consider the sequence of J-holomorphic half-cylinders

$$u_k : [-s_k, \infty) \times S^1 \to W : (s, t) \mapsto u(s + s_k, t).$$

By Lemma 9.3, these maps are uniformly C^1-bounded, so elliptic regularity gives a subsequence converging in C^∞_{loc} on $\mathbb{R} \times S^1$ to a J-holomorphic cylinder

$$u_\infty : \mathbb{R} \times S^1 \to W.$$

Observe that for any $c > 0$, we can write $-s_k/2 \le -c$ for sufficiently large k and thus compute

$$\begin{aligned}\int_{[-c,c]\times S^1} u_\infty^*\omega &= \lim_{k\to\infty} \int_{[-c,c,]\times S^1} u_k^*\omega \le \lim_{k\to\infty} \int_{[-s_k/2,\infty)\times S^1} u_k^*\omega \\ &= \lim_{k\to\infty} \int_{[s_k/2,\infty)\times S^1} u^*\omega = 0\end{aligned}$$

since $\int_{Z_+} u^*\omega < \infty$. This implies $\int_{\mathbb{R}\times S^1} u_\infty^*\omega = 0$, so u_∞ is a constant map to some point $p \in W$, hence after replacing s_k with a subsequence,

$$u(s_k, \cdot) = u_k(0, \cdot) \to p \quad \text{in} \quad C^\infty(S^1, W) \text{ as } k \to \infty.$$

To finish the proof of (9.1), we need to show that one cannot find two sequences $s_k \to \infty$ and $s_k' \to \infty$ such that $u(s_k, \cdot) \to p$ and $u(s_k', \cdot) \to p'$ for distinct points $p \neq p' \in W$. This is an easy consequence of the monotonicity lemma: indeed, if two such sequences exist, then we can find a sequence $s_k'' \to \infty$ for which the loops $u(s_k'', \cdot)$ alternate between arbitrarily small neighborhoods of p and p'. Since u is continuous, it must then pass through $\partial B_{2r}(p)$ infinitely many times for $r > 0$ sufficiently small, and in fact there exists an infinite sequence of pairwise disjoint neighborhoods $\mathcal{U}_k \subset Z_+$ such that each

$$u|_{\mathcal{U}_k} : \mathcal{U}_k \to B_r(q_k)$$

is a proper map passing through some point $q_k \in \partial B_{2r}(p)$. The monotonicity lemma then implies

$$\int_{Z_+} u^*\omega \ge \sum_k \int_{\mathcal{U}_k} u^*\omega \ge \sum_k cr^2 = \infty,$$

a contradiction.

EXERCISE 9.5. Given an area form ω on $S^2 = \mathbb{C} \cup \{\infty\}$ and a finite subset $\Gamma \subset S^2$, show that a holomorphic function $f : S^2 \setminus \Gamma \to \mathbb{C}$ has an essential singularity at one of its punctures if and only if $\int_{\mathbb{C}} f^*\omega = \infty$.

9.2. Finite energy and asymptotics

As further preparation for the compactness discussion, we now prove the long-awaited converse of the fact that asymptotically cylindrical curves have finite energy. We work in the setting described in §6.2: (W, ω) is a symplectic cobordism with stable boundary $\partial W = -M_- \sqcup M_+$ carrying stable Hamiltonian structures

$\mathcal{H}_\pm = (\omega_\pm, \lambda_\pm)$ with induced hyperplane distributions $\xi_\pm = \ker \lambda_\pm$ and Reeb vector fields $R_\pm$. The completion $(\widehat{W}, \omega_h)$ carries the symplectic structure

$$\omega_h := \begin{cases} d\left(h(r)\lambda_+\right) + \omega_+ & \text{on } [0,\infty) \times M_+ \\ \omega & \text{on } W, \\ d\left(h(r)\lambda_-\right) + \omega_- & \text{on } (-\infty, 0] \times M_-, \end{cases}$$

for some C^0-small smooth function $h(r)$ with $h' > 0$ that is the identity near $r = 0$, and for a fixed constant r_0, we define a compact subset

$$W^{r_0} := ([-r_0, 0] \times M_-) \cup_{M_-} W \cup_{M_+} ([0, r_0] \times M_+) \subset \widehat{W},$$

outside of which our ω_h-tame almost complex structures $J \in \mathcal{J}_\tau(\omega_h, r_0, \mathcal{H}_+, \mathcal{H}_-)$ are required to be translation-invariant and compatible with $\mathcal{H}_\pm$. The **energy** of a J-holomorphic curve $u : (\dot{\Sigma}, j) \to (\widehat{W}, J)$ is defined by

$$E(u) := \sup_{f \in \mathcal{T}(h, r_0)} \int_{\dot{\Sigma}} u^* \omega_f,$$

where

$$\mathcal{T}(h, r_0) := \left\{ f \in C^\infty(\mathbb{R}, (-\epsilon, \epsilon)) \;\middle|\; f' > 0 \text{ and } f \equiv h \text{ near } [-r_0, r_0] \right\}.$$

The constant $\epsilon > 0$ should always be assumed sufficiently small so that if $J_\pm \in \mathcal{J}(\mathcal{H}_\pm)$ and $X \in \xi_\pm$,

$$(\omega_\pm + \kappa\, d\lambda_\pm)(X, J_\pm X) > 0 \quad \text{whenever} \quad X \neq 0 \text{ and } \kappa \in (-2\epsilon, 2\epsilon). \tag{9.3}$$

This condition implies that every $J \in \mathcal{J}_\tau(\omega_h, r_0, \mathcal{H}_+, \mathcal{H}_-)$ is tamed by every ω_f for every $f \in \mathcal{T}(h, r_0)$, thus all J-holomorphic curves satisfy $E(u) \geq 0$, with equality if and only if u is constant.

THEOREM 9.6. *Assume all closed Reeb orbits in $(M_+, \mathcal{H}_+)$ and $(M_-, \mathcal{H}_-)$ are nondegenerate, $J \in \mathcal{J}_\tau(\omega_h, r_0, \mathcal{H}_+, \mathcal{H}_-)$, (Σ, j) is a closed Riemann surface with $\dot{\Sigma} = \Sigma \setminus \Gamma$ for some finite subset $\Gamma \subset \Sigma$, and $u : (\dot{\Sigma}, j) \to (\widehat{W}, J)$ is a J-holomorphic curve such that none of the singularities in Γ are removable and $E(u) < \infty$. Then u is asymptotically cylindrical.*

REMARK 9.7. The theorem also holds in the setting of a symplectization $(\mathbb{R} \times M, J)$ with $J \in \mathcal{J}(\mathcal{H})$ for a stable Hamiltonian structure $\mathcal{H} = (\omega, \lambda)$ on M. The only real difference in this case is the slightly simpler definition of energy,

$$E(u) = \sup_{f \in \mathcal{T}} \int_{\dot{\Sigma}} u^* \omega_f,$$

where $\omega_f := d\big(f(r)\lambda\big) + \omega$ and

$$\mathcal{T} = \left\{ f \in C^\infty(\mathbb{R}, (-\epsilon, \epsilon)) \;\middle|\; f' > 0 \right\}.$$

This change necessitates a few trivial modifications to the proof of Theorem 9.6 given below.

Like removal of singularities, Theorem 9.6 is really a local result, so let us formulate a more precise and more general statement in these terms. Let

$$\dot{\mathbb{D}} := \mathbb{D} \setminus \{0\} \subset \mathbb{C}$$

and define the two biholomorphic maps

$$\begin{aligned} \varphi_+ &: Z_+ := [0, \infty) \times S^1 \to \dot{\mathbb{D}} : (s,t) \mapsto e^{-2\pi(s+it)} \\ \varphi_- &: Z_- := (-\infty, 0] \times S^1 \to \dot{\mathbb{D}} : (s,t) \mapsto e^{2\pi(s+it)}. \end{aligned} \tag{9.4}$$

THEOREM 9.8. *Suppose $J \in \mathcal{J}_\tau(\omega_h, r_0, \mathcal{H}_+, \mathcal{H}_-)$ and $u : \dot{\mathbb{D}} \to \widehat{W}$ is a J-holomorphic map with $E(u) < \infty$. Then either the singularity at $0 \in \mathbb{D}$ is removable or u is a proper map. In the latter case the puncture is either positive or negative, meaning that u maps neighborhoods of 0 to neighborhoods of $\{\pm\infty\} \times M_\pm$, and the puncture has a well-defined* ***charge****, defined as*

$$Q = \lim_{\epsilon \to 0^+} \int_{\partial \mathbb{D}_\epsilon} u^* \lambda_\pm,$$

which satisfies $\pm Q > 0$. Moreover, the map

$$(u_\mathbb{R}(s,t), u_M(s,t)) := u \circ \varphi_\pm(s,t) \in \mathbb{R} \times M_\pm \quad \text{for } (s,t) \in Z_\pm \text{ near infinity}$$

satisfies

$$u_\mathbb{R}(s,\cdot) - Ts \to c \quad \text{in} \quad C^\infty(S^1) \text{ as } s \to \pm\infty$$

for $T := |Q|$ and a constant $c \in \mathbb{R}$, while for every sequence $s_k \to \pm\infty$, one can restrict to a subsequence such that

$$u_M(s_k,\cdot) \to \gamma(T\cdot) \quad \text{in} \quad C^\infty(S^1, M_\pm) \text{ as } k \to \infty$$

for some T-periodic Reeb orbit $\gamma : \mathbb{R}/T\mathbb{Z} \to M_\pm$. If γ is nondegenerate or Morse-Bott, then in fact

$$u_M(s,\cdot) \to \gamma(T\cdot) \quad \text{in} \quad C^\infty(S^1, M_\pm) \text{ as } s \to \pm\infty$$

We will not prove this result in its full strength, as in particular the last step (when γ is nondegenerate or Morse-Bott) requires some asymptotic elliptic regularity results that we do not have space to explain here. Note however that most of the above statement does not require any nondegeneracy assumption at all. The price for this level of generality is that if $s_k, s_k' \to \pm\infty$ are two distinct sequences, then we have no guarantee in general that the two Reeb orbits obtained as limits of subsequences of $u_M(s_k,\cdot)$ and $u_M(s_k',\cdot)$ will be the same; at present, neither an example of this rather unpleasant possibility nor any general argument to rule it out is known. If one of these orbits is assumed to be isolated, however—which is always true when the Reeb vector field is nondegenerate—then we will be able to show that both are the same up to parametrization, hence *geometrically*, $u_M(s,t)$ lies in arbitrarily small neighborhoods of the orbit γ as $s \to \pm\infty$. This turns out to be also true in the more general Morse-Bott setting, though it is then much harder to prove since γ need not be isolated. Once $u_M(s,\cdot)$ is localized near γ, one can use the nondegeneracy condition as we did in the Fredholm theory of Lecture 4 to develop asymptotic regularity results that give much finer control over the behavior of u_M as $s \to \pm\infty$, implying in particular that $u_M(s,\cdot) \to \gamma(T\cdot)$ in $C^\infty(S^1, M_\pm)$.

For details on this step, we refer to the original sources: [HWZ96, HWZ01] for the nondegenerate case, and [HWZ96, Bou02] when the Reeb vector field is Morse-Bott. Those papers deal exclusively with the contact case, but the setting of general stable Hamiltonian structures is also dealt with in [Sie08].

Ignoring the final step for now, the proof of Theorem 9.8 will reuse most of the techniques that we already saw in our proof of removal of singularities in §9.1. The main idea is to use a combination of the monotonicity lemma and bubbling analysis to show that unless u has a removable singularity, it is a proper map, and for any sequence $s_k \to \pm\infty$, the holomorphic half-cylinders defined by

$$u_k(s,t) = u \circ \varphi_\pm(s + s_k, t)$$

on a sequence of increasingly large half-cylinders must have a subsequence converging in $C^\infty_{\text{loc}}(\mathbb{R} \times S^1)$ to either a constant map or a trivial cylinder. The first case will turn out to mean (as in Theorem 9.2) that the puncture is removable, and the second implies asymptotic convergence to a closed Reeb orbit.

One major difference between the proof of Theorem 9.8 and removal of singularities is that since $\widehat{W}$ is noncompact, sequences of curves in $\widehat{W}$ with uniformly bounded first derivatives need not be locally C^0-bounded. This issue will arise both in the bubbling argument to prove $|du_k(s,t)| \le C$ and in the analysis of the sequence u_k itself. In such cases, one can use the $\mathbb{R}$-translation action

$$\tau_c : \mathbb{R} \times M_\pm \to \mathbb{R} \times M_\pm : (r, x) \mapsto (r + c, x) \quad \text{for} \quad c \in \mathbb{R} \tag{9.5}$$

on suitable subsets of the cylindrical ends to replace unbounded sequences with uniformly C^1-bounded sequences of curves mapping into $\mathbb{R} \times M_+$ or $\mathbb{R} \times M_-$. These $\mathbb{R}$-translations are the reason why our definition of energy needs to be something slightly more complicated than just the symplectic area $\int_{\dot{\Sigma}} u^*\Omega$ for a single choice of symplectic form. To understand bubbling in the presence of arbitrarily large $\mathbb{R}$-translations, we will need the following lemma.

LEMMA 9.9. *Suppose $J \in \mathcal{J}(\mathcal{H})$ for some stable Hamiltonian structure $\mathcal{H} = (\omega, \lambda)$ on an odd-dimensional manifold M, and $u : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$ is a J-holomorphic curve satisfying*

$$E(u) < \infty \quad \textit{and} \quad \int_{\dot{\Sigma}} u^*\omega = 0.$$

If $\dot{\Sigma} = \mathbb{C}$, then u is constant. If $\dot{\Sigma} = \mathbb{R} \times S^1$, then u either is constant or is biholomorphically equivalent to a trivial cylinder over a closed Reeb orbit.

PROOF. Denote $\xi = \ker \lambda$ and let

$$\pi_\xi : T(\mathbb{R} \times M) \to \xi$$

denote the projection along the subbundle spanned by ∂_r (the unit vector field in the $\mathbb{R}$-direction) and the Reeb vector field R. Then since ω annihilates both ∂_r and R, for any local holomorphic coordinates (s,t) on a subset of $\dot{\Sigma}$, the compatibility of $J|_\xi$ with $\omega|_\xi$ implies

$$u^*\omega(\partial_s, \partial_t) = \omega(\partial_s u, \partial_t u) = \omega(\partial_s u, J\partial_s u) = \omega(\pi_\xi \partial_s u, J\pi_\xi \partial_s u) \ge 0,$$

hence $\int_{\dot{\Sigma}} u^*\omega \geq 0$ for every J-holomorphic curve, and equality means that u is everywhere tangent to the subbundle spanned by ∂_r and R. This implies that $\operatorname{im} u$ is contained in the image of some J-holomorphic plane of the form

$$u_\gamma : \mathbb{C} \to \mathbb{R} \times M : s + it \mapsto (s, \gamma(t)),$$

where $\gamma : \mathbb{R} \to M$ is a (not necessarily periodic) orbit of R. If γ is not periodic, then u_γ is embedded, hence there exists a unique (and necessarily holomorphic) map $\Phi : (\dot{\Sigma}, j) \to (\mathbb{C}, i)$ such that $u = u_\gamma \circ \Phi$. If on the other hand γ is periodic with minimal period $T > 0$, then u_γ descends to an embedding of the cylinder

$$\hat{u}_\gamma : \mathbb{C}/iT\mathbb{Z} \to \mathbb{R} \times M,$$

and we can view u_γ as a covering map to this embedded cylinder. Now there exists a unique holomorphic map $\Phi : \dot{\Sigma} \to \mathbb{C}/iT\mathbb{Z}$ such that $u = \hat{u}_\gamma \circ \Phi$. If $\dot{\Sigma} = \mathbb{C}$, then since $\pi_1(\mathbb{C}) = 0$ implies that Φ can be lifted to a (necessarily holomorphic) map $\widetilde{\Phi} : \mathbb{C} \to \mathbb{C}$ with $u_\gamma \circ \widetilde{\Phi} = u$. Relabeling symbols, we conclude that in general if $\dot{\Sigma} = \mathbb{C}$, then $u = u_\gamma \circ \Phi$ for a holomorphic map $\Phi : \mathbb{C} \to \mathbb{C}$.

Let us consider all cases in which the factorzation $u = u_\gamma \circ \Phi$ exists, where $\Phi : (\dot{\Sigma}, j) \to (\mathbb{C}, i)$ is holomorphic and $\dot{\Sigma} = \Sigma \setminus \Gamma$ for a closed Riemann surface (Σ, j). We will now use the removable singularity theorem for $\Phi : \dot{\Sigma} \to S^2 \setminus \{0\}$ to show that unless Φ is constant, $\int_{\dot{\Sigma}} u^*\omega_f = \infty$ for suitable choices of $f \in \mathcal{T}$. This integral can be rewritten as

$$\int_{\dot{\Sigma}} u^*\omega_f = \int_{\dot{\Sigma}} \Phi^* u_\gamma^* \omega_f = \int_{\dot{\Sigma}} \Phi^* d\left(f(s)\, dt\right) = \int_{\dot{\Sigma}} \Phi^*\left(f'(s)\, ds \wedge dt\right) \tag{9.6}$$

since $\omega_f = d\big(f(r)\,\lambda\big) + \omega$ and $u_\gamma(s,t) = (s, \gamma(t))$. Since $f' > 0$, $f'(s)\, ds \wedge dt$ is an area form on $\mathbb{C}$ with infinite area. We claim now that for suitable choices of $f \in \mathcal{T}$, one can find an area form Ω on $S^2 = \mathbb{C} \cup \{\infty\}$ such that $\Omega \leq f'(s)\, ds \wedge dt$. To see this, let us change coordinates so that ∞ becomes 0: setting $\Psi : \mathbb{C}^* \to \mathbb{C}^* : z \mapsto 1/z$, a slightly tedious but straightforward computation gives

$$\begin{aligned} \Psi^*\left(f'(s)\, ds \wedge dt\right) &= f'(s/|z|^2) \frac{1}{|z|^4} \left(1 + \frac{(2st)^2}{|z|^4}\right) ds \wedge dt \\ &\geq f'(s/|z|^2) \frac{1}{|z|^4}\, ds \wedge dt \quad \text{for} \quad z = s + it \in \mathbb{C} \setminus \{0\}. \end{aligned} \tag{9.7}$$

We need to show that this 2-form can be bounded away from 0 as $z \to 0$. Let us choose $f \in \mathcal{T}$ such that

$$f(r) = \pm\left(\epsilon - \frac{\epsilon}{2r}\right) \quad \text{for} \quad \pm\, r \geq 1 \tag{9.8}$$

and extend f arbitrarily to $[-1, 1]$ such that $f' > 0$. We can then find a constant $c > 0$ such that f' satisfies

$$f'(r) > \min\left\{c, \frac{\epsilon}{2r^2}\right\} \quad \text{for all} \quad r \in \mathbb{R}.$$

Plugging this into (9.7) gives

$$\Psi^* \left(f'(s)\, ds \wedge dt \right) \geq \min \left\{ \frac{c}{|z|^4}, \frac{\epsilon}{2s^2} \right\} ds \wedge dt,$$

which clearly blows up as $|z| \to 0$. With this established, we observe that for any number $C > 0$, the fact that $f'(s)\, ds \wedge dt$ has infinite area implies we can choose an area form Ω on S^2 with

$$\Omega \leq f'(s)\, ds \wedge dt \text{ on } S^2 \setminus \{\infty\} \qquad \text{and} \qquad \int_{S^2} \Omega > C.$$

We now have two possibilities:

(1) If $\int_{\dot{\Sigma}} \Phi^*\Omega < \infty$, then Theorem 9.2 implies that the singularities of $\Phi : \dot{\Sigma} \to \mathbb{C}$ at Γ are all removable, i.e. Φ extends to a holomorphic map $(\Sigma, j) \to (S^2, i)$, which has a well-defined mapping degree $k \geq 0$. Then

$$\int_{\dot{\Sigma}} u^*\omega_f = \int_{\dot{\Sigma}} \Phi^* \left(f'(s)\, ds \wedge dt \right) \geq \int_{\dot{\Sigma}} \Phi^*\Omega = \int_{\Sigma} \Phi^*\Omega = k \int_{S^2} \Omega > kC.$$

Since $C > 0$ can be chosen arbitrarily large, this implies $\int_{\dot{\Sigma}} u^*\omega_f = \infty$ unless $k = 0$, meaning Φ is constant.

(2) If $\int_{\dot{\Sigma}} \Phi^*\Omega = \infty$ (meaning there is an essential singularity, cf. Exercise 9.5), then since $\Phi^* \left(f'(s)\, ds \wedge dt \right) \geq \Phi^*\Omega$, (9.6) implies $\int_{\mathbb{C}} u^*\omega_f = \infty$.

Since u is constant whenever Φ is, this completes the proof for $\dot{\Sigma} = \mathbb{C}$.

If $\dot{\Sigma} = \mathbb{R} \times S^1$, then it remains to deal with the case where the factorization $u = u_\gamma \circ \Phi$ does not exist because γ is periodic. If the minimal period is $T > 0$, then let us in this case redefine u_γ as an embedded J-holomorphic trivial cylinder

$$u_\gamma : \mathbb{R} \times S^1 : (s,t) \mapsto (Ts, \gamma(Tt)).$$

Since the new u_γ is embedded, we can now write $u = u_\gamma \circ \Phi$ for a unique holomorphic map $\Phi : \mathbb{R} \times S^1 \to \mathbb{R} \times S^1$. Identifying $\mathbb{R} \times S^1$ biholomorphically with $S^2 \setminus \{0, \infty\}$, we claim that Φ extends to a holomorphic map $S^2 \to S^2$. Indeed, by the removable singularity theorem, this is true if and only if $\int_{\mathbb{R}\times S^1} \Phi^*\Omega < \infty$ for some area form Ω on S^2. Notice that $u_\gamma^*\omega_f = T^2 \cdot f'(Ts)\, ds \wedge dt$, defines an area form on $\mathbb{R} \times S^1$ with finite area for any $f \in \mathcal{T}$ since $\int_{-\infty}^{\infty} f'(s)\, ds < \infty$; this is equivalent to the observation that trivial cylinders always have finite energy. Using the biholomorphic map $(s,t) \mapsto e^{2\pi(s+it)}$ to identify $\mathbb{R} \times S^1$ with $\mathbb{C}^* = S^2 \setminus \{0, \infty\}$ and using coordinates $z = x + iy$ on the latter, another tedious but straightforward computation gives

$$u_\gamma^*\omega_f = \frac{T^2}{4\pi^2} \frac{f'\left(\frac{T}{2\pi} \log |z|\right)}{|z|^2}\, dx \wedge dy \qquad \text{for} \qquad z = x + iy \in \mathbb{C}^*.$$

Now suppose $f \in \mathcal{T}$ is chosen as in (9.8). Then one can check that the positive function in front of $dx \wedge dy$ in the above formula goes to $+\infty$ as $|z| \to 0$; this means that one can find an area form Ω on $\mathbb{C}$ with $\Omega \leq u_\gamma^*\omega_f$ on $\mathbb{C}^*$. The singularity at $+\infty \in S^2$ can be handled in a similar way, thus we can find an area form Ω on S^2

such that $\Omega \le u_\gamma^* \omega_f$ on $\mathbb{R} \times S^1$. Now since $E(u) < \infty$, we have

$$\int_{\mathbb{R}\times S^1} \Phi^*\Omega \le \int_{\mathbb{R}\times S^1} \Phi^* u_\gamma^* \omega_f = \int_{\mathbb{R}\times S^1} u^*\omega_f < \infty,$$

so by Theorem 9.2, Φ has a holomorphic extension $S^2 \to S^2$, which is then a map of degree $k \ge 0$ with $\Phi^{-1}(\{0,\infty\}) \subset \{0,\infty\}$. If $k = 0$ then Φ is constant, and so is u. Otherwise, Φ is surjective and thus hits both 0 and ∞, but it can only do this at either 0 or ∞, thus it either fixes both or interchanges them. After composing with a biholomorphic map of S^2 preserving $\mathbb{R} \times S^1$, we may assume without loss of generality that $\Phi(0) = 0$ and $\Phi(\infty) = \infty$. This makes Φ a polynomial with only one zero, hence as a map on $\mathbb{C} \cup \{\infty\}$, $\Phi(z) = cz^k$ for some $c \in \mathbb{C}^*$. Up to biholomorphic equivalence, $\Phi(z)$ is then z^k, which appears in cylindrical coordinates as the map $(s,t) \mapsto (ks, kt)$, so u is now the trivial cylinder

$$u(s,t) = u_\gamma(ks, kt) = (kTs, \gamma(kTt))$$

over the k-fold cover of γ. □

REMARK 9.10. It may be useful for some applications to observe that Lemma 9.9 does not require M to be compact. In contrast, the compactness arguments in this lecture almost always depend on the assumption that W and $M_\pm$ are compact—without this, one would need add some explicit assumption to guarantee local C^0-bounds on sequences of holomorphic curves, e.g. the assumption in Theorem 9.2 that $u(\mathbb{D} \setminus \{0\})$ is contained in a compact subset.

Before continuing, it is worth noting that neither of the two definitions of energy stated above (one for curves in $\widehat{W}$ and the other for symplectizations) is unique, i.e. each can be tweaked in various ways such that the results of this section still hold. Indeed, the original definitions appearing in [**Hof93**, **BEH**$^+$**03**] are slightly different, but equivalent to these. The next lemma illustrates one further example of this freedom, which will be useful in some of the arguments below.

LEMMA 9.11. *Given a stable Hamiltonian structure $\mathcal{H} = (\omega, \lambda)$ on M, a sufficiently small constant $\epsilon > 0$ as in (9.3), and $J \in \mathcal{J}(\mathcal{H})$, consider the alternative notion of energy for J-holomorphic curves $u : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$ defined by*

$$E_0(u) = \sup_{f \in \mathcal{T}_0} \int_{\dot{\Sigma}} u^*\omega_f$$

where $\omega_f = d\,(f(r)\,\lambda) + \omega$ and

$$\mathcal{T}_0 = \left\{ f \in C^\infty(\mathbb{R}, (a,b)) \;\middle|\; f' > 0 \right\}$$

for some constants $-\epsilon \le a < b \le \epsilon$. Then if $E(u)$ denotes the energy as written in Remark 9.7, there exists a constant $c > 0$, depending on the data a, b, ϵ and $\mathcal{H}$ but not on u, such that

$$cE(u) \le E_0(u) \le E(u).$$

PROOF. The second of the two inequalities is immediate since $\mathcal{T}_0 \subset \mathcal{T}$. For the first inequality, note that since $\epsilon > 0$ is small, we can assume there exists a constant $c > 1$ such that for every $X \in T(\mathbb{R} \times M)$ and every $\kappa \in [-\epsilon, \epsilon]$,

$$\frac{1}{c}(\omega + \kappa\, d\lambda)(X, JX) \le \omega(X, JX) \le c(\omega + \kappa\, d\lambda)(X, JX). \tag{9.9}$$

This uses (9.3) and the fact that $d\lambda$ annihilates $\ker \omega$. Now suppose $f \in \mathcal{T}$, choose a constant $\delta \in (0, b - a]$ and define $\tilde{f} \in \mathcal{T}_0$ by

$$\tilde{f}(r) = \frac{\delta}{2\epsilon} f(r) + \frac{a + b}{2}.$$

Then $\tilde{f}'(r) = \frac{\delta}{2\epsilon} f'(r)$, and given a J-holomorphic curve $u : \dot{\Sigma} \to \mathbb{R} \times M$, we can write $\omega_f = \omega + f(r)\, d\lambda + f'(r)\, dr \wedge \lambda$ and use (9.9) to estimate

$$\begin{aligned}
\int_{\dot{\Sigma}} u^*\omega_f &= \int_{\dot{\Sigma}} u^* \left(\omega + f(r)\, d\lambda\right) + \int_{\dot{\Sigma}} u^* \left(f'(r)\, dr \wedge \lambda\right) \\
&\le c \int_{\dot{\Sigma}} u^*\omega + \frac{2\epsilon}{\delta} \int_{\dot{\Sigma}} u^* \left(\tilde{f}'(r)\, dr \wedge \lambda\right) \\
&\le c^2 \int_{\dot{\Sigma}} u^* \left(\omega + \tilde{f}(r)\, d\lambda\right) + \frac{2\epsilon}{\delta} \int_{\dot{\Sigma}} u^* \left(\tilde{f}'(r)\, dr \wedge \lambda\right).
\end{aligned}$$

If $c^2 \ge \frac{2\epsilon}{b-a}$, then we can choose $\delta := 2\epsilon/c^2 \le b - a$ and rewrite the last expression as

$$\begin{aligned}
c^2 &\int_{\dot{\Sigma}} u^* \left(\omega + \tilde{f}(r)\, d\lambda\right) + \frac{2\epsilon}{\delta} \int_{\dot{\Sigma}} u^* \left(\tilde{f}'(r)\, dr \wedge \lambda\right) \\
&= c^2 \int_{\dot{\Sigma}} u^* \left(\omega + \tilde{f}(r)\, d\lambda + \tilde{f}'(r)\, dr \wedge \lambda\right) = c^2 \int_{\dot{\Sigma}} u^*\omega_{\tilde{f}} \le c^2 E_0(u).
\end{aligned}$$

On the other hand if $c^2 < \frac{2\epsilon}{b-a}$, we can set $\delta := b - a$ and write

$$\begin{aligned}
c^2 \int_{\dot{\Sigma}} u^* \left(\omega + \tilde{f}(r)\, d\lambda\right) &+ \frac{2\epsilon}{\delta} \int_{\dot{\Sigma}} u^* \left(\tilde{f}'(r)\, dr \wedge \lambda\right) \\
&\le \frac{2\epsilon}{b - a} \int_{\dot{\Sigma}} u^* \left(\omega + \tilde{f}(r)\, d\lambda + \tilde{f}'(r)\, dr \wedge \lambda\right) \\
&= \frac{2\epsilon}{b - a} \int_{\dot{\Sigma}} u^*\omega_{\tilde{f}} \le \frac{2\epsilon}{b - a} E_0(u).
\end{aligned}$$

□

With this preparation out of the way, we now begin in earnest with the proof of Theorem 9.8. Assume $u : \dot{\mathbb{D}} \to \widehat{W}$ is a J-holomorphic punctured disk satisfying $E(u) < \infty$. Using the maps $\varphi_\pm : Z_\pm \to \dot{\mathbb{D}}$ defined in (9.4), we shall write

$$u_\pm := u \circ \varphi_\pm : Z_\pm \to \widehat{W}$$

and observe that these reparametrizations have no impact on the energy, i.e.

$$E(u_\pm) = \sup_{f \in \mathcal{T}(h, r_0)} \int_{Z_\pm} (u \circ \varphi_\pm)^*\omega_f = \sup_{f \in \mathcal{T}(h, r_0)} \int_{\dot{\mathbb{D}}} u^*\omega_f = E(u).$$

Fix a Riemannian metric on $\widehat{W}$ that is translation-invariant on the cylindrical ends, and fix the standard metric on the half-cylinders $Z_\pm$. We will use these metrics implicitly whenever referring to quantities such as $|du_\pm(z)|$.

LEMMA 9.12. *There exists a constant $C > 0$ such that $|du_+(s,t)| \le C$ for all $(s,t) \in Z_+$.*

PROOF. We use a bubbling argument as in the proof of Lemma 9.3. Suppose the contrary, so there exists a sequence $z_k = (s_k, t_k) \in Z_+$ with $R_k := |du_+(z_k)| \to \infty$. Choose a sequence $\epsilon_k > 0$ with $\epsilon_k \to 0$ but $\epsilon_k R_k \to \infty$, and using Lemma 9.4, assume without loss of generality that

$$|du_+(z)| \le 2R_k \quad \text{for all} \quad z \in \mathbb{D}_{\epsilon_k}(z_k).$$

Define a rescaled sequence of J-holomorphic disks by

$$v_k : \mathbb{D}_{\epsilon_k R_k} \to \widehat{W} : z \mapsto u \circ \varphi_+(z_k + z/R_k).$$

These satisfy $|dv_k| \le 2$ on their domains, but they are not necessarily C^1-bounded since their images may escape to infinity. We distinguish three possibilities, at least one of which must hold:

Case 1: $v_k(0)$ has a bounded subsequence.
Then the corresponding subsequence of $v_k : \mathbb{D}_{\epsilon_k R_k} \to \widehat{W}$ is uniformly C^1-bounded on every compact subset and thus (by elliptic regularity) has a further subsequence convergent in $C^\infty_{\mathrm{loc}}(\mathbb{C})$ to a J-holomorphic plane

$$v_\infty : \mathbb{C} \to \widehat{W}$$

with $|dv_\infty(0)| = \lim_{k\to\infty} |dv_k(0)| = 1$. But by the same argument we used in the proof of Lemma 9.3, the fact that $\int_{Z_+} u_+^*\omega_f < \infty$ for any choice of $f \in \mathcal{T}(h, r_0)$ implies

$$\int_{\mathbb{C}} v_\infty^*\omega_f = 0,$$

hence v_∞ is constant, and this is a contradiction.

Case 2: $v_k(0)$ has a subsequence diverging to $\{+\infty\} \times M_+$.
Restricting to this subsequence, suppose

$$v_k(0) \in \{r_k\} \times M_+,$$

so $r_k \to \infty$, and assume without loss of generality that $r_k > r_0$ for all k. Let $\tilde{R}_k \in (0, \epsilon_k R_k]$ for each k denote the largest radius such that $v_k(\mathbb{D}_{\tilde{R}_k}) \subset (r_0, \infty) \times M_+$. Then $\tilde{R}_k \to \infty$ since $|dv_k|$ is bounded. Now using the $\mathbb{R}$-translation maps τ_r defined in (9.5), define

$$\tilde{v}_k := \tau_{-r_k} \circ v_k|_{\mathbb{D}_{\tilde{R}_k}} : \mathbb{D}_{\tilde{R}_k} \to \mathbb{R} \times M_+.$$

Since we're using a translation-invariant metric on $[r_0, \infty) \times M_+$, $\tilde{v}_k$ is now a uniformly C^1_{loc}-bounded sequence of maps into $\mathbb{R} \times M_+$. Elliptic regularity thus provides a subsequence convergent in $C^\infty_{\mathrm{loc}}(\mathbb{C})$ to a plane

$$v_\infty : \mathbb{C} \to \mathbb{R} \times M_+,$$

which is J_+-holomorphic, where $J_+ \in \mathcal{J}(\mathcal{H}_+)$ denotes the restriction of J to $[r_0, \infty) \times M_+$, extended over $\mathbb{R} \times M_+$ by $\mathbb{R}$-invariance. We claim,

$$E(v_\infty) < \infty \quad \text{and} \quad \int_{\mathbb{C}} v_\infty^* \omega_+ = 0, \tag{9.10}$$

where $E(v_\infty)$ is now defined as in Remark 9.7. By Lemma 9.11, the first part of the claim will follow if we can fix a constant $a \in (-\epsilon, \epsilon)$ and establish a uniform bound

$$\int_{\mathbb{C}} v_\infty^* \Omega_f^+ \le C,$$

with $\Omega_f^+ := \omega_+ + d\big(f(r)\,\lambda_+\big)$, for all smooth and strictly increasing functions $f : \mathbb{R} \to (a, \epsilon)$. For convenience in the following, we shall assume $a > h(r_0)$. Now if f is such a function, then for any $R > 0$,

$$\int_{\mathbb{D}_R} v_\infty^* \Omega_f^+ = \lim_{k\to\infty} \int_{\mathbb{D}_R} v_k^* \tau_{-r_k}^* \Omega_f^+ = \lim_{k\to\infty} \int_{\mathbb{D}_R} v_k^* \Omega_{f_k}^+,$$

where $f_k(r) := f(r - r_k)$. Notice that the dependence of the last integral on f_k is limited to the interval (r_0, ∞) since $v_k(\mathbb{D}_R) \subset (r_0, \infty) \times M_+$. Then since $f > a > h(r_0)$ by assumption, there exists for each k a function $h_k \in \mathcal{T}(h, r_0)$ that matches f_k outside some neighborhood of $(-\infty, r_0]$ and thus satisfies

$$\int_{\mathbb{D}_R} v_k^* \Omega_{f_k}^+ = \int_{\mathbb{D}_R} v_k^* \omega_{h_k} \le \int_{\mathbb{D}_{\epsilon_k R_k}} v_k^* \omega_{h_k} = \int_{\mathbb{D}_{\epsilon_k}(z_k)} u_+^* \omega_{h_k} \le \int_{Z_+} u_+^* \omega_{h_k} \le E(u).$$

This is true for every $R > 0$ and thus proves the first part of (9.10). To establish the second part, fix $R > 0$ again and pick any $f \in \mathcal{T}(h, r_0)$. Observe that since we can assume (after perhaps passing to a subsequence) the disks $\mathbb{D}_{\epsilon_k}(z_k)$ are all disjoint,

$$\begin{aligned} 0 &= \lim_{k\to\infty} \int_{\mathbb{D}_{\epsilon_k}(z_k)} u_+^* \omega_f = \lim_{k\to\infty} \int_{\mathbb{D}_{\epsilon_k R_k}} v_k^* \omega_f = \lim_{k\to\infty} \int_{\mathbb{D}_{\epsilon_k R_k}} \tilde{v}_k^* \tau_{r_k}^* \omega_f \\ &\ge \lim_{k\to\infty} \int_{\mathbb{D}_R} \tilde{v}_k^* \tau_{r_k}^* \omega_f = \lim_{k\to\infty} \int_{\mathbb{D}_R} \tilde{v}_k^* \Omega_{f_k}^+, \end{aligned}$$

where now $f_k(r) := f(r + r_k)$. Writing $\Omega_{f_k}^+ = \omega_+ + d\big(f_k(r)\,\lambda_+\big) = \omega_+ + f_k(r)\,d\lambda_+ + f_k'(r)\,dr \wedge \lambda_+$, we can choose f such that $f'(r) = f'(r + r_k) \to 0$ as $k \to \infty$, so the third term contributes nothing to the integral. For the second term, let $f_+ := \lim_{k\to\infty} f_k(r) = \lim_{r\to\infty} f(r)$, so the calculation above becomes

$$0 \ge \int_{\mathbb{D}_R} v_\infty^* \left(\omega_+ + f_+\, d\lambda_+\right).$$

Now observe that since $f_+ \in [-\epsilon, \epsilon]$, condition (9.3) implies that the 2-form $\omega_+ + f_+\,d\lambda_+$ is nondegenerate on ξ_+, and it also annihilates ∂_r and R_+, so the vanishing of this integral implies that v_∞ is everywhere tangent to ∂_r and R_+ over $\mathbb{D}_R$. But $R > 0$ was arbitrary, so this is true on the whole plane, which is equivalent to $\int_{\mathbb{C}} v_\infty^* \omega_+ = 0$. With the claim established, we apply Lemma 9.9 and conclude that v_∞ is constant, contradicting the fact that $|dv_\infty(0)| = 1$.

Case 3: $v_k(0)$ has a subsequence diverging to $\{-\infty\} \times M_-$.
This is simply the mirror image of case 2: writing the restriction of J to $(-\infty, -r_0] \times$

M_- as J_-, one can follow the same bubbling argument but translate up and instead of down, giving rise to a limiting nonconstant J_--holomorphic plane $v_\infty : \mathbb{C} \to \mathbb{R} \times M_-$ that has finite energy but $\int_{\mathbb{C}} v_\infty^* \omega_- = 0$, in contradiction to Lemma 9.9. □

Consider now a sequence $s_k \to \infty$ and construct the J-holomorphic half-cylinders

$$u_k : [-s_k, \infty) \times S^1 \to \widehat{W} : (s,t) \mapsto u_+(s + s_k, t).$$

The derivatives $|du_k|$ are uniformly bounded due to Lemma 9.12, though again, u_k might fail to be uniformly bounded in C^0. We distinguish three cases.

Case 1: $u_k(0,0)$ has a bounded subsequence.
Then the corresponding subsequence of u_k is uniformly C^1-bounded on compact subsets and thus has a further subsequence converging in $C^\infty_{\mathrm{loc}}(\mathbb{R} \times S^1)$ to a J-holomorphic cylinder

$$u_\infty : \mathbb{R} \times S^1 \to \widehat{W}.$$

For any $f \in \mathcal{T}(h, r_0)$ and any $c > 0$, we have

$$\begin{aligned} \int_{[-c,c]\times S^1} u_\infty^* \omega_f &= \lim_{k\to\infty} \int_{[-c,c]\times S^1} u_k^* \omega_f \le \lim_{k\to\infty} \int_{[-s_k/2,\infty)\times S^1} u_k^* \omega_f \\ &= \lim_{k\to\infty} \int_{[s_k/2,\infty)\times S^1} u_+^* \omega_f = 0 \end{aligned} \tag{9.11}$$

since $\int_{Z_+} u_+^* \omega_f < \infty$. It follows that $\int_{\mathbb{R}\times S^1} u_\infty^* \omega_f = 0$, so u_∞ is a constant map to some point $p \in \widehat{W}$, implying that after passing to a subsequence of s_k,

$$u_+(s_k, \cdot) \to p \quad \text{in } C^\infty(S^1, \widehat{W}) \quad \text{as } k \to \infty.$$

Case 2: $u_k(0,0)$ has a subsequence diverging to $\{+\infty\} \times M_+$.
Passing to the corresponding subsequence of u_k, suppose

$$u_k(0,0) \in \{r_k\} \times M_+,$$

so $r_k \to \infty$. Since the derivatives $|du_k|$ are uniformly bounded, we can then find a sequence of intervals $[-R_k^-, R_k^+] \subset [-s_k, \infty)$ such that

$$u_k([-R_k^-, R_k^+] \times S^1) \subset [r_0, \infty) \times M_+ \quad \text{and} \quad R_k^\pm \to \infty.$$

Now the translated sequence

$$\tau_{-r_k} \circ u_k|_{[-R_k^-, R_k^+]\times S^1} : [-R_k^-, R_k^+] \times S^1 \to \mathbb{R} \times M_+$$

is uniformly C^1-bounded on compact subsets and thus has a subsequence coverging in C^∞_{loc} to a J_+-holomorphic cylinder

$$u_\infty : \mathbb{R} \times S^1 \to \mathbb{R} \times M_+,$$

where J_+ again denotes the restriction of J to $[r_0, \infty) \times M_+$, extended over $\mathbb{R} \times M_+$ by $\mathbb{R}$-translation. We claim that this cylinder satisfies

$$E(u_\infty) < \infty \quad \text{and} \quad \int_{\mathbb{R}\times S^1} u_\infty^* \omega_+ = 0.$$

The proof of this should be an easy exercise if you understood the proofs of (9.10) and (9.11) above, so I will leave it as such. Lemma 9.9 now implies that u_∞ is either constant or is a reparametrization of a trivial cylinder

$$u_\gamma : \mathbb{R} \times S^1 \to \mathbb{R} \times M_+ : (s,t) \mapsto (Ts, \gamma(Tt))$$

for some Reeb orbit $\gamma : \mathbb{R}/T\mathbb{Z} \to M_+$ with period $T > 0$. More precisely, all the biholomorphic reparametrizations of $\mathbb{R} \times S^1$ are of the form $(s,t) \mapsto (\pm s + a, \pm t + b)$, thus after shifting the parametrization of γ, we can write u_∞ without loss of generality in the form

$$u_\infty(s,t) = (\pm Ts + a, \gamma(\pm Tt)) \tag{9.12}$$

for some constant $a \in \mathbb{R}$ and a choice of signs to be determined below (see Lemma 9.16).

Case 3: $u_k(0,0)$ has a subsequence diverging to $\{-\infty\} \times M_-$.
Writing $J_- := J|_{(-\infty,-r_0]\times M_-} \in \mathcal{J}(\mathcal{H}_-)$ and imitating the argument for case 2, we suppose $u_k(0,0) \in \{-r_k\} \times M_-$ with $r_k \to \infty$ and obtain a subsequence for which $\tau_{r_k} \circ u_k$ converges in $C^\infty_{\text{loc}}(\mathbb{R} \times S^1)$ to a J_--holomorphic cylinder $u_\infty : \mathbb{R} \times S^1 \to \mathbb{R} \times M_-$, where u_∞ is either a constant or takes the form (9.12) for some orbit Reeb $\gamma : \mathbb{R}/T\mathbb{Z} \to M_-$ of period $T > 0$.

Here is one easy consequence of the discussion so far. Use the Riemannian metric on $\widehat{W}$ to define a metric $\text{dist}_{C^0}(\cdot,\cdot)$ on the space of continuous loops $S^1 \to \widehat{W}$.

LEMMA 9.13. *Given $\delta > 0$, there exists $s_0 \geq 0$ such that for every $s \geq s_0$, the loop $u_+(s,\cdot) : S^1 \to \widehat{W}$ satisfies*

$$\text{dist}_{C^0}(u_+(s,\cdot), \ell) < \delta,$$

where $\ell : S^1 \to \widehat{W}$ either is constant or is a loop of the form $\ell(t) = (r, \gamma(\pm Tt))$ in $[r_0,\infty) \times M_+$ or $(-\infty, r_0] \times M_-$ for some constant $r \in \mathbb{R}$ and Reeb orbit $\gamma : \mathbb{R}/T\mathbb{Z} \to M_\pm$ of period $T > 0$.

PROOF. If not, then there exists a sequence $s_k \to \infty$ such that each of the loops $u_+(s_k,\cdot)$ lies at C^0-distance at least δ away from any loop of the above form. However, the preceding discussion then gives a subsequence for which $u(s_k,\cdot)$ becomes arbitrarily C^∞-close to such a loop, so this is a contradiction. □

LEMMA 9.14. *If $u : \dot{\mathbb{D}} \to \widehat{W}$ is not bounded, then it is proper.*

PROOF. We use the monotonicity lemma. Suppose there exists a sequence $(s_k,t_k) \in Z_+$ such that $u_+(s_k,t_k)$ diverges to $\{+\infty\} \times M_+$. This implies $s_k \to \infty$, and we claim then that for every $R \geq r_0$, there exists $s_0 \geq 0$ such that

$$u_+((s_0,\infty) \times S^1) \subset (R,\infty) \times M_+.$$

If not, then we find $R \geq r_0$ and a sequence $(s_k',t_k') \in Z_+$ with $s_k' \to \infty$ such that $u_+(s_k',t_k') \notin (R,\infty) \times M_+$ for every k. By continuity, we are free to suppose $u_+(s_k',t_k') \in \{R\} \times M_+$ for all k since Lemma 9.13 implies $u_+(\{s_k\} \times S^1) \subset (2R,\infty) \times M_+$ for k sufficiently large. Using Lemma 9.13 again, we also have

$$u_+(\{s_k'\} \times S^1) \subset (R-1, R+1) \times M_+$$

for all k large. Assuming $2R > R + 2$ without loss of generality, we can therefore find infinitely many pairwise disjoint annuli of the form $[s_k', s_j] \times S^1 \subset Z_+$ containing open sets that u maps properly to small balls centered at points in $\{R+2\} \times M_+$. Choosing any $f \in \mathcal{T}(h, r_0)$, the monotonicity lemma implies that each of these contributes at least some fixed amount to $\int_{Z_+} u_+^* \omega_f$, contradicting the assumption that $E(u) < \infty$.[1]

A similar argument works if $u_+(s_k, t_k)$ diverges to $\{-\infty\} \times M_-$, proving that for every $R \geq r_0$, there exists $s_0 \geq 0$ with

$$u_+((s_0, \infty) \times S^1) \subset (-\infty, -R) \times M_-.$$

□

If u is bounded, then the singularity at 0 is removable by Theorem 9.2. If not, then Lemma 9.14 implies that it maps neighborhoods of the puncture to neighborhoods of either $\{+\infty\} \times M_+$ or $\{-\infty\} \times M_-$, and we shall refer to the puncture as *positive* or *negative* accordingly.

LEMMA 9.15. *If the puncture is positive/negative, then the limit*

$$Q := \lim_{s \to \infty} \int_{S^1} u_+(s, \cdot)^* \lambda_\pm \in \mathbb{R}$$

exists.

PROOF. If the puncture is positive, fix $s_0 \geq 0$ such that $u_+([s_0, \infty) \times S^1) \subset [r_0, \infty) \times M_+$. Then by Stokes' theorem, it suffices to show that the integral $\int_{[s_0,\infty)\times S^1} u_+^* d\lambda_+$ exists, which is true if

$$\int_{[s_0,\infty)\times S^1} \left| u_+^* d\lambda_+ \right| < \infty. \tag{9.13}$$

We claim first that $\int_{[s_0,\infty)\times S^1} u_+^* \omega_+ < \infty$. Indeed, for any $s > s_0$ and $f \in \mathcal{T}(h, r_0)$, we have

$$E(u) \geq \int_{[s_0,s]\times S^1} u_+^* \omega_f = \int_{[s_0,s]\times S^1} u_+^* \omega_+ + \int_{[s_0,s]\times S^1} u_+^* d\left(f(r)\,\lambda_+\right).$$

Applying Stokes' theorem, the second term becomes the sum of some number not dependent on s and the integral

$$\int_{S^1} u_+(s, \cdot)^* \left(f(r)\,\lambda_+\right) = \int_{S^1} [f \circ u_+(s, \cdot)]\, u_+(s, \cdot)^* \lambda_+,$$

which is bounded as $s \to \infty$ since f and $|du_+|$ are both bounded. This proves that $\int_{[s_0,s]\times S^1} u_+^* \omega_+$ is also bounded as $s \to \infty$, and since $u_+^* \omega_+ \geq 0$, the claim follows. Now observe that since $d\lambda_+$ annihilates the kernel of ω_+ and the latter tames J on ξ_+, there exists a constant $c > 0$ such that $|u_+^* d\lambda_+| \leq c |u_+^* \omega_+|$, implying (9.13).

An analogous argument works if the puncture is negative. □

[1] The fact that $\widehat{W}$ is noncompact is not a problem for this application of the monotonicity lemma, as we are only using it in the compact subset $W^{2R} \subset \widehat{W}$.

The number $Q \in \mathbb{R}$ defined in the above lemma matches what we referred to in the statement of Theorem 9.8 as the **charge** of the puncture.

LEMMA 9.16. *If the puncture is nonremovable and $Q \neq 0$, then the puncture is positive/negative if and only if $Q > 0$ or $Q < 0$ respectively. In either case, given any sequence $s_k \to \infty$ with $u_+(s_k, 0) \in \{\pm r_k\} \times M_\pm$, one can find a sequence $R_k \in [0, s_k]$ with $R_k \to \infty$ such that u_+ maps $[s_k - R_k, \infty) \times S^1$ into the positive/negative cylindrical end for every k, and the sequence of half-cylinders*

$$u_k : [-R_k, \infty) \times S^1 \to \mathbb{R} \times M_+ \quad \text{or} \quad u_k : (-\infty, R_k] \times S^1 \to \mathbb{R} \times M_-$$

defined by $u_k(s,t) = \tau_{\mp r_k} \circ u_\pm(s \pm s_k, t)$ has a subsequence convergent in $C^\infty_{\mathrm{loc}}(\mathbb{R} \times S^1)$ to a $J_\pm$-holomorphic cylinder of the form

$$u_\infty : \mathbb{R} \times S^1 \to \mathbb{R} \times M_\pm : (s,t) \mapsto (Ts + a, \gamma(Tt))$$

for some constant $a \in \mathbb{R}$ and Reeb orbit $\gamma : \mathbb{R}/T\mathbb{Z} \to M_\pm$ with period $T := \pm Q$.

PROOF. Assume the puncture is either positive or negative and $Q \neq 0$. In the discussion preceding Lemma 9.13, we showed that the sequence $u'(s,t) := \tau_{\mp r_k} \circ u_+(s + s_k, t)$ defined on $[-R_k, \infty) \times S^1$ has a subsequence convergent in C^∞_{loc} to a $J_\pm$-holomorphic cylinder $u'_\infty : \mathbb{R} \times S^1 \to \mathbb{R} \times M_\pm$ which is either constant or of the form

$$u'_\infty(s,t) = (\sigma T s + a, \gamma(\sigma T t)) \tag{9.14}$$

for some $a \in \mathbb{R}$, $\sigma = \pm 1$ and a Reeb orbit $\gamma : \mathbb{R}/T\mathbb{Z} \to M_\pm$ of period $T > 0$. We then have

$$0 \neq Q = \lim_{s \to \infty} \int_{S^1} u_+(s, \cdot)^* \lambda_\pm = \lim_{k \to \infty} \int_{S^1} u'_k(0, \cdot)^* \lambda_\pm = \int_{S^1} u'_\infty(0, \cdot)^* \lambda_\pm,$$

so u'_∞ cannot be constant, and from (9.14) we deduce $Q = \sigma T$, hence $u'_\infty(s,t) = (Qs + a, \gamma(Qt))$. Writing $u_+(s,t) = (u_\mathbb{R}(s,t), u_M(s,t)) \in \mathbb{R} \times M_\pm$ for s sufficiently large, it follows that every sequence $s_k \to \infty$ admits a subsequence for which

$$\partial_s u_\mathbb{R}(s_k, \cdot) \to Q \quad \text{in} \quad C^\infty(S^1, \mathbb{R}),$$

and consequently $\partial_s u_\mathbb{R}(s, \cdot) \to Q$ in $C^\infty(S^1, \mathbb{R})$ as $s \to \infty$. This proves that the sign of Q matches the sign of the puncture whenever $Q \neq 0$. The stated formula for u_∞ now follows by adjusting all the appropriate signs in the case $Q < 0$. □

LEMMA 9.17. *If the puncture is nonremovable, then $Q \neq 0$.*

PROOF. Assume on the contrary that u is a proper map, say with a positive puncture, but $Q = 0$. In this case, the argument of the previous lemma shows that the limiting map $u_\infty : \mathbb{R} \times S^1 \to \mathbb{R} \times M_+$ will always be *constant*, thus for every sequence $s_k \to \infty$, there exists a point $p \in M_+$ such that $u_+(s_k, 0) \in \{r_k\} \times M_+$ with $r_k \to \infty$ and

$$\tau_{-r_k} \circ u_+(s_k, \cdot) \to (0, p) \in \mathbb{R} \times M_+ \quad \text{in} \quad C^\infty(S^1, \mathbb{R} \times M_+) \text{ as } k \to \infty.$$

In particular, this implies that all derivatives of u_+ decay to 0 as $s \to \infty$. Intuitively, this should suggest to you that portions of u_+ near infinity will have improbably small symplectic area, perhaps violating the monotonicity lemma—this will turn out to be true, but we have to be a bit clever with our argument since u_+ is unbounded. We will make this argument precise by translating pieces of u_+ downward so that we only compute its symplectic area in $[0,2] \times M_+$. Fix a function $f : \mathbb{R} \to (-\epsilon, \epsilon)$ with $f' > 0$ and set $\Omega_f^+ = \omega_+ + d\,(f(r)\,\lambda_+)$.

Given a small number $\delta > 0$, we can find $s_0 \ge 0$ such that $|du_+(s,t)| < \delta$ for all $s \ge s_0$ and each of the loops $u_+(s,\cdot)$ for $s \ge s_0$ is δ-close to a constant in $C^1(S^1)$. Assume $u_+(s_0,0) \in \{R\} \times M_+$ and choose $s_1 > s_0$ such that $u_+(s_1,0) \in \{R+2\} \times M_+$, which is possible since $u_+(s,t) \to \{+\infty\} \times M_+$ as $s \to \infty$. Now consider the J_+-holomorphic annulus

$$v_\delta := \tau_{-R} \circ u_+|_{[s_0,s_1]\times S^1} : [s_0,s_1] \times S^1 \to \mathbb{R} \times M_+.$$

We claim that $\int_{[s_0,s_1]\times S^1} v_\delta^* \Omega_f^+$ can be made arbitrarily small by choosing δ suitably small. Indeed, we can use Stokes' theorem to write this integral as

$$\begin{aligned}\int_{[s_0,s_1]\times S^1} v_\delta^* \Omega_f^+ &= \int_{[s_0,s_1]\times S^1} v_\delta^* \omega_+ + \int_{[s_0,s_1]\times S^1} v_\delta^* d\,(f(r)\,\lambda_+) \\ &= \int_{[s_0,s_1]\times S^1} v_\delta^* \omega_+ + \int_{S^1} \left[v_\delta(s_1,\cdot)^*\,(f(r)\,\lambda_+) - v_\delta(s_0,\cdot)^*\,(f(r)\,\lambda_+)\right].\end{aligned}$$

The second term is small because $f(r)$ is bounded and $|v_\delta(s,\cdot)^*\lambda_+|$ is small in proportion to $|dv_\delta(s,t)| = |du_+(s,t)|$ for $s \ge s_0$. For the first term, observe that since both of the loops $v_\delta(s_i,\cdot)$ for $i = 0,1$ are nearly constant, they are contractible and can be filled in with disks $\bar{v}_i : \mathbb{D} \to \mathbb{R} \times M_+$ for which $\left|\int_{\mathbb{D}} \bar{v}_i^* \omega_+\right|$ may be assumed arbitrarily small. Moreover, since all of the loops $v_\delta(s,\cdot)$ are similarly contractible, the union of these two disks with the annulus v_δ defines a closed cycle in M_+ that is trivial in $H_2(M_+)$, hence the integral of the closed 2-form ω_+ over this cycle vanishes, implying

$$\int_{[s_0,s_1]\times S^1} v_\delta^* \omega_+ = \int_{\mathbb{D}} \bar{v}_1^* \omega_+ - \int_{\mathbb{D}} \bar{v}_0^* \omega_+,$$

which is therefore arbitrarily small, and this proves the claim.

To finish, notice that since v_δ maps its boundary components to small neighborhoods of $\{0\} \times M_+$ and $\{2\} \times M_+$, one can fix a suitable choice of radius $r_1 > 0$ such that v_δ must pass through a point in $p \in \{1\} \times M_+$ for which the boundary of v_δ is outside the ball $B_{r_1}(p)$. The monotonicity lemma then bounds the symplectic area of v_δ from below by a constant times r_1^2, but since we can also make this area arbitrarily small by choosing δ smaller, this is a contradiction.

As usual, the case of a negative puncture can be handled similarly. □

We've now proved every statement in Theorem 9.8 up to the final detail about the case where the asymptotic orbit is nondegenerate or Morse-Bott. The complete proof of this part requires delicate analytical results from [**HWZ96**, **HWZ01**, **HWZ96**, **Bou02**], but we can explain the first step for the nondegenerate case. In the following, we say that a closed Reeb orbit $\gamma : \mathbb{R}/T\mathbb{Z} \to M_\pm$ is **isolated** if, after rescaling the domain to write it as an element of $C^\infty(S^1, M_\pm)$, there exists a neighborhood $\gamma \in \mathcal{U} \subset C^\infty(S^1, M_\pm)$ such that all closed Reeb orbits in $\mathcal{U}$ are reparametrizations of γ.

LEMMA 9.18. *Suppose the puncture is nonremovable, write*

$$u_+(s,t) = (u_{\mathbb{R}}(s,t), u_M(s,t)) \in \mathbb{R} \times M_\pm$$

for $s \geq 0$ sufficiently large, and suppose $s_k \to \infty$ is a sequence and $\gamma : \mathbb{R}/T\mathbb{Z} \to M_\pm$ is a Reeb orbit such that

$$u_M(s_k, \cdot) \to \gamma(T\cdot) \quad \text{in} \quad C^\infty(S^1, M_\pm).$$

If γ is isolated, then for every neighborhood $\mathcal{U} \subset C^\infty(S^1, M_\pm)$ of the set of parametrizations $\{\gamma(\cdot + \theta) \mid \theta \in S^1\}$, we have $u_M(s, \cdot) \in \mathcal{U}$ for all sufficiently large s.

PROOF. Note first that if γ is isolated, then its image admits a neighborhood $\operatorname{im}\gamma \subset \mathcal{V} \subset M_\pm$ such that no point in $\mathcal{V} \setminus \operatorname{im}\gamma$ is contained in another Reeb orbit of period T. Indeed, we could otherwise find a sequence of T-periodic Reeb orbits passing through a sequence of points in $\mathcal{V} \setminus \operatorname{im}\gamma$ that converge to a point in $\operatorname{im}\gamma$. Since their derivatives are determined by the Reeb vector field and are therefore bounded, the Arzelà-Ascoli theorem then gives a subsequence of these orbits converging to a reparametrization of γ, contradicting the assumption that γ is isolated.

Arguing by contradiction, suppose now that there exists a sequence $s_k' \to \infty$ with $u_M(s_k, \cdot) \notin \mathcal{U}$ for all k. We can nonetheless restrict to a subsequence for which $u_M(s_k', \cdot)$ converges to some Reeb orbit $\tilde{\gamma} : \mathbb{R}/T\mathbb{Z} \to M_\pm$. Then $\tilde{\gamma}$ is disjoint from γ, and by continuity, one can find a sequence $s_k'' \to \infty$ for which each $u_M(s_k'', 0)$ lies in the region $\mathcal{V}$ some fixed distance away from $\operatorname{im}\gamma$. There must then be a subsequence for which $u_M(s_k'', \cdot)$ converges to another T-periodic orbit, but this is impossible since no such orbits exist in $\mathcal{V} \setminus \operatorname{im}\gamma$. □

9.3. Degenerations of holomorphic curves

To motivate the SFT compactness theorem, we shall now discuss three examples of phenomena that can prevent a sequence of holomorphic curves from having a compact subsequence. The theorem will then tell us that these three things are, in essence, the only things that can go wrong.

Throughout this section and the next, assume $J_k \to J \in \mathcal{J}_\tau(\omega_h, r_0, \mathcal{H}_+, \mathcal{H}_-)$ is a C^∞-convergent sequence of tame almost complex structures on the completed cobordism $\widehat{W}$. More generally, one can also allow the data ω, h and $\mathcal{H}_\pm$ to vary in C^∞-convergent sequences, but let's not clutter the notation too much. We shall denote the restrictions of J to the cylindrical ends by

$$J_+ := J|_{[r_0,\infty)\times M_+} \in \mathcal{J}(\mathcal{H}_+), \qquad J_- := J|_{(-\infty,-r_0]\times M_-} \in \mathcal{J}(\mathcal{H}_-).$$

Suppose

$$u_k := [(\Sigma_k, j_k, \Gamma_k^+, \Gamma_k^-, \Theta_k, u_k)] \in \mathcal{M}_{g,m}(J_k, A_k, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$$

is a sequence of J_k-holomorphic curves in $\widehat{W}$ with fixed genus $g \geq 0$ and $m \geq 0$ marked points, varying relative homology classes $A_k \in H_2(W, \bar{\boldsymbol{\gamma}}^+ \cup \bar{\boldsymbol{\gamma}}^-)$ and fixed collections of asymptotic orbits $\boldsymbol{\gamma}^\pm = (\gamma_1^\pm, \ldots, \gamma_{m_\pm}^\pm)$. Observe that the energies $E(u_k)$ depend only on the orbits $\boldsymbol{\gamma}^\pm$ and relative homology classes A_k, so in particular, $E(u_k)$ is uniformly bounded whenever the relative homology class is also fixed. The fundamental question of this section is:

QUESTION. *If $E(u_k)$ is uniformly bounded and no subsequence of u_k converges to an element of $\mathcal{M}_{g,m}(J, A, \gamma^+, \gamma^-)$ for any $A \in H_2(W, \bar{\gamma}^+ \cup \bar{\gamma}^-)$, what can happen?*

9.3.1. Bubbling. Suppose $(\Sigma_k, j_k, \Gamma_k^+, \Gamma_k^-, \Theta_k) = (\Sigma, j, \Gamma^+, \Gamma^-, \theta)$ is a fixed sequence of domains, and choose Riemannian metrics on $\dot{\Sigma} = \Sigma \setminus \Gamma$ and $\widehat{W}$ that are translation-invariant on the cylindrical ends of both. Suppose there exists a point $\zeta_0 \in \dot{\Sigma}$ such that $u_k(\zeta_0)$ is contained in a compact subset for all k. Suppose also that the maps $u_k : \dot{\Sigma} \to \widehat{W}$ are locally C^1-bounded outside some finite subset

$$\Gamma' = \{\zeta_1, \dots, \zeta_N\} \subset \dot{\Sigma},$$

i.e. for every compact set $K \subset \dot{\Sigma} \setminus \Gamma'$, there exists a constant $C_K > 0$ independent of k such that

$$|du_k| \le C_K \quad \text{on } K.$$

Then elliptic regularity gives a subsequence that converges in $C^\infty_{\text{loc}}(\dot{\Sigma} \setminus \Gamma')$ to a J-holomorphic curve

$$u_\infty : \dot{\Sigma} \setminus \Gamma' \to \widehat{W}$$

with $E(u_\infty) \le \limsup E(u_k) < \infty$, thus all the punctures $\Gamma^+ \cup \Gamma^- \cup \Gamma'$ of u_∞ are either removable or positively or negatively asymptotic to Reeb orbits. We cannot be sure that the asymptotic behavior of u_∞ at $\Gamma^\pm$ is the same as for u_k, but let's assume this for now (§9.3.2 below discusses some things that can happen if this does not hold). Then to complete the picture, we need to understand not only what u_∞ is doing at the additional punctures Γ', but also what is happening to u_k near these points as its first derivative blows up. For this we can apply the familiar rescaling trick: choose for each ζ_i a sequence $z_k^i \to \zeta_i$ such that $|du_k(z_k^i)| =: R_k \to \infty$, along with a sequence $\epsilon_k \to 0$ with $\epsilon_k R_k \to \infty$, and using Lemma 9.4, assume without loss of generality that $|du_k(z)| \le 2R_k$ for all z in the ϵ_k-ball about z_k^i. For convenience, we can choose a holomorphic coordinate system identifying a neighborhood of ζ_i with $\mathbb{D} \subset \mathbb{C}$ and placing ζ_i at the origin, so $z_k^i \to 0$ in these coordinates, and assume without loss of generality that they identify our chosen metric near ζ_i with the Euclidean metric. Now setting

$$v_k^i(z) = u(z_k^i + z/R_k) \quad \text{for} \quad z \in \mathbb{D}_{\epsilon_k R_k}$$

gives a sequence of J_k-holomorphic maps $v_k^i : \mathbb{D}_{\epsilon_k R_k} \to \widehat{W}$ whose energies and first derivatives are both uniformly bounded. As in the arguments of §2, we now have three possibilities:

- If $u_k^i(z_k^i)$ has a bounded subsequence, then the corresponding subsequence of v_k^i converges in $C^\infty_{\text{loc}}(\mathbb{C})$ to a J-holomorphic plane $v_\infty^i : \mathbb{C} \to \widehat{W}$ with finite energy.
- If $u_k(z_k^i)$ has a subsequence diverging to $\{\pm\infty\} \times M_\pm$, then translating v_k^i by the $\mathbb{R}$-action produces a limiting finite-energy plane v_∞^i in the positive/negative symplectization $\mathbb{R} \times M_\pm$.

Viewing $\mathbb{C}$ as the punctured sphere $S^2 \setminus \{\infty\}$, the singularity of v_∞^i at ∞ may be removable, in which case v_∞^i extends to a J-holomorphic sphere and we say that u_k has "bubbled off a sphere" at ζ_i. Alternatively, v_∞^i may be positively or negatively asymptotic to a Reeb orbit at ∞.

Figure 9.2 shows two scenarios that could occur for a sequence in which $|du_k|$ blows up at three points $\Gamma' = \{\zeta_1, \zeta_2, \zeta_3\}$. Both scenarios show u_∞ with ζ_1 and ζ_2 as removable singularities and ζ_3 as a negative puncture, but the behavior of the various v^i_∞ reveals a wide spectrum of possibilities. In the lower-left picture, the points $u_k(z^1_k)$ are bounded and bubble off a sphere $v^1_\infty : S^2 \to \widehat{W}$. The picture shows that v^1_∞ passes through $u_\infty(\zeta_1)$ at some point; this does not follow from our argument so far, but in this situation one can use a more careful analysis of u_k near ζ_1 to show that it must be true, i.e. "bubbles connect". At ζ_3, we have $u_k(z^3_k) \to \{-\infty\} \times M_-$ and v^3_∞ is a plane in $\mathbb{R} \times M_-$ with a positive puncture asymptotic to the same orbit as ζ_3; the coincidence of these orbits is another detail that does not follow from the analysis above but turns out to be true in the general picture. The situation at ζ_2 allows two different interpretations: v^2_∞ could be the plane with negative end in $\mathbb{R} \times M_+$, meaning $u_k(z^2_k) \to \{+\infty\} \times M_+$, and the picture then shows an additional plane in $\widehat{W}$ with a positive end approaching the same asymptotic orbit as v^2_∞ as well as a point passing through $u_\infty(\zeta_2)$. One would need to choose a different rescaled sequence near ζ_2 to find this extra plane, but as we will see, the SFT compactness theorem dictates that some such object must be there. Alternatively, $u_k(z^2_k)$ could also be bounded at ζ_2, in which case v^2_∞ must be the plane in $\widehat{W}$ with positive end, and the extra plane above this is something that one could find via a different choice of rescaled sequence. In general, the range of actual possibilities can involve arbitrarily many additional curves that could be discovered via different choices of rescaled sequences: e.g. there could be entire "bubble trees" as shown in the lower-right picture, where each v^i_∞ is only one of several curves that arise as limits of different parametrizations of u_k near ζ_i. One good place to read about the analysis of bubble trees is [**HWZ03**, §4].

9.3.2. Breaking. Figure 9.2 already shows some phenomena that could be interpreted as "breaking" in the Floer-theoretic sense, but breaking can also happen when no derivatives are blowing up, simply due to the fact that our domains are non-compact. Figures 9.3 and 9.4 show three such scenarios, where we assume again that $(\Sigma_k, j_k, \Gamma^+_k, \Gamma^-_k, \Theta_k) = (\Sigma, j, \Gamma^+, \Gamma^-, \Theta)$ is a fixed sequence of domains, and $\dot{\Sigma} = \Sigma \setminus \Gamma$ and $\widehat{W}$ carry Riemannian metrics that are translation-invariant on the cylindrical ends such that

$$|du_k| \le C \quad \text{everywhere on } \dot{\Sigma}$$

for some constant $C > 0$ independent of k. This is a stronger condition than we had in §9.3.1, and if there exists a point $\zeta_0 \in \dot{\Sigma}$ such that $u_k(\zeta_0)$ is bounded, it implies that u_∞ converges in $C^\infty_{\text{loc}}(\dot{\Sigma})$ to a J-holomorphic map

$$u_\infty : \dot{\Sigma} \to \widehat{W}$$

with $E(u_\infty) \le \limsup E(u_k) < \infty$. Convergence in C^∞_{loc} is, however, not very strong: there may in general be no relation between the asymptotic behavior of u_∞ and u_k at corresponding punctures, e.g. the top scenario in Figure 9.3 shows a case in which a negative puncture of u_k becomes a removable singularity of u_∞. Whenever this happens, there must be more to the story: in this example, one can choose

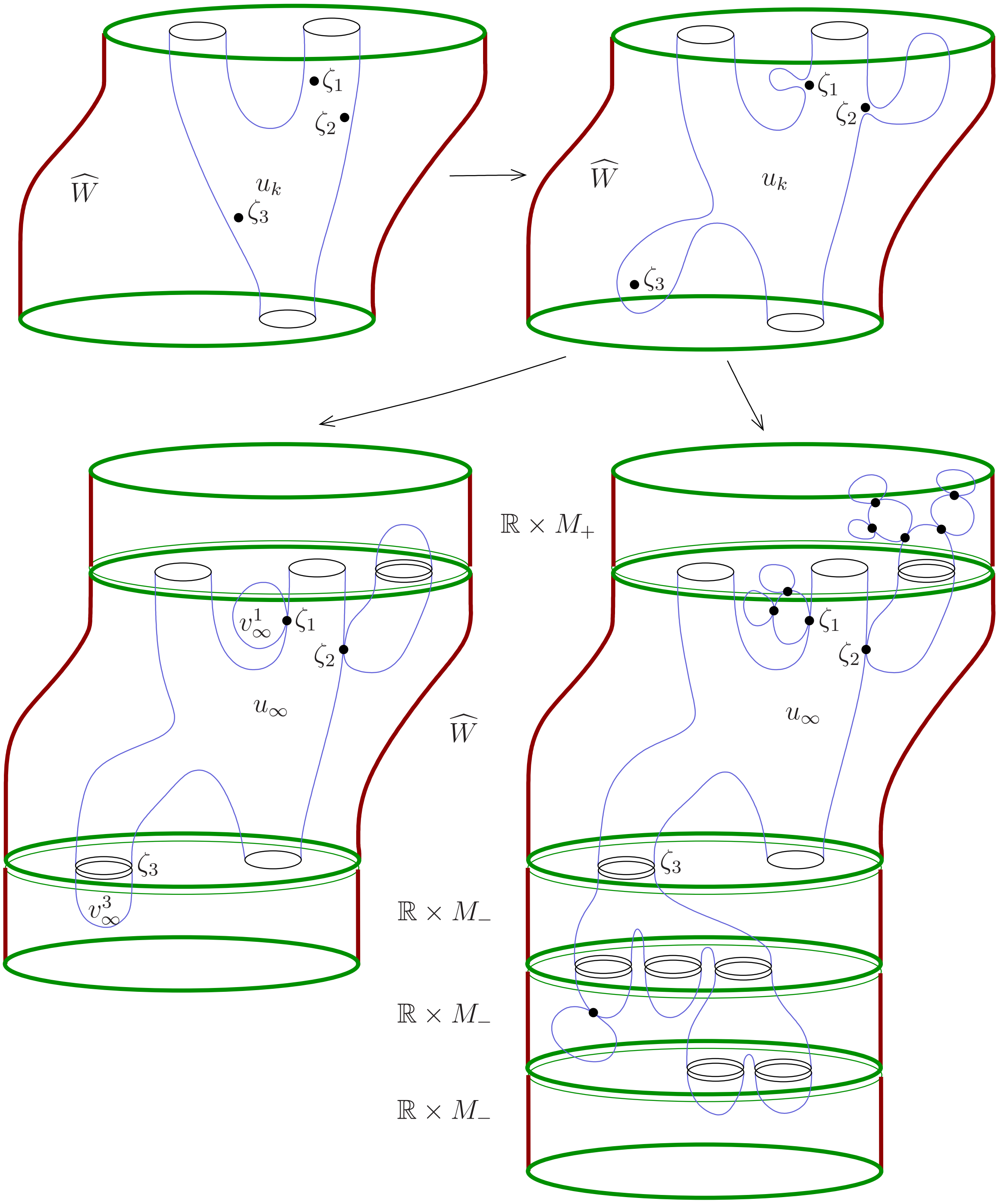

FIGURE 9.2. Two possible pictures of spheres and/or planes that can bubble off when the first derivative blows up near three points.

holomorphic cylindrical coordinates $(s,t) \in (-\infty, 0] \times S^1 \subset \dot{\Sigma}$ near the negative puncture of u_k and find a sequence $s_k \to \infty$ such that the sequence of half-cylinders

$$(-\infty, s_k] \times S^1 \to \widehat{W} : (s,t) \mapsto u_k(s - s_k, t)$$

is uniformly C^1-bounded and thus converges in $C^\infty_{\text{loc}}(\mathbb{R} \times S^1)$ to a finite-energy J-holomorphic cylinder $v_- : \mathbb{R} \times S^1 \to \widehat{W}$. In the picture, v_- turns out to have a removable singularity at $+\infty$ mapping to the same point as the removable singularity of u_∞, and its negative puncture approaches the same orbit as the negative puncture of u_k.

More complicated things can happen in general: the bottom scenario in this same figure shows a case where all three singularities of u_∞ are removable, thus it extends to a closed curve, while at one of the positive cylindrical ends $[0, \infty) \times S^1 \subset \dot{\Sigma}$ of u_k, we can find a sequence $s_k \to \infty$ such that the half-cylinders

$$[-s_k, \infty) \times S^1 \to \widehat{W} : (s, t) \mapsto u_k(s + s_k, t)$$

are uniformly C^1-bounded and converge in $C^\infty_{\text{loc}}(\mathbb{R} \times S^1)$ to a J-holomorphic cylinder $v^1_+ : \mathbb{R} \times S^1 \to \widehat{W}$ with one removable singularity and one positive puncture. At the other positive end, we can perform the same trick in two distinct ways for two sequences $s_k \to \infty$, one diverging faster than the other: the result is a pair of J-holomorphic cylinders $v^2_+, v^3_+ : \mathbb{R} \times S^1 \to \widehat{W}$, the former with both singularities removable (thus forming a holomorphic sphere in the picture), and the latter with one removable singularity and one positive puncture.

It can get weirder. Remember that $\widehat{W}$ is also noncompact!

In each of the above scenarios, we tacitly assumed that all of the various sequences obtained by reparametrizing portions of u_k were locally C^0-bounded, thus all of the limits were curves in $\widehat{W}$. But it may also happen that some of these sequences are C^0_{loc}-bounded while others locally diverge toward $\{\pm\infty\} \times M_\pm$; in fact, two such sequences that both diverge toward, say, $\{+\infty\} \times M_+$, might even locally diverge infinitely far from *each other*, meaning one of them approaches $\{+\infty\} \times M_+$ quantitatively faster than the other. This phenomenon leads to the notion of limiting curves with multiple *levels*.

In Figure 9.4, we see a scenario in which u_k satisfies the same conditions as above, except that instead of $u_k(\zeta_0)$ being bounded, it diverges to $\{+\infty\} \times M_+$. It follows that after applying suitable $\mathbb{R}$-translations, a subsequence converges in $C^\infty_{\text{loc}}(\dot{\Sigma})$ to a J_+-holomorphic curve

$$u_\infty : \dot{\Sigma} \to \mathbb{R} \times M_+$$

with finite energy. In the example, all three of its punctures are nonremovable, but two of them approach orbits that have nothing to do with the asymptotic orbits of u_k. Now observe that since u_k has a negative cylindrical end $(-\infty, 0] \times S^1 \subset \dot{\Sigma}$, one can necessarily find a sequence $s_k \to \infty$ such that $u_k(-s_k, 0)$ is bounded, and the sequence of half-cylinders

$$(-\infty, s_k] \times S^1 \to \widehat{W} : (s, t) \mapsto u_k(s - s_k, t)$$

is then uniformly C^1-bounded and thus has a subsequence convergent in $C^\infty_{\text{loc}}(\mathbb{R} \times S^1)$ to a finite-energy J-holomorphic cylinder $v_0 : \mathbb{R} \times S^1 \to \widehat{W}$. In the picture, v_0 has both a positive and a negative puncture, but its negative end again approaches a different Reeb orbit from the negative ends of u_k, so one can deduce that there must be still more happening near $-\infty$: there exists another sequence $s'_k \to \infty$ with

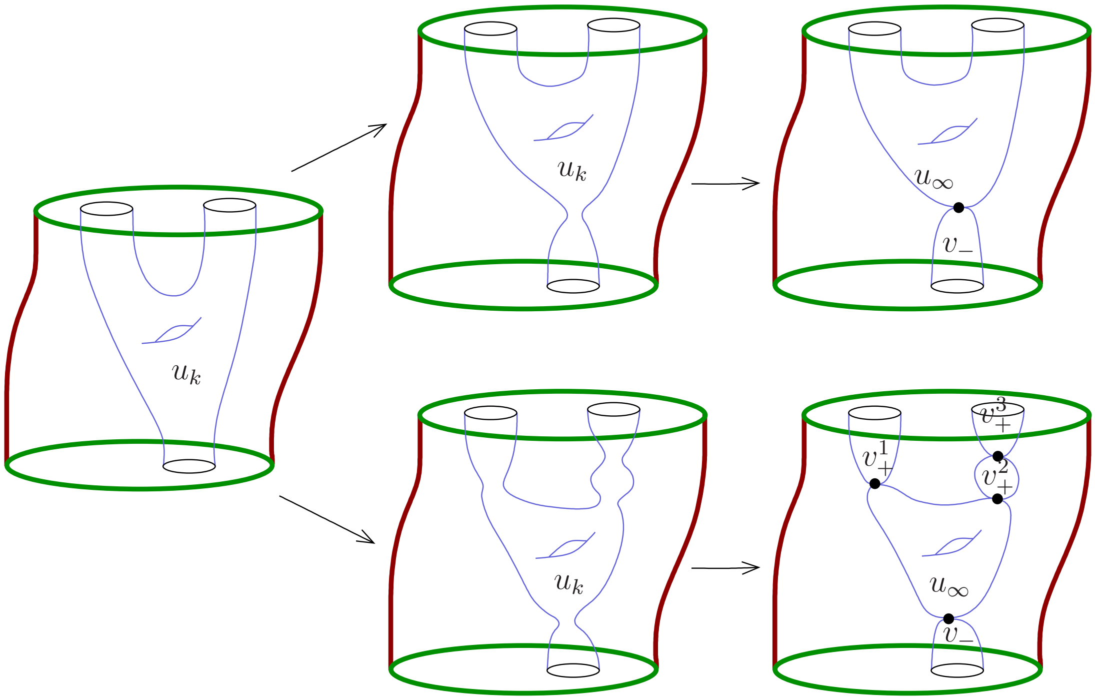

FIGURE 9.3. Even with fixed conformal structures on the domains and without bubbling, a sequence of punctured holomorphic curves in $\widehat{W}$ can break to produce multiple curves in $\widehat{W}$ with extra removable punctures. The picture shows two such scenarios.

$s_k' - s_k \to \infty$ such that suitable $\mathbb{R}$-translations of the half-cylinders

$$(-\infty, s_k] \times S^1 \to (-\infty, -r_0] \times M_- : (s,t) \mapsto u_k(s - s_k', t)$$

define uniformly C^1-bounded maps into $\mathbb{R} \times M_-$, giving a subsequence that converges in $C^\infty_{\mathrm{loc}}(\mathbb{R} \times S^1)$ to a finite-energy J_--holomorphic cylinder

$$v_- : \mathbb{R} \times S^1 \to \mathbb{R} \times M_-.$$

Finally, the fact that u_∞ has a positive asymptotic orbit different from those of u_k indicates that something more must also be happening near $+\infty$: in the example, one of the positive ends $[0,\infty) \times S^1 \subset \dot{\Sigma}$ admits a sequence $s_k \to \infty$ such that $u_k(s_k, 0) \in \{r_k\} \times M_+$ for some $r_k \to \infty$, and suitable $\mathbb{R}$-translations of

$$[-s_k, \infty) \times S^1 \to [r_0, \infty) \times M_+ : (s,t) \mapsto u_k(s + s_k, t)$$

become a uniformly C^1-bounded sequence of half-cylinders in $\mathbb{R} \times M_+$, with a subsequence converging in $C^\infty_{\mathrm{loc}}(\mathbb{R} \times S^1)$ to a finite-energy J_+-holomorphic cylinder

$$v_+^2 : \mathbb{R} \times S^1 \to \mathbb{R} \times M_+$$

that connects the errant asymptotic orbit of u_∞ to the corresponding orbit of u_k. One can now perform the same trick at the other positive end of $\dot{\Sigma}$, as there necessarily also exists a sequence $s_k' \to \infty$ in this end such that $u_k(s_k', 0) \in \{r_k\} \times M_+$ for the same sequence $r_k \to \infty$ as in the above discussion. The resulting limit curve

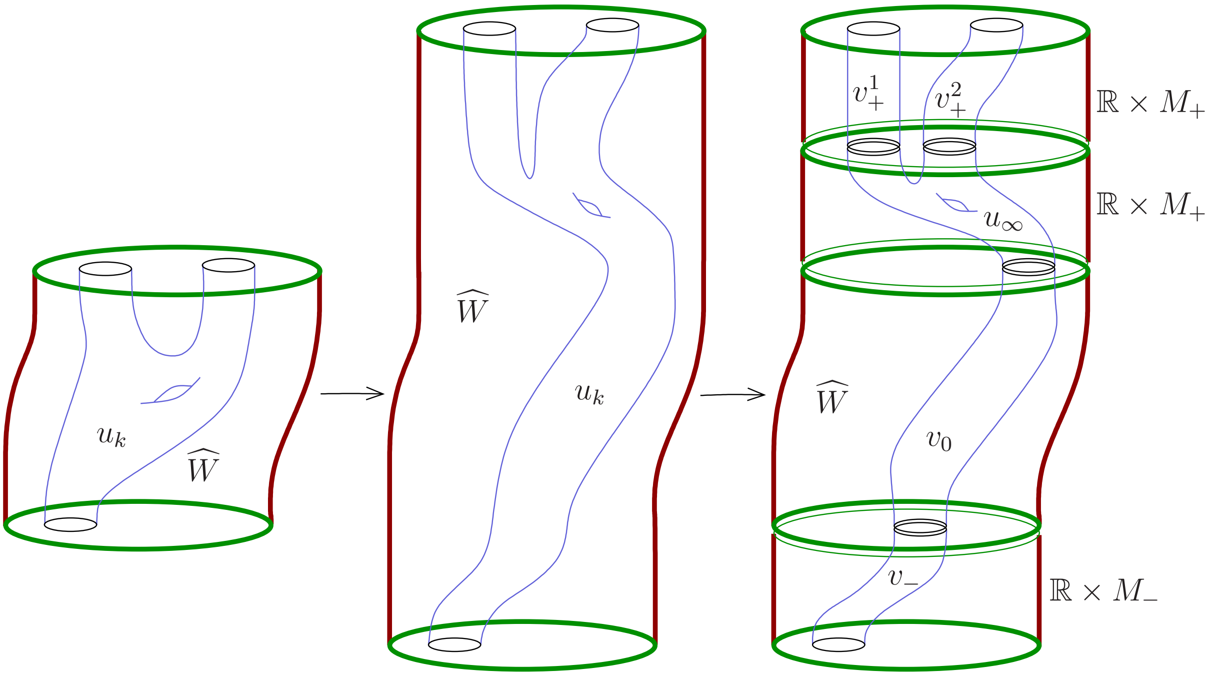

FIGURE 9.4. Different portions of a breaking sequence of curves may also become infinitely far apart in the limit, so that some live in $\widehat{W}$ while others live in the symplectization of M_+ or M_-.

$v_+^1 : \mathbb{R} \times S^1 \to \mathbb{R} \times M_+$ however is not guaranteed to be interesting: in the picture, it turns out to be a trivial cylinder.

The type of degeneration shown in Figure 9.4 happens whenever the sequence u_k does interesting things in multiple regions of its domain that are sent increasingly far away from each other in the image. The usual picture of $\widehat{W}$ that collapses the cylindrical ends to a finite size therefore becomes increasingly inadequate for visualizing u_k as $k \to \infty$: the middle picture in Figure 9.4 deals with this by expanding the scale of the cylindrical ends so that the convergence to upper and lower levels becomes visible.

9.3.3. The Deligne-Mumford space of Riemann surfaces. We next need to relax the assumption that the Riemann surfaces $(\Sigma_k, j_k, \Gamma_k^+ \sqcup \Gamma_k^- \sqcup \Theta_k)$ are fixed. Recall that for integers $g \geq 0$ and $\ell \geq 0$, the moduli space of pointed Riemann surfaces is the space of equivalence classes

$$\mathcal{M}_{g,\ell} = \{(\Sigma, j, \Theta)\} \big/ \sim,$$

where (Σ, j) is a closed connected Riemann surface of genus g, $\Theta \subset \Sigma$ is an ordered set of ℓ distinct points, and $(\Sigma, j, \Theta) \sim (\Sigma', j', \Theta')$ whenever there exists a biholomorphic map $\varphi : (\Sigma, j) \to (\Sigma', j')$ taking Θ to Θ' with the ordering preserved. This space is fairly easy to understand in the finitely many cases with $2g + \ell < 3$, e.g. $\mathcal{M}_{0,\ell}$ is a one-point space for each $\ell \leq 3$. We say that (Σ, j, Θ) is **stable** whenever $\chi(\Sigma \setminus \Theta) < 0$, which means $2g + \ell \geq 3$. In the stable case, one can show that every pointed Riemann surface has a finite automorphism group, and $\mathcal{M}_{g,\ell}$ is a smooth orbifold of dimension $6g - 6 + 2\ell$. It is generally not compact, but it admits a natural compactification

$$\overline{\mathcal{M}}_{g,\ell} \supset \mathcal{M}_{g,\ell},$$

known as the **Deligne-Mumford compactification**. We shall now give a sketch of this construction from the perspective of hyperbolic geometry; for more details, see [**Hum97**, **SS92**].

We recall first the following standard result.

THEOREM (Uniformization theorem). *Every simply connected Riemann surface is biholomorphically equivalent to either the Riemann sphere* $S^2 = \mathbb{C} \cup \{\infty\}$*, the complex plane* $\mathbb{C}$ *or the upper half plane* $\mathbb{H} = \{\operatorname{Im} z > 0\} \subset \mathbb{C}$*.*

The uniformization theorem implies that every Riemann surface can be presented as a quotient of either (S^2, i), $(\mathbb{C}, i)$ or $(\mathbb{H}, i)$ by some freely acting discrete group of biholomorphic transformations. The only punctured surface $\dot{\Sigma} = \Sigma \setminus \Theta$ that has S^2 as its universal cover is S^2 itself. It is almost as easy to see which surfaces are covered by $\mathbb{C}$, as the only biholomorphic transformations on $(\mathbb{C}, i)$ with no fixed points are the translations, so every freely acting discrete subgroup of $\operatorname{Aut}(\mathbb{C}, i)$ is either trivial, a cyclic group of translations or a lattice. The resulting quotients are, respectively, $(\mathbb{C}, i)$, $(\mathbb{R} \times S^1, i) \cong (\mathbb{C} \setminus \{0\}, i)$ and the unpunctured tori (T^2, j). All *stable* pointed Riemann surfaces are thus quotients of $(\mathbb{H}, i)$.

PROPOSITION 9.19. *There exists on* $(\mathbb{H}, i)$ *a complete Riemannian metric* g_P *of constant curvature* -1 *that defines the same conformal structure as* i *and has the property that all conformal transformations on* $(\mathbb{H}, i)$ *are also isometries of* $(\mathbb{H}, g_P)$*.*

PROOF. We define g_P at $z = x + iy \in \mathbb{H}$ by

$$g_P = \frac{1}{y^2} g_E,$$

where g_E is the Euclidean metric. The conformal transformations on $(\mathbb{H}, i)$ are given by fractional linear transformations

$$\begin{aligned}\operatorname{Aut}(\mathbb{H}, i) &= \left\{ \varphi(z) = \frac{az+b}{cz+d} \;\middle|\; a,b,c,d \in \mathbb{R}, \quad ad - bc = 1 \right\} \Big/ \{\pm 1\} \\ &= \mathrm{SL}(2,\mathbb{R})/\{\pm 1\} =: \mathrm{PSL}(2,\mathbb{R}),\end{aligned}$$

and one can check that each of these defines an isometry with respect to g_P. One can also compute that g_P has curvature -1, and the geodesics of g_P are precisely the lines and semicircles that meet $\mathbb{R}$ orthogonally, parametrized so that they exist for all forward and backward time, thus g_P is complete. For more details on all of this, the book by Hummel [**Hum97**] is highly recommended. □

By lifting to universal covers, this implies the following.

COROLLARY 9.20. *For every pointed Riemann surface* (Σ, j, Θ) *such that* $\chi(\Sigma \setminus \Theta) < 0$*, the punctured Riemann surface* $(\Sigma \setminus \Theta, j)$ *admits a complete Riemannian metric* g_j *of constant curvature* -1 *that defines the same conformal structure as* j*, and has the property that all biholomorphic transformations on* $(\Sigma \setminus \Theta, j)$ *are also isometries of* $(\Sigma \setminus \Theta, g_j)$*.*

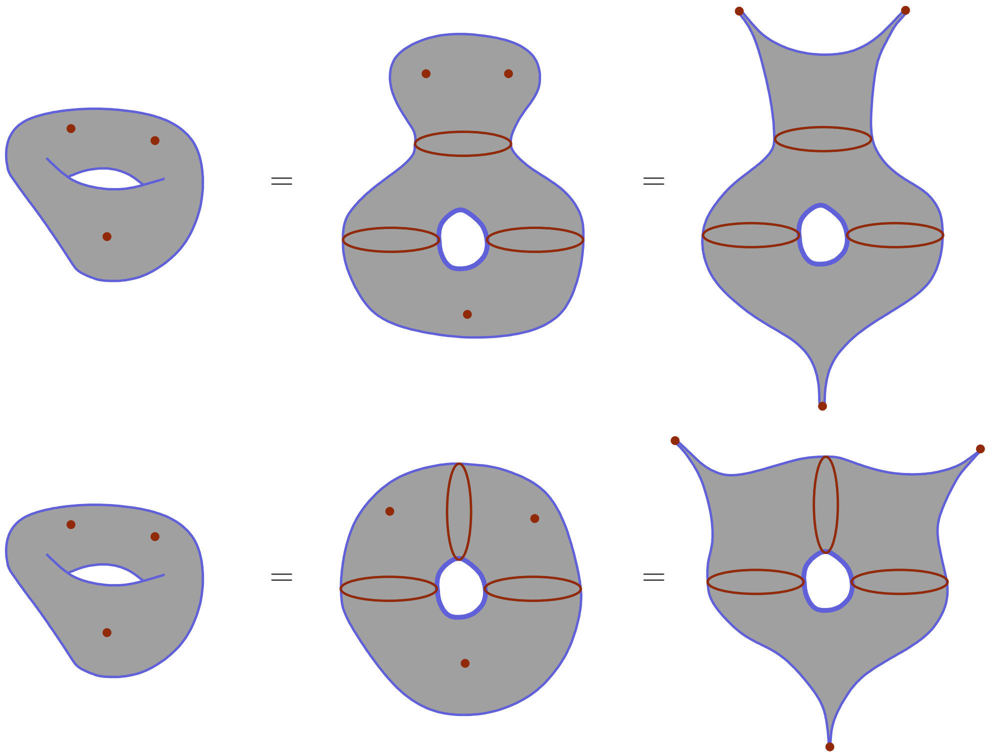

FIGURE 9.5. Two distinct pair-of-pants decompositions for the same genus 1 Riemann surface with three marked points. The decompositions are shown from two perspectives: the pictures at the right are meant to give a more accurate impression of the Poincaré metric, which becomes singular and forms a cusp at each marked point.

The metric g_j in this corollary is often called the **Poincaré metric**. It is uniquely determined by j.

Every class in $\pi_1(\dot{\Sigma})$ contains a unique geodesic for g_j. Now suppose $C \subset \dot{\Sigma}$ is a union of disjoint embedded geodesics such that each connected component of $\dot{\Sigma} \setminus C$ has the homotopy type of a disk with two holes. The components are then called **singular pairs of pants**, and the result is called a **pair-of-pants decomposition** of $(\dot{\Sigma}, j)$. Two examples for the case $g = 1$ and $\ell = 3$ are shown in Figure 9.5.

A pair-of-pants decomposition for (Σ, j, Θ) gives rise to a local parametrization of $\mathcal{M}_{g,\ell}$ near $[(\Sigma, j, \Theta)]$, known as the *Fenchel-Nielsen coordinates*. These consist of two parameters that can be associated to each of the geodesics $\gamma \subset \Sigma$ in the decomposition, namely the length $\ell(\gamma) > 0$ of the geodesic and a *twist* parameter $\theta(\gamma) \in S^1$, which describes how the two neighboring pairs of pants are glued together along γ. Note that by computing Euler characteristics, there are always exactly $-\chi(\Sigma \setminus \Theta) = 2g - 2 + \ell$ pairs of pants in a decomposition, so that the total number of geodesics involved is $[3(2g - 2 + \ell) - \ell]/2 = 3g - 3 + \ell$, thus one can read off the formula $\dim \mathcal{M}_{g,\ell} = 6g - 6 + 2\ell$ from this geometric picture.

One can also see the noncompactness of $\mathcal{M}_{g,\ell}$ in this picture quite concretely: the twist parameters belong to a compact space, but each length parameter can potentially shrink to 0 or blow up to ∞ as j (and hence g_j) is deformed. It turns out that the latter possibility is an illusion, but one may need to switch to a different pair-of-pants decomposition to see why:

Theorem. *For every pair of integers $g \geq 0$ and $\ell \geq 0$ with $2g + \ell \geq 3$, there exists a constant $C = C(g, \ell) > 0$ such that every $[(\Sigma, j, \Theta)] \in \mathcal{M}_{g,\ell}$ admits a pair-of-pants decomposition in which all geodesics bounding the pairs of pants have length at most C.*

This theorem implies that from a hyperbolic perspective, the only meaningful way for stable pointed Riemann surfaces to degenerate is when some of the bounding geodesics in a pair-of-pants decomposition shrink to length zero. Figure 9.6 shows several examples of degenerate Riemann surfaces that can arise in this way for $g = 1$ and $\ell = 3$, giving elements of the space that we will now define as $\overline{\mathcal{M}}_{1,3}$.

Definition 9.21. A **nodal Riemann surface** with $\ell \geq 0$ marked points and $N \geq 0$ **nodes** is a tuple (S, j, Θ, Δ) consisting of:

- A closed but not necessarily connected Riemann surface (S, j);
- An ordered set of ℓ points $\Theta \subset S$;
- An unordered set of $2N$ points $\Delta \subset S \setminus \Theta$ equipped with an involution $\sigma : \Delta \to \Delta$. Each pair $\{z, \sigma(z)\}$ for $z \in \Delta$ is referred to as a **node**.

Let $\widehat{S}$ denote the closed surface obtained by performing connected sums on S at each node $\{z^+, z^-\} \subset \Delta$. We then say that (S, j, Θ, Δ) is **connected** if and only if $\widehat{S}$ is connected, and the genus of $\widehat{S}$ is called the **arithmetic genus** of (S, j, Θ, Δ). We say that (S, j, Θ, Δ) is **stable** if every connected component of $S \setminus (\Theta \cup \Delta)$ has negative Euler characteristic. Finally, two nodal Riemann surfaces (S, j, Θ, Δ) and $(S', j', \Theta', \Delta')$ are considered **equivalent** if there exists a biholomorphic map $\varphi : (S, j) \to (S', j')$ taking Θ to Θ' with the ordering preserved and taking Δ to Δ' such that nodes are mapped to nodes.

The nodes $\{z^+, z^-\} \subset \Delta$ are typically represented in pictures as self-intersections of S, cf. Figure 9.6. We can think of the *stable* nodal surfaces as precisely those which admit (possibly singular) pair-of-pants decompositions. All nodal Riemann surfaces we consider will be assumed connected in the sense defined above unless otherwise noted; note that S itself can nonetheless be disconnected, as is the case in four out of the six nodal surfaces shown in Figure 9.6.

We now introduce some further terminology and notation that will be useful in the next section as well. Whenever $\dot{\Sigma} = \Sigma \setminus \Gamma$ is obtained by puncturing a Riemann surface (Σ, j) at finitely many points $\Gamma \subset \Sigma$, we shall define the **circle compactification**

$$\overline{\Sigma} := \dot{\Sigma} \cup \bigcup_{z \in \Gamma} \delta_z,$$

where for each $z \in \Gamma$, the circle δ_z is defined as a "half-projectivization" of the tangent space at z:

$$\delta_z := (T_z\Sigma \setminus \{0\}) \big/ \mathbb{R}^*_+,$$

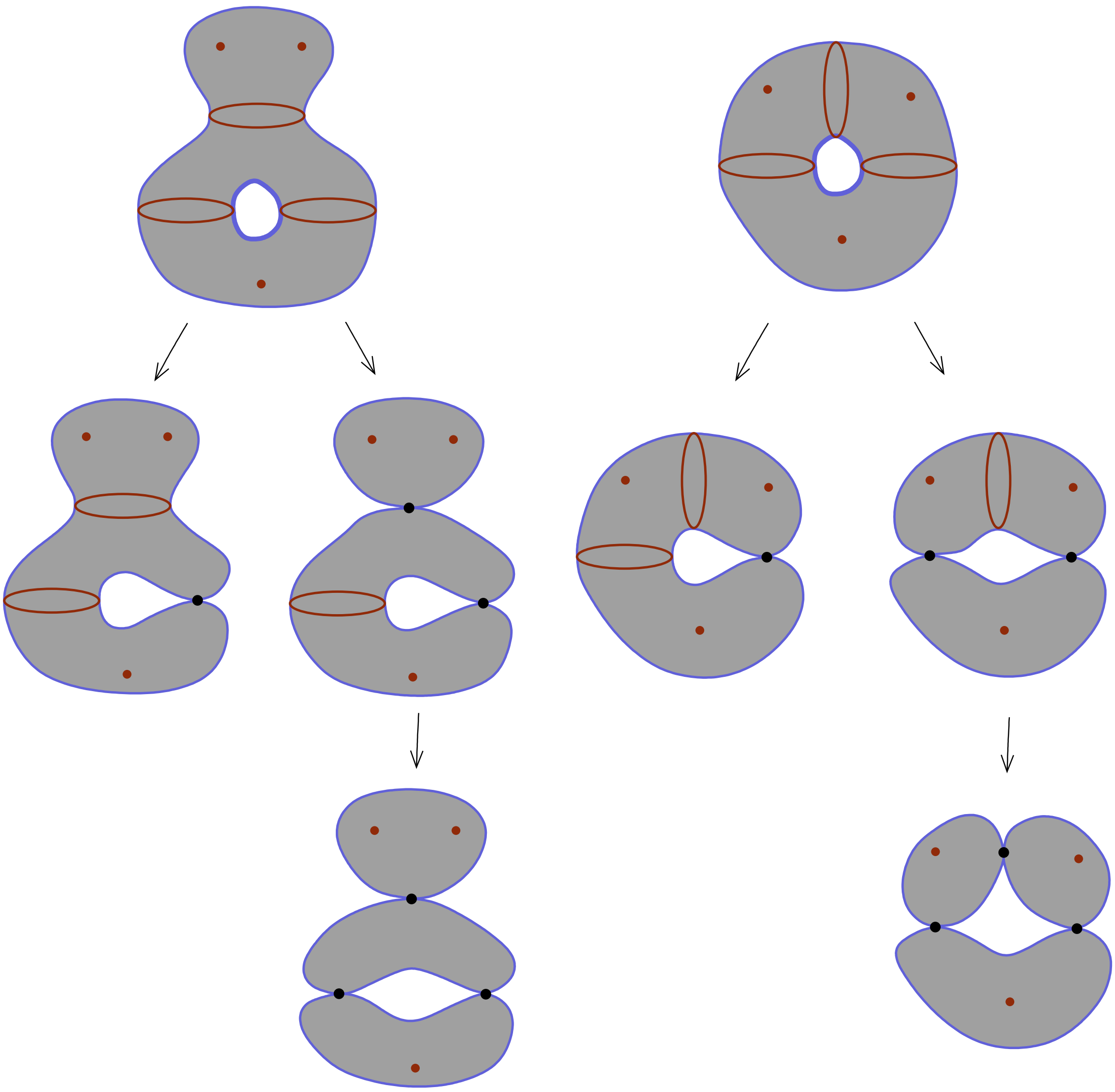

FIGURE 9.6. Starting from each of the pair-of-pants decompositions for the $g = 1$ and $\ell = 3$ case from Figure 9.5, shrinking geodesic lengths to zero produces various examples of stable nodal Riemann surfaces belonging to $\overline{\mathcal{M}}_{1,3}$.

with the positive real numbers $\mathbb{R}_+^*$ acting by scalar multiplication. To understand the topology of $\overline{\Sigma}$, one can equivalently define it by choosing holomorphic cylindrical coordinates $[0, \infty) \times S^1 \subset \dot{\Sigma}$ near each z, and replacing the open half-cylinder with $[0, \infty] \times S^1$, where δ_z is now the **circle at infinity** $\{\infty\} \times S^1$. There is no natural choice of global smooth structure on $\overline{\Sigma}$, but it is homeomorphic to an oriented surface with boundary and carries both smooth and conformal structures on its interior, due to the obvious identification

$$\dot{\Sigma} = \overline{\Sigma} \setminus \bigcup_{z \in \Gamma} \delta_z \subset \overline{\Sigma}.$$

The conformal structure of Σ at each $z \in \Gamma$ does induce on each of the circles δ_z an **orthogonal structure**, meaning a preferred class of homeomorphisms to S^1 that are all related to each other by rotations. One can therefore speak of **orthogonal maps** $\delta_z \to \delta_{z'}$ for $z, z' \in \Gamma$, which are always homeomorphisms and can either preserve or reverse orientation.

Now if (S, j, Θ, Δ) is a nodal Riemann surface, we let $\dot{S} = S \setminus \Delta$ and form the circle compactification $\overline{S}$, which has the topology of a compact oriented surface with boundary. Given a node $\{z^+, z^-\} \subset \Delta$, a **decoration** for $\{z^+, z^-\}$ is a choice of orientation reversing orthogonal map

$$\Phi : \delta_{z^+} \to \delta_{z^-}.$$

We say that (S, j, Θ, Δ) is a **decorated nodal surface** if it is equipped with a choice of decoration Φ for every node, or **partially decorated** if Φ is defined for some subset of the nodes. A partial decoration Φ gives rise to another compact oriented surface

$$\widehat{S}_\Phi := \overline{S}/\sim,$$

where the equivalence relation identifies δ_{z^+} with δ_{z^-} via Φ for each decorated node $\{z^+, z^-\} \subset \Delta$. Note that if every node is decorated, then $\widehat{S}_\Phi$ has the topology of a closed connected and oriented surface whose genus defines the arithmetic genus of (S, j, Θ, Δ) according to Definition 9.21. We shall denote the collection of special circles in $\widehat{S}_\Phi$ where boundray components $\delta_{z^+}, \delta_{z^-} \subset \partial\overline{S}$ have been identified by

$$C_\Phi \subset \widehat{S}_\Phi.$$

Since $\widehat{S}_\Phi \setminus (\partial\widehat{S}_\Phi \cup C_\Phi)$ has a natural identification with $\dot{S}$, it inherits smooth and conformal structures which degenerate along C_Φ and $\partial\widehat{S}_\Phi$. We will say that two partially decorated nodal Riemann surfaces $(S, j, \Theta, \Delta, \Phi)$ and $(S', j', \Theta', \Delta', \Phi')$ are **equivalent** if (S, j, Θ, Δ) and $(S', j', \Theta', \Delta')$ are equivalent via a biholomorphic map $\varphi : (S, j) \to (S', j')$ that extends continuously from $\dot{S} \to \dot{S}'$ to a homeomorphism $\widehat{S}_\Phi \to \widehat{S}'_{\Phi'}$.

Now if $2g + \ell \geq 3$, define $\overline{\mathcal{M}}_{g,\ell}$ as the set of equivalence classes of stable nodal Riemann surfaces with ℓ marked points and arithmetic genus g. There is a natural inclusion

$$\mathcal{M}_{g,\ell} \subset \overline{\mathcal{M}}_{g,\ell}$$

by regarding each pointed Riemann surface (Σ, j, Θ) as a nodal Riemann surface $(\Sigma, j, \Theta, \Delta)$ with $\Delta = \emptyset$. The most important property of $\overline{\mathcal{M}}_{g,\ell}$ is that it admits the structure of a compact metrizable topological space for which the inclusion $\mathcal{M}_{g,\ell} \hookrightarrow \overline{\mathcal{M}}_{g,\ell}$ is continuous onto an open subset. Rather than formulating all of this in precise terms, let us state the main corollary that is important to know in practice.

THEOREM 9.22. *Fix $g \geq 0$ and $\ell \geq 0$ with $2g + \ell \geq 3$. Then for any sequence $[(\Sigma_k, j_k, \Theta_k)] \in \mathcal{M}_{g,\ell}$, there exists a stable nodal Riemann surface $[(S, j, \Theta, \Delta)] \in \overline{\mathcal{M}}_{g,\ell}$ such that after restricting to a subsequence,*

$$[(\Sigma_k, j_k, \Theta_k)] \to [(S, j, \Theta, \Delta)]$$

in the following sense: (S, j, Θ, Δ) *admits a decoration* Φ *such that for sufficiently large* k*, there are homeomorphisms*

$$\varphi : \widehat{S}_\Phi \to \Sigma_k,$$

smooth outside of C_Φ*, which map* Θ *to* Θ_k *preserving the ordering and satisfy*

$$\varphi^* j_k \to j \quad \text{in} \quad C^\infty_{\text{loc}}(\widehat{S}_\Phi \setminus C_\Delta).$$

As one might gather from the above statement, one could just as well define a compact metrizable topology on the space of equivalence classes of *decorated* nodal Riemann surfaces and then characterize the topology of $\overline{\mathcal{M}}_{g,\ell}$ via the natural projection that forgets the decorations.

EXERCISE 9.23. The space $\mathcal{M}_{0,4}$ has a natural identification with $S^2 \setminus \{0, 1, \infty\}$, defined by choosing the unique identification of any 4-pointed Riemann sphere $(S^2, j, (z_1, z_2, z_3, z_4))$ with $\mathbb{C} \cup \{\infty\}$ such that z_1, z_2, z_3 are identified with $0, 1, \infty$ respectively, while z_4 is sent to some point in $S^2 \setminus \{0, 1, \infty\}$. Show that this extends continuously to an identification of $\overline{\mathcal{M}}_{0,4}$ with S^2. What do the three nodal curves in $\overline{\mathcal{M}}_{0,4} \setminus \mathcal{M}_{0,4}$ look like in terms of pair-of-pants decompositions?

9.4. The SFT compactness theorem

We now introduce the natural compactification of $\mathcal{M}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$.

9.4.1. Nodal curves. A punctured J-holomorphic **nodal curve** in $(\widehat{W}, J)$ with $m \ge 0$ marked points consists of the data $(S, j, \Gamma^+, \Gamma^-, \Theta, \Delta, u)$, where

- $(S, j, \Gamma^+ \sqcup \Gamma^- \sqcup \Theta, \Delta)$ is a nodal Riemann surface, with $|\Theta| = m$;
- $u : (\dot{S}, j) \to (\widehat{W}, J)$ for $\dot{S} := S \setminus (\Gamma^+ \cup \Gamma^-)$ is an asymptotically cylindrical J-holomorphic map with positive punctures Γ^+ and negative punctures Γ^- such that for each node $\{z^+, z^-\} \subset \Delta$, $u(z^+) = u(z^-)$.

Equivalence of two nodal curves

$$(S_0, j_0, \Gamma_0^+, \Gamma_0^-, \Theta_0, \Delta_0, u_0) \sim (S_1, j_1, \Gamma_1^+, \Gamma_1^-, \Theta_1, \Delta_1, u_1)$$

is defined as the existence of an equivalence of nodal Riemann surfaces $\varphi : (S_0, j_0, \Gamma_0^+ \sqcup \Gamma_0^- \sqcup \Theta_0, \Delta_0) \to (S_1, j_1, \Gamma_1^+ \sqcup \Gamma_1^- \sqcup \Theta_1, \Delta_1)$ such that $u_0 = u_1 \circ \varphi$. We say that $(S, j, \Gamma^+, \Gamma^-, \Theta, \Delta, u)$ is **connected** if and only if the nodal Riemann surface $(S, j, \Gamma^+ \sqcup \Gamma^- \sqcup \Theta, \Delta)$ is connected, and its **arithmetic genus** is then defined to be the arithmetic genus of the latter. We say that $(S, j, \Gamma^+, \Gamma^-, \Theta, \Delta, u)$ is **stable** if every connected component of $S \setminus (\Gamma^+ \cup \Gamma^- \cup \Theta \cup \Delta)$ on which u is constant has negative Euler characteristic. Note that the underlying nodal Riemann surface $(S, j, \Gamma^+ \sqcup \Gamma^- \sqcup \Theta, \Delta)$ need not be stable in general.

Nodal curves are sometimes also referred to as *holomorphic buildings of height 1.* These are the objects that form the *Gromov compactification* of $\mathcal{M}_{g,m}(J, A)$ when W is a closed symplectic manifold. One can now roughly imagine how the compactness theorem in that setting is proved: given a converging sequence of almost complex structures $J_k \to J$ and a sequence $[(\Sigma_k, j_k, \Theta_k, u_k)] \in \mathcal{M}_{g,m}(J_k, A_k)$ with uniformly bounded energy, we can first add some auxiliary marked points if necessary to assume that $2g + m \ge 3$. Now a subsequence of the domains $[(\Sigma_k, j_k, \Theta_k)] \in \mathcal{M}_{g,m}$ converges to an element of the Deligne-Mumford space $[(S, j, \Theta, \Delta)] \in \overline{\mathcal{M}}_{g,m}$. Concretely, this means that for large k, our sequence in $\mathcal{M}_{g,m}(J_k, A_k)$ admits representatives $(\Sigma, j'_k, \Theta, u'_k)$, with Σ a fixed surface with fixed marked points $\Theta \subset \Sigma$, and (S, j, Θ, Δ) admits decorations Φ so that one can identify $\widehat{S}_\Phi$ with Σ and find

$$j'_k \to j \quad \text{in} \quad C^\infty_{\text{loc}}(\Sigma \setminus C)$$

for some collection of disjoint circles $C \subset \Sigma$. The connected components of $(\Sigma \setminus C, j)$ are then biholomorphically equivalent to the connected components of $(S \setminus \Delta, j)$, and if the newly reparametrized maps $u'_k : \Sigma \to W$ are uniformly C^1_{loc}-bounded on $\Sigma \setminus C$, then a subsequence converges in $C^\infty_{\text{loc}}(\Sigma \setminus C)$ to a limiting finite-energy J-holomorphic map $u_\infty : (S \setminus \Delta, j) \to (W, J)$, whose singularities at Δ are removable. In particularly nice cases, this may be the end of the story, and our subsequence of $[(\Sigma_k, j_k, \Theta_k, u_k)] \in \mathcal{M}_{g,m}(J_k, A_k)$ converges to the nodal curve $[(S, j, \Theta, \Delta, u_\infty)]$; in particular the domain $[(S, j, \Theta, \Delta)]$ in this case is stable and is thus an element of $\overline{\mathcal{M}}_{g,m}$. But more complicated things can also happen, e.g. u'_k might not be C^1-bounded, in which case there is bubbling. The bubbles that arise will be either planes or spheres, so they produce extra domain components with nonnegative Euler characteristic, but since they are never constant, the limiting nodal curve is still considered stable. Similarly, since $\Sigma \setminus C$ is not compact, there can also be breaking as in Figure 9.3, producing more non-stable domain components which can be cylinders in addition to planes and spheres—but again, the limiting map on these components will never be constant.

9.4.2. Holomorphic buildings. Only a small subset of the phenomena observed in §9.3 can be described via nodal curves: we've seen that in general, we also have to allow "broken" curves with multiple "levels". This notion can be formalized as follows.

Given integers $g, m, N_+, N_- \geq 0$, a **holomorphic building of height** $N_-|1|N_+$ with arithmetic genus g and m marked points is a tuple

$$\mathbf{u} = (S, j, \Gamma^+, \Gamma^-, \Theta, \Delta^{\text{nd}}, \Delta^{\text{br}}, L, \Phi, u),$$

with the various data defined as follows:

- The **domain** $(S, j, \Gamma^+ \sqcup \Gamma^- \sqcup \Theta, \Delta^{\text{nd}} \sqcup \Delta^{\text{br}})$ is a connected but not necessarily stable nodal Riemann surface of arithmetic genus g, where $|\Theta| = m$, and the involution on $\Delta^{\text{nd}} \sqcup \Delta^{\text{br}}$ is assumed to preserve the subsets Δ^{nd} and Δ^{br}. Matched pairs in these subsets are called the **nodes** and **breaking pairs** respectively of $\mathbf{u}$. The **marked points** of $\mathbf{u}$ are the points in Θ, while Γ^+ and Γ^- are its positive and negative **punctures** respectively.
- The **level structure** is a locally constant function

$$L : S \to \{-N_-, \dots, -1, 0, 1, \dots, N_+\}$$

that attains every value in $\{-N_-, \dots, N_+\}$ except possibly 0, and satisfies:
  1. $L(z^+) = L(z^-)$ for each node $\{z^+, z^-\} \subset \Delta^{\text{nd}}$;
  2. Each breaking pair $\{z^+, z^-\} \subset \Delta^{\text{br}}$ can be labelled such that $L(z^+) - L(z^-) = 1$;
  3. $L(\Gamma^+) = \{N_+\}$ and $L(\Gamma^-) = \{-N_-\}$.

- The **decoration** is a choice of orientation-reversing orthogonal map
$$\delta_{z^+} \xrightarrow{\Phi} \delta_{z^-}$$
for each breaking pair $\{z^+, z^-\} \subset \Delta^{\mathrm{br}}$.
- The **map** is an asymptotically cylindrical pseudoholomorphic curve
$$u : (\dot{S} := S \setminus (\Gamma^+ \cup \Gamma^- \cup \Delta^{\mathrm{br}}), j) \to \coprod_{N \in \{-N_-, \ldots, N_+\}} (\widehat{W}_N, J_N),$$
where
$$(\widehat{W}_N, J_N) := \begin{cases} (\mathbb{R} \times M_+, J_+) & \text{for } N \in \{1, \ldots, N_+\}, \\ (\widehat{W}, J) & \text{for } N = 0, \\ (\mathbb{R} \times M_-, J_-) & \text{for } N \in \{-N_-, \ldots, -1\}, \end{cases}$$
and u sends $\dot{S} \cap L^{-1}(N)$ into $\widehat{W}_N$ for each N, with positive punctures at Γ^+ and negative punctures at Γ^-. Moreover,
$$u(z^+) = u(z^-) \qquad \text{for every node } \{z^+, z^-\} \subset \Delta^{\mathrm{nd}},$$
and for each breaking pair $\{z^+, z^-\} \subset \Delta^{\mathrm{br}}$ labelled with $L(z^+) - L(z^-) = 1$, u has a positive puncture at z^- and a negative puncture at z^+ asymptotic to the same orbit, such that if $u_+ : \delta_{z^+} \to M_\pm$ and $u_- : \delta_{z^-} \to M_\pm$ denote the induced asymptotic parametrizations of the orbit, then
$$u_+ = u_- \circ \Phi : \delta_{z^+} \to M_\pm.$$

The following additional notation and terminology for the building $\mathbf{u}$ will be useful to keep in mind. For each $N \in \{-N_-, \ldots, 0, \ldots, N_+\}$, denote
$$\dot{S}_N := \big(S \setminus (\Gamma^+ \cup \Gamma^- \cup \Delta^{\mathrm{br}})\big) \cap L^{-1}(N),$$
and denote the restriction of u to this subset by
$$u^N : \dot{S}_N \to \begin{cases} \mathbb{R} \times M_+ & \text{if } N > 0, \\ \widehat{W} & \text{if } N = 0, \\ \mathbb{R} \times M_- & \text{if } N < 0. \end{cases}$$
Including $\Theta \cap L^{-1}(N)$ and $\Delta^{\mathrm{nd}} \cap L^{-1}(N)$ in the data defines u^N as a (generally disconnected) nodal curve with marked points, whose positive punctures are in bijective correspondence with the negative punctures of u^{N+1} if $N < N_+$. We call u_N the N**th level** of $\mathbf{u}$, and all it an **upper** or **lower** level if $N > 0$ or $N < 0$ respectively, and the **main level** if $N = 0$. By convention, every holomorphic building in $\widehat{W}$ has exactly one main level (which lives in $\widehat{W}$ itself) and arbitrary nonnegative numbers of upper and lower levels (which live in the symplectizations $\mathbb{R} \times M_\pm$). One slightly subtle detail is that it is possible for the main level to be *empty*, meaning 0 is not in the image of the level function L. The requirement that L should attain every other value from $-L_-$ to L_+ is a convention to ensure that upper and lower levels are not empty, so e.g. if a building has an empty main level and no lower levels, then the lowest nonempty upper level is always labelled 1 instead of something arbitrary.

The positive punctures of the topmost level of $\mathbf{u}$ are Γ^+, and the negative punctures of the bottommost level are Γ^-, so these give rise to lists of positive/negative asymptotic orbits $\boldsymbol{\gamma}^\pm = (\gamma_1^\pm, \dots, \gamma_{k_\pm}^\pm)$ in $M_\pm$. There is also a relative homology class

$$[\mathbf{u}] \in H_2(W, \bar{\boldsymbol{\gamma}}^+ \cup \bar{\boldsymbol{\gamma}}^-).$$

To define this, recall from §6.2 how it was defined for smooth curves $u : \dot{\Sigma} \to \widehat{W}$: we considered the retraction $\pi : \widehat{W} \to W$ that collapses each cylindrical end to $M_\pm \subset \partial W$, and noted that since u is asymptotically cylindrical, the map $\pi \circ u : \dot{\Sigma} \to W$ extends to a continuous map on the circle compactification,

$$\bar{u} : \overline{\Sigma} \to W,$$

whose relative homology class gives the definition of $[u]$. The conditions on nodes and breaking orbits allow us to perform a similar trick for the building $\mathbf{u}$, using the map

$$\pi : \coprod_{N \in \{-N_-, \dots, N_+\}} \widehat{W}_N \to W$$

which acts as the identity on W but collapses cylindrical ends of $\widehat{W}$ to ∂W and similarly collapses each copy of $\mathbb{R} \times M_\pm$ to $M_\pm \subset \partial W$. Extending the decorations Φ arbitrarily to decorations of the nodes Δ^{nd}, one can then take the circle compactification of $\dot{S} := S \setminus (\Gamma^+ \cup \Gamma^- \cup \Delta^{\text{nd}} \cup \Delta^{\text{br}})$ and glue matching boundary components together along Φ to form a compact surface with boundary $\overline{S}_\Phi$ such that $\pi \circ u : \dot{S} \to W$ extends to a continuous map

$$\bar{u} : \overline{S}_\Phi \to W.$$

Its relative homology class defines $[\mathbf{u}] \in H_2(W, \bar{\boldsymbol{\gamma}}^+ \cup \bar{\boldsymbol{\gamma}}^-)$.

We say that the building $\mathbf{u}$ is **stable** if two properties hold:

(1) Every connected component of $S \setminus (\Gamma^+ \cup \Gamma^- \cup \Theta \cup \Delta^{\text{nd}} \cup \Delta^{\text{br}})$ on which the map u is constant has negative Euler characteristic;
(2) There is no $N \in \{-N_-, \dots, N_+\}$ for which the Nth level consists entirely of a disjoint union of trivial cylinders without any marked points or nodes.

An **equivalence** between two holomorphic buildings

$$\mathbf{u}_i = (S_i, j_i, \Gamma_i^+, \Gamma_i^-, \Theta_i, \Delta_i^{\text{nd}}, \Delta_i^{\text{br}}, L_i, \Phi_i, u_i), \qquad i = 0, 1$$

is defined as an equivalence of partially decorated nodal Riemann surfaces

$$(S_0, j_0, \Gamma_0^+ \sqcup \Gamma_0^+ \sqcup \Theta_0, \Delta_0^{\text{nd}} \sqcup \Delta_0^{\text{br}}, \Phi_0) \xrightarrow{\varphi} (S_1, j_1, \Gamma_1^+ \sqcup \Gamma_1^+ \sqcup \Theta_1, \Delta_1^{\text{nd}} \sqcup \Delta_1^{\text{br}}, \Phi_1)$$

such that $\varphi(\Gamma_0^\pm) = \Gamma_1^\pm$, $\varphi(\Theta_0) = \Theta_1$, $\varphi(\Delta_0^{\text{nd}}) = \Delta_1^{\text{nd}}$, $\varphi(\Delta_0^{\text{br}}) = \Delta_1^{\text{br}}$, $L_1 \circ \varphi = L_0$, and

$$u_1^0 \circ \varphi = u_0^0,$$

while

$$u_1^N \circ \varphi = u_0^N \text{ up to } \mathbb{R}\text{-translation} \quad \text{for each } N \neq 0.$$

Given lists of orbits $\boldsymbol{\gamma}^\pm$ and a relative homology class A, the set of equivalence classes of stable holomorphic buildings in $(\widehat{W}, J)$ with arithmetic genus g and m

marked points, positively/negatively asymptotic to $\boldsymbol{\gamma}^{\pm}$ and homologous to A will be denoted by

$$\overline{\mathcal{M}}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-).$$

Observe that for any $A \neq 0$, there is a natural inclusion $\mathcal{M}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-) \subset \overline{\mathcal{M}}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$ defined by regarding J-holomorphic curves in $\mathcal{M}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$ as buildings with no upper or lower levels and no nodes. Such buildings are always stable if $A \neq 0$ because they are not constant.

9.4.3. Convergence. For a general definition of the topology of $\overline{\mathcal{M}}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$ and the proof that it is both compact and metrizable, we refer to [**BEH+03**] or the more comprehensive treatment in [**Abb14**]. The following statement contains all the details about the topology that one usually needs to know in practice (see Figure 9.7).

THEOREM 9.24. *Fix integers $g \geq 0$ and $m \geq 0$, and assume all Reeb orbits in $(M, \mathcal{H}_+)$ and $(M, \mathcal{H}_-)$ are nondegenerate. Then for any sequence*

$$[(\Sigma_k, j_k, \Gamma_k^+, \Gamma_k^-, \Theta_k, u_k)] \in \mathcal{M}_{g,m}(J_k, A_k, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$$

of nonconstant J_k-holomorphic curves in $\widehat{W}$ with uniformly bounded energy $E(u_k)$, there exists a stable holomorphic building

$$[\mathbf{u}_\infty] = [(S, j, \Gamma^+, \Gamma^-, \Theta, \Delta^{\mathrm{nd}}, \Delta^{\mathrm{br}}, L, \Phi, u_\infty)] \in \overline{\mathcal{M}}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$$

such that after restricting to a subsequence, $[(\Sigma_k, j_k, \Gamma_k^+, \Gamma_k^-, \Theta_k, u_k)] \to [\mathbf{u}_\infty]$ in the following sense. The decorations Φ at Δ^{br} can be extended to decorations at Δ^{nd} so that if $\widehat{S}_\Phi$ denotes the closed oriented topological 2-manifold obtained from $S \setminus (\Delta^{\mathrm{nd}} \cup \Delta^{\mathrm{br}})$ by gluing circle compactifications along Φ, then for k sufficiently large, there exist homeomorphisms

$$\varphi_k : \widehat{S}_\Phi \to \Sigma_k$$

that are smooth outside of C_Φ, map $\Gamma^+ \sqcup \Gamma^- \sqcup \Theta$ to $\Gamma_k^+ \sqcup \Gamma_k^- \sqcup \Theta_k$ with the ordering preserved, and satisfy

$$\varphi_k^* j_k \to j \quad \text{in} \quad C^\infty_{\mathrm{loc}}(\widehat{S}_\Phi \setminus C_\Phi).$$

Moreover for $N = \{-N_-, \ldots, 0, \ldots, N\}$, let

$$v_k^N := u_k \circ \varphi_k|_{\ddot{S}_N} : \ddot{S}_N \to \widehat{W},$$

with $\ddot{S}_N := \left(S \setminus (\Gamma^+ \cup \Gamma^- \cup \Delta^{\mathrm{nd}} \cup \Delta^{\mathrm{br}})\right) \cap L^{-1}(N)$ regarded as a subset of $\widehat{S}_\Phi \setminus C_\Phi$. Then:

(1) $v_k^0 \to u_\infty^N$ in $C^\infty_{\mathrm{loc}}(\ddot{S}_N, \widehat{W})$;
(2) For each $\pm N > 0$, v_k^N has image in the positive/negative cylindrical end for all k sufficiently large, and there exists a sequence $r_k^N \to \pm\infty$ such that the resulting $\mathbb{R}$-translations converge:

$$\tau_{-r_k^N} \circ v_k^N \to u_\infty^N \quad \text{in} \quad C^\infty_{\mathrm{loc}}(\ddot{S}_N, \mathbb{R} \times M_\pm).$$

The rates of divergence of the sequences $r_k^N \to \pm\infty$ *are related by*

$$r_k^{N+1} - r_k^N \to +\infty \quad \text{for all } N < N_+.$$

Finally, let $\overline{S}_\Phi$ *denote the compact topological surface with boundary defined as the circle compactification of* $\widehat{S}_\Phi \setminus (\Gamma^+ \cup \Gamma^-)$*, and let* $\overline{\Sigma}_k$ *denote the circle compactification of* $\dot{\Sigma}_k := \Sigma_k \setminus (\Gamma_k^+ \cup \Gamma_k^-)$*. Then for all* k *large,* φ_k *extends to a continuous map*

$$\bar{\varphi}_k : \overline{S}_\Phi \to \overline{\Sigma}_k$$

such that

$$\bar{u}_k \circ \bar{\varphi}_k \to \bar{u}_\infty \quad \text{in} \quad C^0(\overline{S}_\Phi, W).$$

REMARK 9.25. The theorem is also true under the more general hypothesis that the Reeb vector fields are Morse-Bott. In this case, one can also allow the asymptotic Reeb orbits of the sequence to vary, as long as the sum of their periods is uniformly bounded—such a bound plays the role of an energy bound and guarantees a convergent subsequence of orbits via the Arzelà-Ascoli theorem.

REMARK 9.26. Stability of the limit in Theorem 9.24 is guaranteed for the same reasons as in our discussion of Gromov compactness in §9.4.1: stable domains degenerate to stable nodal domains as geodesics in pair-of-pants decompositions shrink to zero length, while bubbling and breaking produce additional domain components that are not stable but on which the maps are never trivial. Moreover, stability guarantees the *uniqueness* of the limiting building for any convergent sequence, i.e. it is the reason why $\overline{\mathcal{M}}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$ is a Hausdorff space. Indeed, if u_k converges to a stable building $\mathbf{u}_\infty$, then under the notion of convergence described in the theorem, it will also converge to a building $\mathbf{u}'_\infty$ constructed out of $\mathbf{u}_\infty$ by adding to S an extra spherical component, attaching it to the rest by a single node and extending the map u_∞ to be constant on the extra component. One can also insert extra levels into $\mathbf{u}_\infty$ that consist only of trivial cylinders, and u_k will still converge to the resulting building. But these modifications produce buildings that are not stable and thus are not elements of $\overline{\mathcal{M}}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$.

9.4.4. Symplectizations, stretching and so forth. A few minor modifications to the above discussion are necessary to compactify the moduli space of curves in a symplectization $(\mathbb{R} \times M, J)$ for $J \in \mathcal{J}(\mathcal{H})$. It is possible to view this as a special case of a completed symplectic cobordism, but this perspective produces a certain amount of extraneous data that is not meaningful. The key observation is that in the presence of an $\mathbb{R}$-action, one should really compactify $\mathcal{M}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)/\mathbb{R}$ instead of $\mathcal{M}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$. The compactification $\overline{\mathcal{M}}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$ then consists of holomorphic buildings as defined in §9.4.2, but since all levels live in the same symplectization $\mathbb{R} \times M$, there is no longer a distinguished *main level* or any meaningful notion of *upper* vs. *lower* levels; the level structure is simply a function $L : S \to \{1, \ldots, N\}$ for some $N \in \mathbb{N}$, and equivalence of buildings must permit $\mathbb{R}$-translations within each level. For these reasons, the SFT compactness theorem in symplectizations has a few qualitative differences, but is still very much analogous to Theorem 9.24.

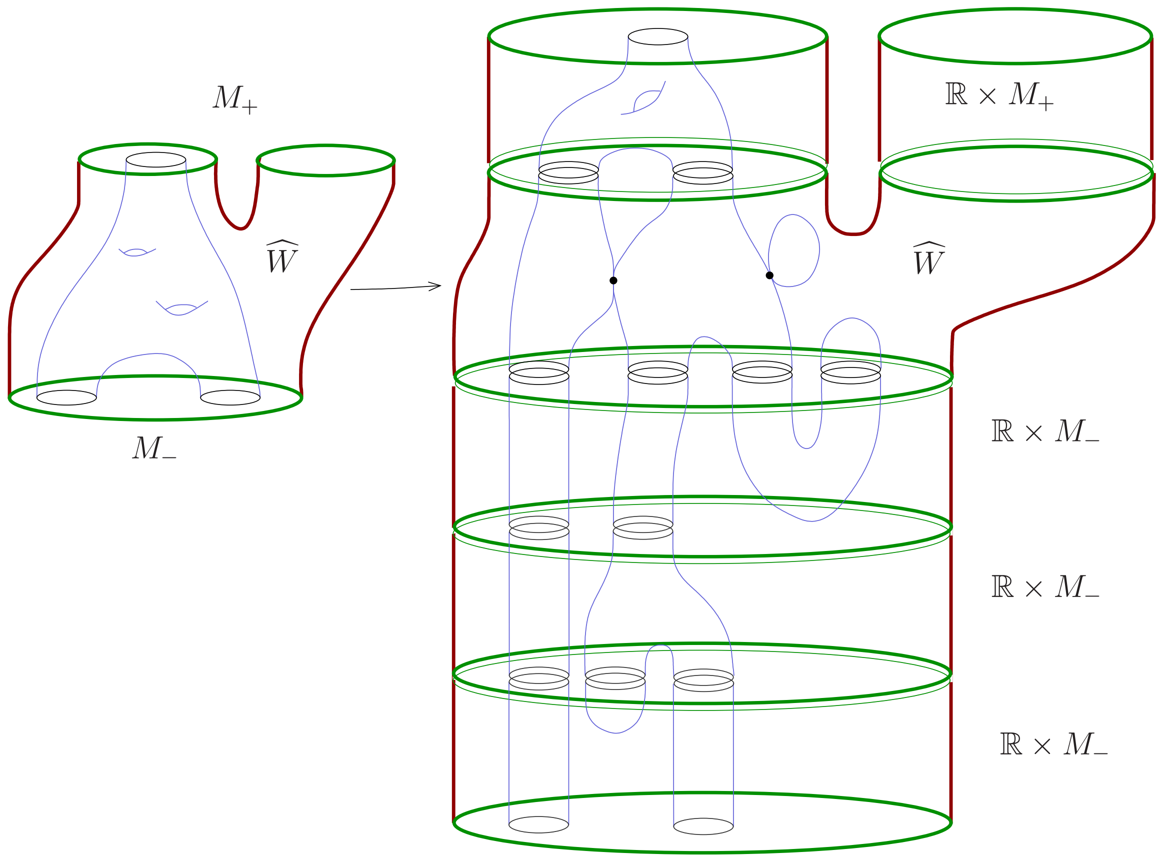

FIGURE 9.7. Convergence to a building with arithmetic genus 2, one upper level and three lower levels.

To complete the picture, we should mention one more type of compactness theorem that appears in [**BEH+03**], which is colloquially described as *stretching the neck*. The geometric idea is as follows: suppose (W, ω) is a closed symplectic manifold and $M \subset W$ is a stable hypersurface that separates W into two pieces $W = W_- \cup_M W_+$, with an induced stable Hamiltonian structure $\mathcal{H} = (\omega, \lambda)$ that orients M as the boundary of W_-.[2] A neighborhood of M in (W, ω) can then be identified symplectically with

$$(\mathcal{N}_\epsilon, \omega_\epsilon) := ((-\epsilon, \epsilon) \times M, d(r\lambda) + \omega)$$

for sufficiently small $\epsilon > 0$. The idea now is to replace $\mathcal{N}_\epsilon$ with larger collars of the form

$$((-T, T) \times M, d\,(f(r)\lambda) + \omega)\,,$$

with C^0-small functions f chosen with $f' > 0$ so that the collar can be glued in smoothly to replace $(\mathcal{N}_\epsilon, \omega_\epsilon)$. This collar looks like a piece of the symplectization of $(M, \mathcal{H})$, thus we are free to choose tame almost complex structures whose restrictions to the inserted collar belong to $\mathcal{J}(\mathcal{H})$. Symplectic manifolds constructed in this

[2]The assumption that $M \subset W$ separates W is inessential, but makes certain details in this discussion more convenient.

way are all symplectomorphic, but their almost complex structures degenerate as one takes $T \to \infty$. Given a sequence $T_k \to \infty$ and a corresponding degenerating sequence J_k, a sequence u_k of J_k-holomorphic curves with bounded energy converges to yet another form of holomorphic building, this time involving a bottom level in $\widehat{W}_- := W_- \cup_M ([0,\infty) \times M)$ with positive punctures approaching orbits in M, some finite number of middle levels that live in the symplectization of M, and a top level that lives in $\widehat{W}_+ := ((-\infty, 0] \times M) \cup_M W_+$ with negative punctures approaching M.

A very popular example for applications arises from Lagrangian submanifolds $L \subset W$. By the Weinstein neighborhood theorem, L always has a neighborhood W_- symplectomorphic to a neighborhood of the zero-section in T^*L, so $M := \partial W_-$ is a contact-type hypersurface contactomorphic to the unit cotangent bundle of L. Stretching the neck then yields T^*L as the completion of W_-, and $W \setminus L$ as the completion of $W_+ := W \setminus \mathring{W}_-$. This construction has often been used in order to study Lagrangian submanifolds via SFT-type methods, see e.g. [**EGH00**, Theorem 1.7.5] and [**Eva10**, **CM**].

LECTURE 10

Cylindrical contact homology and the tight 3-tori

Contents

We've now developed enough of the technical machinery of holomorphic curves to be able to give a rigorous construction of the most basic version of SFT and apply it to a problem in contact topology.

10.1. Contact structures on $\mathbb{T}^3$ and Giroux torsion

As a motivating goal in this lecture, we will prove a result about the classification of contact structures on $\mathbb{T}^3 = S^1 \times S^1 \times S^1$. Denote the three global coordinates on $\mathbb{T}^3$ valued in $S^1 = \mathbb{R}/\mathbb{Z}$ by (ρ, ϕ, θ), and for any $k \in \mathbb{N}$, consider the contact structure

$$\xi_k := \ker \alpha_k, \qquad \text{where} \qquad \alpha_k := \cos(2\pi k\rho)\, d\theta + \sin(2\pi k\rho)\, d\phi.$$

It is an easy exercise to verify that these all satisfy the contact condition $\alpha_k \wedge d\alpha_k > 0$; see Figure 10.1 for a visual representation. The following result is originally due to Giroux [Gir94] and Kanda [Kan97].

Theorem 10.1. *For each pair of positive integers $k \neq \ell$, the contact manifolds $(\mathbb{T}^3, \xi_k)$ and $(\mathbb{T}^3, \xi_\ell)$ are not contactomorphic.*

One of the reasons this result is interesting is that it cannot be proved using any so-called "classical" invariants, i.e. invariants coming from algebraic topology. An example of a classical invariant would be the Euler class of the oriented vector bundle $\xi_k \to \mathbb{T}^3$, or anything else that depends only on the isomorphism class of this

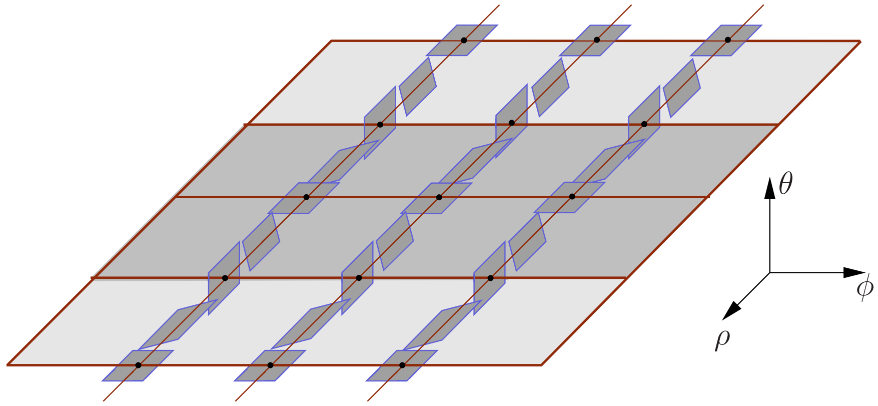

FIGURE 10.1. The contact structures ξ_k on $\mathbb{T}^3$ can be constructed by gluing k copies of the same model $[0,1]\times\mathbb{T}^2$ to each other cyclically.

bundle. The following observation shows that such invariants will never distinguish ξ_k from ξ_ℓ.

PROPOSITION 10.2. *For every $k,\ell \in \mathbb{N}$, ξ_k and ξ_ℓ are homotopic through a smooth family of oriented 2-plane fields on $\mathbb{T}^3$.*

PROOF. In fact, all the ξ_k can be deformed smoothly to $\ker d\rho$, via the homotopy

$$\ker\left[(1-s)\,\alpha_k + s\,d\rho\right], \qquad s \in [0,1].$$

□

REMARK 10.3. One can check in fact that the 1-form in the homotopy given above is contact for every $s \in [0,1)$, so Gray's stability theorem implies that every ξ_k is isotopic to an arbitrarily small perturbation of the foliation $\ker d\rho$. In [Gir94], Giroux used this observation to show that all of them are what we now call *weakly symplectically fillable*. If $\ker d\rho$ were also contact, then Gray's theorem would imply that ξ_k and ξ_ℓ are always isotopic. Thus Theorem 10.1 indicates the impossibility of modifying a homotopy from ξ_k to ξ_ℓ into one that passes only through contact structures.

Let us place this discussion in a larger context. Using the coordinates (ρ,ϕ,θ) on $\mathbb{R}\times\mathbb{T}^2$, a pair of smooth functions $f,g:\mathbb{R}\to\mathbb{R}$ gives rise to a contact form

$$\alpha = f(\rho)\,d\theta + g(\rho)\,d\phi$$

whenever the function $D(\rho) := f(\rho)g'(\rho) - f'(\rho)g(\rho)$ is everywhere positive. Indeed, we have $\alpha\wedge d\alpha = D(\rho)\,d\rho\wedge d\phi\wedge d\theta$, and one easily derives a similar formula for the Reeb vector field,

$$R_\alpha = \frac{1}{D(\rho)}\left[g'(\rho)\,\partial_\theta - f'(\rho)\,\partial_\phi\right].$$

The condition $D > 0$ means geometrically that the path $(f,g):\mathbb{R}\to\mathbb{R}^2$ winds counterclockwise around the origin with its angular coordinate strictly increasing. The simplest special case is the contact form

$$\alpha_{\mathrm{GT}} := \cos(2\pi\rho)\,d\theta + \sin(2\pi\rho)\,d\phi,$$

which matches the formula for α_1 on $\mathbb{T}^3$ given above. Let $\xi_{\mathrm{GT}} := \ker \alpha_{\mathrm{GT}}$ on $\mathbb{R} \times \mathbb{T}^2$.

DEFINITION 10.4. The **Giroux torsion** $\mathrm{GT}(M, \xi) \in \mathbb{N} \cup \{0, \infty\}$ of a contact 3-manifold (M, ξ) is the supremum of the set of positive integers k such that there exists a contact embedding

$$\left([0, k] \times \mathbb{T}^2, \xi_{\mathrm{GT}}\right) \hookrightarrow (M, \xi).$$

We write $\mathrm{GT}(M, \xi) = 0$ if no such embedding exists for any k, and $\mathrm{GT}(M, \xi) = \infty$ if it exists for all k.

EXAMPLE 10.5. The tori $(\mathbb{T}^3, \xi_k)$ for $k \geq \mathbb{Z}$ are contactomorphic to $(\mathbb{R} \times \mathbb{T}^2, \xi_{\mathrm{GT}})/k\mathbb{Z}$, with $k\mathbb{Z}$ acting by translation of the ρ-coordinate. Thus $\mathrm{GT}(\mathbb{T}^3, \xi_k) \geq k - 1$.

A 2-torus $T \subset (M, \xi)$ embedded in a contact 3-manifold is called **pre-Lagrangian** if a neighborhood of T in (M, ξ) admits a contactomorphism to a neighborhood of $\{0\} \times \mathbb{T}^2$ in $(\mathbb{R} \times \mathbb{T}^2, \xi_{\mathrm{GT}})$, identifying T with $\{0\} \times \mathbb{T}^2$. The neighborhood in $\mathbb{R} \times \mathbb{T}^2$ can be arbitrarily small, thus the existence of a pre-Lagrangian torus does not imply $\mathrm{GT}(M, \xi) > 0$; in fact, pre-Lagrangian tori always exist in abundance, e.g. as boundaries of neighborhoods of transverse knots (using the contact model provided by the transverse neighborhood theorem). But given any pre-Lagrangian torus $T \subset (M, \xi)$, one can make a local modification of ξ near T to produce a new contact structure (up to isotopy) with positive Giroux torsion. Define (M', ξ') from (M, ξ) by replacing the small neighborhood $((-\epsilon, \epsilon) \times \mathbb{T}^2, \xi_{\mathrm{GT}})$ with $((-\epsilon, 1+\epsilon) \times \mathbb{T}^2, \xi_{\mathrm{GT}})$, then identify M' with M by a choice of compactly supported diffeomorphism $(-\epsilon, 1+\epsilon) \to (-\epsilon, \epsilon)$. There is now an obvious contact embedding of $([0, 1] \times \mathbb{T}^2, \xi_{\mathrm{GT}})$ into (M, ξ'), hence $\mathrm{GT}(M, \xi') \geq 1$. Moreover, one can adapt the proof of Prop. 10.2 above to show that ξ' is homotopic to ξ through a smooth family of oriented 2-plane fields. The operation changing ξ to ξ' is known as a **Lutz twist** along T. In this language, we see that for each $k \in \mathbb{N}$, $(\mathbb{T}^3, \xi_{k+1})$ is obtained from $(\mathbb{T}^3, \xi_k)$ by performing a Lutz twist along $\{0\} \times \mathbb{T}^2$.

The invariant $\mathrm{GT}(M, \xi)$ is easy to define, but hard to compute in general. The natural guess,

$$\mathrm{GT}(\mathbb{T}^3, \xi_k) = k - 1,$$

turns out to be correct, as was shown in [Gir00], so this is one way to prove Theorem 10.1, but not the approach we will take. The following example shows that one must in any case be careful with such guesses.

EXAMPLE 10.6. For each $k \in \mathbb{N}$, define a model of $S^1 \times S^2$ by

$$S^1 \times S^2 \cong \left([0, k + 1/2] \times \mathbb{T}^2\right) / \sim$$

where the equivalence relation identifies $(\rho, \phi, \theta) \sim (\rho, \phi', \theta)$ for $\rho \in \{0, k+1/2\}$ and every $\theta, \phi, \phi' \in S^1$. Near $\rho = 0$ and $\rho = k+1/2$, this means thinking of (ρ, ϕ) as polar coordinates, so the two subsets $\{\rho = 0\}$ and $\{\rho = k + 1/2\}$ become circles of the form $S^1 \times \{\text{const}\}$ embedded in $S^1 \times S^2$. Since the ϕ-coordinate is singular at these two circles, the contact form α_{GT} needs to be modified slightly in this region before it will descend to a smooth contact form on $S^1 \times S^2$: this can be done by a C^0-small modification of the form $f(\rho)\,d\theta + g(\rho)\,d\phi$, and the resulting contact structure is then uniquely determined up to isotopy. We shall call this contact manifold

$$(S^1 \times S^2, \xi_k).$$

Now observe that for each $k \in \mathbb{N}$, $(S^1 \times S^2, \xi_{k+1})$ is obtained from $(S^1 \times S^2, \xi_k)$ by a Lutz twist. However, both contact manifolds are also **overtwisted**: recall that a contact 3-manifold (M, ξ) is overtwisted whenever it contains an embedded closed 2-disk $\mathcal{D} \subset M$ such that $T(\partial\mathcal{D}) \subset \xi$ but $T\mathcal{D}|_{\partial\mathcal{D}} \pitchfork \xi$. (Exercise: find a disk with this property in $(S^1 \times S^2, \xi_k)$!) Eliashberg's flexibility theorem for overtwisted contact structures [**Eli89**] implies that whenever ξ and ξ' are two contact structures on a closed 3-manifold that are both overtwisted and are homotopic as oriented 2-plane fields, they are actually isotopic. As a consequence, the contact structures ξ_k on $S^1 \times S^2$ defined above for every $k \in \mathbb{N}$ are all isotopic to each other. As tends to be the case with most interesting h-principles, the isotopy is very hard to see concretely, but it must exist.

EXERCISE 10.7. Show that if (M, ξ) is a closed overtwisted contact 3-manifold, then $\mathrm{GT}(M, \xi) = \infty$.

In contrast to the $S^1 \times S^2$ example above, the contact manifolds $(\mathbb{T}^3, \xi_k)$ are not overtwisted, they are **tight**—in fact, the classification of contact structures on $\mathbb{T}^3$ by Giroux [**Gir94**, **Gir99**, **Gir00**] and Kanda [**Kan97**] states that these are *all* of the tight contact structures on $\mathbb{T}^3$ up to contactomorphism. We will use cylindrical contact homology to show that they are not contactomorphic to each other. The reader should keep Example 10.6 in mind and try to spot the reason why the same argument cannot work for $(S^1 \times S^2, \xi_k)$.

REMARK 10.8. It has been conjectured that the converse of Exercise 10.7 might also hold, so every closed tight contact 3-manifold would have finite Giroux torsion. This conjecture is wide open.

10.2. Definition of cylindrical contact homology

10.2.1. Preliminary remarks. Cylindrical contact homology is the natural "first attempt" at using holomorphic curves in symplectizations to define a Floer-type invariant of contact manifolds (M, ξ). The idea is to define a chain complex generated by Reeb orbits in M and a differential ∂ that counts holomorphic cylinders in $\mathbb{R} \times M$. We already know some pretty good reasons why this idea cannot work in general: in order to prove $\partial^2 = 0$, we need to be able to identify the space of rigid "broken" holomorphic cylinders (these are what is counted by ∂^2) with the boundary of the compactified 1-dimensional space of index 2 cylinders (up to $\mathbb{R}$-translation). But this compactified boundary has more than just broken cylinders in it, see Figure 10.2. In order to define cylindrical contact homology, one must therefore restrict to situations in which complicated pictures like Figure 10.2 cannot occur. The first useful remark in this direction is that since we are working with a stable Hamiltonian structure of the form $(d\alpha, \alpha)$ for a contact form α, a certain subset of the scenarios allowed by the SFT compactness theorem can be excluded immediately. Indeed:

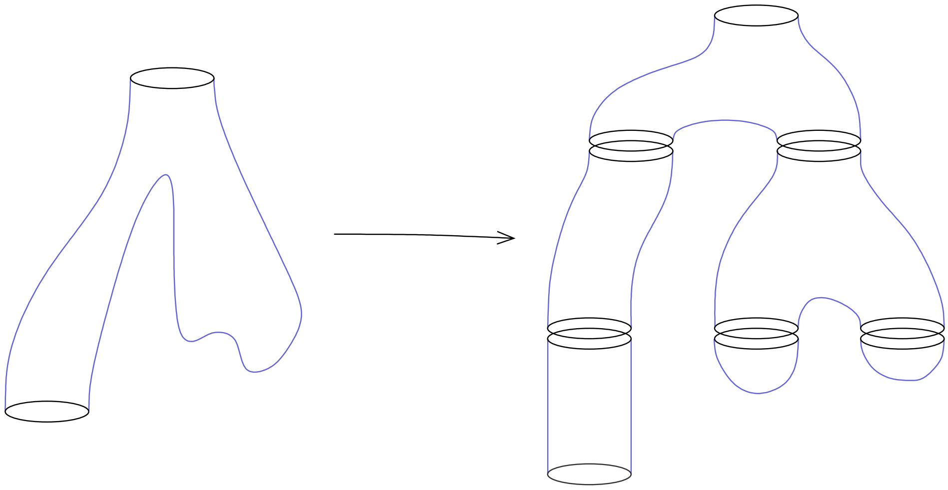

FIGURE 10.2. A family of holomorphic cylinders can converge in the SFT topology to buildings that include more complicated curves than cylinders—this is why cylindrical contact homology is not well defined for all contact manifolds.

PROPOSITION 10.9. *If $J \in \mathcal{J}(\alpha)$ and $u : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$ is an asymptotically cylindrical J-holomorphic curve, then u has at least one positive puncture.*

Let us give two proofs of this result, since both contain useful ideas. As preparation for the first proof, recall the definition of energy for curves in symplectizations of contact manifolds that we wrote down in Lecture 1:

$$E(u) := \sup_{f \in \mathcal{T}} \int_{\dot{\Sigma}} u^* d(e^{f(r)}\,\alpha),$$

where

$$\mathcal{T} := \left\{ f \in C^\infty(\mathbb{R}, (-1,1)) \;\middle|\; f' > 0 \right\}.$$

This formula is not identical to the definition of energy used in Lecture 9, but it is equivalent in the sense that any uniform bounds on one imply similar uniform bounds on the other.

FIRST PROOF OF PROPOSITION 10.9. Denote the positive and negative punctures of $u : \dot{\Sigma} \to \mathbb{R} \times M$ by Γ^+ and Γ^- respectively, and suppose u is asymptotic at $z \in \Gamma^\pm$ to the orbit γ_z with period $T_z > 0$. Choose any $f \in \mathcal{T}$ and denote $f_\pm := \lim_{r \to \pm\infty} f(r) \in [-1, 1]$. Since $d(e^{f(r)}\,\alpha)$ tames $J \in \mathcal{J}(\alpha)$, Stokes' theorem gives

$$0 \le E(u) = e^{f_+} \sum_{z \in \Gamma^+} T_z - e^{f_-} \sum_{z \in \Gamma^-} T_z, \tag{10.1}$$

hence Γ^+ cannot be empty. □

REMARK 10.10. The proof via Stokes' theorem works just as well if instead of $\mathbb{R} \times M$, u lives in the completion of an exact symplectic cobordism (W, ω) with concave boundary $(M_-, \xi_- = \ker \alpha_-)$ and convex boundary $(M_+, \xi_+ = \ker \alpha_+)$.

Recall that this means $\partial W = -M_- \sqcup M_+$, and $\omega = d\lambda$ for a 1-form λ that restricts to positive contact forms $\lambda|_{TM_\pm} = \alpha_\pm$. As in Lecture 1, we will write

$$\mathcal{J}(W, \omega, \alpha_+, \alpha_-) \subset \mathcal{J}(\widehat{W})$$

for the space of almost complex structures J on $\widehat{W} := ((-\infty, 0] \times M_-) \cup_{M_-} W \cup_{M_+} ([0, \infty) \times M_+)$ that are compatible with ω on W and belong to $\mathcal{J}(\alpha_\pm)$ on the cylindrical ends. The energy of a J-holomorphic curve $u : (\dot{\Sigma}, j) \to (\widehat{W}, J)$ is then

$$E(u) := \sup_{f \in \mathcal{T}} \int_{\dot{\Sigma}} u^* d\lambda_f,$$

where $\mathcal{T} := \{f \in C^\infty(\mathbb{R}, (-1, 1)) \mid f' > 0 \text{ and } f(r) = r \text{ near } r = 0\}$ and

$$\lambda_f := \begin{cases} e^{f(r)}\alpha_+ & \text{on } [0, \infty) \times M_+, \\ \lambda & \text{on } W, \\ e^{f(r)}\alpha_- & \text{on } (-\infty, 0] \times M_-. \end{cases}$$

The above proof now generalizes verbatim to show that u must always have a positive puncture. Notice that in both settings, the argument also gives a uniform bound for the energy in terms of the periods of the positive asymptotic orbits.

REMARK 10.11. We can also prove Prop. 10.9 using the fact that $u^* d\alpha \geq 0$ for any $u : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$ with $J \in \mathcal{J}(\alpha)$. Indeed, Stokes' theorem then gives

$$0 \leq \int_{\dot{\Sigma}} u^* d\alpha = \sum_{z \in \Gamma^+} T_z - \sum_{z \in \Gamma^-} T_z. \tag{10.2}$$

The quantity $\int_{\dot{\Sigma}} u^* d\alpha$ is sometimes called the **contact area** of u. This version of the argument however does not easily generalize to arbitrary exact cobordisms.

The second proof is based on the maximum principle for subharmonic functions.

PROPOSITION 10.12. *Suppose $J \in \mathcal{J}(\alpha)$ and $u = (u_{\mathbb{R}}, u_M) : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$ is J-holomorphic, where $\dot{\Sigma}$ has no boundary. Then $u_{\mathbb{R}} : \dot{\Sigma} \to \mathbb{R}$ has no local maxima.*

PROOF. In any local holomorphic coordinates (s, t) on a region in $\dot{\Sigma}$, the nonlinear Cauchy-Riemann equation for u is equivalent to the system of equations

$$\begin{aligned} \partial_s u_{\mathbb{R}} - \alpha(\partial_t u_M) &= 0, \\ \partial_t u_{\mathbb{R}} + \alpha(\partial_s u_M) &= 0, \\ \pi_\xi \, \partial_s u_M + J \pi_\xi \, \partial_t u_M &= 0, \end{aligned}$$

where $\pi_\xi : TM \to \xi$ denotes the projection along the Reeb vector field. This gives

$$\begin{aligned} -\Delta u_{\mathbb{R}} &= -\partial_s^2 u_{\mathbb{R}} - \partial_t^2 u_{\mathbb{R}} = -\partial_s [\alpha(\partial_t u_M)] + \partial_t [\alpha(\partial_s u_M)] \\ &= -d\alpha(\partial_s u_M, \partial_t u_M) = -d\alpha(\pi_\xi \partial_s u_M, J\pi_\xi \partial_s u_M) \leq 0 \end{aligned}$$

since $J|_\xi$ is tamed by $d\alpha|_\xi$, hence $u_{\mathbb{R}}$ is subharmonic. The result thus follows from the maximum principle, see e.g. [**Eva98**]. □

SECOND PROOF OF PROPOSITION 10.9. If $u = (u_{\mathbb{R}}, u_M) : \dot{\Sigma} \to \mathbb{R} \times M$ has no positive puncture then $u_{\mathbb{R}} : \dot{\Sigma} \to \mathbb{R}$ is a proper function bounded above, and therefore has a local maximum, contradicting Proposition 10.12. □

REMARK 10.13. The proof via the maximum principle does not generalize to arbitrary exact cobordisms $(W, d\lambda)$, but it does work in *Stein* cobordisms, i.e. if λ_f and J are related by $\lambda_f = -dF \circ J$ for some plurisubharmonic function $F : \widehat{W} \to \mathbb{R}$, then $F \circ u : \dot{\Sigma} \to \mathbb{R}$ is subharmonic (cf. [CE12]).

With these preliminaries understood, the next two exercises reveal one natural setting in which breaking of cylinders can be kept under control. Both exercises are essentially combinatorial.

EXERCISE 10.14. Suppose $\mathbf{u}$ is a stable J-holomorphic building in a completed symplectic cobordism $\widehat{W}$ with the following properties:

(1) $\mathbf{u}$ has arithmetic genus 0 and exactly one positive puncture;
(2) every connected component of $\mathbf{u}$ has at least one positive puncture.

Show that $\mathbf{u}$ has no nodes, and all of its connected components have *exactly* one positive puncture.

EXERCISE 10.15. Suppose that in addition to the conditions of Exercise 10.14, $\mathbf{u}$ has exactly one negative puncture and no connected component of $\mathbf{u}$ is a plane. Show that every level of $\mathbf{u}$ then consists of a single cylinder with one positive and one negative end.

Exercise 10.15 makes it reasonable to define a Floer-type theory counting only cylinders in any setting where planes can be excluded, for instance because the Reeb vector field has no contractible orbits. This is not always possible, e.g. Hofer [Hof93] proved that on overtwisted contact manifolds, there is *always* a plane (which is why the Weinstein conjecture holds). So the invariant we construct will not be defined in such settings, but it happens to be ideally suited to the study of $(\mathbb{T}^3, \xi_k)$.

10.2.2. A compactness result for cylinders. Fix a closed contact manifold (M, ξ) of dimension $2n-1$ and a primitive homotopy class of loops $h \in [S^1, M]$. By **primitive**, we mean that h is not equal to Nh' for any $h' \in [S^1, M]$ and an integer $N > 1$, and this assumption will be crucial for technical reasons in the following.[1] Given a contact form α for ξ, let

$$\mathcal{P}_h(\alpha)$$

denote the set of closed Reeb orbits homotopic to h, where two Reeb orbits are identified if they differ only by parametrization.

DEFINITION 10.16. Given a contact manifold (M, ξ) and a primitive homotopy class $h \in [S^1, M]$, we will say that a contact form α for ξ is h**-admissible** if:

(1) All orbits in $\mathcal{P}_h(\alpha)$ are nondegenerate;

[1] It is to be expected that cylindrical contact homology can be defined also for non-primitive homotopy classes, but this would require more sophisticated methods to address transversality problems. The assumption that h is primitive allows us to assume that all holomorphic curves in the discussion are somewhere injective, hence they are always regular if J is generic.

(2) There are no contractible closed Reeb orbits.

Similarly, we will say that (M, ξ) is **h-admissible** if a contact form with the above properties exists.

DEFINITION 10.17. Given $h \in [S^1, M]$ and an h-admissible contact form α on (M, ξ), we will say that an almost complex structure $J \in \mathcal{J}(\alpha)$ is **h-regular** if every J-holomorphic cylinder in $\mathbb{R} \times M$ with a positive and a negative end both asymptotic to orbits in $\mathcal{P}_h(\alpha)$ is Fredholm regular.

PROPOSITION 10.18. *If $h \in [S^1, M]$ is a primitive homotopy class of loops and α is h-admissible on (M, ξ), then the space of h-regular almost complex structures is comeager in $\mathcal{J}(\alpha)$.*

PROOF. Since h is primitive, the asymptotic orbits for the relevant holomorphic cylinders cannot be multiply covered, hence all of these cylinders are somewhere injective. The result therefore follows from the standard transversality results proved in Lecture 8 for somewhere injective curves in symplectizations. □

PROPOSITION 10.19. *Given an h-admissible contact form α, an h-regular almost complex structure $J \in \mathcal{J}(\alpha)$ and an orbit $\gamma \in \mathcal{P}_h(\alpha)$, suppose u_k is a sequence of J-holomorphic cylinders in $\mathbb{R} \times M$ with one positive puncture at γ and one negative puncture. Then u_k has a subsequence convergent in the SFT topology to a broken J-holomorphic cylinder, i.e. a stable building $\mathbf{u}_\infty$ whose levels $u_\infty^1, \ldots, u_\infty^{N_+}$ are each cylinders with one positive and one negative puncture. Moreover, each level satisfies $\operatorname{ind}(u_\infty^N) \geq 1$, thus for large k in the convergent subsequence,*

$$\operatorname{ind}(u_k) = \sum_{N=1}^{N_+} \operatorname{ind}(u_\infty^N) \geq N_+.$$

PROOF. Let's start with some bad news: the standard SFT compactness theorem is not applicable in this situation, because we have not assumed that α is nondegenerate, nor even Morse Bott—there is no assumption at all about Reeb orbits in homotopy classes other than h and 0. This fairly loose set of hypotheses is very convenient in applications, as nondegeneracy of a contact form is generally a quite difficult condition to check. The price we pay is that we will have to prove compactness manually instead of applying the big theorem (see Remark 10.20). Fortunately, it is not that hard: the crucial point is that in the situation at hand, there can be no bubbling at all.

Indeed, we claim that the given sequence $u_k : (\mathbb{R} \times S^1, i) \to (\mathbb{R} \times M, J)$ must satisfy a uniform bound

$$|du_k| \leq C$$

with respect to any translation-invariant Riemannian metrics on $\mathbb{R} \times S^1$ and $\mathbb{R} \times M$. To see this, note first that since all the u_k have the same positive asymptotic orbit γ, their energies are uniformly bounded via (10.1). Thus if $|du_k(z_k)| \to \infty$ for some sequence $z_k \in \mathbb{R} \times S^1$, we can perform the usual rescaling trick from Lecture 9 and deduce the existence of a nonconstant finite-energy plane $v_\infty : \mathbb{C} \to \mathbb{R} \times M$. Its singularity at ∞ cannot be removable since this would produce a nonconstant J-holomorphic sphere, violating Proposition 10.9. It follows that v_∞ is asymptotic to a

Reeb orbit at ∞, but this is also impossible since α does not admit any contractible orbits, and the claim is thus proved.

Suppose now that γ has period $T_+ > 0$, and observe that by nondegeneracy, the set

$$\mathcal{P}_h(\alpha, T_+) := \left\{ \gamma \in \mathcal{P}_h(\alpha) \;\middle|\; \gamma \text{ has period at most } T_+ \right\}$$

is finite. Let

$$\mathcal{A}_h(\alpha), \mathcal{A}_h(\alpha, T_+) \subset (0, \infty)$$

denote the set of all periods of orbits in $\mathcal{P}_h(\alpha)$ and $\mathcal{P}_h(\alpha, T_+)$ respectively. By (10.2), the negative asymptotic orbit of each u_k is in $\mathcal{P}_h(\alpha, T_+)$, so we can take a subsequence and assume that these are all the same orbit; call it $\gamma_- \in \mathcal{P}_h(\alpha, T_+)$ and its period $T_- \in \mathcal{A}_h(\alpha, T_+)$. If $T_- = T_+$ then $u_k^* d\alpha \equiv 0$ for all k, implying that all u_k are the trivial cylinder over γ and thus trivially converge. Assume therefore $T_- < T_+$. Then since $u_k^* d\alpha \geq 0$, Stokes' theorem implies that for each k, the function

$$\mathbb{R} \to \mathbb{R} : s \mapsto \int_{S^1} u_k(s, \cdot)^* \alpha$$

is increasing and is a surjective map onto (T_-, T_+). The uniform bound on the derivatives implies that for any sequences $s_k, r_k \in \mathbb{R}$ with $u_k(s_k, 0) \in \{r_k\} \times M$, the sequence[2]

$$v_k : \mathbb{R} \times S^1 \to \mathbb{R} \times M : (s, t) \mapsto \tau_{-r_k} \circ u_k(s + s_k, t)$$

has a subsequence convergent in $C^\infty_{\text{loc}}(\mathbb{R} \times S^1)$ to some finite-energy J-holomorphic cylinder

$$v_\infty : \mathbb{R} \times S^1 \to \mathbb{R} \times M,$$

which necessarily satisfies

$$\int_{S^1} v_\infty(s, \cdot)^* \alpha = \lim_{k \to \infty} \int_{S^1} u_k(s + s_k, \cdot)^* \alpha \in [T_-, T_+]$$

for every $s \in \mathbb{R}$. This proves that v_∞ is nonconstant, with a positive puncture at $s = \infty$ and negative puncture at $s = -\infty$, and both of its asymptotic orbits are in $\mathcal{P}_h(\alpha, T_+)$.[3] If v_∞ is not a trivial cylinder, then it therefore satisfies

$$\int_{\mathbb{R} \times S^1} v_\infty^* d\alpha \geq \delta,$$

where δ is any positive number less than the smallest distance between neighboring elements of $\mathcal{A}_h(\alpha, T_+)$.

Let us call a sequence $s_k \in \mathbb{R}$ *nontrivial* whenever the limiting cylinder v_∞ obtained by the above procedure is not a trivial cylinder, and call two such sequences s_k and s'_k *compatible* if $s_k - s'_k$ is not bounded. We claim now that if $s^1_k, \ldots, s^m_k$ is a collection of nontrivial sequences that are all compatible with each other, then

$$m < \frac{2(T_+ - T_-)}{\delta}.$$

[2] Recall from Lecture 9 that we denote the $\mathbb{R}$-translation action on $\mathbb{R} \times M$ by $\tau_c(r, x) := (r + c, x)$.

[3] For an alternative argument that v_∞ must have a positive puncture at $s = \infty$ and negative at $s = -\infty$, see Figure 10.3.

Indeed, we can assume after ordering our collection appropriately and restricting to a subsequence that $s_k^{N+1} - s_k^N \to \infty$ for each $N = 1, \dots, m-1$, and let $v_\infty^N : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ denote the limits of the corresponding convergent subsequences. Then we can find $R > 0$ such that

$$\int_{[-R,R]\times S^1} (v_\infty^N)^* d\alpha > \frac{\delta}{2}$$

and thus

$$\int_{[s_k^N - R, s_k^N + R]\times S^1} u_k^* d\alpha > \frac{\delta}{2}$$

for each $N = 1, \dots, m$ for sufficiently large k. But these domains are also all disjoint for sufficiently large k, implying

$$T_+ - T_- = \int_{\mathbb{R}\times S^1} u_k^* d\alpha \geq \sum_{N=1}^{m} \int_{[s_k^N - R, s_k^N + R]\times S^1} u_k^* d\alpha > \frac{\delta m}{2}.$$

We've shown that there exists a maximal collection of nontrivial sequences $s_k^1, \dots, s_k^{N_+} \in \mathbb{R}$ satisfying $s_k^{N+1} - s_k^N \to \infty$ for each N, such that if $u_k(s_k^N, 0) \in \{r_k^N\} \times M$, then after restricting to a subsequence, the cylinders

$$v_k^N(s,t) := \tau_{-r_k^N} \circ u_k(s + s_k^N, t)$$

each converge in $C^\infty_{\mathrm{loc}}(\mathbb{R} \times S^1)$ as $k \to \infty$ to a nontrivial J-holomorphic cylinder $u_\infty^N : \mathbb{R} \times S^1 \to \mathbb{R} \times M$. Let $\gamma_N^\pm$ denote the asymptotic orbit of u_∞^N at $s = \pm\infty$. We claim,

$$\gamma_N^+ = \gamma_{N+1}^- \quad \text{for each } N = 1, \dots, N_+ - 1.$$

If $\gamma_N^+ \neq \gamma_{N+1}^-$ for some N, choose a neighborhood $\mathcal{U} \subset M$ of the image of γ_N^+ that does not intersect any other orbit in $\mathcal{P}_h(\alpha, T_+)$. Then since each u_k is continuous, there must exist a sequence $s_k' \in \mathbb{R}$ with

$$s_k' - s_k^N \to \infty \quad \text{and} \quad s_k^{N+1} - s_k' \to \infty$$

such that $u_k(s_k', 0)$ lies in $\mathcal{U}$ for all k but stays a positive distance away from the image of γ_N^+. A subsequence of $(s,t) \mapsto u_k(s + s_k', t)$ then converges after suitable $\mathbb{R}$-translations to a cylinder $u_\infty' : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ that cannot be trivial since $u_\infty'(0,0)$ is not contained in any orbit in $\mathcal{P}_h(\alpha, T_+)$. This contradicts the assumption that our collection $s_k^1, \dots, s_k^{N_+}$ is maximal. A similar argument shows

$$\gamma_1^- = \gamma^- \quad \text{and} \quad \gamma_{N_+}^+ = \gamma,$$

so the curves $u_\infty^1, \dots, u_\infty^{N_+}$ form the levels of a stable holomorphic building $\mathbf{u}_\infty$. A similar argument by contradiction also shows that the sequence u_k must converge in the SFT topology to $\mathbf{u}_\infty$.

Finally, note that since all the breaking orbits in $\mathbf{u}_\infty$ are homotopic to h and J is h-regular, the levels u_∞^N are Fredholm regular. Since all of them also come in 1-parameter families of distinct curves related by the $\mathbb{R}$-action, this implies $\mathrm{ind}(u_\infty^N) \geq 1$ for each $N = 1, \dots, N_+$. □

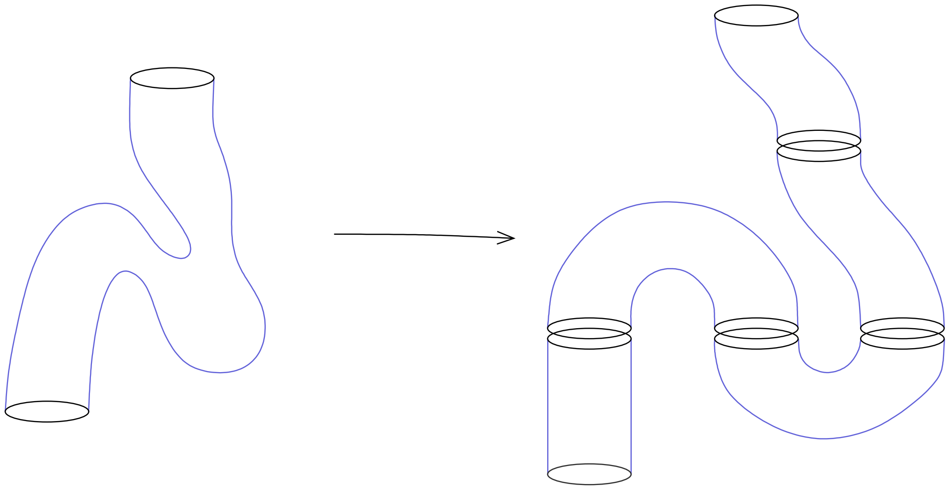

FIGURE 10.3. A degenerating sequence of holomorphic cylinders $u_k : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ cannot have a limiting level with a puncture of the "wrong" sign unless u_k violates the maximum principle for large k.

REMARK 10.20. Nondegeneracy or Morse-Bott conditions are required for several reasons in the proof of SFT compactness, and indeed, the theorem is not true in general without some such assumption. One can see this by considering what happens to a sequence u_k of J_k-holomorphic curves where $J_k \to J_\infty$ is compatible with a sequence of nondegenerate contact forms α_k converging to one that is only Morse-Bott. A compactness theorem for this scenario is proved in [**Bou02**], but it requires more general limiting objects than holomorphic buildings. On the other hand, it is useful for certain kinds of applications to know when one can do without nondegeneracy assumptions and prove compactness anyway. There are two main advantages to knowing that all Reeb orbits are nondegenerate or belong to Morse-Bott families:

(1) It implies that the set of all periods of closed orbits, the so-called **action spectrum** of α, is a *discrete* subset of $(0, \infty)$; in fact, for any $T > 0$, the set of all periods less than T is finite. Using the relations (10.1) and (10.2), this implies lower bounds on the possible energies of limiting components and thus helps show that only finitely many such components can arise.
(2) Curves asymptotic to nondegenerate or Morse-Bott orbits also satisfy exponential convergence estimates proved in [**HWZ96**, **HWZ01**, **HWZ96**, **Bou02**], and similar asymptotic estimates yield a result about "long cylinders with small area" (see [**HWZ02**] and [**BEH$^+$03**, Prop. 5.7]) which helps in proving that neighboring levels connect to each other along breaking orbits.

Our situation in Proposition 10.19 was simple enough to avoid using the "long cylinder" lemma, and we did use the discreteness of the action spectrum, but only needed it for orbits in $\mathcal{P}_h(\alpha)$ since we were able to rule out bubbling in the first step. An alternative would have been to assume that all orbits (in all homotopy classes) with period up to the period of γ are nondegenerate: then (10.2) implies that degenerate orbits never play any role in the main arguments of [BEH+03], so the big theorem becomes safe to use.

10.2.3. The chain complex. We now define a $\mathbb{Z}_2$-graded chain complex with coefficients in $\mathbb{Z}_2$ and generators $\langle\gamma\rangle$ for $\gamma \in \mathcal{P}_h(\alpha)$, i.e.

$$CC_*^h(M, \alpha) := \bigoplus_{\gamma \in \mathcal{P}_h(\alpha)} \mathbb{Z}_2.$$

The degree of each generator $\langle\gamma\rangle \in CC_*^h(M, \alpha)$ is defined by

$$|\langle\gamma\rangle| = n - 3 + \mu_{\mathrm{CZ}}(\gamma) \in \mathbb{Z}_2,$$

where $\mu_{\mathrm{CZ}}(\gamma) \in \mathbb{Z}_2$ denotes the parity of the Conley-Zehnder index with respect to any choice of trivialization. The choice to write $n-3$ in front of this is a convention that will make no difference at all in this lecture, but it is consistent with a $\mathbb{Z}$-grading that we will be able to define under suitable assumptions in Lecture 12. To define the differential on $CC_*^h(M, \alpha)$, choose an h-regular almost complex structure $J \in \mathcal{J}(\alpha)$. Given Reeb orbits $\gamma^+, \gamma^- \in \mathcal{P}_h(\alpha)$ and a number $I \in \mathbb{Z}$, let

$$\mathcal{M}^I(J, \gamma^+, \gamma^-)$$

denote the space of all $\mathbb{R}$-equivalence classes of index I holomorphic cylinders in $(\mathbb{R} \times M, J)$ asymptotic to $\gamma^\pm$ at $\pm\infty$, i.e. the union of all components $\mathcal{M}_{0,0}(J, A, \gamma^+, \gamma^-)/\mathbb{R}$ for which $\operatorname{vir-dim} \mathcal{M}_{0,0}(J, A, \gamma^+, \gamma^-) = I$. Since J is h-regular, all the curves in $\mathcal{M}^I(J, \gamma^+, \gamma^-)$ are Fredholm regular, so if $I \geq 1$, $\mathcal{M}^I(J, \gamma^+, \gamma^-)$ is a smooth manifold with

$$\dim \mathcal{M}^I(J, \gamma^+, \gamma^-) = I - 1.$$

Similarly, $\mathcal{M}^0(J, \gamma^+, \gamma^-)$ only contains trivial cylinders and is thus empty unless $\gamma^+ = \gamma^-$, and $\mathcal{M}^I(J, \gamma^+, \gamma^-)$ is always empty for $I < 0$. In particular, $\mathcal{M}^1(J, \gamma^+, \gamma^-)$ is a discrete set whenever $\gamma^+ \neq \gamma^-$, and by Proposition 10.19, it is also compact, hence finite. We can therefore define

$$\partial\langle\gamma\rangle = \sum_{\gamma' \in \mathcal{P}_h(\alpha)} \#_2 \mathcal{M}^1(J, \gamma, \gamma')\langle\gamma'\rangle,$$

where for any set X, we denote by $\#_2 X$ the cardinality of X modulo 2. The operator ∂ has odd degree with respect to the grading since every index 1 holomorphic cylinder u with asymptotic orbits γ^+ and γ^- satisfies

$$\operatorname{ind}(u) = 1 = \mu_{\mathrm{CZ}}^\tau(\gamma^+) - \mu_{\mathrm{CZ}}^\tau(\gamma^-)$$

for suitable choices of the trivialization τ.

10.2.4. The homology. Following the standard Floer theoretic prescription, the relation $\partial^2 = 0$ should arise by viewing the compactification $\overline{\mathcal{M}}^2(J, \gamma^+, \gamma^-)$ for each $\gamma^+, \gamma^- \in \mathcal{P}_h(\alpha)$ as a compact 1-manifold whose boundary is identified with the set of rigid broken cylinders, as these are what is counted by ∂^2. Here $\overline{\mathcal{M}}^2(J, \gamma^+, \gamma^-)$

is defined as the closure of $\mathcal{M}^2(J, \gamma^+, \gamma^-)$ in the space of all J-holomorphic buildings in $\mathbb{R} \times M$ modulo $\mathbb{R}$-translation. Proposition 10.19 gives a natural inclusion

$$\overline{\mathcal{M}}^2(J, \gamma^+, \gamma^-) \setminus \mathcal{M}^2(J, \gamma^+, \gamma^-) \subset \coprod_{\gamma_0 \in \mathcal{P}_h(\alpha)} \mathcal{M}^1(J, \gamma^+, \gamma_0) \times \mathcal{M}^1(J, \gamma_0, \gamma^-).$$

We therefore need an inclusion in the other direction, and for this we need to say a word about gluing. We have not had time to discuss gluing in earnest in these notes, and we will not do so now either, but the basic idea should be familiar from Floer homology: given $u_+ \in \mathcal{M}^1(J, \gamma^+, \gamma_0)$ and $u_- \in \mathcal{M}^1(J, \gamma_0, \gamma^-)$, one would like to show that there exists a unique (up to $\mathbb{R}$-translation) one-parameter family $\{u_R \in \mathcal{M}^2(J, \gamma^+, \gamma^-)\}_{R \in [R_0, \infty)}$ such that u_R converges as $R \to \infty$ to the building $\mathbf{u}_\infty$ with bottom level u_- and top level u_+. One starts by constructing a family of *preglued* maps

$$\tilde{u}_R : \mathbb{R} \times S^1 \to \mathbb{R} \times M,$$

meaning a smooth family of maps which converge in the SFT topology as $R \to \infty$ to $\mathbf{u}_\infty$ but are only *approximately* J-holomorphic. More precisely, fix parametrizations of u_- and u_+ and a parametrization of the orbit $\gamma_0 : \mathbb{R}/T\mathbb{Z} \to M$ such that

$$\begin{aligned} u_+(s,t) &= \exp_{(Ts, \gamma_0(Tt))} h_+(s,t) && \text{for } s \ll 0, \\ u_-(s,t) &= \exp_{(Ts, \gamma_0(Tt))} h_-(s,t) && \text{for } s \gg 0, \end{aligned}$$

where $h_\pm$ are vector fields along the trivial cylinder satisfying $\lim_{s \to \mp\infty} h_\pm(s,t) = 0$. By interpolating between suitable reparametrizations of h_+ and h_-, one can now define $\tilde{u}_R$ such that

$$\begin{aligned} \tilde{u}_R(s,t) &= \tau_{2RT} \circ u_+(s - 2R, t) && \text{for } s \geq R, \\ \tilde{u}_R(s,t) &\approx (Ts, \gamma_0(Tt)) && \text{for } s \in [-R, R], \\ \tilde{u}_R(s,t) &= \tau_{-2RT} \circ u_-(s + 2R, t) && \text{for } s \leq -R, \\ \bar{\partial}_J \tilde{u}_R &\to 0 && \text{as } R \to \infty. \end{aligned}$$

Given regularity of u_+ and u_-, one can now use a quantitative version of the implicit function theorem (cf. [**MS04**, §3.5]) to show that a distinguished J-holomorphic cylinder u_R close to $\tilde{u}_R$ exists for all R sufficiently large. For a more detailed synopsis of the analysis involved, see [**Nel13**, Chapter 7], and [**AD14**, Chapters 9 and 13] for the analogous story in Floer homology. The result is:

PROPOSITION 10.21. *For an h-admissible α, an h-regular $J \in \mathcal{J}(\alpha)$ and any two orbits $\gamma^+, \gamma^- \in \mathcal{P}_h(\alpha)$, the space $\overline{\mathcal{M}}^2(J, \gamma^+, \gamma^-)$ admits the structure of a compact 1-dimensional manifold with boundary, where its boundary points can be identified naturally with $\coprod_{\gamma_0 \in \mathcal{P}_h(\alpha)} \mathcal{M}^1(J, \gamma^+, \gamma_0) \times \mathcal{M}^1(J, \gamma_0, \gamma^-)$.* □

COROLLARY 10.22. *The homomorphism $\partial : CC_*^h(M, \alpha) \to CC_{*-1}^h(M, \alpha)$ satisfies $\partial^2 = 0$.* □

We shall denote the homology of this chain complex by

$$HC_*^h(M, \alpha, J) := H_*\big(CC_*^h(M, \alpha), \partial\big).$$

The goal of the rest of this section is to prove that up to natural isomorphisms, $HC^h_*(M, \alpha, J)$ depends on (M, ξ) and h but not on the auxiliary data α and J.

10.2.5. Chain maps. For any constant $c > 0$, there is an obvious bijection between the generators of $CC^h_*(M, \alpha)$ and $CC^h_*(M, c\alpha)$, as the rescaling changes periods of orbits but not the set of closed orbits itself. Moreover, if $J \in \mathcal{J}(\alpha)$ and $J_c \in \mathcal{J}(c\alpha)$ are defined to match on ξ, then there is a biholomorphic diffeomorphism

$$(\mathbb{R} \times M, J) \to (\mathbb{R} \times M, J_c) : (r, x) \mapsto (cr, x),$$

thus giving a bijective correspondence between the moduli spaces of J-holomorphic and J_c-holomorphic curves. It follows that our bijection of chain complexes is also a chain map and therefore defines a canonical isomorphism

$$HC^h_*(M, \alpha, J) = HC^h_*(M, c\alpha, J_c). \tag{10.3}$$

Next suppose α_- and α_+ are two distinct contact forms for ξ, hence

$$\alpha_\pm = e^{f_\pm}\alpha$$

for some fixed contact form α and a pair of smooth functions $f_\pm : M \to \mathbb{R}$. After rescaling α_+ by a constant, we are free to assume $f_+ > f_-$ everywhere. Fix h-regular almost complex structures $J_\pm \in \mathcal{J}(\alpha_\pm)$ and let

$$\partial_\pm : CC^h_*(M, \alpha_\pm) \to CC^h_{*-1}(M, \alpha_\pm)$$

denote the resulting differentials on the two chain complexes. The region

$$W := \{(r, x) \in \mathbb{R} \times M \mid f_-(x) \le r \le f_+(x)\}$$

now defines an exact symplectic cobordism from (M, ξ) to itself: more precisely, setting

$$M_\pm := \{(f_\pm(x), x) \in W \mid x \in M\}$$

gives $\partial W = -M_- \sqcup M_+$, and the Liouville form $\lambda := e^r\alpha$ satisfies $\lambda|_{TM_\pm} = \alpha_\pm$. Choose a generic $d\lambda$-compatible almost complex structure J on the completion $\widehat{W}$ that restricts to $J_\pm$ on the cylindrical ends. Now given $\gamma^+ \in \mathcal{P}_h(\alpha_+)$ and $\gamma^- \in \mathcal{P}_h(\alpha_-)$ and a number $I \in \mathbb{Z}$, we shall denote by

$$\mathcal{M}^I(J, \gamma^+, \gamma^-)$$

the union of all components $\mathcal{M}_{0,0}(J, A, \gamma^+, \gamma^-)$ that have virtual dimension I. Note that we are not dividing by any $\mathbb{R}$-action here since J need not be $\mathbb{R}$-invariant. Since $\gamma^\pm$ are still guaranteed to be simply covered, curves in $\mathcal{M}^I(J, \gamma^+, \gamma^-)$ are again always somewhere injective and therefore regular, hence $\mathcal{M}^I(J, \gamma^+, \gamma^-)$ is a smooth manifold with

$$\dim \mathcal{M}^I(J, \gamma^+, \gamma^-) = I$$

if $I \ge 0$, and $\mathcal{M}^I(J, \gamma^+, \gamma^-) = \emptyset$ for $I < 0$. The compactification $\overline{\mathcal{M}}^I(J, \gamma^+, \gamma^-)$ is described via the following straightforward generalization of Proposition 10.19:

PROPOSITION 10.23. *For J as described above, suppose u_k is a sequence of J-holomorphic cylinders in $\widehat{W}$ with one positive puncture at an orbit $\gamma \in \mathcal{P}_h(\alpha_+)$ and one negative puncture. Then u_k has a subsequence convergent in the SFT topology to a broken J-holomorphic cylinder, i.e. a stable building $\mathbf{u}_\infty$ whose levels u_∞^N for $N = -N_-, \ldots, -1, 0, 1, \ldots, N_+$ are each cylinders with one positive and one negative puncture, living in $\mathbb{R} \times M^\pm$ for $\pm N > 0$ and $\widehat{W}$ for $N = 0$. Moreover, the levels satisfy $\operatorname{ind}(u_\infty^0) \geq 0$ and $\operatorname{ind}(u_\infty^N) \geq 1$ for $N \neq 0$, thus for large k in the convergent subsequence,*

$$\operatorname{ind}(u_k) = \sum_{N=-N_-}^{N_+} \operatorname{ind}(u_\infty^N) \geq N_- + N_+.$$

□

It follows that the set $\mathcal{M}^0(J, \gamma^+, \gamma^-)$ is always finite, and we use this to define a map

$$\Phi_J : CC_*^h(M, \alpha_+) \to CC_*^h(M, \alpha_-) : \langle\gamma\rangle \mapsto \sum_{\gamma' \in \mathcal{P}_h(\alpha_-)} \#_2\mathcal{M}^0(J, \gamma, \gamma')\langle\gamma'\rangle.$$

This map preserves degrees since it counts index 0 curves, and we claim that it is a chain map:

$$\Phi_J \circ \partial_+ = \partial_- \circ \Phi_J.$$

This follows from the fact that by Proposition 10.23 (in conjunction with a corresponding gluing theorem), $\overline{\mathcal{M}}^1(J, \gamma^+, \gamma^-)$ is a compact 1-manifold whose boundary consists of two types of broken cylinders, depending whether the index 1 curve appears in an upper or lower level:

$$\partial\overline{\mathcal{M}}^1(J, \gamma^+, \gamma^-) = \bigsqcup_{\gamma_0 \in \mathcal{P}_h(\alpha_+)} \left(\mathcal{M}^1(J_+, \gamma^+, \gamma_0) \times \mathcal{M}^0(J, \gamma_0, \gamma^-)\right)$$
$$\cup \bigsqcup_{\gamma_0 \in \mathcal{P}_h(\alpha_-)} \left(\mathcal{M}^0(J, \gamma^+, \gamma_0) \times \mathcal{M}^1(J_-, \gamma_0, \gamma^-)\right).$$

Counting broken cylinders of the first type gives the coefficient in front of $\langle\gamma^-\rangle$ in $\Phi_J \circ \partial_+(\langle\gamma^+\rangle)$, and the second type gives $\partial_- \circ \Phi_J(\langle\gamma^+\rangle)$.

It follows that Φ_J descends to a homomorphism

$$\Phi_J : HC_*^h(M, \alpha_+, J_+) \to HC_*^h(M, \alpha_-, J_-). \tag{10.4}$$

10.2.6. Chain homotopies. We claim that the map Φ_J in (10.4) does not depend on J. To see this, suppose J_0 and J_1 are two generic choices of compatible almost complex structures on $\widehat{W}$ that both match $J_\pm$ on the cylindrical ends. The space of almost complex structures with these properties is contractible, so we can find a smooth path

$$\{J_s\}_{s\in[0,1]}$$

connecting them. For $I \in \mathbb{Z}$, consider the parametric moduli space

$$\mathcal{M}^I(\{J_s\}, \gamma^+, \gamma^-) := \left\{(s, u) \;\middle|\; s \in [0,1],\ u \in \mathcal{M}^I(J_s, \gamma^+, \gamma^-)\right\}.$$

As we observed in Remark 7.4, a generic choice of the homotopy $\{J_s\}$ makes $\mathcal{M}^I(\{J_s\})$ a smooth manifold with

$$\dim \mathcal{M}^I(\{J_s\}, \gamma^+, \gamma^-) = I + 1$$

whenever $I \ge -1$, and $\mathcal{M}^I(\{J_s\}, \gamma^+, \gamma^-) = \emptyset$ when $I < -1$. Adapting Proposition 10.23 to allow for a converging sequence of almost complex structures, it implies that $\mathcal{M}^{-1}(\{J_s\}, \gamma^+, \gamma^-)$ is compact and thus finite, so we can use it to define a homomorphism of odd degree by

$$H : CC_*^h(M, \alpha_+) \to CC_{*+1}^h(M, \alpha_-) : \langle\gamma\rangle \mapsto \sum_{\gamma' \in \mathcal{P}_h(\alpha_-)} \#\mathcal{M}^{-1}(\{J_s\}, \gamma, \gamma')\langle\gamma'\rangle.$$

We claim that this is a chain homotopy between Φ_{J_0} and Φ_{J_1}, i.e.

$$\Phi_{J_1} - \Phi_{J_0} = \partial_- \circ H + H \circ \partial_+.$$

This follows by looking at the boundary of the compactified 1-dimensional space $\overline{\mathcal{M}}^0(\{J_s\}, \gamma^+, \gamma^-)$, which consists of four types of objects:

(1) Pairs $(0, u)$ with $u \in \mathcal{M}^0(J_0, \gamma^+, \gamma^-)$, which are counted by Φ_{J_0}.
(2) Pairs $(1, u)$ with $u \in \mathcal{M}^0(J_1, \gamma^+, \gamma^-)$, which are counted by Φ_{J_1}.
(3) Pairs $(s, \mathbf{u})$ with $\mathbf{u}$ a broken cylinder with upper level $u_+ \in \mathcal{M}^1(J_+, \gamma^+, \gamma_0)$ and main level $u_0 \in \mathcal{M}^{-1}(J_s, \gamma_0, \gamma^-)$ for some $s \in (0, 1)$; these are counted by $H \circ \partial_+$.
(4) Pairs $(s, \mathbf{u})$ with $\mathbf{u}$ a broken cylinder with lower level $u_- \in \mathcal{M}^1(J_-, \gamma_0, \gamma^-)$ and main level $u_0 \in \mathcal{M}^{-1}(J_s, \gamma^+, \gamma_0)$ for some $s \in (0, 1)$; these are counted by $\partial_- \circ H$.

The sum $\Phi_{J_0} + \Phi_{J_1} + \partial_- \circ H + H \circ \partial_+$ therefore counts (modulo 2) the boundary points of a compact 1-manifold, so it vanishes.

Since the action of Φ_J on homology no longer depends on J, we will denote it from now on by

$$\Phi : HC_*^h(M, \alpha_+, J_+) \to HC_*^h(M, \alpha_-, J_-).$$

It is well defined for any pair of h-admissible contact forms $\alpha_\pm$ and h-regular $J_\pm \in \mathcal{J}(\alpha_\pm)$ since one can first rescale α_+ to assume $\alpha_\pm = e^{f_\pm}\alpha$ with $f_+ > f_-$, using the canonical isomorphism (10.3).

10.2.7. Proof of invariance. We claim that for any h-admissible α and h-regular $J \in \mathcal{J}(\alpha)$, the cobordism map

$$\Phi : HC_*^h(M, \alpha, J) \to HC_*^h(M, \alpha, J)$$

is the identity. Indeed, the literal meaning of this statement is that for any $c > 1$, the composition of the canonical isomorphism (10.3) with the map

$$\Phi : HC_*^h(M, c\alpha, J_c) \to HC_*^h(M, \alpha, J)$$

defined by counting index 0 cylinders in a trivial cobordism from (M, α, J) to $(M, c\alpha, J_c)$ is the identity. Writing $c = e^a$ for $a > 0$, the Liouville cobordism in question is simply

$$(W, d\lambda) = ([0, a] \times M, d(e^r\alpha)),$$

and one can choose a compatible almost complex structure on this which matches J and J_c on ξ while taking ∂_r to $g(r)R_\alpha$ for a suitable function g with $g(r) = 1$ near $r = 0$ and $g(r) = 1/c$ near $r = a$. The resulting almost complex manifold is biholomorphically diffeomorphic to the usual symplectization $(\mathbb{R} \times M, J)$, so our count of index 0 cylinders is equivalent to the count of such cylinders in $(\mathbb{R} \times M, J)$. The latter are simply the trivial cylinders, all of which are Fredholm regular, so counting these defines the identity map on the chain complex.

Finally, we need to show that for any three h-admissible pairs (α_i, J_i) with $i = 0, 1, 2$, the cobordism maps $\Phi_{ij} : HC_*^h(M, \alpha_j, J_j) \to HC_*^h(M, \alpha_i, J_i)$ satisfy

$$\Phi_{21} \circ \Phi_{10} = \Phi_{20}. \tag{10.5}$$

We will only sketch this part: the idea is to use a stretching construction. After rescaling, suppose without loss of generality that $\alpha_i = e^{f_i}\alpha$ with $f_2 > f_1 > f_0$. Then the cobordism

$$W_{20} := \left\{ (r, x) \;\middle|\; f_0(x) \le r \le f_2(x) \right\}$$

contains a contact-type hypersurface

$$M_1 := \left\{ (f_1(x), x) \;\middle|\; x \in M \right\} \subset W_{20}.$$

As described at the end of Lecture 9, one can now choose a sequence of compatible almost complex structures $\{J_{20}^N\}_{N \in \mathbb{N}}$ on $\widehat{W}_{20}$ that are fixed outside a neighborhood of M_1 but degenerate in this neighborhood as $N \to \infty$, equivalent to replacing a small tubular neighborhood of M_1 with increasingly large collars $[-N, N] \times M$ in which J_{20}^N belongs to $\mathcal{J}(\alpha_1)$. The resulting chain maps

$$\Phi_{J_{20}^N} : CC_*^h(M, \alpha_2, J_2) \to CC_*^h(M, \alpha_0, J_0)$$

are chain homotopic for all N, but as $N \to \infty$, the index 0 cylinders counted by these maps converge to buildings with two levels, the top one an index 0 cylinder in the completion of a cobordism from (M, α_1, J_1) to (M, α_2, J_2), while the bottom one also has index 0 and lives in a cobordism from (M, α_0, J_0) to (M, α_1, J_1). The composition $\Phi_{21} \circ \Phi_{10}$ counts these broken cylinders, so this proves (10.5).

In particular, we conclude now that each of the cobordism maps

$$\Phi : HC_*^h(M, \alpha_+, J_+) \to HC_*^h(M, \alpha_-, J_-)$$

is an isomorphism, since composing it with a cobordism map in the opposite direction must give the identity. The isomorphism class of $HC_*^h(M, \alpha, J)$ is therefore independent of the auxiliary data (α, J), and will be denoted by

$$HC_*^h(M, \xi).$$

This is the **cylindrical contact homology** of (M, ξ) in the homotopy class h. It is defined for any primitive homotopy class $h \in [S^1, M]$ and closed contact manifold that is h-admissible in the sense of Definition 10.16. It is also invariant under contactomorphisms in the following sense:

PROPOSITION 10.24. *Suppose $\varphi : (M_0, \xi_0) \to (M_1, \xi_1)$ is a contactomorphism with $\varphi_* h_0 = h_1$, where $h_0 \in [S^1, M]$ is a primitive homotopy class of loops, and (M_1, ξ_1) is h_1-admissible. Then (M_0, ξ_0) is h_0-admissible, and $HC_*^{h_0}(M_0, \xi_0) \cong HC_*^{h_1}(M_1, \xi_1)$.*

PROOF. Given an h_1-admissible contact form α_1 on (M_1, ξ_1) and an h_1-regular $J_1 \in \mathcal{J}(\alpha_1)$, the contact form $\alpha_0 := \varphi^*\alpha_1$ on M_0 is h_0-admissible since φ defines a bijection from $\mathcal{P}_{h_0}(\alpha_0)$ to $\mathcal{P}_{h_1}(\alpha_1)$ and also a bijection between the sets of contractible Reeb orbits for α_0 and α_1. Since $\varphi_*\xi_0 = \xi_1$, α_0 is a contact form for (M_0, ξ_0), hence the latter is h_0-admissible. The diffeomorphism $\tilde{\varphi} := \mathrm{Id} \times \varphi : \mathbb{R} \times M_0 \to \mathbb{R} \times M_1$ then maps ∂_r to ∂_r, R_{α_0} to R_{α_1} and ξ_0 to ξ_1, thus $J_0 := \tilde{\varphi}^* J_1 \in \mathcal{J}(\alpha_0)$, so $\tilde{\varphi}$ defines a biholomorphic map $(\mathbb{R} \times M_0, J_0) \to (\mathbb{R} \times M_1, J_1)$ and thus a bijection between the sets of holomorphic cylinders in each. It follows that J_0 is h_0-regular, and the bijection $\mathcal{P}_{h_0}(\alpha_0) \to \mathcal{P}_{h_1}(\alpha_1)$ defines an isomorphism between the chain complexes defining $HC_*^{h_0}(M_0, \alpha_0, J_0)$ and $HC_*^{h_1}(M_1, \alpha_1, J_1)$. □

10.3. Computing $HC_*(\mathbb{T}^3, \xi_k)$

10.3.1. The Morse-Bott setup. The contact form α_k on $\mathbb{T}^3$ defined at the beginning of this lecture has Reeb vector field

$$R_k(\rho, \phi, \theta) = \cos(2\pi k\rho)\, \partial_\theta + \sin(2\pi k\rho)\, \partial_\phi.$$

Its Reeb orbits therefore preserve and define linear foliations on each of the tori $\{\rho\} \times \mathbb{T}^2$. In particular, none of the closed orbits are contractible, though all of them are also degenerate, as they all come in S^1-parametrized families foliating $\{\text{const}\} \times \mathbb{T}^2$. For certain homotopy classes $h \in [S^1, \mathbb{T}^3]$, this yields a very easy computation of $HC_*^h(\mathbb{T}^3, \xi_k)$, namely whenever h contains no periodic orbits:

THEOREM 10.25. *Suppose $h \in [S^1, \mathbb{T}^3]$ is any primitive homotopy class of loops such that the projection $p : \mathbb{T}^3 \to S^1 : (\rho, \phi, \theta) \mapsto \rho$ satisfies $p_*h \neq 0 \in [S^1, S^1]$. Then α_k is h-admissible and the resulting contact homology $HC_*^h(\mathbb{T}^3, \xi_k)$ is trivial.* □

Now for the interesting part. Every primitive class $h \in [S^1, \mathbb{T}^3]$ not covered by Theorem 10.25 contains closed orbits of R_k, all of them degenerate since they come in S^1-parametrized families foliating the tori $\{\text{const}\} \times \mathbb{T}^2$. This makes it not immediately clear whether $(\mathbb{T}^3, \xi_k)$ is h-admissible, though the following observation in conjunction with Proposition 10.24 shows that if $HC_*^h(\mathbb{T}^3, \xi_k)$ can be defined, it will be the same for all the homotopy classes under consideration.

LEMMA 10.26. *Suppose $h_0, h_1 \in [S^1, \mathbb{T}^3]$ are primitive homotopy classes that are both mapped to the trivial class under the projection $\mathbb{T}^3 \to S^1 : (\rho, \phi, \theta) \mapsto \rho$. Then there exists a contactomorphism $\varphi : (\mathbb{T}^3, \xi_k) \to (\mathbb{T}^3, \xi_k)$ satisfying $\varphi_* h_0 = h_1$.*

PROOF. We can represent h_i for $i = 0, 1$ by loops of the form $\gamma_i(t) = (0, \beta_i(t)) \in S^1 \times \mathbb{T}^2$, where the loops $\beta_i : S^1 \to \mathbb{T}^2$ are embedded and thus represent generators of $\pi_1(\mathbb{T}^2) = \mathbb{Z}^2$. One can thus find a matrix $\begin{pmatrix} m & n \\ p & q \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ such that the diffeomorphism

$$\varphi : \mathbb{T}^3 \to \mathbb{T}^3 : (\rho, \phi, \theta) \mapsto (\rho, m\phi + n\theta, p\phi + q\theta)$$

satisfies $\varphi_* h_0 = h_1$. We have

$$\begin{aligned} \varphi^*\alpha_k &= [q\cos(2\pi k\rho) + n\sin(2\pi k\rho)]\, d\theta + [p\cos(2\pi k\rho) + m\sin(2\pi k\rho)]\, d\phi \\ &=: F(\rho)\, d\theta + G(\rho)\, d\phi. \end{aligned}$$

The loop $(F, G) : S^1 \to \mathbb{R}^2$ satisfies

$$\begin{pmatrix} F(\rho) \\ G(\rho) \end{pmatrix} = \begin{pmatrix} q & n \\ p & m \end{pmatrix} \begin{pmatrix} \cos(2\pi k\rho) \\ \sin(2\pi k\rho) \end{pmatrix},$$

where $\begin{pmatrix} q & n \\ p & m \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$, thus (F, G) winds k times about the origin. Any choice of homotopy from (F, G) to $(\cos(2\pi k\rho), \sin(2\pi k\rho))$ through loops $(F_s, G_s) : S^1 \to \mathbb{R}^2$ winding k times about the origin with positive rotational velocity then gives rise to a homotopy from $\varphi^*\alpha_k$ to α_k through contact forms $F_s(\rho)\, d\theta + G_s(\rho)\, d\phi$. Gray's stability theorem therefore yields a contactomorphism $\psi : (\mathbb{T}^3, \xi_k) \to (\mathbb{T}^3, \ker \varphi^*\alpha_k)$ with ψ smoothly isotopic to the identity. The map $\varphi \circ \psi$ is thus a contactomorphism of $(\mathbb{T}^3, \xi_k)$ with $(\varphi \circ \psi)_* h_0 = \varphi_* \psi_* h_0 = \varphi_* h_0 = h_1$. □

In light of the lemma, we are free from now on to restrict our attention to the particular homotopy class

$$h := [t \mapsto (0, 0, t)],$$

which is the homotopy class of the 1-periodic orbits foliating the k tori

$$T_m := \{m/k\} \times \mathbb{T}^2, \qquad m = 0, \ldots, k-1$$

since $R_k(m/k, \phi, \theta) = \partial_\theta$. Though the orbits on these tori are degenerate, it is not hard to show that they all satisfy the Morse-Bott condition; in fact, α_k is a Morse-Bott contact form. We will explain a self-contained computation of $HC_*^h(\mathbb{T}^3, \xi_k)$ in the next two sections without using the Morse-Bott condition—but first, it seems worthwhile to sketch how one can guess the answer using Morse-Bott data.

Bourgeois's thesis [**Bou02**] gives a prescription for calculating contact homology in Morse-Bott settings, i.e. for deducing what orbits and what holomorphic curves will appear under certain standard ways of perturbing the Morse-Bott contact form to make it nondegenerate. Notice first that the only orbits in $\mathcal{P}_h(\alpha_k)$ are the ones that foliate the k tori $T_0, \ldots, T_{k-1}$, and they all have period 1. By (10.2), it follows that for any $J \in \mathcal{J}(\alpha_k)$, there can be no nontrivial J-holomorphic cylinders connecting two orbits in $\mathcal{P}_h(\alpha_k)$. This makes the calculation of $HC_*^h(\mathbb{T}^3, \xi_k)$ sound trivial, but of course there is more to the story since α_k is not admissible; indeed, the chain complex $CC_*(\mathbb{T}^3, \alpha_k)$ is not even well defined. The prescription in [**Bou02**] now gives the following. Each of the families of orbits in $T_0, \ldots, T_{k-1}$ is parametrized by S^1, and by a standard perturbation technique, any choice of a Morse function $f_m : S^1 \to \mathbb{R}$ for $m = 0, \ldots, k-1$ yields a contact form α_k' that is C^∞-close to α_k, matches it outside a neighborhood of T_m, but has a nondegenerate Reeb orbit on T_m for each critical point of f_m, while every other closed orbit in the perturbed region can be assumed to have arbitrarily large period. Moreover, there is a corresponding perturbation from $J \in \mathcal{J}(\alpha_k)$ to $J' \in \mathcal{J}(\alpha_k')$ such that every gradient flow line of the function $f_m : S^1 \to \mathbb{R}$ gives rise to a J'-holomorphic cylinder in $\mathbb{R} \times \mathbb{T}^3$ connecting the corresponding nondegenerate Reeb orbits along T_m. In the present situation, since no J-holomorphic cylinders of the relevant type exist before the perturbation, the only ones after the perturbation are those that come from gradient flow lines.

Now imagine performing a similar perturbation near every $T_0, \ldots, T_{k-1}$, using Morse functions $f_0, \ldots, f_{k-1} : S^1 \to \mathbb{R}$ that each have exactly two critical points.

For the perturbed contact form α'_k, $\mathcal{P}_h(\alpha'_k)$ now consists of exactly $2k$ orbits

$$\gamma_0^\pm, \dots, \gamma_{k-1}^\pm \in \mathcal{P}_h(\alpha'_k),$$

where we denote by γ_m^+ and γ_m^- the orbits on T_m corresponding to the maximum and minimum of f_m respectively. For the obvious choice of trivialization τ for the contact bundle along $\gamma_m^\pm$, one can relate the Conley-Zehnder indices to the Morse indices of the corresponding critical points, giving

$$\mu_{\mathrm{CZ}}^\tau(\gamma_m^+) = 0, \qquad \mu_{\mathrm{CZ}}^\tau(\gamma_m^-) = 1, \qquad m = 0, \dots, k-1.$$

Moreover, the two gradient flow lines connecting maximum and minimum for each f_m give rise two exactly two holomorphic cylinders in $\mathcal{M}^1(J', \gamma_m^-, \gamma_m^+)$ for each $m = 0, \dots, k-1$, and these are all the curves that are counted for the differential on $CC_*^h(\mathbb{T}^3, \alpha'_k, J')$. Counting modulo 2, we thus have

$$\partial \langle \gamma_m^\pm \rangle = 0 \quad \text{for all} \quad m = 0, \dots, k-1,$$

implying

$$HC_*^h(\mathbb{T}^3, \alpha'_k, J') = \begin{cases} \mathbb{Z}_2^k & * = \text{odd}, \\ \mathbb{Z}_2^k & * = \text{even}. \end{cases}$$

Let us state this as a theorem.

THEOREM 10.27. *Suppose $h \in [S^1, \mathbb{T}^3]$ is a primitive homotopy class that maps to the trivial class under the projection $\mathbb{T}^3 \to S^1 : (\rho, \phi, \theta) \mapsto \rho$. Then $(\mathbb{T}^3, \xi_k)$ is h-admissible and*

$$HC_*^h(\mathbb{T}^3, \xi_k) \cong \begin{cases} \mathbb{Z}_2^k & * = \textit{odd}, \\ \mathbb{Z}_2^k & * = \textit{even}. \end{cases}$$

Theorem 10.1 is an immediate corollary of this: indeed, if $\varphi : (\mathbb{T}^3, \xi_k) \to (\mathbb{T}^3, \xi_\ell)$ is a contactomorphism, choose any $h \in [S^1, \mathbb{T}^3]$ for which Theorem 10.27 applies, and let $h_0 := \varphi^* h \in [S^1, \mathbb{T}^3]$. Then $HC_*^h(\mathbb{T}^3, \xi_\ell) \cong \mathbb{Z}_2^{2\ell}$ implies via Proposition 10.24 that $HC_*^{h_0}(\mathbb{T}^3, \xi_k) \cong \mathbb{Z}_2^{2\ell}$. But Theorems 10.25 and 10.27 imply that the latter is also either 0 or $\mathbb{Z}_2^{2k}$, hence $k = \ell$.

10.3.2. A digression on the Floer equation. In preparation for giving a self-contained proof of Theorem 10.27, we now explain a general procedure for relating holomorphic cylinders in a symplectization to solutions of the Floer equation. This idea is loosely inspired by arguments in [**EKP06**].

To motivate what follows, notice that on a neighborhood of $T_0 = \{0\} \times \mathbb{T}^2 \subset (\mathbb{T}^3, \xi_k)$, we can write

$$\alpha_k = \cos(2\pi k \rho)\,(d\theta + \beta)\,,$$

where $\beta := \tan(2\pi k\rho)\, d\phi$ defines a Liouville form on the annulus $\mathbb{A} := [-1/8, 1/8] \times S^1$ with coordinates (ρ, ϕ). This makes the neighborhood $\mathbb{A} \times S^1 \subset (\mathbb{T}^3, \xi_k)$ a special case of the following general construction.

DEFINITION 10.28. Suppose V is a $2n$-dimensional manifold with an exact symplectic form $d\beta$. The contact manifold $(V \times S^1, \ker(d\theta + \beta))$ is then called the **contactization** of (V, β).[4] Here θ denotes the coordinate on the S^1 factor.

It's easy to check that $d\theta + \beta$ is indeed a contact form on $V \times S^1$ whenever $d\beta$ is symplectic on V: the latter means $(d\beta)^n > 0$ on V, so

$$(d\theta + \beta) \wedge [d(d\theta + \beta)]^n = (d\theta + \beta) \wedge (d\beta)^n = d\theta \wedge (d\beta)^n > 0.$$

Now here's a cute trick one can play with contactizations. For the rest of this subsection, assume

$$(V, d\beta)$$

is an arbitrary compact $2n$-dimensional exact symplectic manifold with boundary. Fix a smooth function

$$H : V \times S^1 \to \mathbb{R},$$

which we shall think of in the following as a time-dependent Hamiltonian $H_\theta := H(\cdot, \theta) : V \to \mathbb{R}$ on $(V, d\beta)$. The 2-form on $V \times S^1$ defined by

$$\Omega = d\beta + d\theta \wedge dH = d(\beta - H\, d\theta)$$

is then *fiberwise symplectic*, meaning its restriction to each of the fibers of the projection map $V \times S^1 \to S^1$ is symplectic. We claim that for every $\epsilon > 0$ sufficiently small,

$$\lambda_\epsilon := d\theta + \epsilon(\beta - H\, d\theta)$$

defines a contact form on $V \times S^1$. This is a variation on the construction that was used by Thurston and Winkelnkemper [**TW75**] to define contact forms out of open book decompositions, and the proof is simple enough: since $d\lambda_\epsilon = \epsilon\Omega$, we just need to check that $\lambda_\epsilon \wedge \Omega^n > 0$ for $\epsilon > 0$ sufficiently small, and indeed,

$$\lambda_\epsilon \wedge \Omega^n = d\theta \wedge (d\beta)^n + \epsilon(\beta - H\, d\theta) \wedge \Omega^n > 0$$

since the first term is a volume form and ϵ is small. To see the relation between λ_ϵ and the contactization, we can write

$$\lambda_\epsilon = (1 - \epsilon H)\, d\theta + \epsilon\beta = (1 - \epsilon H)\left(d\theta + \frac{\epsilon}{1 - \epsilon H}\beta\right)$$

and observe that $\frac{\epsilon}{1-\epsilon H}\beta$ is also a Liouville form on V whenever H is θ-independent and $\epsilon > 0$ is sufficiently small.

The Reeb vector fields R_ϵ for λ_ϵ vary with ϵ, but their directions do not, since $d\lambda_\epsilon = \epsilon\Omega$ has the same kernel for every ϵ. Moreover, while λ_ϵ ceases to be a contact form when $\epsilon \to 0$, the Reeb vector fields still have a well-defined limit: they converge as $\epsilon \to 0$ to the unique vector field R_0 satisfying

$$d\theta(R_0) \equiv 1 \quad \text{and} \quad \Omega(R_0, \cdot) \equiv 0.$$

The latter can be written more explicitly as

$$R_0 = \partial_\theta + X_\theta,$$

[4] Elsewhere in the literature, the contactization is also often defined as $V \times \mathbb{R}$ instead of $V \times S^1$. The usage here is consistent with [**MNW13**].

where X_θ is the time-dependent Hamiltonian vector field determined by H_θ, i.e. via the condition

$$d\beta(X_\theta, \cdot) = -dH_\theta.$$

As one can easily compute, the reason for this nice behavior as $\epsilon \to 0$ is that the R_ϵ are also the Reeb vector fields for a smooth family of stable Hamiltonian structures:

PROPOSITION 10.29. *The pairs $\mathcal{H}_\epsilon := (\Omega, \lambda_\epsilon)$ for $\epsilon \geq 0$ sufficiently small define a smooth family of stable Hamiltonian structures whose Reeb vector fields are R_ϵ.* □

We shall write the hyperplane distributions induced by $\mathcal{H}_\epsilon$ as

$$\Xi_\epsilon := \ker \lambda_\epsilon \subset T(V \times S^1).$$

These are contact structures for $\epsilon > 0$ small, and the space $\mathcal{J}(\mathcal{H}_\epsilon)$ of $\mathbb{R}$-invariant almost complex structures on $\mathbb{R} \times (V \times S^1)$ compatible with $\mathcal{H}_\epsilon$ is then identical to $\mathcal{J}(\lambda_\epsilon)$. On the other hand for $\epsilon = 0$, $\Xi_0 = \ker d\theta$ is a foliation, namely it is the vertical subbundle of the trivial fibration $V \times S^1 \to S^1$. To interpret $\mathcal{H}_0$, notice that its closed Reeb orbits in the homotopy class of $\gamma : S^1 \to V \times S^1 : t \mapsto (\text{const}, t)$ are all of the form $\gamma(t) = (x(t), t)$ where $x : S^1 \to V$ is a contractible 1-periodic orbit of X_θ. Moreover, suppose $J \in \mathcal{J}(\mathcal{H}_0)$, which is equivalent to a choice of compatible complex structure on the symplectic bundle $(\Xi_0, \Omega|_{\Xi_0})$, or in other words, an S^1-parametrized family of $d\beta$-compatible almost complex structures $\{J_\theta\}_{\theta \in S^1}$ on V. Then if

$$u = (f, v, g) : \mathbb{R} \times S^1 \to \mathbb{R} \times (V \times S^1)$$

is a J-holomorphic cylinder asymptotic at $\{\pm\infty\} \times S^1$ to two orbits of the form described above, the nonlinear Cauchy-Riemann equation for u turns out to imply that $(f, g) : \mathbb{R} \times S^1 \to \mathbb{R} \times S^1$ is a holomorphic map with degree 1 sending $\{\pm\infty\} \times S^1$ to $\{\pm\infty\} \times S^1$, and we can therefore choose a unique biholomorphic reparametrization of u so that (f, g) becomes the identity map. Having done this, the equation satisfied by $v : \mathbb{R} \times S^1 \to V$ is now

$$\partial_s v + J_t(v)(\partial_t v - X_t(v)) = 0,$$

in other words, the Floer equation for the data $\{J_\theta\}_{\theta \in S^1}$ and $\{H_\theta\}_{\theta \in S^1}$.

To complete the analogy, notice that since Ω is exact, we can write down a natural symplectic action functional with respect to each $\mathcal{H}_\epsilon$ as

$$\mathcal{A}_\epsilon : C^\infty(S^1, V \times S^1) \to \mathbb{R} : \gamma \mapsto \int_{S^1} \gamma^*(\beta - H\, d\theta).$$

For loops of the form $\gamma(t) = (x(t), t)$ with $x : S^1 \to V$ contractible, this reduces (give or take a sign—see Remark 10.32) to the usual formula for the Floer action functional

$$\mathcal{A}_H(\gamma) = \int_{S^1} x^*\beta - \int_{S^1} H(x(t))\, dt = \int_{\mathbb{D}} \bar{x}^* d\beta - \int_{S^1} H(x(t))\, dt, \tag{10.6}$$

where $\bar{x} : \mathbb{D} \to V$ is any map satisfying $\bar{x}|_{\partial\mathbb{D}} = x$. Stokes' theorem gives an easy relation between the action and the so-called *Ω-energy* if $u : \mathbb{R} \times S^1 \to \mathbb{R} \times (V \times S^1)$

is a J-holomorphic curve for $J \in \mathcal{J}(\mathcal{H}_\epsilon)$ and is positively/negatively asymptotic to orbits $\gamma^\pm : S^1 \to V \times S^1$ at $s = \pm\infty$: we have

$$0 \le \int_{\mathbb{R}\times S^1} u^*\Omega = \mathcal{A}_\epsilon(\gamma^+) - \mathcal{A}_\epsilon(\gamma^-).$$

If $u(s,t) = (s, v(s,t), t)$, then the left hand side is identical to the definition of energy in Floer homology, namely

$$E_H(v) := \int_{\mathbb{R}\times S^1} d\beta(\partial_s v, \partial_t v - X_t(v))\, ds \wedge dt = \int_{\mathbb{R}\times S^1} d\beta(\partial_s v, J_t(v)\partial_s v)\, ds \wedge dt,$$

thus giving the familiar relation

$$E_H(v) = \mathcal{A}_H(\gamma^+) - \mathcal{A}_H(\gamma^-). \tag{10.7}$$

To relate this to the usual notion of energy with respect to a stable Hamiltonian structure, we write the usual formula

$$E_\epsilon(u) := \sup_{\varphi\in\mathcal{T}} \int_{\dot{\Sigma}} u^* \left[d\big(\varphi(r)\lambda_\epsilon\big) + \Omega \right],$$

with $\mathcal{T} := \left\{ \varphi \in C^\infty(\mathbb{R}, (-\epsilon_0, \epsilon_0)) \;\middle|\; \varphi' > 0 \right\}$ for some constant $\epsilon_0 > 0$ sufficiently small. Notice first that for any fixed ϵ, Stokes' theorem gives a bound for $E_\epsilon(u)$ in terms of the asymptotic orbits of u since Ω is exact. Finally, in the case $\epsilon = 0$ with $u(s,t) = (s, v(s,t), t)$, we find

$$E_0(u) = \sup_{\varphi\in\mathcal{T}} \int_{\mathbb{R}\times S^1} \varphi'(s)\, ds \wedge dt + \int_{\mathbb{R}\times S^1} u^*\Omega = 2\epsilon_0 + E_H(v),$$

so bounds on $E_0(u)$ are equivalent to bounds on the Floer homological energy $E_H(v)$. The basic fact that Floer trajectories $v : \mathbb{R} \times S^1 \to V$ with $E_H(v) < \infty$ are asymptotic to contractible 1-periodic Hamiltonian orbits can now be regarded as a corollary of our Theorem 9.6.

The above discussion gives a one-to-one correspondence between a certain moduli space of unparametrized J-holomorphic cylinders in $\mathbb{R} \times (V \times S^1)$ and the moduli space of Floer trajectories between contractible 1-periodic orbits in $(V, d\beta)$ with Hamiltonian function H. If we can adequately understand the moduli space of Floer trajectories—in particular if we can classify them and prove that they are regular—then the idea will be to extend this classification via the implicit function theorem to any $J_\epsilon \in \mathcal{J}(\lambda_\epsilon)$ sufficiently close to J for $\epsilon > 0$ small. As the reader may be aware, classifying Floer trajectories is also not easy in general, but it does become easy under certain conditions. Simple examples of contractible 1-periodic Hamiltonian orbits are furnished by the constant loops $\gamma(t) = x$ at critical points $x \in \mathrm{Crit}(H)$, and for each such orbit, $\gamma^*\Xi_0$ has a canonical homotopy class of unitary trivializations, the so-called **constant trivialization**. The following fundamental result is commonly used in proving the isomorphism from Hamiltonian Floer homology to singular homology.

THEOREM 10.30. *Suppose $H : V \to \mathbb{R}$ is a smooth Morse function with no critical points on the boundary, J is a fixed $d\beta$-compatible almost complex structure on V, and the gradient flow of H with respect to the metric $d\beta(\cdot, J\cdot)$ is Morse-Smale* *and transverse to ∂V. Given $\delta > 0$, let $H^\delta := \delta H : V \to \mathbb{R}$, with Hamiltonian vector field $X_{H^\delta} = \delta X_H$, and consider the stable Hamiltonian structure*

$$\mathcal{H}_0^\delta := (d\beta + d\theta \wedge dH^\delta, d\theta)$$

on $V \times S^1$ with induced Reeb vector field $R_0^\delta = \partial_\theta + X_{H^\delta}$. Then for all $\delta > 0$ sufficiently small, the following statements hold.

(1) *The 1-periodic R_0^δ-orbit $\gamma_x : S^1 \to V \times S^1 : t \mapsto (x,t)$ arising from any critical point $x \in \mathrm{Crit}(H)$ is nondegenerate, and its Conley-Zehnder index relative to the constant trivialization τ is related to the Morse index $\mathrm{ind}(x) \in \{0, \ldots, 2n\}$ by*

$$\mu_{CZ}^\tau(\gamma_x) = n - \mathrm{ind}(x). \tag{10.8}$$

(2) *Any trajectory $\gamma : \mathbb{R} \to V$ satisfying the negative gradient flow question $\dot\gamma = -\nabla H^\delta(\gamma)$ gives rise to a Fredholm regular solution $v : \mathbb{R} \times S^1 \to V : (s,t) \mapsto \gamma(s)$ of the time-independent Floer equation*

$$\partial_s v + J(v)(\partial_t v - X_{H^\delta}(v)) = 0, \tag{10.9}$$

and the virtual dimensions of the spaces of Floer trajectories near v and gradient flow trajectories near γ are the same.

(3) *Every 1-periodic orbit of X_{H^δ} in $\mathring{V}$ is a constant loop at a critical point of H.*

(4) *Every finite-energy solution $v : \mathbb{R} \times S^1 \to \mathring{V}$ of (10.9) is of the form $v(s,t) = \gamma(s)$ for some negative gradient flow trajectory $\gamma : \mathbb{R} \to V$.*

PROOF. The following proof is based on arguments in [SZ92], see in particular Theorem 7.3.

For the first statement, let $\gamma(t) = (x,t)$ for $x \in \mathrm{Crit}(H)$ and recall from Lecture 3 the formula for the asymptotic operator of a 1-periodic orbit,

$$\mathbf{A}_\gamma : \Gamma(\gamma^*\Xi_0) \to \Gamma(\gamma^*\Xi_0) : \eta \mapsto -J\left(\nabla_t \eta - \nabla_\eta R_0^\delta\right),$$

where ∇ is any symmetric connection on $V \times S^1$. Identifying $\Gamma(\gamma^*\Xi_0)$ in the natural way with $C^\infty(S^1, T_x V)$, using the trivial connection and writing $R_0^\delta(z,\theta) = \partial_\theta + X_{H^\delta}(z) = \partial_\theta + \delta J(z)\nabla H(z)$, $\mathbf{A}_\gamma$ becomes the operator

$$\mathbf{A}_\gamma = -J\partial_t - \delta\nabla^2 H(x)$$

on $C^\infty(S^1, T_x V)$, where $\nabla^2 H(x) : T_x V \to T_x V$ denotes the Hessian of H at x. Choosing a unitary basis for $T_x V$ identifies this with $-J_0\partial_t - \delta S$ for some symmetric $2n$-by-$2n$ matrix S and the standard complex structure $J_0 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$, so $\ker \mathbf{A}_\gamma$ corresponds to the space of 1-periodic solutions to $\dot\eta = \delta J_0 S\eta$. The Morse condition implies that S is nonsingular, so the eigenvalues of $\delta J_0 S$ are all nonzero, but they are also small since δ is small. It follows that nontrivial solutions of $\dot\eta = \delta J_0 S\eta$ cannot be 1-periodic if S is nonsingular and δ is sufficiently small, thus proving that $\ker \mathbf{A}_\gamma$ is trivial, hence γ is nondegenerate.

To calculate $\mu^\tau_{\mathrm{CZ}}(\gamma)$, note that $\lambda \in \sigma(\mathbf{A}_\gamma)$ if and only if there exists a nontrivial 1-periodic solution η to the equation

$$\dot{\eta} = J_0(\delta S + \lambda)\eta.$$

If δ and λ are both close to 0, then the same argument again implies that no such solution exists unless $\delta S + \lambda$ is singular, meaning $\lambda \in \sigma(-\delta S)$. On the other hand, any constant loop $\eta(t) \in \ker(\lambda + \delta S)$ furnishes an element of the λ-eigenspace of $\mathbf{A}_\gamma$, so we obtain a bijection between the spectra of $\mathbf{A}_\gamma$ and $-\delta S$ in some neighborhood of 0. It follows that if $S_\pm$ denotes a pair of nonsingular symmetric matrices defining asymptotic operators $\mathbf{A}_\pm = -J_0\partial_t - \delta S_\pm$, then the spectral flows are related by

$$\mu^{\mathrm{spec}}(\mathbf{A}_-, \mathbf{A}_+) = -\mu^{\mathrm{spec}}(S_-, S_+)$$

when $\delta > 0$ is sufficiently small. Denoting the maximal negative-definite subspace of $S_\pm$ by $E^-(S_\pm)$, this relation implies

$$\dim E^-(S_+) - \dim E^-(S_-) = \mu_{\mathrm{CZ}}(\mathbf{A}_-) - \mu_{\mathrm{CZ}}(\mathbf{A}_+).$$

Now suppose S_+ is a coordinate expression for the Hessian $\nabla^2 H(x)$, hence $\dim E^-(S_+) = \operatorname{ind}(x)$ and $\mu_{\mathrm{CZ}}(\mathbf{A}_+) = \mu^\tau_{\mathrm{CZ}}(\gamma)$. Choosing $S_- = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$ then gives $\dim E^-(S_-) = n$ and $\mu_{\mathrm{CZ}}(\mathbf{A}_-) = 0$ by definition, so $\mu^\tau_{\mathrm{CZ}}(\gamma) = n - \operatorname{ind}(x)$ follows.

The second statement follows in a similar manner by writing down and comparing the linearized operators for the Floer equation and the negative gradient flow equation. Let's leave this as an exercise.

For the third statement, suppose we have a sequence $\delta_k \to 0$ and a sequence of loops $x_k : S^1 \to \mathring{V}$ satisfying $\dot{x}_k = X_{H^{\delta_k}}(x_k) = \delta_k X_H(x_k)$. Pick a number $c > 0$ small enough for part (1) of the theorem to hold with $\delta = c$, choose a sequence of integers $N_k \in \mathbb{N}$ such that

$$N_k \delta_k \to c,$$

and consider the loops $y_k : S^1 \to \mathring{V} : t \mapsto x_k(N_k t)$. These satisfy

$$\dot{y}_k = N_k \delta_k X_H(y_k),$$

and since X_H is C^∞-bounded on V and $N_k\delta_k$ is also bounded, the Arzelà-Ascoli theorem provides a subsequence with

$$y_k \to y_\infty \quad \text{in} \quad C^\infty(S^1, V),$$

where $y_\infty : S^1 \to V$ satisfies $\dot{y}_\infty = X_{H^c}(y_\infty)$ for $H^c := cH : V \to \mathbb{R}$. But y_∞ is also constant: indeed, since $y_k(t + 1/N_k) = y_k(t)$ and $N_k \to \infty$, we can find for any $t \in S^1$ a sequence $q_k \in \mathbb{Z}$ satisfying $q_k/N_k \to t$, so

$$y_\infty(t) = \lim_{k\to\infty} y_k(q_k/N_k) = \lim_{k\to\infty} y_k(0) = y_\infty(0). \tag{10.10}$$

Since the constant orbit y_∞ is nondegenerate by part (1) of the theorem, there can only be one sequence of solutions to $\dot{y}_k = X_{H^{N_k\delta_k}}(y_k)$ converging to y_∞, and we conclude that y_k is also constant for all k sufficiently large.

We will now use a similar trick to prove the fourth statement in the theorem. We shall work under the additional assumption that

$$|\operatorname{ind}(x) - \operatorname{ind}(y)| \le 1 \quad \text{for all pairs} \quad x, y \in \operatorname{Crit}(H), \tag{10.11}$$

which suffices for the application in §10.3.3 below.[5]

Suppose to the contrary that there exists a sequence of positive numbers $\delta_k \to 0$ with finite-energy solutions $v_k : \mathbb{R} \times S^1 \to \mathring{V}$ of the equation $\partial_s v_k + J(v_k)(\partial_t v_k - X_{H^{\delta_k}}(v_k)) = 0$, where each $v_k(s,t)$ is not t-independent. By part (3) of the theorem, we can restrict to a subsequence and assume each v_k for large k is asymptotic to a fixed pair of critical points $x_\pm = \lim_{s\to\pm\infty} v_k(s,\cdot) \in \mathrm{Crit}(H)$, and $x_+ \neq x_-$ since v_k would otherwise by constant and therefore t-independent. Choose a sequence $N_k \in \mathbb{N}$ with

$$N_k \to \infty \quad \text{and} \quad N_k\delta_k \to c,$$

where $c > 0$ is chosen sufficiently small for the first three statements in the theorem to hold with $\delta = c$. Define $w_k : \mathbb{R} \times S^1 \to V$ by

$$w_k(s,t) = v_k(N_k s, N_k t).$$

Then w_k satisfies another time-independent Floer equation,

$$\partial_s w_k + J(w_k)\left(\partial_t w_k - X_{H^{N_k\delta_k}}(w_k)\right) = 0, \tag{10.12}$$

where the Hamiltonian functions $H^{N_k\delta_k}$ converge to H^c. The standard compactness theorem for Floer trajectories should now imply that a subsequence of w_k converges to a broken Floer trajectory whose levels will be t-independent. Since the setting may seem a bit nonstandard, here are some details.

The sequence w_k is uniformly C^0-bounded since V is compact. We claim that it is also C^1-bounded. If not, then there is a sequence $z_k = (s_k, t_k) \in \mathbb{R} \times S^1$ with $|dw_k(z_k)| =: R_k \to \infty$, and we can use the usual rescaling trick from Lecture 9 to define a sequence

$$f_k : \mathbb{D}_{\epsilon_k R_k} \to V : z \mapsto w_k(z_k + z/R_k)$$

for a suitable sequence $\epsilon_k \to 0$ with $\epsilon_k R_k \to \infty$ and $|dw_k(z)| \le 2R_k$ for all $z \in \mathbb{D}_{\epsilon_k}(z_k)$. The latter implies that f_k satisfies a local C^1-bound independent of k, and since

$$\partial_s f_k + J(f_k)\left(\partial_t f_k - \frac{1}{R_k} J(f_k) X_{H^{N_k\delta_k}}(f_k)\right),$$

elliptic regularity (see Remark 10.31 below) provides a subsequence for which f_k converges in $C^\infty_{\mathrm{loc}}(\mathbb{C}, V)$ to a J-holomorphic plane $f_\infty : \mathbb{C} \to V$, which is nonconstant since

$$|df_\infty(0)| = \lim_{k\to\infty} |df_k(0)| = 1.$$

Since v_k and therefore w_k are all asymptotic to fixed constant orbits $x_\pm$, we have a uniform bound on the Floer energies of w_k,

$$E_{H^{N_k\delta_k}}(w_k) = \mathcal{A}_{H^{N_k\delta_k}}(x_+) - \mathcal{A}_{H^{N_k\delta_k}}(x_-) = N_k\delta_k \left[H(x_-) - H(x_+)\right], \tag{10.13}$$

[5]Lifting this assumption requires gluing, whereas we shall only need the usual implicit function theorem for Fredholm regular solutions of the Floer equation.

where the right hand side is bounded since $N_k\delta_k \to c$. Using change of variables and the fact that $d\beta(\partial_s f_k, J(f_k)\,\partial_s f_k) \ge 0$, this implies a uniform bound

$$\begin{aligned}\int_{\mathbb{D}_{\epsilon_k R_k}} d\beta(\partial_s f_k, J(f_k)\,\partial_s f_k)\, ds\wedge dt &= \int_{\mathbb{D}_{\epsilon_k}(z_k)} d\beta(\partial_s v_k, J(v_k)\,\partial_s v_k)\, ds\wedge dt \\ &\le \int_{\mathbb{R}\times S^1} d\beta(\partial_s v_k, J(v_k)\,\partial_s v_k)\, ds\wedge dt = E_{H^{N_k\delta_k}}(w_k) \le C,\end{aligned}$$

thus

$$\int_{\mathbb{C}} f_\infty^* d\beta = \int_{\mathbb{C}} d\beta(\partial_s f_\infty, \partial_t f_\infty)\, ds\wedge dt = \int_{\mathbb{C}} d\beta(\partial_s f_\infty, J(f_\infty)\,\partial_s f_\infty)\, ds\wedge dt < \infty.$$

The removable singularity theorem now extends f_∞ to a nonconstant J-holomorphic sphere $f_\infty : S^2 \to V$, but this violates Stokes' theorem since J is tamed by an exact symplectic form.

We've now shown that the sequence $w_k : \mathbb{R}\times S^1 \to V$ is uniformly C^1-bounded, and it has bounded energy due to (10.13). Pick any sequence $s_k \in \mathbb{R}$ and consider the sequence of translated Floer trajectories

$$\tilde{w}_k(s,t) := w_k(s + s_k, t).$$

These are also uniformly C^1-bounded, so by elliptic regularity (see Remark 10.31 again), a subsequence converges in $C^\infty_{\text{loc}}(\mathbb{R}\times S^1)$ to a map $w_\infty : \mathbb{R}\times S^1 \to V$ satisfying

$$\partial_s w_\infty + J(w_\infty)\,(\partial_t w_\infty - X_{H^c}(w_\infty)) = 0,$$

and it has finite energy $E_{H^c}(w_\infty) < \infty$ due to (10.13), implying that w_∞ is asymptotic to a pair of 1-periodic orbits of X_{H^c} as $s \to \pm\infty$. By the same argument used in (10.10) above, w_∞ is also t-independent. It follows that $w_\infty(s,t) = \gamma_\infty(s)$ for some nonconstant gradient flow trajectory $\gamma_\infty : \mathbb{R} \to \mathring{V}$. Depending on the choice of sequence s_k, this trajectory may or may not be constant, but we can always choose s_k to guarantee that γ_∞ is not constant: indeed, since each w_k is asymptotic to two separate critical points at $\pm\infty$, $s_k \in \mathbb{R}$ can be chosen such that $w_k(s_k, 0)$ stays a fixed distance away from every critical point of H, and then

$$w_\infty(0,0) = \lim_{k\to\infty} w_k(s_k, 0) \notin \operatorname{Crit}(H^c).$$

One can now adapt the argument of Proposition 10.19 to find various sequences $s_k \in \mathbb{R}$ that yield potentially separate limiting trajectories forming the levels of a broken trajectory, which is the limit of w_k in the Floer topology. But since all the levels are t-independent and the gradient flow of H^c is Morse-Smale, condition (10.11) implies that the most complicated (and therefore the only) limit possible involves a single level $w_\infty(s,t) = \gamma(s)$, which is a gradient flow trajectory between critical points whose Morse indices differ by 1. This trajectory is Fredholm regular and has index 1 due to part (2) of the theorem, thus by the implicit function theorem, the only solutions to (10.12) that can converge to w_∞ are the obvious reparametrizations of γ, i.e. they are also t-independent. This is a contradiction. □

REMARK 10.31. In previous lectures we've used the theorem that "C^1-bounds imply C^∞-bounds" to prove compactness for J-holomorphic curves, but not for solutions of inhomogeneous Cauchy-Riemann type equations such as the Floer trajectories w_k and rescalings f_k in the above proof. There is an easy trick to reduce these to our standard setup: as we've already seen, solutions of the Floer equation are equivalent to honest pseudoholomorphic curves in the symplectization of a certain stable Hamiltonian structure, which is a manifold of two dimensions higher. A similar trick can be used for any inhomogeneous Cauchy-Riemann type equation $\bar{\partial}_J f = \nu$, reducing it to an honest Cauchy-Riemann type equation at the cost of adding two dimensions. This trick was used already by Gromov, see [**Gro85**, 1.4.C].

REMARK 10.32. You may notice with some horror that (10.8) differs by a sign from what is stated in [**SZ92**]. As far as I can tell, the discrepancy arises from the fact that while Floer homology is traditionally defined in terms of a negative gradient flow for the action functional, SFT is based on a *positive* gradient flow—this is also why the action functional in (10.6) differs by a sign from what we saw in Lecture 1. If one takes as an axiom that the Conley-Zehnder index should serve as a "relative Morse index" for the action functional, then changing the sign of the functional also reverses the signs of Conley-Zehnder indices, so as a result there appear to be two parallel sign conventions for Conley-Zehnder indices in different sectors of the literature. I'm sorry. It's not my fault.

Returning now to the family $\mathcal{H}_\epsilon$, choose $\delta > 0$ sufficiently small for Theorem 10.30 to hold and define a modified family of stable Hamiltonian structures on $V \times S^1$ by
$$\mathcal{H}_\epsilon^\delta = (\Omega^\delta, \lambda_\epsilon^\delta),$$
where
$$\Omega^\delta := d\beta + d\theta \wedge dH^\delta \quad \text{and} \quad \lambda_\epsilon^\delta := d\theta + \epsilon(\beta - H^\delta\, d\theta).$$
Denote the induced hyperplane distributions and Reeb vector fields by Ξ_ϵ^δ and R_ϵ^δ respectively. We have only changed the Hamiltonian H by rescaling, so all previous statements about $\mathcal{H}_\epsilon$ also apply to $\mathcal{H}_\epsilon^\delta$, in particular λ_ϵ^δ is contact and $\mathcal{J}(\mathcal{H}_\epsilon^\delta) = \mathcal{J}(\lambda_\epsilon^\delta)$ for all $\epsilon > 0$ sufficiently small, though the upper bound for the allowed range of ϵ may now depend on δ. Once $\delta > 0$ is fixed by the requirements of Theorem 10.30, we are still free to take $\epsilon > 0$ is small as we like.

THEOREM 10.33. *Assume the same hypotheses as in Theorem 10.30, including (10.11), and denote the unique extension of J to an $\mathbb{R}$-invariant almost complex structure in $\mathcal{J}(\mathcal{H}_0^\delta)$ by J_0. Given δ sufficiently small and any smooth family of compatible $\mathbb{R}$-invariant almost complex structures $J_\epsilon \in \mathcal{J}(\mathcal{H}_\epsilon^\delta)$ matching J_0 at $\epsilon = 0$, there exists $\epsilon_0 > 0$ such that every critical point $x \in \mathrm{Crit}(H)$ gives rise to a smooth family of nondegenerate closed R_ϵ^δ-orbits*
$$x^\epsilon : S^1 \to V \times S^1 \qquad \epsilon \in [0, \epsilon_0]$$
with $x^0(t) = (x, t)$, and every gradient flow trajectory $\gamma : \mathbb{R} \to V$ for H gives rise to a smooth family of Fredholm regular J_ϵ-holomorphic cylinders
$$u_\gamma^\epsilon : \mathbb{R} \times S^1 \to \mathbb{R} \times (V \times S^1) \qquad \epsilon \in [0, \epsilon_0]$$
with $u_\gamma^0(s,t) = (s, \gamma(\delta s), t)$. Moreover, for all $\epsilon \in [0, \epsilon_0]$, every closed R_ϵ^δ-orbit homotopic to $t \mapsto (\mathrm{const}, t)$ belongs to one of the families x^ϵ up to parametrization, and

every J_ϵ-holomorphic cylinder with a positive and a negative end asymptotic to orbits of this type belongs to one of the families u^ϵ_γ, up to biholomorphic parametrization.

PROOF. The first part is immediate from the implicit function theorem since the orbits $x^0(t) = (x,t)$ are nondegenerate and the curves $u^0_\gamma(s,t) = (s, \gamma(\delta s), t)$ are Fredholm regular by Theorem 10.30. For the uniqueness statement, observe that if $\epsilon_k \to 0$ and γ_k is a sequence of $R^\delta_{\epsilon_k}$-orbits in the relevant homotopy class, then their periods are uniformly bounded, so Arzelà-Ascoli gives a subsequence convergent to a closed R^δ_0-orbit, which is a nondegenerate orbit of the form $x^0(t) = (x,t)$ for $x \in \mathrm{Crit}(H)$ by Theorem 10.30, so sequences converging to this orbit are unique by the implicit function theorem. A similar argument proves uniqueness of J_ϵ-holomorphic cylinders: if $\epsilon_k \to 0$ and u_k is a J_{ϵ_k}-holomorphic sequence, then first by the uniqueness of the orbits, we can extract a subsequence for which all u_k are asymptotic at both ends to orbits in fixed families $x^{\epsilon_k}_\pm$ converging to $x^0_\pm(t) = (x_\pm, t)$ as $k \to \infty$. Since Ω is exact, Stokes' theorem then gives a uniform bound on the energies $E_{\epsilon_k}(u_k)$. Since all R^δ_0-orbits in the relevant homotopy class are nondegenerate and none are contractible, one can now prove as in Proposition 10.19 that u_k has a subsequence convergent to a finite-energy stable J_0-holomorphic building $\mathbf{u}_\infty$ consisting only of cylinders. Its levels are asymptotic to orbits of the form $x(t) = (x,t)$ for $x \in \mathrm{Crit}(H)$, thus they can be parametrized as $(s,t) \mapsto (s, v(s,t), t)$ for $v : \mathbb{R} \times S^1 \to V$ satisfying the H^δ-Floer equation, hence $v(s,t) = \gamma(\delta s)$ by Theorem 10.30. Now since ∇H is Morse-Smale and indices of critical points can only differ by at most 1, the building $\mathbf{u}_\infty$ can have at most one nontrivial level $u_\infty(s,t) = (s, \gamma(\delta s), t)$, implying $u_k \to u_\infty$. Since u_∞ is Fredholm regular, the implicit function theorem does the rest. □

10.3.3. Admissible data for $(\mathbb{T}^3, \xi_k)$. We now complete the computation of the cylindrical contact homology $HC^h_*(\mathbb{T}^3, \xi_k)$. We can assume via Lemma 10.26 that h is the homotopy class of the orbits in the special set of tori

$$T_m = \{m/k\} \times \mathbb{T}^2 \subset \mathbb{T}^3, \qquad m = 0, \dots, k-1.$$

Let's focus for now on the case $k = 1$, as the general case will simply be a k-fold cover of this. Thanks to the Morse-Bott discussion in §10.3.1, we know what we're looking for: we want an h-admissible contact form α for $(\mathbb{T}^3, \xi_1)$ such that $\mathcal{P}_h(\alpha)$ contains exactly two orbits, both in $T_0 \subset \mathbb{T}^3$, along with an h-regular $J \in \mathcal{J}(\alpha)$ such that the differential on $CC^h_*(\mathbb{T}^3, \alpha)$ counts exactly two J-holomorphic cylinders that connect the two orbits in T_0. Let $\mathbb{A}$ denote the annulus

$$\mathbb{A} = [-1, 1] \times S^1$$

with coordinates (ρ, ϕ). This will play the role of the Liouville manifold $(V, d\beta)$ from the previous section, and we set

$$\beta := \rho \, d\phi.$$

For the Hamiltonian $H : \mathbb{A} \to \mathbb{R}$, choose a Morse function with the following properties:

(1) H has a minimum at $x_0 = (0,0)$, an index 1 critical point at $x_1 = (0, 1/2)$, and no other critical points;
(2) $H(\rho, \phi) = |\rho|$ for $1/2 \leq |\rho| \leq 1$;

(3) The gradient flow of H with respect to the standard Euclidean metric on $[-1,1]\times S^1$ is Morse-Smale.

Fix a number $\delta>0$ sufficiently small so that Theorem 10.30 applies for Floer trajectories of $H^\delta := \delta H$ in $\mathbb{A}$, and since it will turn out to be useful in Lemma 10.34 below, assume without loss of generality

$$\delta\in\mathbb{Q}.$$

Then following the prescription described above, we consider the family of stable Hamiltonian structures $\mathcal{H}^\delta_\epsilon = (\Omega^\delta, \lambda^\delta_\epsilon)$ on $\mathbb{A}\times S^1$ for $\epsilon\ge 0$ small, where

$$\lambda^\delta_\epsilon = (1-\epsilon\delta H)\,d\theta + \epsilon\rho\,d\phi, \qquad \Omega^\delta = d\rho\wedge d\phi + \delta\,d\theta\wedge dH,$$

with induced Reeb vector fields R^δ_ϵ and hyperplane distributions $\Xi^\delta_\epsilon := \ker\lambda^\delta_\epsilon$. Choose $J_\epsilon\in\mathcal{J}(\mathcal{H}^\delta_\epsilon)$ to be any smooth family such that $J_0|_{\Xi^\delta_0}$ matches the standard complex structure on $\mathbb{A}$ defined by $J_0\partial_\rho = \partial_\phi$. Then for all $\epsilon>0$ sufficiently small, Theorems 10.30 and 10.33 give a complete classification of all closed R^δ_ϵ-orbits in $\mathbb{A}\times S^1$ homotopic to $t\mapsto(0,0,t)$, as well as a classification of all J_ϵ-holomorphic cylinders asymptotic to them. Up to parametrization, there are exactly two such orbits,

$$\gamma^\epsilon_i : S^1\to\mathbb{A}\times S^1, \qquad i=0,1,$$

which correspond to the Morse critical points x_0 and x_1 and thus by (10.8) have Conley-Zehnder indices

$$\mu^\tau_{\mathrm{CZ}}(\gamma^\epsilon_i) = 1-\operatorname{ind}(x_i) = 1-i\in\{0,1\}$$

relative to the constant trivialization τ. There are also exactly two J_ϵ-holomorphic cylinders

$$u^\epsilon_\pm : \mathbb{R}\times S^1\to\mathbb{R}\times(\mathbb{A}\times S^1),$$

corresponding to the two negative gradient flow lines that descend from x_1 to x_0, thus the $u^\epsilon_\pm$ are index 1 curves with a negative end approaching γ^ϵ_1 and a positive end approaching γ^ϵ_0. If we can suitably embed this model into $(\mathbb{T}^3,\xi_1)$ and show that all the orbits and curves needing to be counted are contained in the model, then we will have a complete description of $HC^h_*(\mathbb{T}^3,\xi_1)$, with two generators $\langle\gamma^\epsilon_0\rangle$ and $\langle\gamma^\epsilon_1\rangle$, of even and odd degree respectively, satisfying

$$\partial\langle\gamma^\epsilon_0\rangle = 2\langle\gamma^\epsilon_1\rangle = 0 \quad\text{ and }\quad \partial\langle\gamma^\epsilon_1\rangle = 0$$

since the former counts two curves and the latter counts none.

LEMMA 10.34. *For any $\epsilon>0$ sufficiently small, there exists a contact embedding of*

$$(\mathbb{A}\times S^1, \ker\lambda^\delta_\epsilon)\hookrightarrow(\mathbb{T}^3,\xi_1)$$

identifying the homotopy class of the loops $t\mapsto(0,0,t)$ in $\mathbb{A}\times S^1$ with h. Moreover, the contact form λ^δ_ϵ and almost complex structure $J_\epsilon\in\mathcal{J}(\mathcal{H}^\delta_\epsilon)$ can then be extended to an h-admissible contact form α on $(\mathbb{T}^3,\xi_1)$ and an h-regular almost complex structure $J\in\mathcal{J}(\alpha)$ such that γ^ϵ_0 and γ^ϵ_1 are the only orbits in $\mathcal{P}_h(\alpha)$, and all J-holomorphic cylinders with a positive and a negative end asymptotic to either of these orbits are contained in the interior of $\mathbb{A}\times S^1$.

PROOF. We've chosen β and H so that in the region $1/2 \le |\rho| \le 1$,

$$\alpha := \lambda_\epsilon^\delta = (1 - \epsilon\delta|\rho|)\, d\theta + \epsilon\rho\, d\phi =: f(\rho)\, d\theta + g(\rho)\, d\phi,$$

so the Reeb vector field on this region has the form $\frac{1}{D(\rho)}(g'(\rho)\,\partial_\theta - f'(\rho)\,\partial_\phi)$. Notice that

$$\frac{f'(\rho)}{g'(\rho)} = \mp\frac{\epsilon\delta}{\epsilon} = \mp\delta,$$

and we assumed $\delta \in \mathbb{Q}$, so the Reeb orbits in this region are all periodic. Next, pick a large number $N \gg 1$ and extend α to a contact form on $[-N, N] \times S^1 \times S^1$ via the same formula. Now extend the path $(f, g) : [-N, N] \to \mathbb{R}^2$ to $\mathbb{R}$ such that it has period $2N + 2$ and winds once around the origin over the interval $[-N - 1, N + 1]$, with positive angular velocity. This produces a contact form α on

$$\mathbb{T}^3_N := \left(\mathbb{R}\big/(2N + 2)\mathbb{Z}\right) \times S^1 \times S^1$$

which takes the form $f(\rho)\, d\theta + g(\rho)\, d\phi$ outside of $|\rho| \le 1/2$. We claim in fact that α is homotopic through contact forms to one that takes this form globally, where (f, g) may be assumed to be a smooth loop winding once around the origin. To see this, one need only homotop H in the region $|\rho| \le 1/2$ to a Morse-Bott function that depends only on the ρ-coordinate; the contact condition holds for all Hamiltonians in this homotopy as long as $\epsilon > 0$ is sufficiently small. With this understood, the obvious diffeomorphism

$$\mathbb{T}^3_N \to \mathbb{T}^3 : (\rho, \phi, \theta) \mapsto \left(\frac{\rho}{2N + 2}, \phi, \theta\right)$$

pushes $\ker\alpha$ forward to a contact structure isotopic to one of the form $F(\rho)\, d\theta + G(\rho)\, d\phi$ for a loop $(F, G) : S^1 \to \mathbb{R}^2$ winding once around the origin, so taking a homotopy of this loop to $(\cos(2\pi\rho), \sin(2\pi\rho))$ and applying Gray's stability theorem produces a contactomorphism

$$(\mathbb{T}^3_N, \ker\alpha) \to (\mathbb{T}^3, \xi_1)$$

that is isotopic to the above diffeomorphism.

The construction clearly guarantees that no closed Reeb orbit of α outside $\mathbb{A} \times S^1$ is homotopic to the preferred class h, and there are also no contractible orbits, so α is an h-admissible contact form on $\mathbb{T}^3_N$. Choose any extension of J_ϵ to some $J \in \mathcal{J}(\alpha)$ on $\mathbb{T}^3_N$. We claim now that if N is chosen sufficiently large, then no J-holomorphic cylinder in $\mathbb{R} \times \mathbb{T}^3_N$ with one positive end at either of the orbits γ_i^ϵ can ever venture outside the region $\mathbb{R} \times (-1/2, 1/2) \times \mathbb{T}^2$. Suppose in particular that u is such a curve and its image intersects $\mathbb{R} \times \{1/2\} \times \mathbb{T}^2$. Since the entire region $[1/2, N] \times \mathbb{T}^2$ is foliated by closed Reeb orbits, we can define Υ to be the set of Reeb orbits γ in that region for which the image of u intersects $\mathbb{R} \times \gamma$. This is a closed subset of the connected topological space of all Reeb orbits in $[1/2, N] \times \mathbb{T}^2$: indeed, if $\gamma_k \in \Upsilon$ is a sequence converging to some orbit γ_∞, then $u(z_k) \in \mathbb{R} \times \gamma_k$ for some sequence $z_k \in \mathbb{R} \times S^1$, which must be contained in a compact subset since the asymptotic orbits of u lie outside of $[1/2, N] \times \mathbb{T}^2$, hence z_k has a convergent subsequence $z_k \to z_\infty \in \mathbb{R} \times S^1$ with $u(z_\infty) \in \mathbb{R} \times \gamma_\infty$, proving $\gamma_\infty \in \Upsilon$. We claim that Υ is also an open subset of the space of orbits in $[1/2, N] \times \mathbb{T}^2$. This follows from positivity of intersections, as every $\mathbb{R} \times \gamma$ is also a J-holomorphic curve: if $u(z) \in \mathbb{R} \times \gamma$, then for every other closed orbit γ' close enough to γ, there is a point $z' \in \mathbb{R} \times S^1$ near z with $u(z') \in \mathbb{R} \times \gamma'$. This proves that, in fact, u passes through $\mathbb{R} \times \gamma$ for *every* orbit γ in the region $[1/2, N] \times \mathbb{T}^2$. We will now use this to show that if N is sufficiently large, the contact area of u will be larger than is allowed by Stokes' theorem.

Let us write

$$u(s,t) = (r(s,t), \rho(s,t), \phi(s,t), \theta(s,t)) \in \mathbb{R} \times \big(\mathbb{R}/(2N+2)\mathbb{Z}\big) \times S^1 \times S^1$$

and choose two points $\rho_1 \in [1/2, 1]$ and $\rho_2 \in [N-1, N]$ which are both regular values of the function $\rho : \mathbb{R} \times S^1 \to \mathbb{R}/(2N+2)\mathbb{Z}$. The intersections of u with the orbits in $[1/2, N] \times \mathbb{T}^2$ imply that the function $\rho(s,t)$ attains every value in $[1/2, N]$, and since the asymptotic limits of u lie outside this region,

$$\mathcal{U} := \rho^{-1}([\rho_1, \rho_2]) \subset \mathbb{R} \times S^1$$

is then a nonempty and compact smooth submanifold with boundary

$$\partial \mathcal{U} = -C_1 \sqcup C_2,$$

where $C_i := \rho^{-1}(\rho_i)$ for $i = 1, 2$. Restricting u to the multicurves C_i then gives a pair of smooth maps

$$w_i : C_i \to \mathbb{T}^2 : (s,t) \mapsto (\phi(s,t), \theta(s,t)), \qquad i = 1, 2,$$

which are homologous to each other. Denote the generators of $H_1(\mathbb{T}^2)$ corresponding to the ϕ- and θ-coordinates by ℓ_ϕ and ℓ_θ respectively, and suppose $[w_i] = m\ell_\phi + n\ell_\theta$ for $m, n \in \mathbb{Z}$. The key observation now is that the restriction of α to each of the tori $\{\rho_i\} \times \mathbb{T}^2$ is a closed 1-form, thus for each $i = 1, 2$, $\int_{C_i} u^*\alpha$ depends only on the homology class $m\ell_\phi + n\ell_\theta \in H_1(\mathbb{T}^2)$ and not any further on the maps w_i. In particular,

$$\int_{C_i} u^*\alpha = f(\rho_i)n + g(\rho_i)m$$

for $i = 1, 2$. We now compute,

$$\begin{aligned} \int_{\mathcal{U}} u^* d\alpha &= \int_{C_2} u^*\alpha - \int_{C_1} u^*\alpha = n[f(\rho_2) - f(\rho_1)] + m[g(\rho_2) - g(\rho_1)] \\ &= n[(1 - \epsilon\delta\rho_2) - (1 - \epsilon\delta\rho_1)] + m[\epsilon\rho_2 - \epsilon\rho_1] \\ &= \epsilon(\rho_2 - \rho_1)(m - n\delta) \end{aligned}$$

This integral has to be positive since $u^*d\alpha \geq 0$ and u is not a trivial cylinder, thus $m - n\delta > 0$. Moreover, δ was assumed rational, so if $\delta = p/q$ for some $p, q \in \mathbb{N}$, we have

$$m - n\delta = \frac{1}{q}(mq - np) \geq \frac{1}{q},$$

implying

$$\int_{\mathbb{R} \times S^1} u^* d\alpha \geq \int_{\mathcal{U}} u^* d\alpha \geq \frac{\epsilon}{q}(\rho_2 - \rho_1) \geq \frac{\epsilon(N-2)}{q}.$$

Having chosen δ (which determines q) and ϵ in advance, we are free to make N as large as we like. But by (10.2), $\int_{\mathbb{R} \times S^1} u^* d\alpha$ cannot be any larger than the period of its positive asymptotic orbit, which does not depend on N. So this gives a contradiction, proving that u cannot touch the region $\{\rho \geq 1/2\}$. The mirror image of this argument shows that u also cannot touch the region $\{\rho \leq -1/2\}$. □

With Lemma 10.34 in hand, the calculation of $HC_*^h(\mathbb{T}^3_N, \alpha, J)$ for sufficiently large N is straightforward: there is one odd generator and one even generator, with a trivial differential, giving

$$HC_*^h(\mathbb{T}^3, \xi_1) \cong \begin{cases} \mathbb{Z}_2 & * = \text{odd}, \\ \mathbb{Z}_2 & * = \text{even}. \end{cases}$$

This calculation can now be extended to $(\mathbb{T}^3, \xi_k)$ by a cheap trick: using the contactomorphism $(\mathbb{T}^3_N, \ker \alpha) \to (\mathbb{T}^3, \xi_1)$, let us identify $\mathbb{T}^3_N$ with $\mathbb{T}^3$ and write $\alpha = F\alpha_1$ for some function $F : \mathbb{T}^3 \to (0, \infty)$. Then the k-fold covering map

$$\Phi_k : \mathbb{T}^3 \to \mathbb{T}^3 : (\rho, \phi, \theta) \mapsto (k\rho, \phi, \theta)$$

maps the homotopy class h to itself and pulls back ξ_1 to ξ_k, so $\Phi_k^*\alpha$ is a contact form for ξ_k. It is also h-admissible: indeed, $\Phi_k^*\alpha$ admits no contractible orbits since they would project down to contractible orbits on $(\mathbb{T}^3, \alpha)$, and every orbit in $\mathcal{P}_h(\Phi_k^*\alpha)$ projects to one in $\mathcal{P}_h(\alpha)$, hence they are all nondegenerate. The almost complex structure $\Phi_k^*J \in \mathcal{J}(\Phi_k^*\alpha)$ then makes the map $\mathrm{Id} \times \Phi_k : (\mathbb{R} \times \mathbb{T}^3, \Phi_k^*J) \to (\mathbb{R} \times \mathbb{T}^3, J)$ holomorphic, so every Φ_k^*J-holomorphic cylinder counted by $HC_*^h(\mathbb{T}^3, \Phi_k^*\alpha, \Phi_k^*J)$ projects to a J-holomorphic cylinder counted by $HC_*^h(\mathbb{T}^3, \alpha, J)$, and conversely, each orbit in $\mathcal{P}_h(\alpha)$ and each J-holomorphic cylinder has exactly k lifts to the cover. The generators of $CC_*^h(\mathbb{T}^3, \Phi_k^*\alpha)$ thus consist of $2k$ orbits, k odd and k even, with $2k$ connecting Φ_k^*J-holomorphic cylinders that cancel each other in pairs, giving a trivial differential. In summary:

$$HC_*^h(\mathbb{T}^3, \xi_k) \cong \begin{cases} \mathbb{Z}_2^k & * = \text{odd}, \\ \mathbb{Z}_2^k & * = \text{even}. \end{cases}$$

LECTURE 11

Coherent orientations

Contents

11.1. Gluing maps and coherence

This lecture will be concerned with orienting the moduli spaces

$$\mathcal{M}(J) := \mathcal{M}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$$

of J-holomorphic curves in a completed symplectic cobordism $\widehat{W}$, in cases where they are smooth. We assume as usual that all Reeb orbits are nondegenerate so that the usual linearized Cauchy-Riemann operators are Fredholm.

For SFT and other Floer-type theories, it is not enough to know that each component of $\mathcal{M}(J)$ is orientable—relations like $\partial^2 = 0$ rely on having certain compatibility conditions between the orientations on different components. The point is that whenever a space of broken curves is meant to be interpreted as the boundary of some other compactified moduli space, we need to make sure that it carries the boundary orientation. This compatibility is what is known as *coherence*, and in order to define it properly, we need to return to the subject of gluing.

Our discussion of gluing in Lecture 10 was fairly simple because it was limited to somewhere injective holomorphic cylinders that could only break along simply covered Reeb orbits. Recall however that more general holomorphic buildings carry a certain amount of extra structure that was not relevant in that simple case. Even in a building $\mathbf{u}$ that has only two nontrivial levels u_- and u_+, the breaking punctures carry *decorations*: i.e. if $\{z^+, z^-\}$ is a breaking pair in $\mathbf{u}$, then the decoration defines an orientation-reversing orthogonal map

$$\delta_{z^+} \xrightarrow{\Phi} \delta_{z^-}$$

between the two “circles at infinity” $\delta_{z^\pm}$ associated to the punctures $z^\pm$ (see §9.3.3). This extra information is uniquely determined if the breaking orbit is simply covered, but at a multiply covered breaking orbit there is ambiguity, and the decoration cannot be deduced from knowledge of u_- and u_+ alone. We therefore need to consider moduli spaces of curves with a bit of extra structure.

For each Reeb orbit γ in M_+ or M_-, choose a point on its image

$$p_\gamma \in \operatorname{im}\gamma \subset M_\pm.$$

For a J-holomorphic curve $u : (\dot{\Sigma} = \Sigma \setminus (\Gamma^+ \cup \Gamma^-), j) \to (\widehat{W}, J)$ with a puncture $z \in \Gamma^\pm$ asymptotic to γ, an **asymptotic marker** is a choice of a ray $\ell \subset T_z\Sigma$ such that

$$\lim_{t\to 0^+} u(c(t)) = (\pm\infty, p_\gamma)$$

for any smooth path $c(t) \in \Sigma$ with $c(0) = z$ and $0 \neq \dot{c}(0) \in \ell$. If γ has covering multiplicity $m \in \mathbb{N}$, then there are exactly m choices of asymptotic markers at z, related to each other by the action on $T_z\Sigma$ by the mth roots of unity. We shall denote

$$\mathcal{M}^{\$}(J) := \mathcal{M}^{\$}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-) := \big\{(\Sigma, j, \Gamma^+, \Gamma^-, \Theta, u, \ell)\big\} \big/ \sim,$$

where $(\Sigma, j, \Gamma^+, \Gamma^-, \Theta, u)$ represents an element of $\mathcal{M}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$, ℓ denotes an assignment of asymptotic markers to every puncture $z \in \Gamma^\pm$, and

$$(\Sigma_0, j_0, \Gamma_0^+, \Gamma_0^-, \Theta_0, u_0, \ell_0) \sim (\Sigma_1, j_1, \Gamma_1^+, \Gamma_1^-, \Theta_1, u_1, \ell_1)$$

means the existence of a biholomorphic map $\psi : (\Sigma_0, j_0) \to (\Sigma_1, j_1)$ which defines an equivalence of $(\Sigma_0, j_0, \Gamma_0^+, \Gamma_0^-, \Theta_0, u_0)$ with $(\Sigma_1, j_1, \Gamma_1^+, \Gamma_1^-, \Theta_1, u_1)$ and satisfies $\psi_*\ell_0 = \ell_1$. There is a natural surjection

$$\mathcal{M}^{\$}(J) \to \mathcal{M}(J)$$

defined by forgetting the markers. We will say that an element $u \in \mathcal{M}^{\$}(J)$ is Fredholm regular whenever its image under the map to $\mathcal{M}(J)$ is regular. Let

$$\mathcal{M}^{\$,\mathrm{reg}}(J) = \mathcal{M}^{\$,\mathrm{reg}}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-) \subset \mathcal{M}^{\$}(J)$$

denote the open subset consisting of Fredholm regular curves with asymptotic markers. Note that components of $\mathcal{M}(J)$ and $\mathcal{M}^{\$}(J)$ consisting of closed curves are identical spaces; components with punctures have the following simple relationship to each other.

PROPOSITION 11.1. *Each component of $\mathcal{M}^{\$,\mathrm{reg}}(J)$ consisting of curves with at least one puncture admits the structure of a smooth manifold, whose dimension on each connected component matches that of $\mathcal{M}^{\mathrm{reg}}(J)$. Moreover, the natural map*

$$\mathcal{M}^{\$,\mathrm{reg}}(J) \to \mathcal{M}^{\mathrm{reg}}(J)$$

is smooth, and the preimage of a curve $u \in \mathcal{M}^{\mathrm{reg}}(J)$ with asymptotic orbits $\{\gamma_z\}_{z\in\Gamma}$ of covering multiplicities $\{\kappa_z\}_{z\in\Gamma}$ contains exactly

$$\frac{\prod_{z\in\Gamma}\kappa_z}{|\operatorname{Aut}(u)|}$$

distinct elements.

PROOF. The smooth structure of $\mathcal{M}^{\$,\mathrm{reg}}(J)$ arises from the same argument we used in Lecture 7 for $\mathcal{M}^{\mathrm{reg}}(J)$, supplemented by the following remarks: first, every nontrivial automorphism $\psi \in \mathrm{Aut}(u)$ for $u \in \mathcal{M}(J)$ acts nontrivially on the asymptotic markers. Indeed, ψ is required to fix each of the punctures and is a biholomorphic map with $\psi^k \equiv \mathrm{Id}$ for some $k \in \mathbb{N}$, thus it takes the form $z \mapsto e^{2\pi i m/k}$ in suitable holomorphic coordinates near each puncture for suitable integers $m, k \in \mathbb{Z}$. If $m = 0$, then unique continuation implies $\psi \equiv \mathrm{Id}$, and otherwise ψ changes the asymptotic marker at every puncture. With this understood, one can define as in §7.4 a local identification of $\mathcal{M}^{\$}(J)$ with $\bar{\partial}_J^{-1}(0)/\mathrm{Aut}(\Sigma, j_0, \Gamma \cup \Theta)$, where $\bar{\partial}_J^{-1}(0)$ includes information about asymptotic markers and is a smooth manifold by the implicit function theorem, but $\mathrm{Aut}(\Sigma, j_0, \Gamma \cup \Theta)$ acts on it *freely*, producing a quotient with no isotropy.

Finally, if $(\Sigma, j, \Gamma \cup \Theta, u)$ represents an element of $\mathcal{M}(J)$ with asymptotic orbits $\{\gamma_z\}_{z\in\Gamma}$, then the number of possible choices of asymptotic markers is precisely $\prod_{z\in\Gamma} \kappa_z$. However, not all of these produce inequivalent elements of $\mathcal{M}^{\$}(J)$: indeed, the previous paragraph shows that $\mathrm{Aut}(u)$ acts freely on the set of all choices of markers, so that the total number of inequivalent choices is as stated. □

Suppose u_+ and u_- are two (possibly disconnected and/or nodal) holomorphic curves, with asymptotic markers, such that the number of negative punctures of u_+ equals the number of positive punctures of u_-, and the asymptotic orbit of u_+ at its ith negative puncture matches that of u_- at its ith positive puncture for every i. Then the pair (u_-, u_+) naturally determines a holomorphic building: indeed, the breaking punctures admit unique decorations determined by identifying the markers on u_+ with the markers at corresponding punctures of u_-.

Let us now consider a concrete example of a gluing scenario. Figure 11.1 shows the degeneration of a sequence of curves in $\mathcal{M}_{3,4}(J, A_k, (\gamma_4, \gamma_5), \boldsymbol{\gamma}^-)$ to a building $\mathbf{u} \in \overline{\mathcal{M}}_{3,4}(J, A + B + C, (\gamma_4, \gamma_5), \boldsymbol{\gamma}^-)$ with one main level and one upper level. The main level is a connected curve $u_A \in \mathcal{M}_{1,2}(J, A, (\gamma_1, \gamma_2, \gamma_3), \boldsymbol{\gamma}^-)$, and the upper level consists of two connected curves

$$u_B \in \mathcal{M}_{1,1}(J_+, B, \gamma_4, (\gamma_1, \gamma_2)), \qquad u_C \in \mathcal{M}_{0,1}(J_+, C, \gamma_5, \gamma_3).$$

One can endow each of these curves with asymptotic markers compatible with the decoration of $\mathbf{u}$; this is a non-unique choice, but e.g. if one chooses markers for u_A arbitrarily, then the markers at the negative punctures of u_B and u_C are uniquely determined. Now if all three curves are Fredholm regular, then a substantial generalization of the gluing procedure outlined in Lecture 10 provides open neighborhoods $\mathcal{U}_A^{\$}$ and $\mathcal{U}_{BC}^{\$}$,

$$u_A \in \mathcal{U}_A^{\$} \subset \mathcal{M}_{1,2}^{\$}(J, A, (\gamma_1, \gamma_2, \gamma_3)),$$
$$[(u_B, u_c)] \in \mathcal{U}_{BC}^{\$} \subset \left(\mathcal{M}_{1,1}^{\$}(J_+, B, \gamma_4, (\gamma_1, \gamma_2)) \times \mathcal{M}_{0,1}^{\$}(J_+, C, \gamma_5, \gamma_3)\right) \Big/ \mathbb{R}$$

which are smooth manifolds of dimensions

$$\dim \mathcal{U}_A^{\$} = \text{vir-dim}\, \mathcal{M}_{1,2}(J, A, (\gamma_1, \gamma_2, \gamma_3)),$$
$$\dim \mathcal{U}_{BC}^{\$} = \text{vir-dim}\, \mathcal{M}_{1,1}(J_+, B, \gamma_4, (\gamma_1, \gamma_2)) + \text{vir-dim}\, \mathcal{M}_{0,1}(J_+, C, \gamma_5, \gamma_3) - 1,$$

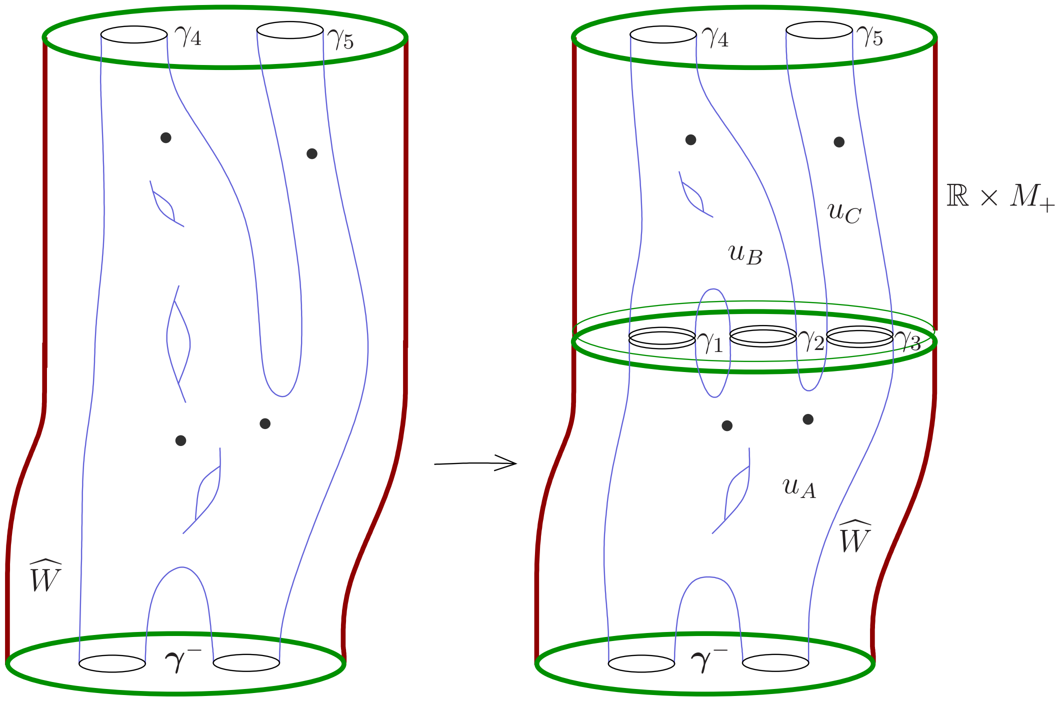

FIGURE 11.1. The degeneration scenario behind the gluing map (11.1)

along with a smooth embedding

$$\Psi : [R_0, \infty) \times \mathcal{U}_A^{\$} \times \mathcal{U}_{BC}^{\$} \hookrightarrow \mathcal{M}_{3,4}^{\$}(J, A + B + C, (\gamma_4, \gamma_5), \boldsymbol{\gamma}^-), \tag{11.1}$$

defined for $R_0 \gg 1$. This is an example of a **gluing map**: it has the property that for any $u \in \mathcal{U}_A^{\$}$ and $v \in \mathcal{U}_{BC}^{\$}$, $\Psi(R, u, v)$ converges in the SFT topology as $R \to \infty$ to the unique building (with asymptotic markers) having main level u and upper level v, and moreover, every sequence of smooth curves degenerating in this way is eventually in the image of Ψ.

In analogous ways one can define gluing maps for buildings with a main level and a lower level, or more than two levels, or multiple levels in a symplectization (always dividing symplectization levels by the $\mathbb{R}$-action). It's important to notice that in all such scenarios, the domain and target of the gluing map have the same dimension, e.g. the dimension of both sides of (11.1) is the sum of the virtual dimensions of the three moduli spaces concerned.

DEFINITION 11.2. A set of orientations for the connected components of $\mathcal{M}^{\$}(J)$ and $\mathcal{M}^{\$}(J_\pm)$ is called **coherent** if all gluing maps are orientation preserving.

Stated in this way, this definition is based on the pretense that we never have to worry about non-regular curves in any components of $\mathcal{M}^{\$}(J)$, and that is of course false—sometimes regularity cannot be achieved, in particular for multiply covered curves. As we'll see though in §11.4, the question of orientations can be reframed in a way that completely disjoins it from the question of regularity, thus we will later be able to state a more general version of the above definition that is independent of regularity (see Definition 11.14). The main result whose proof we will outline in the next few sections is then:

THEOREM 11.3. *Coherent orientations exist.*

But there is also some bad news. The space $\mathcal{M}^{\$}(J)$ with asymptotic markers is not actually the space we want to orient. In fact, even the usual moduli space $\mathcal{M}(J)$ has a certain amount of extra information in it that we'd rather not keep track of when we don't have to, for instance the ordering of the punctures. Can we forget this information without forgetting the orientation of the moduli space? Not always:

PROPOSITION 11.4. *Suppose $\hat{\boldsymbol{\gamma}}^+ = (\gamma_1^+, \dots, \gamma_{k_+}^+)$, and $\check{\boldsymbol{\gamma}}^+$ is a similar ordered list of Reeb orbits obtained from $\hat{\boldsymbol{\gamma}}^+$ by exchanging γ_j^+ with γ_k^+ for some $1 \le j < k \le k_+$. Then for any choice of coherent orientations, the natural map*

$$\mathcal{M}^{\$}_{g,m}(J, A, \hat{\boldsymbol{\gamma}}^+, \boldsymbol{\gamma}^-) \to \mathcal{M}^{\$}_{g,m}(J, A, \check{\boldsymbol{\gamma}}^+, \boldsymbol{\gamma}^-)$$

defined by permuting the corresponding punctures $z_j^+, z_k^+ \in \Gamma^+$ along with their asymptotic markers is orientation reversing if and only if the numbers

$$n - 3 + \mu_{CZ}(\gamma_i^+)$$

for $i = j, k$ are both odd. A similar statement holds for permutations of negative punctures.

This result is the reason for the super-commutative algebra that we will see in the next lecture. What about forgetting the markers? It turns out that we can sometimes do that as well, but again not always.

PROPOSITION 11.5. *Suppose $\mathcal{M}^{\$}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-) \to \mathcal{M}^{\$}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$ is the map defined by multiplying the asymptotic marker by $e^{2\pi i/m}$ at one of the punctures for which the asymptotic orbit is an m-fold cover γ^m of a simple orbit γ. For any choice of coherent orientations, this map reverse orientation if and only if m is even and $\mu_{CZ}(\gamma^m) - \mu_{CZ}(\gamma)$ is odd.*

Note that in both of the above propositions, only the odd/even parity of the Conley-Zehnder indices matters, so there is no need to choose trivializations. Proposition 11.5 motivates one of the more mysterious technical definitions in SFT.

DEFINITION 11.6. A closed nondegenerate Reeb orbit γ is called a **bad** orbit if it is an m-fold cover of some simple orbit γ' where m is even and $\mu_{\mathrm{CZ}}(\gamma) - \mu_{\mathrm{CZ}}(\gamma')$ is odd. Orbits that are not bad are called **good**.

The upshot is that coherent orientations can be defined on the union of all components $\mathcal{M}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$ for which all of the orbits in the lists $\boldsymbol{\gamma}^+$ and $\boldsymbol{\gamma}^-$ are good. This does not mean that moduli spaces involving bad orbits cannot be dealt with—in fact, such moduli spaces have the convenient property that the number of distinct choices of asymptotic markers is always even, and every such choice can be cancelled by an alternative choice that induces the opposite orientation. For this reason, while bad orbits certainly can appear in breaking of holomorphic curves, we will see that they do not need to serve as generators of SFT.

11.2. Permutations of punctures and bad orbits

Before addressing the actual construction of coherent orientations, we can already give heuristic proofs of Propositions 11.4 and 11.5. They are not fully rigorous because they are based on the same pretense as Definition 11.2, namely that all curves we ever have to worry about (including multiple covers) are regular. But we will be able to turn these into precise arguments in §11.7, after discussing the determinant line bundle.

HEURISTIC PROOF OF PROPOSITION 11.4. To simplify the notation, suppose $\hat{\boldsymbol{\gamma}}^+$ consists of only two orbits, so $\hat{\boldsymbol{\gamma}}^+ = (\gamma_1, \gamma_2)$ and $\check{\boldsymbol{\gamma}}^+ = (\gamma_2, \gamma_1)$. Consider the gluing scenario shown in Figure 11.2, where $u \in \mathcal{M}^{\$}_{g,m}(J, A, (\gamma_1, \gamma_2), \boldsymbol{\gamma}^-)$ needs to be glued to a disjoint union of two planes

$$u_B \in \mathcal{M}^{\$}_{0,0}(J_+, B, \emptyset, \gamma_1), \qquad u_C \in \mathcal{M}^{\$}_{0,0}(J_+, C, \emptyset, \gamma_2).$$

You might object that there's no guarantee that such planes must exist in $\mathbb{R} \times M_+$, e.g. the orbits γ_1 and γ_2 might not even be contractible. This concern is valid so far as it goes, but it misses the point: since we're talking about gluing rather than compactness, we do not need any seriously global information about $\widehat{W}$ and M_+, as the gluing process doesn't depend on anything outside a small neighborhood of the curves we're considering. Thus we are free to change the global structure of M_+ elsewhere so that the planes u_B and u_C will exist.[1] If you still can't imagine how one might do this, try not to worry about it and just think of Figure 11.2 as a thought-experiment: it's a situation that certainly does sometimes happen, so when it does, let's see what it implies about orientations.

Assuming all three curves in the picture are regular, there will be smooth open neighborhoods

$$u \in \mathcal{U}_{12} \subset \mathcal{M}^{\$}_{g,m}(J, A, (\gamma_1, \gamma_2), \boldsymbol{\gamma}^-)$$
$$[(u_B, u_C)] \in \mathcal{U}_{BC} \subset \left(\mathcal{M}^{\$}_{0,0}(J_+, B, \emptyset, \gamma_1) \times \mathcal{M}^{\$}_{0,0}(J_+, C, \emptyset, \gamma_2)\right) \Big/ \mathbb{R}$$

and a gluing map

$$\Psi_{BC} : [R_0, \infty) \times \mathcal{U}_{12} \times \mathcal{U}_{BC} \hookrightarrow \mathcal{M}^{\$}_{g,m}(J, A + B + C, \emptyset, \boldsymbol{\gamma}^-),$$

which must be orientation preserving by assumption. But reversing the order of the product $\mathcal{M}^{\$}_{0,0}(J_+, B, \emptyset, \gamma_1) \times \mathcal{M}^{\$}_{0,0}(J_+, C, \emptyset, \gamma_2)$ and letting $u' \in \mathcal{M}^{\$}_{g,m}(J, A, (\gamma_2, \gamma_1), \boldsymbol{\gamma}^-)$ denote the image of u under the map that switches the order of its positive punctures, there are also smooth open neighborhoods

$$u' \in \mathcal{U}_{21} \subset \mathcal{M}^{\$}_{g,m}(J, A, (\gamma_2, \gamma_1), \boldsymbol{\gamma}^-)$$
$$[(u_C, u_B)] \in \mathcal{U}_{CB} \subset \left(\mathcal{M}^{\$}_{0,0}(J_+, C, \emptyset, \gamma_2) \times \mathcal{M}^{\$}_{0,0}(J_+, B, \emptyset, \gamma_1)\right) \Big/ \mathbb{R}$$

and a gluing map

$$\Psi_{CB} : [R_0, \infty) \times \mathcal{U}_{21} \times \mathcal{U}_{CB} \hookrightarrow \mathcal{M}^{\$}_{g,m}(J, A + B + C, \emptyset, \boldsymbol{\gamma}^-).$$

[1] Of course by the maximum principle, planes with only negative ends will not exist in $\mathbb{R} \times M_+$ if this is the symplectization of a contact manifold. But we could also change the contact data to a stable Hamiltonian structure for which such planes are allowed.

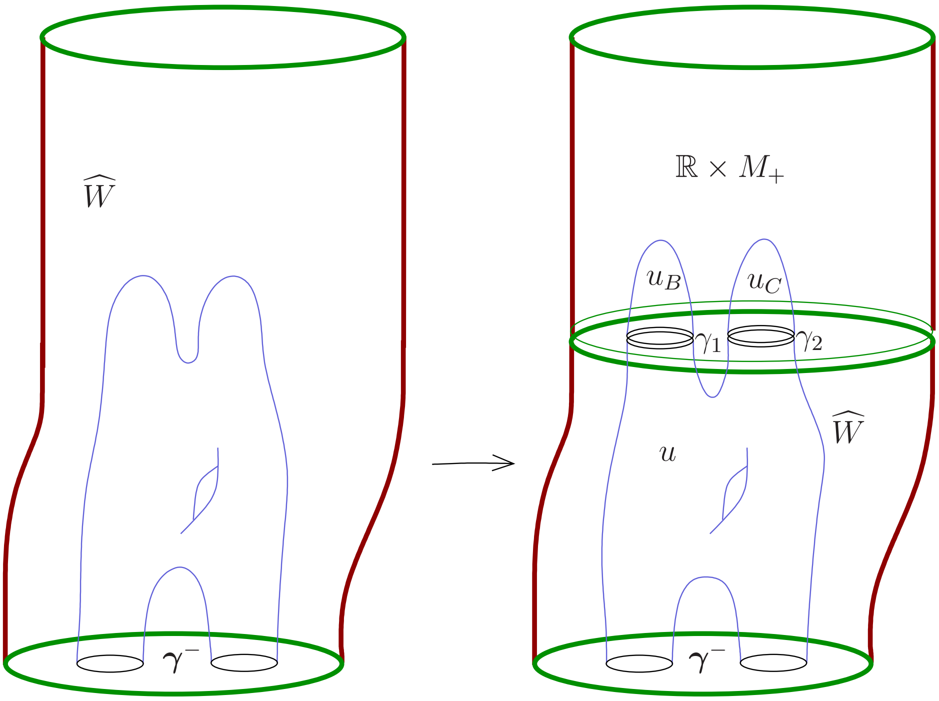

FIGURE 11.2. The gluing thought-experiment used for proving Propositions 11.4 and 11.5.

If both of these gluing maps preserve orientation, then the effect on orientations of the map from $\mathcal{M}^\$_{g,m}(J, A, (\gamma_1, \gamma_2), \boldsymbol{\gamma}^-)$ to $\mathcal{M}^\$_{g,m}(J, A, (\gamma_2, \gamma_1), \boldsymbol{\gamma}^-)$ defined by interchanging the positive punctures must be the same as that of the map

$$\mathcal{M}^\$_{0,0}(J_+, B, \emptyset, \gamma_1) \times \mathcal{M}^\$_{0,0}(J_+, C, \emptyset, \gamma_2) \to \mathcal{M}^\$_{0,0}(J_+, C, \emptyset, \gamma_2) \times \mathcal{M}^\$_{0,0}(J_+, B, \emptyset, \gamma_1)$$
$$(u_B, u_C) \mapsto (u_C, u_B).$$

The latter is orientation reversing if and only if both moduli spaces of planes are odd dimensional, which means $n - 3 + \mu_{\mathrm{CZ}}(\gamma_i)$ is odd for $i = 1, 2$. □

HEURISTIC PROOF OF PROPOSITION 11.5. Let us reuse the thought-experiment of Figure 11.2, but with different details in focus. Suppose γ_1 in the picture is an m-fold covered orbit γ^m, where γ is simply covered, and suppose that u_B is also an m-fold cover, taking the form

$$u_B(z) = v(z^m)$$

for a somewhere injective plane $v \in \mathcal{M}_{0,0}(J_+, B_0, \emptyset, \gamma)$. We're going to assume again that all curves in the discussion are regular, including the multiple cover u_B; while this doesn't sound very plausible, we will see once the determinant line bundle enters the picture in §11.4 that it is an irrelevant detail. Now, u_B has a cyclic automorphism group

$$\mathrm{Aut}(u_B) = \mathbb{Z}_m \subset \mathrm{U}(1)$$

which acts freely on the set of m choices of asymptotic marker for u_B. Then if we act with the same element of $\mathbb{Z}_m$ on u_B and on the corresponding asymptotic marker for u, the building is unchanged, as it has the same decoration. Coherence therefore implies that the effect on orientations of the map from $\mathcal{M}^{\$}_{g,m}(J, A, (\gamma_1, \gamma_2), \boldsymbol{\gamma}^-)$ to itself defined by acting with the canonical generator of $\mathbb{Z}_m \subset \mathrm{U}(1)$ on the marker at γ_1 is the same as the effect of the map $\mathcal{M}^{\$}_{0,0}(J_+, mB_0, \emptyset, \gamma^m) \to \mathcal{M}^{\$}_{0,0}(J_+, mB_0, \emptyset, \gamma^m)$ defined by composing $u_B : \mathbb{C} \to \mathbb{R} \times M_+$ with $\psi(z) := e^{2\pi i/m} z$.

The derivative of this map from $\mathcal{M}^{\$}_{0,0}(J_+, mB_0, \emptyset, \gamma^m)$ to itself at u_B defines a linear self-map

$$\Psi : T_{u_B}\mathcal{M}_{0,0}(J_+, mB_0, \emptyset, \gamma^m) \to T_{u_B}\mathcal{M}_{0,0}(J_+, mB_0, \emptyset, \gamma^m)$$

with $\Psi^m = \mathbb{1}$. The latter implies that Ψ cannot reverse orientation if m is odd. If m is even, observe that the representation theory of $\mathbb{Z}_m$ gives a decomposition

$$T_{u_B}\mathcal{M}_{0,0}(J_+, mB_0, \emptyset, \gamma^m) = V_1 \oplus V_{-1} \oplus V_{\mathrm{rot}},$$

where Ψ acts on $V_{\pm 1}$ as $\pm\mathbb{1}$, and V_{rot} is a direct sum of real 2-dimensional subspaces on which Ψ acts by rotations (and therefore preserves orientations). Thus Ψ reverses the orientation of $T_{u_B}\mathcal{M}_{0,0}(J_+, mB_0, \emptyset, \gamma^m)$ if and only if $\dim V_{-1}$ is odd. As we will review in the next section, $T_{u_B}\mathcal{M}_{0,0}(J_+, mB_0, \emptyset, \gamma^m)$ is a space of holomorphic sections of $u_B^* T(\mathbb{R} \times M_+)$ modulo a subspace defined via the linearized automorphisms of $\mathbb{C}$, so V_1 consists of precisely those sections η that satisfy $\eta = \eta \circ \psi$, meaning they are m-fold covers of sections of $v^* T(\mathbb{R} \times M_+)$. This defines a bijective correspondence between V_1 and $T_v\mathcal{M}_{0,0}(J_+, B_0, \emptyset, \gamma)$, so

$$\dim V_{-1} = \dim \mathcal{M}_{0,0}(J_+, mB_0, \emptyset, \gamma^m) - \dim \mathcal{M}_{0,0}(J_+, B_0, \emptyset, \gamma) \qquad (\mathrm{mod}\ 2).$$

The result then comes from plugging in the dimension formulas for these two moduli spaces. □

11.3. Orienting moduli spaces in general

We now discuss concretely what is involved in orienting a moduli space of J-holomorphic curves.

Recall from Lecture 7 that whenever a curve $u_0 : (\dot{\Sigma} = \Sigma \setminus \Gamma, j_0) \to (\widehat{W}, J)$ with marked points $\Theta \subset \dot{\Sigma}$ is Fredholm regular, a neighborhood of u_0 in $\mathcal{M}(J)$ can be identified with

$$\bar{\partial}_J^{-1}(0)/G_0,$$

where $G_0 = \mathrm{Aut}(\Sigma, j_0, \Gamma \cup \Theta)$ and $\bar{\partial}_J$ is the smooth Fredholm section

$$\mathcal{T} \times \mathcal{B}^{k,p,\delta} \to \mathcal{E}^{k-1,p,\delta} : (j, u) \mapsto Tu + J \circ Tu \circ j,$$

defined on the product of a G_0-invariant Teichmüller slice $\mathcal{T}$ through j_0 with a Banach manifold $\mathcal{B}^{k,p,\delta}$ of $W^{k,p}$-smooth maps $\dot{\Sigma} \to \widehat{W}$ satisfying an exponential decay condition at the cylindrical ends. Here G_0 acts on $\bar{\partial}_J^{-1}(0)$ by

$$G_0 \times \bar{\partial}_J^{-1}(0) \to \bar{\partial}_J^{-1}(0) : (\varphi, (j, u)) \mapsto (\varphi^* j, u \circ \varphi). \tag{11.2}$$

Regularity means that the linearization $D\bar{\partial}_J(j_0, u_0) : T_{j_0}\mathcal{T} \oplus T_{u_0}\mathcal{B}^{k,p,\delta} \to \mathcal{E}^{k-1,p,\delta}_{(j_0,u_0)}$ is surjective, and the implicit function theorem then gives a natural identification

$$T_{u_0}\mathcal{M}(J) = \ker D\bar{\partial}_J(j_0, u_0)/\mathfrak{aut}(\Sigma, j_0, \Gamma \cup \Theta),$$

where $\mathfrak{aut}(\Sigma, j_0, \Gamma \cup \Theta)$ denotes the Lie algebra of G_0, which acts on $\ker D\bar{\partial}_J(j_0, u_0)$ by differentiating (11.2).[2] This action actually defines an *inclusion* of $\mathfrak{aut}(\Sigma, j_0, \Gamma \cup \Theta)$ into $\ker D\bar{\partial}_J(j_0, u_0)$ whenever u_0 is not constant, thus we can regard $\mathfrak{aut}(\Sigma, j_0, \Gamma \cup \Theta)$ as a subspace of $\ker D\bar{\partial}_J(j_0, u_0)$.

As outlined in Proposition 11.1, the space $\mathcal{M}^{\$}(J)$ with asymptotic markers admits a similar local description: here one only needs to enhance the structure of the Banach manifold $\mathcal{B}^{k,p,\delta}$ with information about asymptotic markers at each puncture, so the Banach manifold needed to describe $\mathcal{M}^{\$}(J)$ is a finite covering space of $\mathcal{B}^{k,p,\delta}$. The rest of the discussion is identical, except for the fact that when markers are included, G_0 always acts freely on $\bar{\partial}_J^{-1}(0)$.

We now make a useful observation about the spaces $\mathfrak{aut}(\Sigma, j_0, \Gamma \cup \Theta)$ and $T_{j_0}\mathcal{T}$: namely, they both carry natural complex structures and are thus canonically oriented. This follows from the fact that both the automorphism group G_0 and the Teichmüller space $\mathcal{T}(\Sigma, \Gamma \cup \Theta) = \mathcal{J}(\Sigma) / \mathrm{Diff}_0(\Sigma, \Gamma \cup \Theta)$ are naturally complex manifolds. On the linearized level, one way to see it is via the fact—mentioned previously in §6.3—that $\mathfrak{aut}(\Sigma, j_0, \Gamma \cup \Theta)$ and $T_{[j_0]}\mathcal{T}(\Sigma, \Gamma \cup \Theta)$ can be naturally identified with the kernel and cokernel respectively of the natural linear Cauchy-Riemann type operator on (Σ, j_0),

$$\mathbf{D}_{\mathrm{Id}} : W^{k,p}_{\Gamma \cup \Theta}(T\Sigma) \to W^{k-1,p}(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma)), \tag{11.3}$$

which is the linearization at Id of the nonlinear operator that detects holomorphic maps $(\Sigma, j_0) \to (\Sigma, j_0)$. This operator is equivalent to the operator that defines the holomorphic structure of $T\Sigma$, thus it is complex linear. To handle the punctures and marked points, one needs to restrict the nonlinear operator to the space of $W^{k,p}$-smooth maps $\Sigma \to \Sigma$ that fix every point in $\Gamma \cup \Theta$, thus the domain of the linearization becomes the finite-codimensional subspace

$$W^{k,p}_{\Gamma \cup \Theta}(T\Sigma) := \left\{ X \in W^{k,p}(T\Sigma) \;\middle|\; X|_{\Gamma \cup \Theta} = 0 \right\}.$$

This subspace is still complex, thus so is (11.3), and its kernel and cokernel inherit natural complex structures.

The complex structure on $\mathfrak{aut}(\Sigma, j_0, \Gamma \cup \Theta)$ means that defining an orientation on the tangent space $T_{u_0}\mathcal{M}^{\$}(J)$ is equivalent to defining one on $\ker D\bar{\partial}_J(j_0, u_0)$. The latter operator takes the form

$$D\bar{\partial}_J(j_0, u_0) : T_{j_0}\mathcal{T} \oplus T_{u_0}\mathcal{B}^{k,p,\delta} \to \mathcal{E}^{k-1,p,\delta}_{(j_0,u_0)} : (y, \eta) \mapsto J \circ Tu_0 \circ y + \mathbf{D}_{u_0}\eta,$$

where $\mathbf{D}_{u_0} : W^{k,p,\delta}(u_0^* T\widehat{W}) \oplus V_\Gamma \to W^{k-1,p,\delta}(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, u_0^* T\widehat{W}))$ is the usual linearized Cauchy-Riemann operator at u_0, with V_Γ denoting a complex $(\#\Gamma)$-dimensional space of smooth sections that are constant near infinity.. The remarks above and the fact that u_0 is J-holomorphic imply that the first term in this operator,

$$T_{j_0}\mathcal{T} \to \mathcal{E}^{k-1,p,\delta}_{(j_0,u_0)} : y \mapsto J \circ Tu_0 \circ y$$

[2] The presence of $\mathfrak{aut}(\Sigma, j_0, \Gamma \cup \Theta)$ in this discussion is only relevant in the finite set of "non-stable" cases where $\chi(\dot{\Sigma} \setminus \Theta) \geq 0$, since otherwise G_0 is finite and thus $\mathfrak{aut}(\Sigma, j_0, \Gamma \cup \Theta)$ is trivial.

is a complex-linear map. Now if $\mathbf{D}_{u_0}$ happens also to be a complex-linear map, then we are done, because $\ker D\bar{\partial}_J(j_0, u_0)$ will then be a complex vector space and inherit a natural orientation.

In general, $\mathbf{D}_{u_0}$ is not complex linear, though it does have a *complex-linear part*,

$$\mathbf{D}_{u_0}^{\mathbb{C}}\eta := \frac{1}{2}\left(\mathbf{D}_{u_0}\eta - J\mathbf{D}_{u_0}(J\eta)\right),$$

which is also a Cauchy-Riemann type operator. The space of all Cauchy-Riemann type operators on a fixed vector bundle is affine, so one can interpolate from $\mathbf{D}_{u_0}$ to $\mathbf{D}_{u_0}^{\mathbb{C}}$ through a path of Cauchy-Riemann type operators, though they may not all be Fredholm—this depends on the asymptotic operators at the punctures. In the special case however where there are no punctures, one can easily imagine making use of this idea: if $\dot{\Sigma} = \Sigma$ is a closed surface, then the obvious homotopy from $\mathbf{D}_{u_0}$ to its complex-linear part yields a homotopy from $D\bar{\partial}_J(j_0, u_0)$ to its complex-linear part, and if every operator along this homotopy happens to be surjective, then the canonical orientation defined on the kernel of the complex-linear operator determines an orientation on $\ker D\bar{\partial}_J(j_0, u_0)$.

There are two obvious problems with the above discussion:

(1) We have no way to ensure that every operator in the homotopy from $D\bar{\partial}_J(j_0, u_0)$ to its complex-linear part is surjective;
(2) If there are punctures, then we cannot even expect every operator in this homotopy to be Fredholm.

The first problem motivates the desire to define a notion of orientations for a Fredholm operator $\mathbf{T}$ that does not require $\mathbf{T}$ to be surjective but reduces to the usual notion of orienting $\ker \mathbf{T}$ whenever it is. The solution to this problem is the *determinant line bundle*, which we will discuss in the next section. With this object in hand, the above discussion for the case of closed curves can be made rigorous, so that all smooth moduli spaces of closed J-holomorphic curves inherit canonical orientations. One of the advantages of using the determinant line bundle is that the question of orientations becomes entirely disjoined from the question of transversality: if one can orient the determinant line bundle then moduli spaces of regular curves inherit orientations, but orienting the determinant bundle does not require knowing in advance whether the curves are regular.

The second problem is obviously significant because in the punctured case, moduli spaces of J-holomorphic curves sometimes have *odd* real dimension, making it clearly impossible to homotop $D\bar{\partial}_J(j_0, u_0)$ through Fredholm operators to one that is complex linear. The solution in this case will be to define orientations algorithmically via the coherence condition, and we will describe a suitable algorithm for this in §11.6.

11.4. The determinant line bundle

Fix real Banach spaces X and Y and let $\mathrm{Fred}_{\mathbb{R}}(X, Y)$ denote the space of real-linear Fredholm operators, viewed as an open subset of the Banach space $\mathscr{L}_{\mathbb{R}}(X, Y)$ of all bounded linear operators. We'll use the following notation throughout: if V

is an n-dimensional real vector space, then the top-dimensional exterior power of V is denoted by

$$\Lambda^{\max}V := \Lambda^n V.$$

This 1-dimensional real vector space is spanned by any wedge product of the form $v_1 \wedge \ldots \wedge v_n$ where $(v_1, \ldots, v_n)$ is a basis of V. Denoting the dual space of V by V^*, note that there is a canonical isomorphism $(\Lambda^{\max}V)^* = \Lambda^{\max}V^*$. If $\dim V = 0$, then we adopt the convention $\Lambda^{\max}V = \mathbb{R}$.

DEFINITION 11.7. Given $\mathbf{T} \in \mathrm{Fred}_{\mathbb{R}}(X, Y)$, the **determinant line** of $\mathbf{T}$ is the real 1-dimensional vector space

$$\det(\mathbf{T}) = (\Lambda^{\max}\ker\mathbf{T}) \otimes (\Lambda^{\max}\operatorname{coker}\mathbf{T})^*.$$

Our main goal in this section is to prove:

THEOREM 11.8. *There exists a topological vector bundle* $\det(X, Y) \xrightarrow{\pi} \mathrm{Fred}_{\mathbb{R}}(X, Y)$ *of real rank* 1 *such that* $\pi^{-1}(\mathbf{T}) = \det(\mathbf{T})$ *for each* $\mathbf{T} \in \mathrm{Fred}_{\mathbb{R}}(X, Y)$.

Observe that whenever $\mathbf{T} \in \mathrm{Fred}_{\mathbb{R}}(X, Y)$ is surjective, $\det(\mathbf{T}) = \Lambda^{\max}\ker\mathbf{T}$, so an orientation of $\det(\mathbf{T})$ is equivalent to an orientation of $\ker\mathbf{T}$. More generally, an orientation of $\det(\mathbf{T})$ is equivalent to an orientation for $\ker\mathbf{T} \oplus \operatorname{coker}\mathbf{T}$. If $\mathbf{T}$ is an isomorphism, then $\det(\mathbf{T})$ is simply $\mathbb{R}$, so an orientation of $\det(\mathbf{T})$ amounts to a choice of sign ± 1.

To construct local trivializations of $\det(X, Y) \to \mathrm{Fred}_{\mathbb{R}}(X, Y)$, we start with the case where X and Y are both finite dimensional. Note that in this case, every linear map is Fredholm, including the zero map, and its determinant is simply $\Lambda^{\max}X \otimes (\Lambda^{\max}Y)^*$.

LEMMA 11.9. *Suppose X and Y are real vector spaces of finite dimensions n and m respectively. Then for every $\mathbf{T} \in \mathscr{L}_{\mathbb{R}}(X, Y)$, there exists a canonical isomorphism*

$$(\Lambda^{\max}\ker\mathbf{T}) \otimes (\Lambda^{\max}\operatorname{coker}\mathbf{T})^* = (\Lambda^{\max}V) \otimes (\Lambda^{\max}W)^*.$$

PROOF. Suppose $\dim\ker\mathbf{T} = k$ and $\dim\operatorname{coker}\mathbf{T} = \ell$, so $\operatorname{ind}(\mathbf{T}) = k - \ell = n - m$, thus $n - k = m - \ell$. We define a linear map $\Phi : (\Lambda^n X) \otimes (\Lambda^m Y)^* \to \left(\Lambda^k \ker\mathbf{T}\right) \otimes \left(\Lambda^\ell \operatorname{coker}\mathbf{T}\right)^*$ via the following procedure. Fix $\mathbf{x} \in \Lambda^n X$ and $\mathbf{y}^* \in (\Lambda^m Y)^*$ and suppose both are nontrivial. Then for any nontrivial element $\mathbf{k} \in \Lambda^k \ker\mathbf{T}$, there exists a unique element $\mathbf{v} \in \Lambda^{n-k}(X/\ker\mathbf{T})$ such that for any subspace $V \subset X$ complementary to $\ker\mathbf{T}$, the element $\tilde{\mathbf{v}} \in \Lambda^{n-k}V \subset \Lambda^{n-k}X$ obtained from $\mathbf{v}$ by inverting the natural isomorphism $V \to X/\ker\mathbf{T}$ induced by the projection $X \to X/\ker\mathbf{T}$ satisfies

$$\mathbf{k} \wedge \tilde{\mathbf{v}} = \mathbf{x}.$$

The map $\mathbf{T}$ descends to an isomorphism $X/\ker\mathbf{T} \to \operatorname{im}\mathbf{T}$ and thus induces an isomorphism $\Lambda^{n-k}(X/\ker\mathbf{T}) \to \Lambda^{m-\ell}(\operatorname{im}\mathbf{T}) \subset \Lambda^{m-\ell}Y$, which takes $\mathbf{v}$ to a nontrivial element $\mathbf{T}\mathbf{v}$. There is then a unique element $\mathbf{c} \in \Lambda^\ell \operatorname{coker}\mathbf{T} = \Lambda^\ell(Y/\operatorname{im}\mathbf{T})$ such that for any subspace $W \subset Y$ complementary to $\operatorname{im}\mathbf{T}$, the element $\tilde{\mathbf{c}} \in \Lambda^\ell W \subset \Lambda^\ell Y$ obtained from $\mathbf{c}$ by inverting the isomorphism $W \to Y/\operatorname{im}\mathbf{T}$ induced by the projection $Y \to Y/\operatorname{im}\mathbf{T}$ satisfies

$$\mathbf{y}^*(\tilde{\mathbf{c}} \wedge \mathbf{T}\mathbf{v}) = 1.$$

Now define Φ as the unique linear map such that

$$\Phi(\mathbf{x} \otimes \mathbf{y}^*) = \mathbf{k} \otimes \mathbf{c}^*,$$

where $\mathbf{c}^* \in (\Lambda^\ell \operatorname{coker} \mathbf{T})^*$ is defined by $\mathbf{c}^*(\mathbf{c}) = 1$. It is straightforward to check that this definition does not depend on any choices: indeed, if we replace $\mathbf{k}$ by $\lambda\mathbf{k}$ for some $\lambda \in \mathbb{R} \setminus \{0\}$ in the above procedure, then $\mathbf{v}$ is replaced by $\frac{1}{\lambda}\mathbf{v}$, hence $\mathbf{Tv}$ becomes $\frac{1}{\lambda}\mathbf{Tv}$, $\mathbf{c}$ becomes $\lambda\mathbf{c}$ and $\mathbf{c}^*$ therefore becomes $\frac{1}{\lambda}\mathbf{c}^*$, so that $\mathbf{k}\otimes\mathbf{c}^*$ is replaced by

$$(\lambda\mathbf{k}) \otimes \left(\frac{1}{\lambda}\mathbf{c}^*\right) = \mathbf{k} \otimes \mathbf{c}^*.$$

□

To construct local trivializations of $\det(X, Y)$ in the infinite-dimensional case, recall the following construction from Lecture 3. Given $\mathbf{T}_0 \in \operatorname{Fred}_{\mathbb{R}}(X, Y)$, we can write $X = V \oplus K$ and $Y = W \oplus C$ where $K = \ker \mathbf{T}_0$, $C \cong \operatorname{coker} \mathbf{T}_0$, $W = \operatorname{im} \mathbf{T}_0$ and $\mathbf{T}_0|_V : V \to W$ is an isomorphism. We shall use these splittings to write any other operator $\mathbf{T} \in \operatorname{Fred}_{\mathbb{R}}(X, Y)$ as

$$\mathbf{T} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}$$

and let $\mathcal{U} \subset \operatorname{Fred}_{\mathbb{R}}(X, Y)$ denote the open neighborhood of $\mathbf{T}_0$ for which the block $\mathbf{A} : V \to W$ is invertible. This gives rise to a pair of smooth maps

$$\Phi : \mathcal{U} \to \mathscr{L}_{\mathbb{R}}(K, C) : \mathbf{T} \mapsto \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$$

and

$$F : \mathcal{U} \to \mathscr{L}_{\mathbb{R}}(V \oplus K) = \mathscr{L}_{\mathbb{R}}(X) : \mathbf{T} \mapsto \begin{pmatrix} \mathbb{1} & -\mathbf{A}^{-1}\mathbf{B} \\ 0 & \mathbb{1} \end{pmatrix},$$

such that $F(\mathbf{T})$ is always invertible and maps $\{0\} \oplus \ker \Phi(\mathbf{T})$ isomorphically to $\ker \mathbf{T}$. Similarly, there is a smooth map

$$G : \mathcal{U} \to \mathscr{L}_{\mathbb{R}}(W \oplus C) = \mathscr{L}_{\mathbb{R}}(Y) : \mathbf{T} \mapsto \begin{pmatrix} \mathbb{1} & 0 \\ -\mathbf{C}\mathbf{A}^{-1} & \mathbb{1} \end{pmatrix}$$

such that $G(\mathbf{T})$ is always invertible and maps $\operatorname{im} \mathbf{T}$ isomorphically to $W \oplus \operatorname{im} \Phi(\mathbf{T})$, so it descends to an isomorphism of $\operatorname{coker} \mathbf{T}$ to $\operatorname{coker} \Phi(\mathbf{T})$. Given the canonical isomorphism $\det(\Phi(\mathbf{T})) = \Lambda^{\max} K \otimes (\Lambda^{\max} C)^* = \det(\mathbf{T}_0)$ from Lemma 11.9, the resulting smooth families of isomorphisms $\ker \mathbf{T} \to \ker \Phi(\mathbf{T})$ and $\operatorname{coker} \mathbf{T} \to \operatorname{coker} \Phi(\mathbf{T})$ determine a local trivialization

$$\det(X, Y)|_{\mathcal{U}} \to \mathcal{U} \times \det(\mathbf{T}_0).$$

I will leave it as an exercise for the reader to check that the resulting transition maps are continuous.[3]

[3]This detail should not be underestimated, e.g. [MW, §7.4] observes that the local trivializations constructed in [MS04, §A.2] are, unfortunately, not continuously compatible. See [Zin] for further discussion of this point. If you discover that my local trivializations are also not continuously compatible, please let me know.

EXERCISE 11.10. Show that if X and Y are complex Banach spaces, then the restriction of $\det(X, Y)$ to the subspace of complex-linear Fredholm operators $\mathrm{Fred}_{\mathbb{C}}(X, Y) \subset \mathrm{Fred}_{\mathbb{R}}(X, Y)$ admits a canonical orientation compatible with the complex structures of $\ker \mathbf{T}$ and $\operatorname{coker} \mathbf{T}$ for each $\mathbf{T} \in \mathrm{Fred}_{\mathbb{C}}(X, Y)$. Show also that whenever $\mathbf{T} \in \mathrm{Fred}_{\mathbb{C}}(X, Y)$ is an isomorphism, the canonical orientation of $\det(\mathbf{T})$ agrees with the standard orientation of $\mathbb{R}$.

The orientation of $\det(\mathbf{T})$ for $\mathbf{T} \in \mathrm{Fred}_{\mathbb{C}}(X, Y)$ described in Exercise 11.10 is called the **complex orientation**.

11.5. Determinant bundles of moduli spaces

Combining ideas from the previous two sections, let

$$\det(J) \to \mathcal{M}^{\$}(J)$$

denote the topological line bundle that associates to any $u \in \mathcal{M}^{\$}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$ the determinant line of the Fredholm operator

$$\mathbf{D}_u : W^{k,p,\delta}(u^* T\widehat{W}) \oplus V_\Gamma \to W^{k-1,p,\delta}(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, u^* T\widehat{W})).$$

One can construct local trivializations for this bundle using Theorem 11.8 and any choice of local trivializations for the Banach space bundles $T\mathcal{B}^{k,p,\delta}$ and $\mathcal{E}^{k-1,p,\delta}$.

PROPOSITION 11.11. *Any orientation of* $\det(J) \to \mathcal{M}^{\$}(J)$ *canonically determines an orientation of* $\mathcal{M}^{\mathrm{reg}}(J)$.

PROOF. As explained in §11.3, an orientation of $\mathcal{M}^{\mathrm{reg}}(J)$ near a particular curve $u_0 : (\dot{\Sigma}, j_0) \to (\widehat{W}, J)$ is equivalent to a continuously varying choice of orientations for the kernels

$$\ker D\bar{\partial}_J(j, u) \subset T_j\mathcal{T} \oplus T_u\mathcal{B}^{k,p,\delta}$$

for all $(j, u) \in \bar{\partial}_J^{-1}(0)$, where $\mathcal{T}$ is a Teichmüller slice through j_0. The operator $D\bar{\partial}_J(j, u)$ is of the form

$$\mathbf{L}(y, \eta) := J \circ Tu \circ y + \mathbf{D}_u\eta$$

and thus is homotopic through Fredholm operators to

$$\mathbf{L}^0(y, \eta) := \mathbf{D}_u\eta,$$

namely via the homotopy $\mathbf{L}^s(y, \eta) := sJ \circ Tu \circ y + \mathbf{D}_u\eta$ for $s \in [0, 1]$. The kernel and cokernel of $\mathbf{L}^0$ are $T_j\mathcal{T} \oplus \ker \mathbf{D}_u$ and $\operatorname{coker} \mathbf{D}_u$ respectively, and since $T_j\mathcal{T}$ carries a complex structure, the orientation of $\det(\mathbf{D}_u)$ naturally determines an orientation of $\det(\mathbf{L}^0)$. Using the homotopy $\mathbf{L}^s$, this determines orientations of $\det(D\bar{\partial}_J(j, u))$ and thus orientations of $\ker D\bar{\partial}_J(j, u)$ for all (j, u) near (j_0, u_0), and this orientation does not depend on the choice of Teichmüller slice since the operators $\mathbf{D}_u$ also do not. □

From now on, when we speak of an **orientation of** $\mathcal{M}^{\$}(J)$, we will actually mean an orientation of the bundle $\det(J) \to \mathcal{M}^{\$}(J)$. The above proposition implies that this is equivalent to what we want in applications, but one advantage of talking about $\det(J)$ is that there is no need to limit the discussion to curves that are regular, i.e. the notion of an orientation of $\mathcal{M}^{\$}(J)$ now makes sense even though $\mathcal{M}^{\$}(J)$ is not globally a smooth object.

PROPOSITION 11.12. *Suppose all Reeb orbits in $\boldsymbol{\gamma}^{\pm}$ have the property that their asymptotic operators are complex linear. Then $\mathcal{M}^{\$}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$ admits a natural orientation, known as the* ***complex orientation****.*

PROOF. Having complex-linear asymptotic operators implies that the obvious homotopy from each Cauchy-Riemann operator $\mathbf{D}_u$ to its complex-linear part does not change the asymptotic operators and is therefore a homotopy through Fredholm operators. We therefore have a continuously varying homotopy of each of the relevant fibers of $\det(J)$ to the determinant bundle over a family of complex-linear operators, which inherit the complex orientation described in Exercise 11.10. ☐

Proposition 11.12 applies in particular to all moduli spaces of closed J-holomorphic curves, and thus solves the orientation problem in that case.

11.6. An algorithm for coherent orientations

We now briefly describe the construction of coherent orientations due to Bourgeois and Mohnke [BM04]. A slightly different construction is described in [EGH00], though it appears to have minor errors in some details.

Recall from Lecture 4 the notion of an *asymptotically Hermitian* vector bundle (E, J) over a punctured Riemann surface $(\dot{\Sigma}, j)$. Here $(\dot{\Sigma}, j)$ is endowed with the extra structure of fixed cylindrical ends $(\dot{\mathcal{U}}_z, j) \cong (Z_{\pm}, i)$ for each puncture $z \in \Gamma^{\pm}$, which determines a choice of asymptotic markers. Likewise, the bundle E comes with an asymptotic bundle $(E_z, J_z, \omega_z) \to S^1$ associated to each puncture, carrying compatible complex and symplectic structures. We shall now endow E with a bit more structure that is always naturally present in the case $E = u^*T\widehat{W}$: namely, assume each of the asymptotic bundles comes with a splitting

$$(E_z, J_z, \omega_z) = (\mathbb{C} \oplus \widehat{E}_z, i \oplus \hat{J}_z, \omega_0 \oplus \hat{\omega}_z), \tag{11.4}$$

where ω_0 is the standard symplectic structure on the trivial complex line bundle $(\mathbb{C}, i)$ over S^1, and $(\widehat{E}_z, \hat{J}_z, \hat{\omega}_z) \to S^1$ is another Hermitian bundle. Fix a choice $\{\mathbf{A}_z\}_{z\in\Gamma}$ of nondegenerate asymptotic operators on each of the bundles $(\widehat{E}_z, \hat{J}_z, \hat{\omega}_z)$, and define the topological space

$$\mathcal{CR}(E, \{\mathbf{A}_z\}_{z\in\Gamma})$$

to consist of all Cauchy-Riemann type operators on E that are asymptotic at the punctures $z \in \Gamma$ to the asymptotic operators

$$(-i\partial_t) \oplus \mathbf{A}_z : \Gamma(\mathbb{C} \oplus \widehat{E}_z) \to \Gamma(\mathbb{C} \oplus \widehat{E}_z).$$

This is an affine space, so it is contractible, and if $\delta > 0$ is sufficiently small and $V_\Gamma \subset \Gamma(E)$ denotes a complex $(\#\Gamma)$-dimensional space of smooth sections that take constant values in $\mathbb{C} \oplus \{0\} \subset E_z$ near each puncture z, then every $\mathbf{D} \in \mathcal{CR}(E, \{\mathbf{A}_z\}_{z\in\Gamma})$ determines a Fredholm operator

$$\mathbf{D} : W^{k,p,\delta}(E) \oplus V_\Gamma \to W^{k-1,p,\delta}(\overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, E)).$$

It follows that a choice of orientation of the determinant line for any one of these operators determines an orientation for all of them. The point of this construction is that every $u \in \mathcal{M}^{\$}(J)$ determines an operator $\mathbf{D}_u$ belonging to a space of this form.

We now construct a gluing operation for Cauchy-Riemann operators that linearizes the gluing maps described in §11.1. Suppose $(E^i, J^i) \to (\dot{\Sigma}_i = \Sigma_i \setminus \Gamma_i, j_i)$ for $i = 0, 1$ is a pair of asymptotically Hermitian bundles of the same rank, endowed with asymptotic splittings as in (11.4) and asymptotic operators $\{\mathbf{A}_z\}_{z \in \Gamma_i}$, and that there exists a pair of punctures $z_0 \in \Gamma_0^+$ and $z_1 \in \Gamma_1^-$ such that some unitary bundle isomorphism

$$\widehat{E}^1_{z_1} \xrightarrow{\cong} \widehat{E}^0_{z_0}$$

identifies $\mathbf{A}_{z_1}$ with $\mathbf{A}_{z_0}$. Note that such an isomorphism is uniquely determined up to homotopy whenever it exists. For $R > 0$, we can define a family of glued Riemann surfaces

$$(\dot{\Sigma}_R = \Sigma_R \setminus \Gamma_R, j_R)$$

by cutting off the ends $(R, \infty) \times S^1 \subset \dot{\mathcal{U}}_{z_0}$ and $(-\infty, -R) \times S^1 \subset \dot{\mathcal{U}}_{z_1}$ and gluing $\{R\} \times S^1 \subset \dot{\Sigma}_0$ to $\{-R\} \times S^1 \subset \dot{\Sigma}_1$. The glued Riemann surface contains an annulus biholomorphic to $([-R, R] \times S^1, i)$ in place of the infinite cylindrical ends at the punctures z_0 and z_1. The unitary isomorphism $\widehat{E}^1_{z_1} \to \widehat{E}^0_{z_0}$ then determines an isomorphism $E^1_{z_1} \to E^0_{z_0}$ via the splitting (11.4) and hence an asymptotically Hermitian bundle

$$(E^R, J^R) \to (\dot{\Sigma}_R, J_R).$$

Using cutoff functions in the neck $[-R, R] \times S^1$, any Cauchy-Riemann operators $\mathbf{D}_i \in \mathcal{CR}(E^i, \{\mathbf{A}_z\}_{z \in \Gamma_i})$ for $i = 0, 1$ now determine a family of operators

$$\mathbf{D}_R \in \mathcal{CR}(E^R, \{\mathbf{A}_z\}_{z \in \Gamma_R})$$

uniquely up to homotopy. Analogously to the gluing maps in §11.1, one can arrange this construction so that the operators $\mathbf{D}_R$ converge in some sense to the pair $(\mathbf{D}_0, \mathbf{D}_1)$ as $R \to \infty$, which has the following consequence:

Lemma 11.13 ([BM04, Corollary 7]). *For $R > 0$ sufficiently large, there is a natural isomorphism*

$$\det(\mathbf{D}_0) \otimes \det(\mathbf{D}_1) \to \det(\mathbf{D}_R)$$

that is defined up to homotopy. □

Up to some additional direct sums and quotients by finite-dimensional complex vector spaces, this isomorphism should be understood as the linearization of a gluing map between moduli spaces, generalized to a setting in which the holomorphic curves involved need not be regular. To orient $\mathcal{M}^{\$}(J)$ coherently, it now suffices to choose orientations for the operators in $\mathcal{CR}(E, \{\mathbf{A}_z\}_{z \in \Gamma})$ that vary continuously under deformations of j and E and are preserved by the isomorphisms of Lemma 11.13. This motivates the following generalization of Definition 11.2.

Definition 11.14. A system of **coherent orientations** is an assignment to each asymptotically Hermitian bundle $(E, J) \to (\dot{\Sigma}, j)$ with asymptotic splittings as in (11.4) and asymptotic operators $\{\mathbf{A}_z\}_{z \in \Gamma}$ of an orientation for the determinant line of each $\mathbf{D} \in \mathcal{CR}(E, \{\mathbf{A}_z\})$, such that these orientations vary continuously with $\mathbf{D}$ as well as the data j and J, and such that the isomorphisms in Lemma 11.13 are always orientation preserving.

The prescription of [BM04] to construct such systems is now as follows.

(1) For any trivial bundle E over $\dot{\Sigma} = \mathbb{C}$ with ∞ as a negative puncture and any asymptotic operator $\mathbf{A}_\infty$, choose an arbitrary continuous family of orientations for the operators in $\mathcal{CR}(E, \{\mathbf{A}_\infty\})$, subject only to the requirement that these should match the complex orientation whenever $\mathbf{A}_\infty$ is complex linear.

(2) For any trivial bundle E_- over $\dot{\Sigma} = \mathbb{C}$ with ∞ as a positive puncture, any asymptotic operator $\mathbf{A}_\infty$ and any $\mathbf{D}_- \in \mathcal{CR}(E_-, \{\mathbf{A}_\infty\})$, let E_+ denote the trivial bundle over $\mathbb{C}$ with a negative puncture as in step (1), choose any $\mathbf{D}_+ \in \mathcal{CR}(E_+, \{\mathbf{A}_\infty\})$ and construct the resulting family of glued operators

$$\mathbf{D}_R \in \mathcal{CR}(E^R),$$

where the E^R are trivial bundles over S^2. Since S^2 has no punctures, $\mathbf{D}_R$ has a natural complex orientation, so define the orientation of $\mathbf{D}_-$ to be the one that is compatible via Lemma 11.13 with this and the orientation chosen for $\mathbf{D}_+$ in step (1).

(3) For an arbitrary $(E, J) \to (\dot{\Sigma}, j)$, glue positive and negative planes to $\dot{\Sigma}$ to produce a bundle over a closed surface $\widehat{\Sigma}$, and define the orientation of any $\mathbf{D} \in \mathcal{CR}(E, \{\mathbf{A}_z\}_{z\in\Gamma})$ to be compatible via Lemma 11.13 with the choices in steps (1) and (2) and the complex orientation for operators over $\widehat{\Sigma}$.

It should be easy to convince yourself that if we now vary the bundle $(E, J) \to (\dot{\Sigma}, j)$ or the operators on this bundle (but *not* the asymptotic operators!) continuously, the capping procedure described in step (3) above produces a continuous family of Cauchy-Riemann type operators on bundles over closed Riemann surfaces. Since these all carry the complex orientation, the resulting orientations of the original operators vary continuously. It is similarly clear from the construction that any Cauchy-Riemann operator whose asymptotic operators are all complex linear will end up with the complex orientation. Bourgeois and Mohnke use this fact to prove that any system of orientations constructed in this way is compatible with *all* possible linear gluing maps arising from Lemma 11.13. The idea is to reduce it to the complex-linear case by gluing cylinders to the ends of any asymptotically Hermitian bundle so that the asymptotic operators can be changed at will; see [BM04, Proposition 8].

11.7. Permutations and bad orbits revisited

The heuristic proofs in §11.2 can now be made precise in the following way.

Suppose $\mathbf{D} \in \mathcal{CR}(E, \{\mathbf{A}_z\}_{z\in\Gamma})$, and $\mathbf{D}'$ is the same operator after interchanging two of the punctures in Γ. Imagine gluing $(E, J) \to (\dot{\Sigma}, j)$ to trivial bundles E^1 and E^2 over planes in order to cap off the two punctures that are being interchanged, and choose Cauchy-Riemann operators $\mathbf{D}_1$ and $\mathbf{D}_2$ on these planes to form a glued operator on the capped surface. This capping procedure is done one plane at a time, and the order of the two punctures determines which plane is glued first. Compatibility with the isomorphisms of Lemma 11.13 then dictates that the orientations of $\det(\mathbf{D})$ and $\det(\mathbf{D}')$ match if and only if the orientations of $\det(\mathbf{D}_1) \otimes \det(\mathbf{D}_2)$ and $\det(\mathbf{D}_2) \otimes \det(\mathbf{D}_1)$ match. Since orientations of $\det(\mathbf{D}_i)$ for $i = 1, 2$ are equivalent to orientations of $\ker \mathbf{D}_i \oplus \operatorname{coker} \mathbf{D}_i$, reversing the order of the tensor product changes orientations if and only if both of these direct sums are odd dimensional, which means $\operatorname{ind}(\mathbf{D}_1)$ and $\operatorname{ind}(\mathbf{D}_2)$ are both odd. If the bundles have complex rank n and the asymptotic operators are $\mathbf{A}_i$ for $k = 1, 2$, we have

$$\operatorname{ind}(\mathbf{D}_i) = n\chi(\mathbb{C}) \pm \mu_{\mathrm{CZ}}((-i\partial_t \oplus \mathbf{A}_i) \pm \delta) = n - 1 \pm \mu_{\mathrm{CZ}}(\mathbf{A}_i),$$

which matches $n - 3 + \mu_{\mathrm{CZ}}(\mathbf{A}_i)$ modulo 2. This proves Proposition 11.4.

Similarly for Proposition 11.5, we consider the action of the generator $\psi \in \mathbb{Z}^m$ on $\det(\mathbf{D})$ where ψ rotates the cylindrical end by $1/m$ at some puncture where the trivialized asymptotic operator $\mathbf{A}$ is of the form $-i\partial_t - S(mt)$ for a loop of symmetric matrices $S(t)$. Capping off this puncture with a plane carrying a Cauchy-Riemann operator $\mathbf{D}_\infty$, coherence dictates that the same transformation must act the same way on the orientation of $\det(\mathbf{D}_\infty)$. Since $\psi^m = 1$, ψ cannot reverse this orientation if m is odd. To understand the case of m even, note first that we are free to choose $\mathbf{D}_\infty$ so that it is an m-fold cover, meaning it is related to the branched cover $\varphi : \mathbb{C} \to \mathbb{C} : z \mapsto z^m$ by

$$\mathbf{D}_\infty(\eta \circ \varphi) = \varphi^* \widehat{\mathbf{D}}_\infty \eta$$

for some other Cauchy-Riemann operator $\widehat{\mathbf{D}}_\infty$, which is asymptotic to $\hat{\mathbf{A}} := -i\partial_t - S(t)$. Now the group $\mathbb{Z}_m$ generated by ψ acts on $\ker \mathbf{D}_\infty$ and $\operatorname{coker} \mathbf{D}_\infty$, so representation theory tells us

$$\ker \mathbf{D}_\infty = V_1 \oplus V_{-1} \oplus V_{\mathrm{rot}}$$
$$\operatorname{coker} \mathbf{D}_\infty = W_1 \oplus W_{-1} \oplus W_{\mathrm{rot}},$$

where ψ acts on $V_{\pm 1}$ and $W_{\pm 1}$ as $\pm \mathbb{1}$ and acts as orientation-preserving rotations on V_{rot} and W_{rot}. It follows that ψ reverses the orientation of $\ker \mathbf{D}_\infty \oplus \operatorname{coker} \mathbf{D}_\infty$ if and only if $\dim V_{-1} - \dim W_{-1}$ is odd. Now observe that there are natural isomorphisms

$$V_1 = \ker \widehat{\mathbf{D}}_\infty, \qquad W_1 = \operatorname{coker} \widehat{\mathbf{D}}_\infty,$$

hence

$$\dim V_{-1} - \dim W_{-1} = \operatorname{ind}(\mathbf{D}_\infty) - \operatorname{ind}(\widehat{\mathbf{D}}_\infty) \pmod 2.$$

This difference in Fredholm indices is precisely $\mu_{\mathrm{CZ}}(\mathbf{A}) - \mu_{\mathrm{CZ}}(\hat{\mathbf{A}})$ up to a sign, and this completes the proof of Proposition 11.5.

LECTURE 12

The generating function of SFT

Contents

It is time to begin deriving algebraic consequences from the analytical results of the previous lectures. We saw the simplest possible example of this in Lecture 10, where the behavior of holomorphic cylinders in symplectizations of contact manifolds without contractible Reeb orbits led to a rudimentary version of cylindrical contact homology $HC_*(M, \xi)$ with $\mathbb{Z}_2$ coefficients. Unfortunately, the condition on contractible orbits means that this version of $HC_*(M, \xi)$ cannot always be defined, and even when it can, it only counts cylinders—we would only expect it to capture a small fragment of the information contained in more general moduli spaces of holomorphic curves. Extracting information from these general moduli spaces will require enlarging our algebraic notion of what a Floer-type theory can look like.

12.1. Some important caveats on transversality

For this and the next lecture, we fix the following fantastically optimistic assumption:

Assumption 12.1 (science fiction). *One can choose suitably compatible almost complex structures so that all pseudoholomorphic curves are Fredholm regular.*

This assumption held in Lecture 10 for the curves we were interested in, because they were all guaranteed for topological reasons to be somewhere injective. It can also be shown to hold under some very restrictive conditions on Conley-Zehnder indices in dimension three, see [**Nel15**, **Nel13**]. Both of those are very lucky situations, and as we've discussed before, the assumption cannot generally be achieved merely by perturbing J generically—it *must* sometimes fail for curves that are multiply covered, and such curves always exist (see §12.4 for more on this). The only way in reality to ensure something like Assumption 12.1 is to perturb the nonlinear Cauchy-Riemann equation more abstractly, e.g. by replacing $\bar{\partial}_J u = 0$ with an inhomogeneous equation of the form

$$\bar{\partial}_J u = \nu$$

for a generic perturbation ν. This is the standard technique in certain versions of Gromov-Witten theory, see e.g. [**RT95**, **RT97**]. Alternatively, one can allow J to depend generically on points in the domain rather than just points in the target, as in [**MS04**, §7.3]. Both approaches eliminate the initial problem with multiple covers, but they both also run into serious and subtle difficulties concerning the relationship between $\mathcal{M}(J)$ and the strata of its compactification $\overline{\mathcal{M}}(J)$. As observed in [**Sal99**, §5], the possibility of symmetry in strata of $\overline{\mathcal{M}}(J)$ makes it necessary for any sufficiently general abstract perturbation scheme to involve *multivalued* perturbations, and it is important for these perturbations to be "coherent" in a sense analogous to our discussion of orientations in the previous lecture. These notions have not yet all been developed in a sufficiently consistent and general way to give a rigorous definition of SFT, though there has been much progress: this is the main objective of the long-running *polyfold* project by Hofer-Wysocki-Zehnder [**Hof06**]. Recently, a quite different and much more topological approach has been proposed by John Pardon [**Par**].

For most of this lecture we will ignore these subtleties and simply adopt Assumption 12.1 as a convenient fiction, thus pretending that all components of $\mathcal{M}(J)$ are smooth orbifolds of the correct dimension and all gluing maps are smooth. All "theorems" stated under this assumption should be read with the caveat that they are only true in a fictional world in which the assumption holds. Even if it is a fiction, one can get quite far with this point of view: it is still possible not only to deduce the essential structure of what we assume will someday be a rigorously defined polyfold-based SFT, but also to infer the existence of certain contact invariants that have interesting rigorous applications requiring only well-established techniques, e.g. the cobordism obstructions discovered in [**LW11**].

12.2. Auxiliary data, grading and supercommutativity

The goal is to define an invariant of closed $(2n-1)$-dimensional contact manifolds (M, ξ) with closed nondegenerate Reeb orbits as generators and a Floer-type differential counting J-holomorphic curves in the symplectization $(\mathbb{R} \times M, d(e^r \alpha))$. The auxiliary data we choose must obviously therefore include a nondegenerate contact form α and a generic $J \in \mathcal{J}(\alpha)$, for which we shall assume Assumption 12.1 holds. For convenience, we will also assume throughout most of this lecture:

ASSUMPTION 12.2. *$H_1(M)$ is torsion free.*

This is needed mainly in order to be able to define an integer grading, though without this assumption, it is still always possible to define a $\mathbb{Z}_2$-grading—see §12.7.1 for more on what to do when Assumption 12.2 does not hold. We now supplement the auxiliary data (α, J) with the following additional choices:

(1) Coherent orientations as in Lecture 11 for the moduli spaces $\mathcal{M}^{\$}(J)$ with asymptotic markers.
(2) A collection of **reference curves**

$$S^1 \cong C_1, \dots, C_r \subset M$$

whose homology classes form a basis of $H_1(M)$.
(3) A unitary trivialization of ξ along each of the reference curves $C_1, \dots, C_r$, denoted collectively by τ.
(4) A **spanning surface** C_γ for each periodic Reeb orbit γ: this is a smooth map of a compact and oriented surface with boundary into M such that

$$\partial C_\gamma = \sum_i m_i [C_i] - [\gamma]$$

in the sense of singular 2-chains, where $m_i \in \mathbb{Z}$ are the unique coefficients with $[\gamma] = \sum_i m_i [C_i] \in H_1(M)$.

These choices determine the following. To any collections of Reeb orbits $\boldsymbol{\gamma}^\pm = (\gamma_1^\pm, \dots, \gamma_{k_\pm}^\pm)$ and any relative homology class $A \in H_2(M, \bar{\boldsymbol{\gamma}}^+ \cup \bar{\boldsymbol{\gamma}}^-)$ with $\partial A = \sum_i [\gamma_i^+] - \sum_j [\gamma_j^-]$, we can now associate a cycle in absolute homology,

$$A + \sum_i C_{\gamma_i^+} - \sum_j C_{\gamma_j^-} \in H_2(M).$$

Indeed, the boundary of this real 2-chain is a sum of linear combinations of the reference curves C_i, which add up to zero because $\sum_i [\gamma_i^+]$ and $\sum_j [\gamma_j^-]$ are homologous. We shall abuse notation and use this correspondence to associate the absolute homology class

$$[u] \in H_2(M)$$

to any asymptotically cylindrical holomorphic curve u in $\mathbb{R} \times M$. Adapting the previous notation,

$$\mathcal{M}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$$

for $A \in H_2(M)$ will now denote a moduli space of curves whose relative homology classes glue to the chosen capping surfaces to form A.

Secondly, the chosen trivializations τ along the reference curves can be pulled back and extended over every capping surface C_γ, giving trivializations of ξ along every orbit γ uniquely up to homotopy. We shall define

$$\mu_{\mathrm{CZ}}(\gamma) \in \mathbb{Z}$$

from now on to mean the Conley-Zehnder index of γ relative to this trivialization.

EXERCISE 12.3. Show that if $H_1(M)$ has no torsion and $u : \dot{\Sigma} \to \mathbb{R} \times M$ is asymptotically cylindrical, then its relative first Chern number with respect to the trivializations τ described above satisfies

$$c_1^\tau(u^* T(\mathbb{R} \times M)) = c_1([u]),$$

where $c_1([u])$ denotes the evaluation of $c_1(\xi) \in H^2(M)$ on $[u] \in H_2(M)$.

By Exercise 12.3, the index of a curve $u : (\dot{\Sigma} = \Sigma \setminus \Gamma, j) \to (\mathbb{R} \times M, J)$ with $[u] = A \in H_2(M)$ and asymptotic orbits $\{\gamma_z\}_{z\in\Gamma^\pm}$ can now be written as

$$\operatorname{ind}(u) = -\chi(\dot{\Sigma}) + 2c_1(A) + \sum_{z\in\Gamma^+} \mu_{\mathrm{CZ}}(\gamma_z) - \sum_{z\in\Gamma^-} \mu_{\mathrm{CZ}}(\gamma_z). \tag{12.1}$$

In order to keep track of homology classes of holomorphic curves algebraically, we can define our theory to have coefficients in the group ring $\mathbb{Q}[H_2(M)]$, or more generally,

$$R := \mathbb{Q}[H_2(M)/G]$$

for a given subgroup $G \subset H_2(M)$. Elements of R will be written as finite sums

$$\sum_i c_i e^{A_i} \in R, \qquad c_i \in \mathbb{Q},\ A_i \in H_2(M)/G,$$

where the multiplicative structure of the group ring is derived from the additive structure of $H_2(M)/G$ by $e^A e^B := e^{A+B}$. The most common examples of G are $H_2(M)$ and the trivial subgroup, giving $R = \mathbb{Q}$ or $R = \mathbb{Q}[H_2(M)]$ respectively. We will see a geometrically meaningful example in between these two extremes in the next lecture.

Finally, we define certain formal variables which have degrees in $\mathbb{Z}$ or $\mathbb{Z}_{2N}$ for some $N \in \mathbb{N}$, and will serve as generators in our graded algebra. To each closed Reeb orbit γ we associate two variables, q_γ, p_γ, whose integer-valued degrees are

$$|q_\gamma| = n - 3 + \mu_{\mathrm{CZ}}(\gamma), \qquad |p_\gamma| = n - 3 - \mu_{\mathrm{CZ}}(\gamma).$$

To remember these numbers, think of the index of a J-holomorphic plane u positively or negatively asymptotic to γ, with $[u] = 0$.

We also assign an integer grading to the group ring $\mathbb{Q}[H_2(M)]$ such that rational numbers have degree 0 and

$$|e^A| = -2c_1(A), \qquad \text{for} \qquad A \in H_2(M).$$

If $c_1(A) = 0$ for every $A \in G$, in particular if $c_1(\xi) = 0$, then this descends to an integer grading on the ring $R = \mathbb{Q}[H_2(M)/G]$. Otherwise, R inherits a $\mathbb{Z}_{2N}$-grading, where

$$N := \min\left\{ c_1(A) > 0 \;\middle|\; A \in G \right\}.$$

A $\mathbb{Z}_2$-grading is well defined in every case.

The algebra will include one additional formal variable $\hbar$, which is defined to have degree

$$|\hbar| = 2(n-3).$$

The degrees of $\hbar$ and the p_γ and q_γ variables should all be interpreted modulo $2N$ if $c_1(\xi)|_G \neq 0$.

The algebra of SFT uses monomials in the variables p_γ and q_γ respectively to encode sets of positive and negative asymptotic orbits of holomorphic curves, while the group ring $R = \mathbb{Q}[H_2(M)/G]$ is used to keep track of the homology classes of such curves, and powers of $\hbar$ are used to keep track of their genus. More precisely, given $g \ge 0$, $A \in H_2(M)$ and ordered lists of Reeb orbits $\boldsymbol{\gamma}^\pm = (\gamma_1^\pm, \dots, \gamma_{k_\pm}^\pm)$, we encode the moduli space $\mathcal{M}_{g,0}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$ formally via the product

$$e^A \hbar^{g-1} q^{\boldsymbol{\gamma}^-} p^{\boldsymbol{\gamma}^+} := e^A \hbar^{g-1} q_{\gamma_1^-} \dots q_{\gamma_{k_-}^-} p_{\gamma_1^+} \dots p_{\gamma_{k_+}^+}, \tag{12.2}$$

where we are abusing notation by identifying A with its equivalence class in $H_2(M)/G$ if G is nontrivial. Notice that according to the above definitions, this expression has degree

(12.3)

$$\begin{aligned}
|e^A \hbar^{g-1} q^{\boldsymbol{\gamma}^-} p^{\boldsymbol{\gamma}^+}| &= |e^A| + (g-1)|\hbar| + \sum_{i=1}^{k_-} \left[(n-3) + \mu_{\mathrm{CZ}}(\gamma_i^-)\right] \\
&\quad + \sum_{i=1}^{k_+} \left[(n-3) - \mu_{\mathrm{CZ}}(\gamma_i^+)\right] \\
&= -2c_1(A) + (2g - 2 + k_+ + k_-)(n-3) - \sum_{i=1}^{k_+} \mu_{\mathrm{CZ}}(\gamma_i^+) + \sum_{i=1}^{k_-} \mu_{\mathrm{CZ}}(\gamma_i^-) \\
&= -\operatorname{vir-dim} \mathcal{M}_{g,0}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-),
\end{aligned}$$

interpreted modulo $2N$ if $c_1(\xi)|_G \ne 0$. The orientation results in Lecture 11 suggest introducing a *supercommutativity* relation for the variables q_γ and p_γ: defining the graded commutator bracket by

$$[F, G] := FG - (-1)^{|F||G|} GF, \tag{12.4}$$

we define a relation on the set of all monomials of the form $q^{\boldsymbol{\gamma}^-} p^{\boldsymbol{\gamma}^+}$ by setting

$$[q_{\gamma_1}, q_{\gamma_2}] = [p_{\gamma_1}, p_{\gamma_2}] = 0 \tag{12.5}$$

for all pairs of orbits γ_1 and γ_2. As a consequence, permuting the orbits in the lists $\boldsymbol{\gamma}^\pm$ changes the sign of the monomial (12.2) if and only if it changes the orientation of the corresponding moduli space. In particular, any product that includes multiple copies of an odd generator q_γ or p_γ is identified with 0. This accounts for the fact that any rigid moduli space $\mathcal{M}_{g,0}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$ with two copies of γ among its positive or negative asymptotic orbits contains zero curves when counted with the correct signs: every curve is cancelled by a curve that looks identical except for a permutation of two of its punctures.

12.3. The definition of H and commutators

To write down the SFT generating function, let

$$\mathcal{M}^\sigma(J) := \mathcal{M}(J)/\sim$$

denote the space of equivalence classes where two curves are considered equivalent if they have parametrizations that differ only in the ordering of the punctures. This space is in some sense more geometrically natural than $\mathcal{M}(J)$ or $\mathcal{M}^\$(J)$, but due to the orientation results in the previous lecture, less convenient for technical reasons.

Given $u : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$ representing a nonconstant element of $\mathcal{M}^\sigma(J)$ with no marked points, it is natural to define

$$\mathrm{Aut}^\sigma(u) \subset \mathrm{Aut}(\Sigma, j)$$

as the (necessarily finite) group of biholomorphic transformations $\varphi : (\Sigma, j) \to (\Sigma, j)$ satisfying $u = u \circ \varphi$; in particular, elements of $\mathrm{Aut}^\sigma(u)$ are allowed to permute the punctures, so $\mathrm{Aut}^\sigma(u)$ is generally a larger group than the usual $\mathrm{Aut}(u)$. For $k \in \mathbb{Z}$, let

$$\mathcal{M}^\sigma_k(J) \subset \mathcal{M}^\sigma(J)$$

denote the subset consisting of index k curves that have no marked points and whose asymptotic orbits are all *good* (see Definition 11.6 in Lecture 11).

We now define the **SFT generating function** as a formal power series

$$\mathbf{H} = \sum_{u \in \mathcal{M}^\sigma_1(J)/\mathbb{R}} \frac{\epsilon(u)}{|\mathrm{Aut}^\sigma(u)|} \hbar^{g-1} e^A q^{\boldsymbol{\gamma}^-} p^{\boldsymbol{\gamma}^+}, \tag{12.6}$$

where the terms of each monomial are determined by $u \in \mathcal{M}^\sigma_1(J)$ as follows:

- g is the genus of u;
- A is the equivalence class of $[u] \in H_2(M)$ in $H_2(M)/G$;
- $\boldsymbol{\gamma}^\pm = (\gamma_1^\pm, \ldots, \gamma_{k_\pm}^\pm)$ are the asymptotic orbits of u after arbitrarily fixing orderings of its positive and negative punctures;
- $\epsilon(u) \in \{1, -1\}$ is determined by the chosen coherent orientations on $\mathcal{M}^\$(J)$. Specifically, given the chosen ordering of the punctures and an arbitrary choice of asymptotic markers at each puncture, u determines a 1-dimensional connected component of $\mathcal{M}^\$(J)$, and we define $\epsilon(u) = +1$ if and only if the coherent orientation of $\mathcal{M}^\$(J)$ matches its tautological orientation determined by the $\mathbb{R}$-action.

Note that while both $\epsilon(u)$ and the corresponding monomial $q^{\boldsymbol{\gamma}^-} p^{\boldsymbol{\gamma}^+}$ depend on a choice of orderings of the punctures, their product does not depend on this choice. Moreover, $\epsilon(u)$ does not depend on the choice of asymptotic markers since curves with bad asymptotic orbits are excluded from $\mathcal{M}^\sigma_1(J)$. Since every monomial in $\mathbf{H}$ corresponds to a holomorphic curve of index 1, (12.3) implies

$$|\mathbf{H}| = -1.$$

There are various combinatorially more elaborate ways to rewrite $\mathbf{H}$. For any Reeb orbit γ, let

$$\kappa_\gamma := \mathrm{cov}(\gamma) \in \mathbb{N}$$

denote its covering multiplicity, and for a finite list of orbits $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_k)$, let

$$\kappa_{\boldsymbol{\gamma}} := \prod_{i=1}^k \kappa_{\gamma_i}.$$

Given $u \in \mathcal{M}^\sigma(J)$ with $k_\pm \geq 0$ positive/negative punctures asymptotic to the set of orbits $\boldsymbol{\gamma}^\pm = (\gamma^1_\pm, \ldots, \gamma^{k_\pm}_\pm)$, there are $k_+! k_-! \kappa_{\boldsymbol{\gamma}^+} \kappa_{\boldsymbol{\gamma}^-}$ ways to order the punctures and choose asymptotic markers, but some of them are equivalent since (by an easy variation on Proposition 11.1) the finite group $\mathrm{Aut}^\sigma(u)$ acts freely on this set of choices. As a result, (12.6) is the same as

$$\mathbf{H} = \sum_{u \in \mathcal{M}_1^\$(J)/\mathbb{R}} \frac{\epsilon(u)}{k_+! k_-! \kappa_{\gamma^+} \kappa_{\gamma^-}} \hbar^{g-1} e^A q^{\gamma^-} p^{\gamma^+}, \tag{12.7}$$

where $\mathcal{M}_1^\$(J)$ denotes the space of all index 1 curves without marked points in $\mathcal{M}^\$(J)$, and the rest of the mononomial is determined by the condition that u belongs to $\mathcal{M}_{g,0}^\$(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$, with no need for any arbitrary choices. Another way of writing this is

$$\mathbf{H} = \sum_{g,A,\boldsymbol{\gamma}^+,\boldsymbol{\gamma}^-} \frac{\#\left(\mathcal{M}_{g,0}^\$(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)/\mathbb{R}\right)}{k_+! k_-! \kappa_{\gamma^+} \kappa_{\gamma^-}} \hbar^{g-1} e^A q^{\gamma^-} p^{\gamma^+}, \tag{12.8}$$

where the sum ranges over all integers $g \geq 0$, homology classes $A \in H_2(M)$ and ordered tuples of Reeb orbits $\boldsymbol{\gamma}^\pm = (\gamma_1^\pm, \ldots, \gamma_{k_\pm}^\pm)$, and $\#\left(\mathcal{M}_{g,0}^\$(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)/\mathbb{R}\right) \in \mathbb{Z}$ is the signed count of index 1 connected components in $\mathcal{M}_{g,0}^\$(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$. For fixed g and $\boldsymbol{\gamma}^\pm$, the union of these spaces for all $A \in H_2(M)$ is finite due to SFT compactness, as the energy of curves in $(\mathbb{R} \times M, d(e^t\alpha))$ is computed by integrating exact symplectic forms and thus (by Stokes) admits a uniform upper bound in terms of $\boldsymbol{\gamma}^+$. For this reason, (12.8) defines a formal power series in the p variables and in $\hbar$, with coefficients that are *polynomials* in the q variables and the group ring R.

We played a slightly sneaky trick in writing down (12.7) and (12.8): these summations to not exclude bad orbits, whereas (12.6) was a sum over curves u that are not asymptotic to any bad orbits—a necessary exclusion in that case because $\epsilon(u)$ would otherwise depend on choices of asymptotic markers. The reason bad orbits are allowed in (12.8) is that their total contribution adds up to zero: indeed, bad orbits are always multiple covers with even multiplicity, so whenever $u \in \mathcal{M}^\$(J)$ has a puncture approaching a bad orbit with multiplicity $2m$, there are exactly $2m - 1$ other elements of $\mathcal{M}^\$(J)$ that differ only by adjustment of the marker at that one puncture, and by Proposition 11.5, half of these cancel out the other half in the signed count. We've already seen that a similar remark explains the harmless absence from (12.8) of terms with multiple factors of any odd generator q_γ or p_γ.

Remark 12.4. Readers famliar with Floer homology may see a resemblance between the group ring $R = \mathbb{Q}[H_2(M)/G]$ and the Novikov rings that often appear in Floer homology, though R is not a Novikov ring since it only allows finite sums. In Floer homology, the Novikov ring sometimes must be included because counts of curves may fail to be finite, though they only do so if the energies of those curves blow up. The situation above is somewhat different: since the symplectization is an exact symplectic manifold, Stokes' theorem implies that energy cannot blow up if the positive asymptotic orbits are fixed, and one therefore obtains well-defined curve counts no matter the choice of the coefficient ring R. The use of the group ring is convenient however for two reasons: first, without it one cannot always define an integer grading, and second, different choices of coefficients can sometimes be used to detect different geometric phenomena via SFT. We will see an example of the latter in Lecture 13.

The compactness and gluing theory of SFT is encoded algebraically by viewing $\mathbf{H}$ as an element on a noncommutative operator algebra determined by the commutator relations

$$\begin{aligned} [p_\gamma, q_\gamma] &= \kappa_\gamma \hbar \\ [p_\gamma, q_{\gamma'}] &= 0 \quad \text{ if } \gamma \neq \gamma'. \end{aligned} \tag{12.9}$$

Here $[\ ,\ ]$ again denotes the graded commutator (12.4), so "commuting" generators actually anticommute whenever they are both odd. The rest of the multiplicative structure of this algebra is determined by requiring all elements of R and powers of $\hbar$ (all of which are even generators) to commute with everything, meaning all operators are $R[[\hbar]]$-linear.

One concrete representation of this operator algebra is as follows: let $\mathcal{A}$ denote the graded supercommutative unital algebra over R generated by the set

$$\left\{ q_\gamma \;\middle|\; \gamma \text{ a good Reeb orbit} \right\}.$$

The ring of formal power series $\mathcal{A}[[\hbar]]$ is then an $R[[\hbar]]$-module. Define each of the generators q_γ to be $R[[\hbar]]$-linear operators on $\mathcal{A}[[\hbar]]$ via multiplication from the left, and define $p_\gamma : \mathcal{A}[[\hbar]] \to \mathcal{A}[[\hbar]]$ by

$$p_\gamma = \kappa_\gamma \hbar \frac{\partial}{\partial q_\gamma}. \tag{12.10}$$

Here the $R[[\hbar]]$-linear partial derivative operator is defined via

$$\frac{\partial}{\partial q_\gamma} q_\gamma = 1, \qquad \frac{\partial}{\partial q_\gamma} q_{\gamma'} = 0 \quad \text{for } \gamma \neq \gamma'$$

and the graded Leibniz rule

$$\frac{\partial}{\partial q_\gamma}(FG) = \frac{\partial F}{\partial q_\gamma} G + (-1)^{|q_\gamma||F|} F \frac{\partial G}{\partial q_\gamma}$$

for all homogeneous elements $F, G \in \mathcal{A}[[\hbar]]$.

EXERCISE 12.5. Check that the operator $p_\gamma : \mathcal{A}[[\hbar]] \to \mathcal{A}[[\hbar]]$ defined above has the correct degree and satisfies the commutation relations (12.5) and (12.9).

Notice that while $\mathbf{H}$ contains terms of order -1 in $\hbar$, every term also contains at least one p_γ variable since all index 1 holomorphic curves in $(\mathbb{R} \times M, d(e^t \alpha))$ have at least one positive puncture. The substitution (12.10) thus produces a differential operator in which every term contains a nonnegative power of $\hbar$, giving a well-defined $R[[\hbar]]$-linear operator

$$\mathbf{D}_{\mathrm{SFT}} : \mathcal{A}[[\hbar]] \xrightarrow{\mathbf{H}} \mathcal{A}[[\hbar]].$$

The following may be regarded as the fundamental theorem of SFT.

THEOREM 12.6. $\mathbf{H}^2 = 0$.

We will discuss in §12.6 how this relation follows from the compactness and gluing theory of punctured holomorphic curves, and we will use it in Lecture 13 to define various Floer-type contact invariants. The first and most obvious of these is the homology

$$H_*^{\mathrm{SFT}}(M,\xi) := H_*(\mathcal{A}[[\hbar]], \mathbf{D}_{\mathrm{SFT}}),$$

which will turn out to be an invariant of (M,ξ) in the sense that any two choices of α, J and the other auxiliary data described in §12.2 gives rise to a functorial isomorphism between the two graded homology groups. Notice that while $\mathcal{A}[[\hbar]]$ is an algebra, its product structure does not descend to $H_*^{\mathrm{SFT}}(M,\xi)$ since $\mathbf{D}_{\mathrm{SFT}}$ is not a derivation—indeed, it is a formal sum of differential operators of all orders, not just order one. In the next lecture we will discuss various ways to produce homological invariants out of $\mathbf{H}$ with nicer algebraic structures.

On the other hand, it is fairly easy to understand the geometric meaning of the complex $(\mathcal{A}[[\hbar]], \mathbf{D}_{\mathrm{SFT}})$ in Floer-theoretic terms. Each individual curve $u \in \mathcal{M}_1^\sigma(J)$ with genus g, homology class $A \in H_2(M)$ and asymptotic orbits $\boldsymbol{\gamma}^\pm = (\gamma_1^\pm, \dots, \gamma_{k_\pm}^\pm)$ contributes to $\mathbf{D}_{\mathrm{SFT}}$ the differential operator

$$\frac{\epsilon(u)}{|\operatorname{Aut}^\sigma(u)|}\kappa_{\boldsymbol{\gamma}^+}\hbar^{g+k_+-1}e^A q_{\gamma_1^-}\dots q_{\gamma_{k_-}^-}\frac{\partial}{\partial q_{\gamma_1^+}}\dots\frac{\partial}{\partial q_{\gamma_{k_+}^+}}.$$

Applying this operator to a monomial $q_{\gamma_1}\dots q_{\gamma_m} \in \mathcal{A}[[\hbar]]$ that does not contain all of the generators $q_{\gamma_1^+},\dots,q_{\gamma_{k_+}^+}$ will produce zero, and its effect on a product that does contain all of these generators will be to eliminate them and multiply $q_{\gamma_1^-}\dots q_{\gamma_{k_-}^-}$ by whatever remains, plus some combinatorial factors and signs that may arise from differentiating by the same q_γ more than once. Ignoring the combinatorics and signs for the moment, this operation on $q_{\gamma_1}\dots q_{\gamma_m}$ has a geometric interpretation: it counts all *potentially disconnected* J-holomorphic curves of index 1 (i.e. disjoint unions of u with trivial cylinders) that have $\gamma_1,\dots,\gamma_m$ as their positive asymptotic orbits; see Figure 12.1. In other words, the action of $\mathbf{D}_{\mathrm{SFT}}$ on each monomial $q^{\boldsymbol{\gamma}}$ for $\boldsymbol{\gamma} = (\gamma_1,\dots,\gamma_m)$ is determined by a formula of the form

$$\mathbf{D}_{\mathrm{SFT}}q^{\boldsymbol{\gamma}} = \sum_{g=0}^{\infty}\sum_{A\in H_2(M)}\sum_{\boldsymbol{\gamma}'}\sum_{k=1}^{m}\hbar^{g+k-1}e^A n_g(\boldsymbol{\gamma},\boldsymbol{\gamma}',k)q^{\boldsymbol{\gamma}'}, \tag{12.11}$$

where $n_g(\boldsymbol{\gamma},\boldsymbol{\gamma}',k)$ is a product of some combinatorial factors with a signed count of generally disconnected index 1 holomorphic curves of genus g and homology class A with positive ends at $\boldsymbol{\gamma}$ and negative ends at $\boldsymbol{\gamma}'$, such that the nontrivial connected component has exactly k positive ends. The presence of the combinatorial factors hidden in $n_g(\boldsymbol{\gamma},\boldsymbol{\gamma}',k)$ is a slightly subtle point which we will try to clarify in the following sections.

12.4. Interlude: How to count points in an orbifold

As in all versions of Floer theory, the proof that $\mathbf{H}^2 = 0$ is based on the fact that certain moduli spaces are compact oriented 1-dimensional manifolds with boundary, and the signed count of their boundary points is therefore zero. We must be

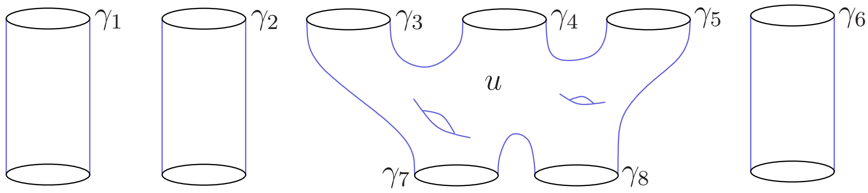

FIGURE 12.1. Counting disjoint unions of index 1 curves $u \in \mathcal{M}_{2,0}(J, A, (\gamma_3, \gamma_4, \gamma_5), (\gamma_7, \gamma_8))$ with some trivial cylinders contributes a multiple of $\hbar^4 e^A q_{\gamma_1} q_{\gamma_2} q_{\gamma_7} q_{\gamma_8} q_{\gamma_6}$ to $\mathbf{D}_{\mathrm{SFT}}(q_{\gamma_1} q_{\gamma_2} q_{\gamma_3} q_{\gamma_4} q_{\gamma_5} q_{\gamma_6})$.

careful of course because, strictly speaking, $\overline{\mathcal{M}}(J)$ is not a manifold even when Assumption 12.1 holds—it is an orbifold, with the possibility of singularities at multiply covered curves with nontrivial automorphism groups. On the other hand, one can show that (after excluding curves with bad asymptotic orbits) it is an *oriented* orbifold, and oriented 1-dimensional orbifolds happen to be very simple objects: since smooth finite group actions on $\mathbb{R}$ cannot be nontrivial without reversing orientation, all oriented 1-dimensional orbifolds are actually manifolds, suggesting the simple formula

$$\text{“}\#\partial\overline{\mathcal{M}}_1(J) = 0.\text{”}$$

I have placed this formula in quotation marks for a reason. The reality of the situation is somewhat more complicated.

This is in fact where it becomes important to remember that Assumption 12.1, in the way that we stated it, really is not just science fiction but *fantasy*: transversality is sometimes impossible to achieve for multiple covers, and we must therefore at least have a sensible back-up plan for such cases. To see the problem, remember that our local structure theorem for $\mathcal{M}(J)$ was proved by identifying it in a neighborhood of any curve $u_0 : (\dot{\Sigma}, j_0) \to (\mathbb{R} \times M, J)$ with a set of the form

$$\bar{\partial}_J^{-1}(0)/G,$$

where $\bar{\partial}_J : \mathcal{T} \times \mathcal{B}^{k,p,\delta} \to \mathcal{E}^{k-1,p,\delta}$ is a smooth section of a Banach space bundle $\mathcal{E}^{k-1,p,\delta}$ over the product of a Teichmüller slice $\mathcal{T}$ through j_0 with a Banach manifold $\mathcal{B}^{k,p,\delta}$ of maps $\dot{\Sigma} \to \mathbb{R} \times M$, and G is the group of automorphisms of j_0, whose action on the base[1]

$$G \times (\mathcal{T} \times \mathcal{B}^{k,p,\delta}) \to \mathcal{T} \times \mathcal{B}^{k,p,\delta} : (\psi, (j, u)) \mapsto (\psi^* j, u \circ \psi)$$

preserves $\bar{\partial}_J^{-1}(0)$. In fact, the action of G on $\mathcal{T} \times \mathcal{B}^{k,p,\delta}$ is covered by a natural action on the bundle $\mathcal{E}^{k-1,p,\delta}$, and the reason for it preserving the zero-set is that $\bar{\partial}_J$ is an equivariant section,

$$\bar{\partial}_J(\psi^* j, u \circ \psi) = \psi^* \bar{\partial}_J(j, u).$$

[1]As you may know if you've ever heard a talk about polyfolds, there are some analytical problems with this discussion if G is a Lie group of positive dimension: its action on the infinite-dimensional manifold $\mathcal{B}^{k,p,\delta}$ of non-smooth maps cannot then be considered smooth in any conventional sense. This problem leads to the introduction of *sc-smooth* structures, cf. [**HWZ07**]. There is no problem however if G is finite, e.g. if the underlying Riemann surface is stable, which we may as well assume for this discussion.

If G is finite, then another way to say this is that $\bar{\partial}_J$ is a smooth Fredholm section of the infinite-dimensional **orbibundle** $\mathcal{E}^{k-1,p,\delta}/G$ over the orbifold $(\mathcal{T} \times \mathcal{B}^{k,p,\delta})/G$, whose isotropy group at (j_0, u_0) is $\mathrm{Aut}(u_0)$. This section is transverse to the zero-section if and only if the usual regularity condition holds, making $\bar{\partial}_J^{-1}(0)/G$ a suborbifold of $(\mathcal{T} \times \mathcal{B}^{k,p,\delta})/G$ whose isotropy group at (j_0, u_0) is some quotient of $\mathrm{Aut}(u_0)$.

REMARK 12.7. Most sensible definitions of the term **orbifold** (cf. [**ALR07**, **Dav**, **FO99**]) require local models of the form $\mathcal{U}/G$, where $\mathcal{U}$ is a G-invariant open subset of a vector space on which the finite group G acts smoothly and *effectively*—the latter condition is necessary in order to have isotropy groups that are well-defined up to isomorphism at every point. In the above example, G acts effectively on $\mathcal{T} \times \mathcal{B}^{k,p,\delta}$ but might have a nontrivial subgroup $H \subset G$ of transformations that fix every element of $\bar{\partial}_J^{-1}(0)$, in which case the G-action on $\bar{\partial}_J^{-1}(0)$ can be replaced by an effective action of G/H. The isotropy group of $(j_0, u_0) \in \bar{\partial}_J^{-1}(0)/G$ is then $\mathrm{Aut}(u_0)/(\mathrm{Aut}(u_0) \cap H)$.

Now to see just how unreasonably optimistic Assumption 12.1 is, notice that it's easy to think up examples of smooth orbibundles in which zeroes of sections can *never* be regular if they have nontrivial isotropy.

EXAMPLE 12.8. Let $M = \mathbb{C}/\mathbb{Z}_2$ with $\mathbb{Z}_2$ acting as the antipodal map, and consider the trivial complex line bundle $E = M \times \mathbb{C} = (\mathbb{C} \times \mathbb{C})/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ action on $\mathbb{C} \times \mathbb{C}$ identifies (z, v) with $(-z, v)$. A smooth function $f : \mathbb{C} \to \mathbb{C}$ then represents a section of the orbibundle $E \to M$ if and only if $f(z) = f(-z)$ for all z. This implies that if $f(0) = 0$, then $df(0) = 0$. It is possible to perturb f generically to a section that is transverse to the zero-section, but such a perturbation can never have zeroes at 0.

Of course, we do know how to assign $\mathbb{Z}$-valued orders to degenerate zeroes of sections, e.g. $f(z) = z^2$ defines a section of $E \to M$ with a zero of order 2 at 0. Notice however that if we perturb this to $f_\epsilon(z) = z^2 + \epsilon$ for $\epsilon > 0$ small, then f_ϵ has two simple zeroes at points near the origin, but they are actually *the same point* in $\mathbb{C}/\mathbb{Z}_2$, giving a count of only 1 zero. This means that if we give the zero of f at the origin its full weight, then we are counting wrongly—the resulting count will not be homotopy invariant. The correct algebraic count of zeroes is evidently

$$\#f^{-1}(0) := \sum_{z \in f^{-1}(0) \subset M} \frac{\mathrm{ord}(f; z)}{\kappa_z} \in \mathbb{Q}, \tag{12.12}$$

where $\mathrm{ord}(f; z) \in \mathbb{Z}$ is the order of the zero (computed in the usual way as a winding number, or in higher dimensions as the degree of a map of spheres, cf. [**Mil97**]), and $\kappa_z \in \mathbb{N}$ denotes the order of the isotropy group at z.

EXERCISE 12.9. Convince yourself that for any smooth oriented orbibundle $E \to M$ of real rank m over a compact, smooth and oriented m-dimensional orbifold M without boundary, the count (12.12) gives the same result for any section with isolated zeroes.[2]
Hint: The space of sections of an orbibundle is still a vector space, so any two

[2] If you're still not sure what an orbibundle is, a definition can be found in [**FO99**, Chapter 1].

are homotopic. Since M and $[0,1]$ are both compact, it suffices to focus on small perturbations of a single section on a single orbifold chart.

For a slightly different perspective on (12.12), consider the special case of a closed orbifold that is the quotient of a closed manifold $\widetilde{M}$ by an effective orientation-preserving finite group action,

$$M = \widetilde{M}/G.$$

Suppose $\widetilde{E} \to \widetilde{M}$ is an oriented vector bundle with rank equal to $\dim M$, and G also acts on $\widetilde{E}$ by orientation-preserving linear bundle maps that cover its action on $\widetilde{M}$, so the quotient

$$E = \widetilde{E}/G \to M$$

is an orbibundle. A section $f : M \to E$ is then equivalent to a G-equivariant section $\tilde{f} : \widetilde{M} \to \widetilde{E}$, and the signed count of zeroes

$$\#\tilde{f}^{-1}(0) = \sum_{z \in \tilde{f}^{-1}(0) \subset \widetilde{M}} \operatorname{ord}(\tilde{f}; z) \in \mathbb{Z}$$

is of course the same for any section that has only isolated zeroes. It can also be expressed in terms of f since any $z \in f^{-1}(0) \subset M$ has exactly $|G|/\kappa_z$ lifts to points in $\tilde{f}^{-1}(0) \subset \widetilde{M}$, implying

$$\#\tilde{f}^{-1}(0) = \sum_{z \in f^{-1}(0) \subset M} \frac{|G|}{\kappa_z} \operatorname{ord}(f; z)$$

and thus $\# f^{-1}(0) = \frac{1}{|G|} \#\tilde{f}^{-1}(0)$. The invariance of (12.12) is now an immediate consequence of the invariance of $\#\tilde{f}^{-1}(0)$, which follows from the standard argument as in [Mil97].

Now, if you enjoyed reading [Mil97] as much as I did, then it may seem tempting to try proving invariance of (12.12) in general by choosing a generic homotopy $H : [0,1] \times M \to E$ between two generic sections f_0 and f_1 and showing that $H^{-1}(0) \subset [0,1] \times M$ is a compact oriented 1-dimensional orbifold with boundary. As we observed at the beginning of this section, $H^{-1}(0)$ is then actually a manifold, so the signed count of its boundary points should be zero. But this would give the wrong result: it would suggest that $\sum_{z \in f^{-1}(0) \subset M} \operatorname{ord}(f; z)$ should be homotopy invariant, without the rational weights, and we've already seen that this is not true. What is going on here? The answer is that the homogopy H cannot in general be made transverse to the zero-section, now matter how generically we perturb it! It is an illustration of the fundamental conflict between the notions of *genericity* and *equivariance*.

EXAMPLE 12.10. Let $M = \mathbb{C}/\mathbb{Z}_2$ as in Example 12.8, but define the complex orbibundle $E \to M$ by

$$E = (\mathbb{C} \times \mathbb{C})\big/(z, v) \sim (-z, -v),$$

i.e. the $\mathbb{Z}_2$-action also acts antipodally on fibers. Now a smooth function $f : \mathbb{C} \to \mathbb{C}$ defines a section of E if and only if $f(-z) = -f(z)$, hence *all* such sections have a zero at the origin. Compare the two sections

$$f_0(x+iy) = x+iy, \qquad f_1(x+iy) = (x^3-x)+iy.$$

They have qualitatively the same behavior near infinity, meaning in particular that they are homotopic through a family of sections whose zeroes are confined to some compact subset, thus we expect the algebraic count of zeroes to be the same for both. This is true if the count is defined by (12.12): we have $\#f_0^{-1}(0) = \#f_1^{-1}(0) = \frac{1}{2}$, in particular the negative zero of f_1 at the origin counts for $-1/2$ while the positive zero at $(1,0) \sim (-1,0)$ counts for 1. We see that the inclusion of the rational weights $\frac{1}{\kappa_x}$ is crucial for this result. Notice that if $H : [0,1] \times M \to E$ is a homotopy of sections from f_0 to f_1, then $H(\tau,0) = 0$ for all τ, thus $\partial_\tau H(\tau,0)$ vanishes and

$$dH(\tau,0) = df_\tau(0)$$

where $f_\tau = H(\tau,\cdot)$. But $df_\tau(0)$ cannot be an isomorphism for all $\tau \in (0,1)$ since $df_0(0)$ preserves orientation while $df_1(0)$ reverses it. This is not a problem that can be fixed by making H more generic—the homotopy will never be transverse to the zero-section, no matter what we do.

The need to address issues of the type raised by the above examples leads naturally to the notion of **multisections** as outlined in [**Sal99**, §5] and [**FO99**], and this is a major feature of the analysis under development by Hofer-Wysocki-Zehnder, see for example [**HWZ10**]. In Example 12.10 for instance, one can consider functions

$$f : \mathbb{C} \to \mathrm{Sym}_2(\mathbb{C}) := (\mathbb{C} \times \mathbb{C})/(z_1,z_2) \sim (z_2,z_1),$$

which can be regarded as doubly-valued sections of $E \to M$ if f is $\mathbb{Z}_2$-equivariant for the antipodal action of $\mathbb{Z}_2$ on the symmetric product $\mathrm{Sym}_2(\mathbb{C})$. Such a section is considered single-valued at any point z where $f(z)$ is of the form $[(v,v)]$, so one can now imagine homotopies from f_0 to f_1 through doubly-valued sections. One advantage of this generalization is that f can now take nonzero values of the form $[(v,-v)]$ at the origin, e.g. if $g : \mathbb{C} \to \mathbb{C}$ is any odd function, then

$$f(z) := [(g(z)+c, g(z)-c)]$$

is a well-defined multisection for every $c \in \mathbb{C}$.

EXERCISE 12.11. Find a homotopy between the sections f_0 and f_1 of Example 12.10 through doubly-valued sections, such that the homotopy is transverse to the zero-section.

You may notice if you work out Exercise 12.11 that the zero set of the homotopy in $[0,1] \times M$ is still not submanifold or suborbifold. Instead, it naturally carries the structure of a *weighted branched manifold with boundary*. The rational weights attached to every point in this object can be used to explain the weights appearing in (12.12) and thus give a Milnor-style proof that $\#f^{-1}(0) \in \mathbb{Q}$ is invariant.

We will not discuss multisections or weighted branched manifolds any further, but the main takeaway from this discussion should be that the "right" way to count 0-dimensional orbifolds algebraically is always some version of (12.12), and the count in general is a rational number, not an integer. We've discussed this above from the perspective of obtaining a homotopy-invariant count, but the same logic applies to any Floer-type theory since the relation $\partial^2 = 0$ is typically based on similar arguments via 1-dimensional moduli spaces with boundary. While a more simplistic notion of counting may produce well-defined homology theories in isolated cases where Assumption 12.1 holds (e.g. in [Nel15]), we cannot expect it to generalize beyond these cases, due to the fundamental conflict between transversality and equivariance. On the other hand, it will be possible in our situation to remove isotropy from the picture by lifting to moduli spaces with asymptotic markers; the moduli space we're interested in is always the quotient of this larger space by a finite group action, so the situation is analogous to replacing an orbibundle $E = \widetilde{E}/G \to \widetilde{M}/G$ by an ordinary vector bundle $\widetilde{E}$ over a manifold $\widetilde{M}$. In the infinite-dimensional setting, transversality is still a hard problem, but having lifted to a manifold and thus removed the need for equivariance, there is no longer any *a priori* reason why it cannot be solved by choosing sufficiently generic perturbations. This makes counting curves with rational weights seem a much more promising method for defining invariants, and we will adopt this perspective in the discussion to follow.

12.5. Cylindrical contact homology revisited

Under an extra assumption on the complex $(\mathcal{A}[[\hbar]], \mathbf{D}_{\mathrm{SFT}})$, we can recover from it a more general version of the cylindrical contact homology we saw in Lecture 10. Suppose in particular that there are no index 1 holomorphic planes in $\mathbb{R} \times M$, so every term in $\hbar\mathbf{H}$ has at least one factor of either $\hbar$ or one of the q_γ variables. Then

$$\mathbf{D}_{\mathrm{SFT}} = \sum_{\gamma,\gamma',A} \kappa_\gamma \left(\sum_{u \in \mathcal{M}_{0,0}(J,A,\gamma,\gamma')/\mathbb{R}} \frac{\epsilon(u)}{|\operatorname{Aut}(u)|} e^A q_{\gamma'} \frac{\partial}{\partial q_\gamma} \right) + \dots,$$

where the first sum is over all pairs of good Reeb orbits γ and γ', and the ellipsis is a sum of terms that all include at least a positive power of $\hbar$ or two q_γ variables or two partial derivatives. Let us abbreviate the spaces $\mathcal{M}_{0,0}(J,A,\gamma,\gamma')/\mathbb{R}$ of $\mathbb{R}$-equivalence classes of J-holomorphic cylinders by $\mathcal{M}_A(\gamma,\gamma')$, and notice that for any $u \in \mathcal{M}_A(\gamma,\gamma')$, the automorphism group is a cyclic group of order equal to the covering multiplicity

$$|\operatorname{Aut}(u)| = \kappa_u := \operatorname{cov}(u) \in \mathbb{N}.$$

Thus for any single generator q_γ, we have

$$\mathbf{D}_{\mathrm{SFT}} q_\gamma = \partial_{\mathrm{CCH}} q_\gamma + O(|q|^2, \hbar),$$

where

$$\partial_{\mathrm{CCH}} q_\gamma := \kappa_\gamma \sum_{\gamma',A} \left(\sum_{u \in \mathcal{M}_A(\gamma,\gamma')} \frac{\epsilon(u)}{\kappa_u} \right) e^A q_{\gamma'}. \tag{12.13}$$

The fact that $\mathbf{D}_{\mathrm{SFT}}^2 = 0$ thus implies

$$\partial_{\mathrm{CCH}}^2 = 0,$$

and the homology of the graded R-module generated by $\{q_\gamma \mid \gamma \text{ good}\}$ with differential ∂_{CCH} is an obvious generalization of the cylindrical contact homology from Lecture 10. What we saw there was a special case of this where the combinatorial factor κ_γ/κ_u did not appear because we were restricting to a homotopy class in which all orbits were simply covered, and all holomorphic cylinders were thus somewhere injective.

The presence of the factor κ_γ/κ_u deserves further comment. According to the above formula, we have

$$\partial^2_{\mathrm{CCH}} q_\gamma = \sum_{\gamma',\gamma'',A,A'} \sum_{u \in \mathcal{M}_A(\gamma,\gamma')} \sum_{v \in \mathcal{M}_{A'}(\gamma',\gamma'')} e^{A+A'} \frac{\kappa_\gamma \kappa_{\gamma'} \epsilon(u)\epsilon(v)}{\kappa_u \kappa_v} q_{\gamma''},$$

hence $\partial^2_{\mathrm{CCH}} = 0$ holds if and only if for all $A \in H_2(M)$ and all pairs of good orbits γ_+, γ_-,

$$\sum_{\gamma_0} \sum_{B+C=A} \left(\sum_{(u,v) \in \mathcal{M}_B(\gamma_+,\gamma_0) \times \mathcal{M}_C(\gamma_0,\gamma_-)} \frac{\kappa_{\gamma_0}}{\kappa_u \kappa_v} \epsilon(u)\epsilon(v) \right) = 0. \tag{12.14}$$

If γ_+ and γ_- happen to be simply covered orbits, then u and v in this expression always have trivial automorphism groups and it is clear what this sum means: every such pair $(u, v) \in \mathcal{M}_B(\gamma_+, \gamma_0) \times \mathcal{M}_C(\gamma_0, \gamma_-)$ corresponds to exactly κ_{γ_0} distinct holomorphic buildings obtained by different choices of decoration, so (12.14) is the count of boundary points of the compactified 1-dimensional manifold of index 2 cylinders $\mathcal{M}_A(\gamma_+, \gamma_-)/\mathbb{R}$. This sum skips over all bad orbits γ_0, but this is fine because whenever the breaking orbit is bad, there are evenly many choices of decoration such that half of these choices cancel the other half when counted with the correct signs.

To understand why this formula is still correct in the presence of automorphisms, let us outline two equivalent approaches.

The easiest option is to instead consider moduli spaces with asymptotic markers, which never have automorphisms: removing unnecessary factors of κ_{γ_+} and κ_{γ_-} then transforms (12.14) into

$$\sum_{\gamma_0} \sum_{B+C=A} \frac{1}{\kappa_{\gamma_0}} \#\mathcal{M}^\$_B(\gamma_+, \gamma_0) \cdot \#\mathcal{M}^\$_C(\gamma_0, \gamma_-) = 0.$$

Now since each pair $(u, v) \in \mathcal{M}^\$_B(\gamma_+, \gamma_0) \times \mathcal{M}^\$_C(\gamma_0, \gamma_-)$ carries a canonical decoration and thus determines a holomorphic building, the division by κ_{γ_0} accounts for the fact that $\#\mathcal{M}^\$_B(\gamma_+, \gamma_0) \cdot \#\mathcal{M}^\$_C(\gamma_0, \gamma_-)$ overcounts the set of broken cylinders from γ_+ to γ_- with asymptotic markers at $\gamma_\pm$ by precisely this factor, as a simultaneous adjustment of the marker at γ_0 in both $u \in \mathcal{M}^\$_B(\gamma_+, \gamma_0)$ and $v \in \mathcal{M}^\$_C(\gamma_0, \gamma_-)$ produces the same decoration and therefore the same building.

The following alternative perspective will be more useful when we generalize beyond cylinders in the next section. We can directly count points in $\partial\overline{\mathcal{M}}_A(\gamma_+, \gamma_-)$, though as we saw in §12.4, rational weights should be included in the count whenever there is isotropy. Let us write

$$\mathcal{M}_A(\gamma_+, \gamma_-) = \mathcal{M}^\$_A(\gamma_+, \gamma_-)/G,$$

where $G \cong \mathbb{Z}_{\kappa_{\gamma_+}} \times \mathbb{Z}_{\kappa_{\gamma_-}}$ is a finite group acting by adjustment of the asymptotic markers. Since $\overline{\mathcal{M}}^\$_A(\gamma_+, \gamma_-)$ is a compact oriented 1-manifold with boundary under Assumption 12.1, the signed count of its boundary points is 0. We can ignore buildings broken along bad orbits in this count, since these always come in cancelling pairs. Let us now transform this into a count of buildings $(u|\Phi|v) \in \partial\overline{\mathcal{M}}_A(\gamma_+,\gamma_-)$ broken along good orbits γ_0: here $u \in \mathcal{M}_B(\gamma_+,\gamma_0)$ and $v \in \mathcal{M}_C(\gamma_0,\gamma_-)$ for some homology classes with $B + C = A$, and Φ is a decoration which describes how to glue the ends of u and v at γ_0. The automorphism group of such a building is the subgroup

$$\mathrm{Aut}(u|\Phi|v) \subset \mathrm{Aut}(u) \times \mathrm{Aut}(v)$$

consisting of all pairs $(\varphi,\psi) \in \mathrm{Aut}(u) \times \mathrm{Aut}(v)$ that define the same rotation at the two punctures asymptotic to γ_0; note that this group does not actually depend on the decoration Φ. Since we're talking about cylinders, we can be much more specific: we have $\mathrm{Aut}(u) = \mathbb{Z}_{\kappa_u}$ and $\mathrm{Aut}(v) = \mathbb{Z}_{\kappa_v}$, and if both are regarded as subgroups of $\mathrm{U}(1)$,

$$\mathrm{Aut}(u|\Phi|v) = \mathbb{Z}_{\kappa_u} \cap \mathbb{Z}_{\kappa_v} = \mathbb{Z}_{\gcd(\kappa_u,\kappa_v)},$$

which is injected into $\mathrm{Aut}(u)\times\mathrm{Aut}(v)$ by $\psi \mapsto (\psi,\psi)$. The boundary of $\overline{\mathcal{M}}^{\$}_A(\gamma_+,\gamma_-)$ can be understood likewise as a space of equivalence classes

$$[(u,v)] \in \left(\mathcal{M}^{\$}_B(\gamma_+,\gamma_0) \times \mathcal{M}^{\$}_C(\gamma_0,\gamma_-)\right) / \sim,$$

where two such pairs are equivalent if their asymptotic markers at the ends asymptotic to γ_0 determine the same decoration. Now observe that the group $G \cong \mathbb{Z}_{\kappa_{\gamma_+}} \times \mathbb{Z}_{\kappa_{\gamma_-}}$ also acts on buildings in $\partial\overline{\mathcal{M}}^{\$}_A(\gamma_+,\gamma_-)$, and the stabilizer of this action at (u,v) is $\mathrm{Aut}(u|\Phi|v)$, hence each $(u|\Phi|v) \in \partial\overline{\mathcal{M}}_A(\gamma_+,\gamma_-)$ gives rise to $\frac{|G|}{\gcd(\kappa_u,\kappa_v)}$ terms in the count of $\partial\overline{\mathcal{M}}^{\$}_A(\gamma_+,\gamma_-)$, implying

$$\sum_{(u|\Phi|v)\in\partial\overline{\mathcal{M}}_A(\gamma_+,\gamma_-)} \frac{\epsilon(u)\epsilon(v)}{\gcd(\kappa_u,\kappa_v)} = 0. \tag{12.15}$$

Finally, notice that while each pair $(u,v) \in \mathcal{M}_B(\gamma_+,\gamma_0) \times \mathcal{M}_C(\gamma_0,\gamma_-)$ determines buildings with κ_{γ_0} distinct choices of decoration, some of these buildings may be equivalent: every pair of automorphisms $(\varphi,\psi) \in \mathrm{Aut}(u) \times \mathrm{Aut}(v)$ transforms a building $(u|\Phi|v)$ by potentially changing the decoration Φ, thus producing an equivalent building. This action on buildings is trivial if and only if $(\varphi,\psi) \in \mathrm{Aut}(u|\Phi|v)$, hence every pair $(u,v) \in \mathcal{M}_B(\gamma_+,\gamma_0) \times \mathcal{M}_C(\gamma_0,\gamma_-)$ gives rise to exactly

$$\frac{\kappa_{\gamma_0}}{\left|(\mathrm{Aut}(u)\times\mathrm{Aut}(v)) / \mathrm{Aut}(u|\Phi|v)\right|} = \frac{\kappa_{\gamma_0}\gcd(\kappa_u,\kappa_v)}{\kappa_u\kappa_v}$$

elements of $\partial\overline{\mathcal{M}}_A(\gamma_+,\gamma_-)$, so that (12.15) becomes

$$\begin{aligned}\sum_{\gamma_0}\sum_{B+C=A}&\left(\sum_{(u,v)\in\mathcal{M}_B(\gamma_+,\gamma_0)\times\mathcal{M}_C(\gamma_0,\gamma_-)} \frac{\epsilon(u)\epsilon(v)}{\gcd(\kappa_u,\kappa_v)}\frac{\kappa_{\gamma_0}\gcd(\kappa_u,\kappa_v)}{\kappa_u\kappa_v}\right)\\ &= \sum_{\gamma_0}\sum_{B+C=A}\left(\sum_{(u,v)\in\mathcal{M}_B(\gamma_+,\gamma_0)\times\mathcal{M}_C(\gamma_0,\gamma_-)} \frac{\epsilon(u)\epsilon(v)\kappa_{\gamma_0}}{\kappa_u\kappa_v}\right) = 0,\end{aligned}$$

reproducing (12.14).

12.6. Combinatorics of gluing

Now let's try to justify the formula $\mathbf{H}^2 = 0$. The product of $\mathbf{H}$ with itself is the formal sum over all pairs of index 1 curves $u, v \in \mathcal{M}_1^\sigma(J)/\mathbb{R}$ of certain monomials: in particular if these two curves respectively have genus g_u and g_v, homology classes A_u and A_v, and asymptotic orbits $\boldsymbol{\gamma}_u^\pm$ and $\boldsymbol{\gamma}_v^\pm$, then the corresponding term in $\mathbf{H}^2$ is

$$\frac{\epsilon(u)\epsilon(v)}{|\operatorname{Aut}^\sigma(u)||\operatorname{Aut}^\sigma(v)|}\hbar^{g_u+g_v-2}e^{A_u+A_v}q^{\boldsymbol{\gamma}_u^-}p^{\boldsymbol{\gamma}_u^+}q^{\boldsymbol{\gamma}_v^-}p^{\boldsymbol{\gamma}_v^+}.$$

Before we can add up all monomials of this form, we need to put all the q and p variables in the same order: within each of the products $q^{\boldsymbol{\gamma}_u^-}$, $p^{\boldsymbol{\gamma}_u^+}$ and so forth this is simply a matter of permuting the variables and changing signs as appropriate, but the interesting part is the product $p^{\boldsymbol{\gamma}_u^+}q^{\boldsymbol{\gamma}_v^-}$, for which we can apply the commutation relations (12.9) to put all q variables before all p variables. Before discussing how this works in general, let us consider a more specific example.

Assume γ_i for $i = 1, 2$ are two specific orbits with $n - 3 + \mu_{\mathrm{CZ}}(\gamma_i)$ even, so the corresponding q and p variables have even degree, and suppose

$$\boldsymbol{\gamma}_u^+ = (\gamma_1, \gamma_1, \gamma_2), \qquad \boldsymbol{\gamma}_v^- = (\gamma_1, \gamma_1).$$

After applying the relation $p_{\gamma_1}q_{\gamma_1} = q_{\gamma_1}p_{\gamma_1} + \kappa_{\gamma_1}\hbar$ a total of five times, one obtains the expansion

$$p_{\gamma_1}p_{\gamma_1}p_{\gamma_2}q_{\gamma_1}q_{\gamma_1} = q_{\gamma_1}^2p_{\gamma_1}^2p_{\gamma_2} + 4\kappa_{\gamma_1}\hbar q_{\gamma_1}p_{\gamma_1}p_{\gamma_2} + 2\kappa_{\gamma_1}^2\hbar^2p_{\gamma_2},$$

thus contributing a total of three terms to $\mathbf{H}^2$, namely the products of the factor $\frac{\epsilon(u)\epsilon(v)}{|\operatorname{Aut}(u)||\operatorname{Aut}(v)|}e^{A_u+A_v}$ with each of the expressions

$$\hbar^{g_u+g_v-2}q^{\boldsymbol{\gamma}_u^-}q_{\gamma_1}^2p_{\gamma_1}^2p_{\gamma_2}p^{\boldsymbol{\gamma}_v^+}, \tag{12.16}$$

$$4\kappa_{\gamma_1}\hbar^{g_u+g_v-1}q^{\boldsymbol{\gamma}_u^-}q_{\gamma_1}p_{\gamma_1}p_{\gamma_2}p^{\boldsymbol{\gamma}_v^+}, \tag{12.17}$$

$$2\kappa_{\gamma_1}^2\hbar^{g_u+g_v}q^{\boldsymbol{\gamma}_u^-}p_{\gamma_2}p^{\boldsymbol{\gamma}_v^+}. \tag{12.18}$$

As shown in Figure 12.2, this sum of three terms can be interpreted as the count of all possible holomrphic buildings obtained by gluing v on top of u together with a collection of trivial cylinders. Indeed, since $\boldsymbol{\gamma}_u^+$ and $\boldsymbol{\gamma}_v^-$ include two matching orbits (which also happen to be the same one), there are several choices to be made:

(1) The top-right picture shows what we might call the "stupid gluing," in which no ends of u are matched with any ends of v, but all are instead glued to trivial cylinders, thus producing a disconnected building. This possibility is encoded by (12.16), and we will see that in the total sum forming $\mathbf{H}^2$, this term gets cancelled out by a similar term for the stupid gluing of u on top of v.

(2) The lower-left picture shows the building obtained by gluing one end of u to an end of v along the matching orbit γ_1. This option is encoded by (12.17), where the factor $4\kappa_{\gamma_1}$ appears because there are precisely $4\kappa_{\gamma_1}$ distinct buildings of this type: indeed, there are four choices of which end of u should be glued to which end of v, and for each of these, a further κ_{γ_1}

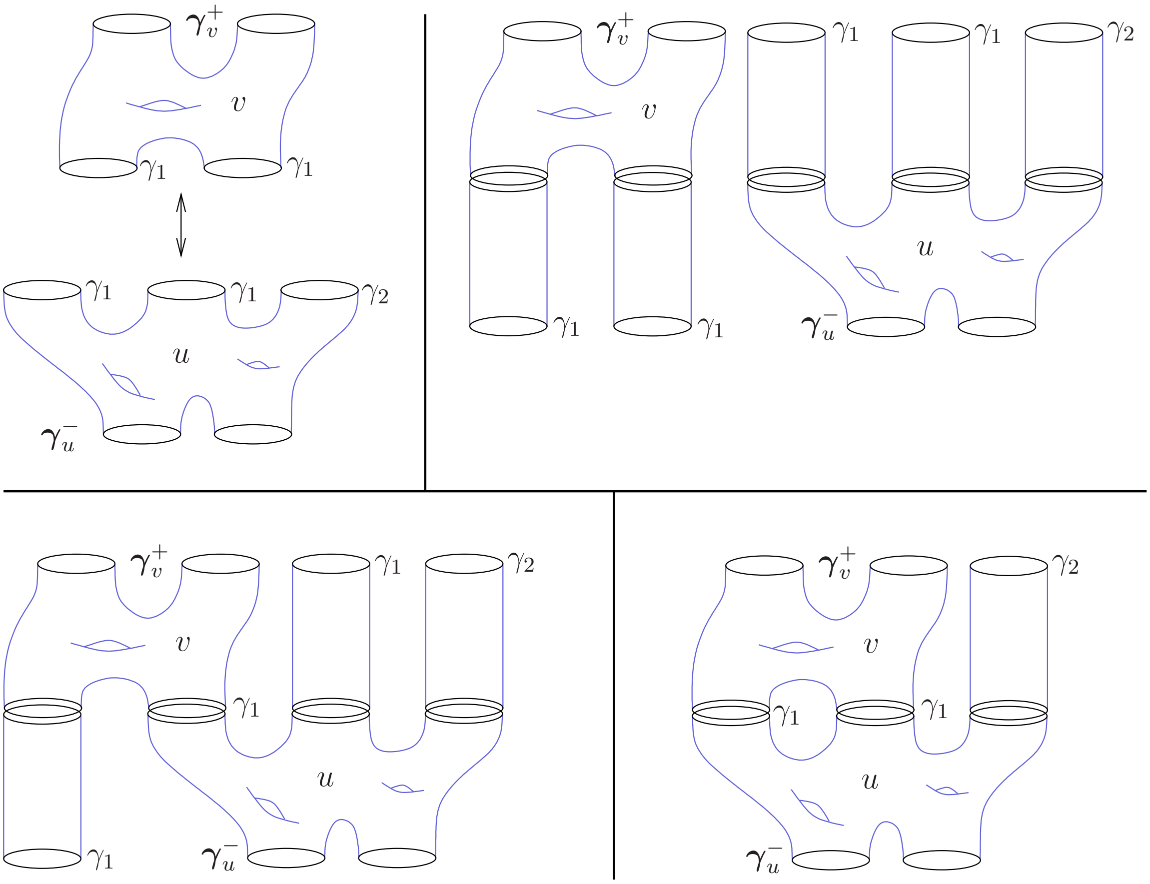

FIGURE 12.2. Three possible ways of gluing the curves u and v along with trivial cylinders to form index 2 curves.

choices of the decoration. The arithmetic genus of the resulting building is $g_u + g_v$, as represented by the factor $\hbar^{g_u+g_v-1}$.

(3) The lower-right picture is encoded by (12.18): here there are two choices of bijections between the two pairs of punctures asymptotic to γ_1, and taking the choices of decoration at each breaking orbit into account, we obtain the combinatorial factor $2\kappa_{\gamma_1}^2$. The presence of two nontrivial breaking orbits increases the arithmetic genus to g_u+g_v+1, as encoded in the factor $\hbar^{g_u+g_v}$.

You may now be able to extrapolate from the above example why the commutator algebra we've defined encodes gluing of holomorphic curves in the symplectization and thus leads to the relation $\mathbf{H}^2 = 0$. Think of the algorithm by which you change $q^{\gamma_u^-} p^{\gamma_u^+} q^{\gamma_v^-} p^{\gamma_u^+}$ into a sum of products with all q's appearing before p's: for the first q you see appearing after a p, move it past each p for different orbits (changing signs as necessary) until it encounters a p for the *same* orbit. Now you replace $p_\gamma q_\gamma$ with $(-1)^{|p_\gamma||q_\gamma|} q_\gamma p_\gamma + \kappa_\gamma \hbar$, turning one product into a sum of two. This represents a choice between two options: either you move q_γ past p_γ and apply the usual sign change, or you eliminate them both but replace them with the combinatorial factor κ_γ and an extra $\hbar$. Then you continue this process until all q's appear before all p's.

The key point is that the process of gluing v on top of u in all possible ways is governed by *exactly the same algorithm*: first consider the disjoint union of the two curves as a single disconnected curve, with its punctures ordered in the same way in which their orbits appear in the monomial. Now reorder negative punctures of v and positive punctures of u, changing orientations as appropriate, until you see two such punctures next to each other approaching the same orbit γ. Here you have two options: either glue them together, or don't glue them but exchange their order. If you exchange the order, then you may again have to change orientations (depending on the parity of $n - 3 + \mu_{CZ}(\gamma)$), but if you glue, then you have κ_γ distinct choices of decoration and will also increase the arithmetic genus of the eventual building by 1. In this way, every individual term in the final expansion of $q^{\gamma_u^-} p^{\gamma_u^+} q^{\gamma_v^-} p^{\gamma_u^+}$ represents a particular choice of which positive of ends of u should or should not be glued to which negative ends of v. Additional factors of $\hbar$ appear to keep track of the increase in arithmetic genus, and covering multiplicities of the breaking orbits also appear due to distinct choices of decorations. At the end these must still be divided by orders of automorphism groups in order to avoid counting equivalent buildings separately. Fleshing out these details leads to the following explanation for the relation $\mathbf{H}^2 = 0$:

PROPOSITION 12.12. *Let $\partial\overline{\mathcal{M}}_2^\sigma(J)$ denote the space of two-level holomorphic buildings in $\overline{\mathcal{M}}(J)$ that have total index 2 and no bad asymptotic or breaking orbits, divided by the equivalence relation that forgets the order of the punctures. Then*

$$\mathbf{H}^2 = \sum_{\mathbf{u} \in \partial\overline{\mathcal{M}}_2^\sigma(J)} \frac{\epsilon(\mathbf{u})}{|\operatorname{Aut}^\sigma(\mathbf{u})|} \hbar^{g-1} e^A q^{\boldsymbol{\gamma}^-} p^{\boldsymbol{\gamma}^+},$$

where the terms in each monomial are determined by $\mathbf{u} \in \partial\overline{\mathcal{M}}_2^\sigma(J)$ as follows:

(1) g is the arithmetic genus of $\mathbf{u}$;
(2) A is the equivalence class of $[\mathbf{u}] \in H_2(M)$ in $H_2(M)/G$;
(3) $\boldsymbol{\gamma}^\pm = (\gamma_1^\pm, \ldots, \gamma_{k_\pm}^\pm)$ are the asymptotic orbits of $\mathbf{u}$ after arbitrarily fixing orderings of its positive and negative punctures;
(4) $\epsilon(\mathbf{u}) \in \{1, -1\}$ is the boundary orientation at $\mathbf{u}$ determined by the chosen coherent orientations on $\mathcal{M}^\$(J)$. Specifically, given the chosen ordering of the punctures and an arbitrary choice of asymptotic markers at each puncture, $\mathbf{u}$ determines a boundary point of a 1-dimensional connected component of $\overline{\mathcal{M}}^\$(J)$, and we define $\epsilon(\mathbf{u}) = +1$ if and only if the orientation of $\overline{\mathcal{M}}^\$(J)$ at this point is outward.

Once again $\epsilon(\mathbf{u})$ and $q^{\boldsymbol{\gamma}^-} p^{\boldsymbol{\gamma}^+}$ change signs in the same way under any reordering of the punctures, so their product is well defined, and there is no dependence on choices of markers since bad orbits have been excluded.

Proof of Proposition 12.12. Our original formula for $\mathbf{H}$ gives rise to an expansion

$$\mathbf{H}^2 = \sum_{(u,v)\in\mathcal{M}_1^\sigma(J)/\mathbb{R}\times\mathcal{M}_1^\sigma(J)/\mathbb{R}} \frac{\epsilon(u)\epsilon(v)}{|\operatorname{Aut}^\sigma(u)||\operatorname{Aut}^\sigma(v)|}\hbar^{g_u+g_v-2}e^{A_u+A_v}q^{\gamma_u^-}p^{\gamma_u^+}q^{\gamma_v^-}p^{\gamma_v^+}.$$

As explained in the previous paragraph, the process of reordering $p^{\gamma_u^+}q^{\gamma_v^-}$ to put all q's before p's produces an expansion, each term of which can be identified with a specific choice of which positive punctures of u should be glued to which negative punctures of v. If k punctures are glued, then the resulting power of $\hbar$ is $g_u + g_v - 2 + k$, corresponding to the fact that the resulting building has arithmetic genus g_u+g_v+k-1. We claim that the term for $k = 0$ is cancelled out by the corresponding term of $\mathbf{H}^2$ that has the roles of u and v reversed. To see this, imagine first the case where u and v have no asymptotic orbits in common, hence no nontrivial gluings are possible and all the q and p variables in the expression supercommute with each other. Then since both curves have index 1, the monomials $q^{\gamma_u^-}p^{\gamma_u^+}$ and $q^{\gamma_v^-}p^{\gamma_v^+}$ must both have odd degree, implying

$$q^{\gamma_u^-}p^{\gamma_u^+}q^{\gamma_v^-}p^{\gamma_v^+} = -q^{\gamma_v^-}p^{\gamma_v^+}q^{\gamma_u^-}p^{\gamma_u^+}$$

and thus the desired cancellation. If u and v do have orbits in common, then the result for the $k = 0$ terms is still not any different from this: all signs still change in the same way when applying $[p_\gamma, q_\gamma] = \kappa_\gamma\hbar$ to change $p_\gamma q_\gamma$ into $q_\gamma p_\gamma$, we simply ignore the extra term $\kappa_\gamma\hbar$ since it is only relevant for gluings with $k > 0$. This proves the claim, and consequently, that the expansion resulting from the curves u and v has no term containing $\hbar^{g_u+g_v-2}$.

The combinatorial factors can be explained as follows. The commutator expansion for $p^{\gamma_u^+}q^{\gamma_v^-}$ automatically produces combinatorial factors that count the different possible gluings, but if u and v have automorphisms, then not all of these give inequivalent buildings. This part of the discussion is a straightforward extension of what we did for cylindrical contact homology at the end of §12.5. Indeed, the actual set of inequivalent buildings is the quotient of this larger set by an action of

$$(\operatorname{Aut}^\sigma(u) \times \operatorname{Aut}^\sigma(v)) \,/\, \operatorname{Aut}^\sigma(\mathbf{u}),$$

where for a building $\mathbf{u}$ formed by endowing the pair (u, v) with decorations, $\operatorname{Aut}^\sigma(\mathbf{u})$ denotes the subgroup consisting of pairs $(\varphi, \psi) \in \operatorname{Aut}^\sigma(u) \times \operatorname{Aut}^\sigma(v)$ that preserve pairs of breaking punctures along with their decorations. This is what changes the factor $\frac{1}{|\operatorname{Aut}^\sigma(u)||\operatorname{Aut}^\sigma(v)|}$ into $\frac{1}{|\operatorname{Aut}^\sigma(\mathbf{u})|}$ as in the statement of the proposition. □

The theorem that $\mathbf{H}^2 = 0$ now follows once you believe the propaganda from §12.4, arguing that $\sum_{\mathbf{u}\in\partial\overline{\mathcal{M}}_2^\sigma(J)} \frac{\epsilon(\mathbf{u})}{|\operatorname{Aut}^\sigma(\mathbf{u})|}$ is the correct way to count the boundary points of $\overline{\mathcal{M}}_2^\sigma(J)$. As we did with cylindrical contact homology, we can use the obvious projection $\overline{\mathcal{M}}^\$(J) \to \overline{\mathcal{M}}^\sigma(J)$ to reduce this to the fact that if the 1-dimensional components of $\overline{\mathcal{M}}^\$(J)$ are manifolds (which is true if Assumption 12.1 holds), then the integer-valued signed count of their boundary points vanishes.

12.7. Some remarks on torsion, coefficients, and conventions

12.7.1. What if $H_1(M)$ has torsion? The main consequence for SFT if $H_1(M)$ has torsion is that one cannot define an integer grading, though there is always a canonical $\mathbb{Z}_2$-grading.[3] The setup in §12.2 must now be modified as follows. The **reference curves**

$$C_1, \dots, C_r \subset M$$

are required to form a basis of $H_1(M)/\text{torsion}$, so for every integral homology class $[\gamma]$, there is a unique collection of integers $m_1, \dots, m_r$ such that $[\gamma] = \sum_i m_i[C_i] \in H_1(M;\mathbb{Q})$. Instead of spanning surfaces for each orbit, one can define **spanning chains** C_γ, which are singular 2-chains with rational coefficients satisfying

$$\partial C_\gamma = \sum_i m_i[C_i] - [\gamma]$$

for the aforementioned set of integers $m_i \in \mathbb{Z}$. Note that C_γ must in general have nonintegral coefficients since $\sum_i m_i[C_i]$ and $[\gamma]$ might not be homologous in $H_1(M;\mathbb{Z})$, so C_γ cannot always be represented by a smooth map of a surface. One consequence of this is that the absolute homology class associated to an asymptotically cylindrical holomorphic curve $u : \dot{\Sigma} \to \mathbb{R} \times M$ will now be rational,

$$[u] \in H_2(M;\mathbb{Q}),$$

and we must therefore take G to be a linear subspace

$$G \subset H_2(M;\mathbb{Q}).$$

Another consequence is that we cannot use capping chains to transfer trivializations from the reference curves to the orbits, so there is no natural way to define $\mu_{\mathrm{CZ}}(\gamma)$ as an integer. The easiest thing to do instead is to take the mod 2 Conley-Zehnder index

$$\mu_{\mathrm{CZ}}(\gamma) \in \mathbb{Z}_2$$

and define all degrees of generators as either even or odd with no further distinction. In particular, we now have

$$|q_\gamma| = n - 3 + \mu_{\mathrm{CZ}}(\gamma) \in \mathbb{Z}_2, \qquad |p_\gamma| = n - 3 - \mu_{\mathrm{CZ}}(\gamma) \in \mathbb{Z}_2,$$

while $\hbar$ and all elements of $R = \mathbb{Q}[H_2(M;\mathbb{Q})/G]$ are even. With these modifications, the rest of the discussion also becomes valid for the case where $H_1(M)$ has torsion, and leads to $\mathbb{Z}_2$-graded contact invariants.

12.7.2. Combinatorial conventions. The combinatorial factors appearing in our definition of $\mathbf{H}$ may at first look slightly different from what appears elsewhere in the literature. Actually, most papers seem to agree on this detail, but various subtle differences and ambiguities in notation mean that it sometimes requires intense concentration to recognize this fact.

The original propaganda paper [**EGH00**] expresses everything in terms of moduli spaces with asymptotic markers, and the formula for $\mathbf{H}$ in §2.2.3 of that paper (which

[3] In fact there is a bit more than a $\mathbb{Z}_2$-grading, see [**EGH00**, §2.9.1].

is expressed in a slightly more general form involving marked points) agrees with our (12.8).

Cieliebak and Latschev [**CL09**, §2] write down the same formula in terms of moduli spaces that have no asymptotic markers but remember the order of the punctures, thus it includes some factorials that do not appear in (12.6) but is missing the κ_γ terms of (12.8). The notation $n_g(\Gamma^-, \Gamma^+)$ used in [**CL09**] for curve counts must be understood implicitly to include rational weights arising from automorphisms (or multivalued perturbations, as the case may be).

My paper with Latschev [**LW11**] uses moduli spaces with asymptotic markers and attempts to write down the same formula as in [**EGH00**, **CL09**], but gets it slightly wrong due to some missing κ_γ terms that should appear in front of each $\frac{\partial}{\partial q_\gamma}$. Mea culpa.

For cylindrical contact homology, the combinatorial factors in §12.5 also agree with what appears in [**Bou03**]. As observed by Nelson [**Nel13**, Remark 8.3], there are other conventions for ∂_{CCH} that appear in the literature and lead to equivalent theories: in particular it is possible to replace (12.13) with

$$\partial_{\text{CCH}} q_\gamma := \sum_{\gamma', A} \kappa_{\gamma'} \left(\sum_{u \in \mathcal{M}_A(\gamma, \gamma')} \frac{\epsilon(u)}{\kappa_u} \right) e^A q_{\gamma'}.$$

One can derive this from the same definition of $\mathbf{H}$ by applying a "change of coordinates" to the algebra $\mathcal{A}[[\hbar]]$, or equivalently, by choosing a slightly different representation of the operator algebra defined by the p_γ and q_γ variables. To avoid confusion, let us write the generators of $\mathcal{A}$ as x_γ instead of q_γ, and then define the operators q_γ and p_γ on $\mathcal{A}[[\hbar]]$ by

$$q_\gamma = \kappa_\gamma x_\gamma, \qquad p_\gamma = \hbar \frac{\partial}{\partial x_\gamma}.$$

These operators still satisfy $[p_\gamma, q_\gamma] = \kappa_\gamma \hbar$ and thus define an equivalent theory, but the resulting differential operator $\mathbf{D}_{\text{SFT}}$ on $\mathcal{A}[[\hbar]]$ now includes extra factors of κ_γ for the negative punctures instead of the positive punctures.

12.7.3. Coefficients: $\mathbb{Q}$, $\mathbb{Z}$ or $\mathbb{Z}_2$? While we were able to use $\mathbb{Z}_2$ coefficients for cylindrical contact homology in a primitive homotopy class in Lecture 10, a quick glance at any version of the formula for $\mathbf{H}$ should make the reader very skeptical about doing this for more general versions of SFT. The existence of curves with automorphisms means that $\mathbf{H}$ always contains terms with rational (but nonintegral) coefficients. And this is only what is true in the fictional world of Assumption 12.1: in the general version of the theory, we expect to have to replace expressions like $\sum_u \frac{\epsilon(u)}{|\text{Aut}(u)|}$ with counts of 0-dimensional weighted branched orbifolds with rational weights, arising as zero-sets of generic multisections. In this case we not only obtain rational counts but may also lose all control over the sizes of the denominators.

A similar phenomenon occurs in general versions of Gromov-Witten theory. For instance, in the approach of Cieliebak-Mohnke [**CM07**] for the rational Gromov-Witten invariants of a closed symplectic manifold (W^{2n}, ω) with $[\omega] \in H^2(W; \mathbb{Q})$, the invariants are defined by replacing the usual moduli space $\mathcal{M}_{0,m}(J, A)$ by a space

$\mathcal{M}_{0,m+N}(J, A; Y)$ consisting of J-holomorphic spheres $u : S^2 \to W$ with some large number of auxiliary marked points $\zeta_1, \ldots, \zeta_N$ required to satisfy the condition

$$u(\zeta_i) \in Y, \qquad i = 1, \ldots, N.$$

Here $Y^{2n-2} \subset W^{2n}$ is a J-holomorphic hypersurface with $[Y] = D \cdot \mathrm{PD}([\omega]) \in H_{2n-2}(W)$ for some degree $D \in \mathbb{N}$, and the number of extra marked points is determined by

$$N = A \cdot [Y] = D\langle [\omega], A\rangle,$$

so positivity of intersections implies that u *only* intersects Y at the auxiliary marked points. These auxiliary points are convenient for technical reasons involving transversality—their role is vaguely analogous to the way that asymptotic markers get rid of isotropy in SFT—but they are not geometrically meaningful, as we'd actually prefer to count curves in $\mathcal{M}_{0,m}(J, A)$. Every such curve has N intersections with Y, so accounting for permutations, it lifts to $N!$ distinct elements of $\mathcal{M}_{0,m+N}(J, A; Y)$, and the correct count is therefore obtained as an integer count of curves in the latter space divided by $N!$. Perturbing to achieve transversality breaks the symmetry, however, so there is no guarantee that counting curves in $\mathcal{M}_{0,m+N}(J, A; Y)$ will produce a multiple of $N!$, and moreover, N could be arbitrarily large since one needs to take hypersurfaces of arbitrarily large degree in order to show that the invariants don't depend on this choice. For these reasons, the resulting Gromov-Witten invariants are rational numbers rather than integers, and their denominators cannot be predicted or bounded.

The upshot of this discussion is that there is probably no hope of defining SFT with integer coefficients in general, much less with $\mathbb{Z}_2$ coefficients—for this reason the inclusion of orientations in the picture is unavoidable. That is the bad news.

The good news however is that whenever formulas like $\sum_u \frac{\epsilon(u)}{|\mathrm{Aut}(u)|}$ can be taken literally as a count of curves, the chain complex $(\mathcal{A}[[\hbar]], \mathbf{D}_{\mathrm{SFT}})$ can in fact be defined with $\mathbb{Z}$ coefficients, and one can even reduce to a $\mathbb{Z}_2$ version in order to ignore signs. A special case of this was observed for cylindrical contact homology in [**Nel15**, Remark 1.5], and you may notice it already when you look at the formula (12.13) for ∂_{CCH}: the factor κ_γ/κ_u is always an integer since the multiplicity of a holomorphic cylinder always divides the covering multiplicity of both its asymptotic orbits. Surprisingly, something similar turns out to be true for the much larger chain complex of SFT. The following result is stated under Assumption 12.1 for safety's sake, but in light of the discussion in §12.4, we should expect it to hold somewhat more generally.

PROPOSITION 12.13. *If Assumption 12.1 holds then the rational coefficients* $n_g(\boldsymbol{\gamma}, \boldsymbol{\gamma}', k)$ *in the formula* (12.11) *for* $\mathbf{D}_{\mathrm{SFT}} q^{\boldsymbol{\gamma}}$ *are all integers.*

COROLLARY 12.14. *Under Assumption 12.1, there exist well-defined chain complexes*

$$(\mathcal{A}_{\mathbb{Z}}[[\hbar]], \mathbf{D}_{\mathrm{SFT}}) \quad \text{and} \quad (\mathcal{A}_{\mathbb{Z}_2}[[\hbar]], \mathbf{D}_{\mathrm{SFT}}),$$

where for a general commutative ring $\mathscr{R}$, $\mathcal{A}_{\mathscr{R}}$ *denotes the graded supercommutative unital algebra over* $\mathscr{R}[H_2(M)/G]$ *generated by the* q_γ *variables for good Reeb orbits* γ. *The differentials* $\mathbf{D}_{\mathrm{SFT}}$ *on* $\mathcal{A}_{\mathbb{Z}}[[\hbar]]$ *and* $\mathcal{A}_{\mathbb{Z}_2}[[\hbar]]$ *are defined by the same formula as on* $\mathcal{A}[[\hbar]]$, *where in the* $\mathbb{Z}_2$ *case we are free to set all signs* $\epsilon(u)$ *equal to* 1.

PROOF OF PROPOSITION 12.13. We need to show that expressions of the form

$$\frac{\kappa_{\boldsymbol{\gamma}^+}}{|\operatorname{Aut}^\sigma(u)|}\frac{\partial}{\partial q_{\gamma_1^+}}\cdots\frac{\partial}{\partial q_{\gamma_{k_+}^+}}q^{\boldsymbol{\gamma}}$$

produce integer coefficients for every holomorphic curve u with asymptotic orbits $\boldsymbol{\gamma}^\pm = (\gamma_1^\pm, \ldots, \gamma_{k_\pm}^\pm)$ and every tuple $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_m)$. It suffices to consider the special case $\boldsymbol{\gamma} = \boldsymbol{\gamma}^+$, as the derivative in question is only nonzero on monomials that are divisible by $q^{\boldsymbol{\gamma}^+}$. Up to a sign change, we can reorder the orbits and write $\boldsymbol{\gamma}^+$ in the form

$$\boldsymbol{\gamma}^+ = (\underbrace{\gamma_1, \ldots, \gamma_1}_{m_1}, \ldots, \underbrace{\gamma_N, \ldots, \gamma_N}_{m_N})$$

for some finite set of distinct orbits $\gamma_1, \ldots, \gamma_N$ and numbers $m_i \in \mathbb{N}$, $i = 1, \ldots, N$. We then have

(12.19)

$$\begin{aligned}\frac{\kappa_{\boldsymbol{\gamma}^+}}{|\operatorname{Aut}^\sigma(u)|}\frac{\partial}{\partial q_{\gamma_1^+}}\cdots\frac{\partial}{\partial q_{\gamma_{k_+}^+}}q^{\boldsymbol{\gamma}^+} &= \frac{\kappa_{\gamma_1}^{m_1}\ldots\kappa_{\gamma_N}^{m_N}}{|\operatorname{Aut}^\sigma(u)|}\left(\frac{\partial}{\partial q_{\gamma_1}}\right)^{m_1}\cdots\left(\frac{\partial}{\partial q_{\gamma_N}}\right)^{m_N}\left(q_{\gamma_1}^{m_1}\ldots q_{\gamma_N}^{m_N}\right)\\ &= \pm\frac{\kappa_{\gamma_1}^{m_1}\ldots\kappa_{\gamma_N}^{m_N}m_1!\ldots m_N!}{|\operatorname{Aut}^\sigma(u)|}.\end{aligned}$$

We claim that this number is always an integer. Indeed, if $\operatorname{Aut}^\sigma(u)$ is nontrivial, then $u : \dot{\Sigma} \to \mathbb{R} \times M$ is a multiple cover $u = v \circ \varphi$ for some holomorphic branched cover $\varphi : (\Sigma, j) \to (\Sigma', j')$ and somewhere injective curve $v : (\dot{\Sigma}' = \Sigma' \setminus \Gamma', j') \to (\mathbb{R} \times M, J)$. Automorphisms $\psi \in \operatorname{Aut}^\sigma(u)$ thus define biholomorphic maps on (Σ, j) that permute each of the sets of punctures asymptotic to the same orbit. Given any puncture $z \in \Gamma$ where u is asymptotic to γ_i, the $\operatorname{Aut}^\sigma(u)$-orbit of z consists of $\ell \le m_i$ other punctures also asymptotic to γ_i, and its stabilizer is a cyclic subgroup of order $k = |\operatorname{Aut}^\sigma(u)|/\ell$, acting on a neighborhood of z by biholomorphic rotations. It follows that κ_{γ_i} is divisible by k, hence

$$\frac{\kappa_{\gamma_i}\ell}{|\operatorname{Aut}^\sigma(u)|} \in \mathbb{N},$$

and (12.19) is a multiple of this. □

REMARK 12.15. Since $1 = -1$ in $\mathcal{A}_{\mathbb{Z}_2}$, anticommuting elements of $\mathcal{A}_{\mathbb{Z}_2}[[\hbar]]$ actually commute, so unless one imposes extra algebraic conditions in the case of $\mathbb{Z}_2$ coefficients, higher powers of odd generators p_γ and q_γ do not vanish. Nonetheless, these powers still do not appear in $\mathbf{H}$, so the complex $(\mathcal{A}_{\mathbb{Z}_2}[[\hbar]], \mathbf{D}_{\mathrm{SFT}})$ ignores curves with multiple ends approaching an orbit of odd degree (and also bad orbits, for that matter).

LECTURE 13

Contact invariants

Contents

In the previous lecture, we introduced an operator algebra defined via the supercommutators $[p_\gamma, q_\gamma] = \kappa_\gamma \hbar$, then we defined the SFT generating function

$$\mathbf{H} = \sum_{u \in \mathcal{M}_1^\sigma(J)/\mathbb{R}} \frac{\epsilon(u)}{|\operatorname{Aut}^\sigma(u)|} \hbar^{g-1} e^A q^{\gamma^-} p^{\gamma^+}$$

and proved (modulo transversality) that $\mathbf{H}^2 = 0$. The generating function is a formal power series whose coefficients are rational counts of holomorphic curves, and these counts are strongly dependent on the choices of contact form α, almost complex structure $J \in \mathcal{J}(\alpha)$ and further auxiliary data such as coherent orientations. Thus in contrast to Gromov-Witten theory, the generating function does not define an invariant, but one can follow the standard prescription of Floer-type theories and define invariants via homology. We saw that for the natural representation $\mathcal{A}[[\hbar]]$ of the operator algebra defined by setting $p_\gamma = \kappa_\gamma \hbar \frac{\partial}{\partial q_\gamma}$, $\mathbf{H}$ defines a differential operator $\mathbf{D}_{\mathrm{SFT}} : \mathcal{A}[[\hbar]] \to \mathcal{A}[[\hbar]]$ with $\mathbf{D}_{\mathrm{SFT}}^2 = 0$. One of our goals in this lecture will be to explain (again modulo transversality) why the resulting homology

$$H_*^{\mathrm{SFT}}(M, \xi; R) = H_*(\mathcal{A}[[\hbar]], \mathbf{D}_{\mathrm{SFT}})$$

is an invariant of the contact structure. We will then use it to define simpler numerical invariants that detect symplectic fillability properties of contact manifolds.

But first, $\mathcal{A}[[\hbar]]$ is not the only possible representation of the operator algebra of SFT: other choices lead to different invariants with different algebraic structures. Let's begin by describing the original hierarchy of contact invariants that were outlined in [**EGH00**].

REMARK 13.1. Throughout this lecture, we assume for simplicity that $H_2(M)$ has no torsion, and the same assumption is made about cobordisms in §13.2.2. Only minor changes are necessary if this condition is lifted, e.g. one could then replace all instances of $H_2(M)$ with $H_2(M;\mathbb{Q})$ and assume always that the grading is $\mathbb{Z}_2$; see §12.7.1.

13.1. The Eliashberg-Givental-Hofer package

In the following, (M,ξ) is a $(2n-1)$-dimensional closed contact manifold with a contact form α and almost complex structure $J \in \mathcal{J}(\alpha)$ for which the optimistic transversality condition (Assumption 12.1) of Lecture 12 is assumed to hold. We fix also the auxiliary data described in §12.2, plus a choice of subgroup $G \subset H_2(M)$ which determines the coefficient ring

$$R = \mathbb{Q}[H_2(M)/G].$$

Each of the differential graded algebras described below then carries the same grading that was described in that lecture, i.e. there is always at least a $\mathbb{Z}_2$-grading, and it lifts to $\mathbb{Z}$ if $H_1(M)$ is torsion free and $c_1(\xi)|_G = 0$, or possibly $\mathbb{Z}_{2N}$ if $N \in \mathbb{N}$ is the smallest possible value for $c_1(A)$ with $A \in G$.

13.1.1. Full SFT as a Weyl superalgebra.

We start with some seemingly trivial algebraic observations. First, the relation $\mathbf{H}^2 = 0$ is equivalent to

$$[\mathbf{H},\mathbf{H}] = 0.$$

Remember that $[\ ,\ ]$ is a *super*-commutator, so $[\mathbf{F},\mathbf{F}] = 0$ holds automatically for operators $\mathbf{F}$ with even degree, but $\mathbf{H}$ is odd, and for odd operators the commutator is defined by $[\mathbf{F},\mathbf{G}] = \mathbf{F}\mathbf{G} + \mathbf{G}\mathbf{F}$, hence $[\mathbf{H},\mathbf{H}] = 2\mathbf{H}^2$. Formally speaking $[\ ,\ ]$ is a **super Lie bracket** and thus satisfies the "super Jacobi identity":

$$\big[\mathbf{F},[\mathbf{G},\mathbf{K}]\big] + (-1)^{|\mathbf{F}||\mathbf{G}|+|\mathbf{F}||\mathbf{K}|}\big[\mathbf{G},[\mathbf{K},\mathbf{F}]\big] + (-1)^{|\mathbf{F}||\mathbf{K}|+|\mathbf{G}||\mathbf{K}|}\big[\mathbf{K},[\mathbf{F},\mathbf{G}]\big] = 0. \tag{13.1}$$

A consequence of this is that in order to create a homology theory out of $\mathbf{H}$, we don't absolutely need to find a representation of the entire operator algebra: it suffices to find a representation of the induced super Lie algebra. Indeed, suppose V is a graded $R[[\hbar]]$-module and L is a linear grading-preserving map that associates to operators $\mathbf{F}$ (expressed as power series functions of p's, q's and $\hbar$ with coefficients in R) an $R[[\hbar]]$-linear map

$$L_{\mathbf{F}} : V \to V$$

such that

$$L_{[\mathbf{F},\mathbf{G}]} = L_{\mathbf{F}}L_{\mathbf{G}} - (-1)^{|\mathbf{F}||\mathbf{G}|}L_{\mathbf{G}}L_{\mathbf{F}}$$

for every pair of operators $\mathbf{F},\mathbf{G}$. Then the $R[[\hbar]]$-linear map $L_{\mathbf{H}} : V \to V$ satisfies

$$L_{\mathbf{H}}^2 = \frac{1}{2}[L_{\mathbf{H}},L_{\mathbf{H}}] = \frac{1}{2}L_{[\mathbf{H},\mathbf{H}]} = 0,$$

hence $(V, L_{\mathbf{H}})$ is a chain complex. The complex $(\mathcal{A}[[\hbar]],\mathbf{D}_{\mathrm{SFT}})$ was a special case of this, in which we represented the super Lie algebra via a faithful representation of the whole operator algebra.

EXERCISE 13.2. Verify (13.1).

Remark 13.3 (supersymmetric sign rules). To see where the signs in (13.1) come from, it suffices to know the following basic rule of superalgebra: for any pair of $\mathbb{Z}_2$-graded vector spaces V and W, the natural "commutation" isomorphism $c : V \otimes W \to W \otimes V$ is defined on homogeneous elements by

$$c(v \otimes w) = (-1)^{|v||w|} w \otimes v.$$

For any permutation of a finite tuple of $\mathbb{Z}_2$-graded vector spaces, one can derive the appropriate isomorphism from this: in particular the cyclic permutation isomorphism $\sigma : X \otimes Y \otimes Z \to Y \otimes Z \otimes X$ takes the form

$$\sigma = (\mathbb{1} \otimes c_{23}) \circ (c_{12} \otimes \mathbb{1}) : x \otimes y \otimes z \mapsto (-1)^{|x||y|+|x||z|} y \otimes z \otimes x.$$

Writing the Jacobi identity as $[\cdot, [\cdot, \cdot]] \circ (\mathbb{1} + \sigma + \sigma^2) = 0$ then produces (13.1). In this sense, it only differs from the usual Jacobi identity in being based on a different definition of the commutation isomorphism $V \otimes W \to W \otimes V$. For more on this perspective, see [**Var04**, §3.1].

Now here is a different kind of example, where the representation does not respect the product structure of the operator algebra but does respect its Lie bracket. Let $\mathfrak{W}$ denote the graded unital algebra consisting of formal power series

$$\sum_{\boldsymbol{\gamma},k} f_{\boldsymbol{\gamma},k}(q) \hbar^k p^{\boldsymbol{\gamma}},$$

where the sum ranges over all integers $k \geq 0$ and all ordered sets $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_m)$ of good Reeb orbits for $m \geq 0$, and the $f_{\boldsymbol{\gamma},k}$ are polynomial functions of the q_γ variables, with coefficients in R. Note that the case of the empty set of orbits is included here, which means $p^{\boldsymbol{\gamma}} = 1$. The multiplicative structure of $\mathfrak{W}$ is defined via the usual (super)commutation relations, and its elements can be interpreted as operators. If we now associate to each $\mathbf{F} \in \mathfrak{W}$ the $R[[\hbar]]$-linear map

$$D_{\mathbf{F}} : \mathfrak{W} \to \mathfrak{W} : \mathbf{G} \mapsto [\mathbf{F}, \mathbf{G}],$$

then the Jacobi identity (13.1) implies

$$D_{[\mathbf{F},\mathbf{G}]} = D_{\mathbf{F}} D_{\mathbf{G}} - (-1)^{|\mathbf{F}||\mathbf{G}|} D_{\mathbf{G}} D_{\mathbf{F}}.$$

This is just the graded version of the standard *adjoint representation* of a Lie algebra. The only problem in applying this idea to define a differential

$$D_{\mathbf{H}} : \mathfrak{W} \to \mathfrak{W} : \mathbf{F} \mapsto [\mathbf{H}, \mathbf{F}] \tag{13.2}$$

is that $\mathbf{H}$ is not technically an element of $\mathfrak{W}$: indeed, $\mathbf{H}$ contains terms of order -1 in $\hbar$, thus

$$\mathbf{H} \in \frac{1}{\hbar}\mathfrak{W}.$$

On the other hand, the failure of supercommutativity in $\mathfrak{W}$ is a "phenomenon of order $\hbar$," i.e. since every nontrivial commutator contains a factor of $\hbar$, we have

$$[\mathbf{F}, \mathbf{G}] = \mathcal{O}(\hbar) \quad \text{for all} \quad \mathbf{F}, \mathbf{G} \in \mathfrak{W}.$$

Here and in the following we use the symbol

$$\mathcal{O}(\hbar^k)$$

to denote any element of the form $\hbar^k \mathbf{F}$ for $\mathbf{F} \in \mathfrak{W}$. As a consequence, $[\mathbf{H}, \mathbf{F}] \in \mathfrak{W}$ whenever $\mathbf{F} \in \mathfrak{W}$, hence (13.2) is well defined, and the Jacobi identity now implies

$$D_{\mathbf{H}}^2 = 0.$$

The homology of the resulting chain complex gives another version of what is often called **full SFT**,

$$H_*^{\mathfrak{W}}(M, \xi; R) := H_*(\mathfrak{W}, D_{\mathbf{H}}).$$

A proof (modulo transversality) that this defines a contact invariant is outlined in [**EGH00**, §2], but it is algebraically somewhat more involved than for $H_*^{\mathrm{SFT}}(M, \xi; R)$, so I will skip it since I don't have any applications of $H_*^{\mathfrak{W}}(M, \xi; R)$ in mind. As far as I am aware, no contact topological applications of this invariant or computations of it (outside the trivial case—see §13.1.4 below) have yet appeared in the literature. This is a pity, because $H_*^{\mathfrak{W}}(M, \xi; R)$ actually has much more algebraic structure than $H_*^{\mathrm{SFT}}(M, \xi; R)$. Indeed, using the identities

$$\begin{aligned} [\mathbf{F}, \mathbf{G}\mathbf{K}] &= [\mathbf{F}, \mathbf{G}]\mathbf{K} + (-1)^{|\mathbf{F}||\mathbf{G}|}\mathbf{G}[\mathbf{F}, \mathbf{K}], \\ [\mathbf{F}\mathbf{G}, \mathbf{K}] &= \mathbf{F}[\mathbf{G}, \mathbf{K}] + (-1)^{|\mathbf{G}||\mathbf{K}|}[\mathbf{F}, \mathbf{K}]\mathbf{G}, \end{aligned} \tag{13.3}$$

one sees that $D_{\mathbf{H}} : \mathfrak{W} \to \mathfrak{W}$ satisfies a graded Leibniz rule,

$$D_{\mathbf{H}}(\mathbf{F}\mathbf{G}) = (D_{\mathbf{H}}\mathbf{F})\mathbf{G} + (-1)^{|\mathbf{F}|}\mathbf{F}\, D_{\mathbf{H}}\mathbf{G}.$$

It follows that $D_{\mathbf{H}} : \mathfrak{W} \to \mathfrak{W}$ is also a derivation with respect to the bracket structure on $\mathfrak{W}$, i.e.

$$D_{\mathbf{H}}[\mathbf{F}, \mathbf{G}] = [D_{\mathbf{H}}\mathbf{F}, \mathbf{G}] + (-1)^{|\mathbf{F}|}[\mathbf{F}, D_{\mathbf{H}}\mathbf{G}]$$

for all $\mathbf{F}, \mathbf{G} \in \mathfrak{W}$. As a consequence, the product and bracket structures on $\mathfrak{W}$ descend to $H_*^{\mathfrak{W}}(M, \xi; R)$, giving it the structure of a *Weyl superalgebra*.

As a matter of interest, we observe that $(\mathfrak{W}, D_{\mathbf{H}})$, as with $(\mathcal{A}[[\hbar]], \mathbf{D}_{\mathrm{SFT}})$ in the previous lecture, can be defined with $\mathbb{Z}$ or $\mathbb{Z}_2$ coefficients whenever the transversality results are good enough to take the usual expression $\sum_u \frac{\epsilon(u)}{|\mathrm{Aut}^\sigma(u)|}$ literally as a count of holomorphic curves. This result is of limited interest since it cannot hold in general cases where transversality for multiple covers is impossible without multivalued perturbations—nonetheless I find it amusing.[1]

PROPOSITION 13.4. *If Assumption 12.1 in Lecture 12 holds, then $D_{\mathbf{H}}$ is also well defined if the ring $R = \mathbb{Q}[H_2(M)/G]$ is replaced by $\mathbb{Z}[H_2(M)/G]$ or $\mathbb{Z}_2[H_2(M)/G]$.*

PROOF. Since $D_{\mathbf{H}}$ is a derivation, it suffices to check that for every good Reeb orbit γ, $D_{\mathbf{H}} q_\gamma$ and $D_{\mathbf{H}} p_\gamma$ are each sums of monomials of the form $c e^A \hbar^k q^{\gamma^-} p^{\gamma^+}$ with

---

[1] The same arguments used to define SFT chain complexes over the integers can also be applied to the chain maps involved in the proof of invariance (see §13.3.1), so the SFT invariants *should* be defined over the integers if transversality can be achieved for multiple covers. There are known situations however in which this cannot hold: even if the chain complexes are well defined over $\mathbb{Z}$, invariance may hold only over $\mathbb{Q}$, due to the failure of transversality in cobordisms. See [**Hut**].

coefficients $c \in \mathbb{Z}$. Suppose $u \in \mathcal{M}_1(J)$ is an index 1 holomorphic curve with positive and/or negative asymptotic orbits

$$\boldsymbol{\gamma}^\pm = (\underbrace{\gamma_1^\pm, \dots, \gamma_1^\pm}_{m_1^\pm}, \dots, \underbrace{\gamma_{k_\pm}^\pm, \dots, \gamma_{k_\pm}^\pm}_{m_{k_\pm}^\pm}),$$

where $\gamma_i^\pm \neq \gamma_j^\pm$ for $i \neq j$. We can assume all the orbits $\gamma_i^\pm$ are good and that $m_i^\pm = 1$ whenever $n - 3 + \mu_{\mathrm{CZ}}(\gamma_i^\pm)$ is odd. Up to a sign and factors of e^A and $\hbar$ which are not relevant to this discussion, u then contributes a monomial

$$\mathbf{H}_u := \frac{1}{|\operatorname{Aut}^\sigma(u)|} q_{\gamma_1^-}^{m_1^-} \dots q_{\gamma_{k_-}^-}^{m_{k_-}^-} p_{\gamma_1^+}^{m_1^+} \dots p_{\gamma_{k_+}^+}^{m_{k_+}^+}$$

to $\mathbf{H}$. The commutator $[\mathbf{H}_u, q_\gamma]$ vanishes unless γ is one of the orbits $\gamma_1^+, \dots, \gamma_{k_+}^+$, so suppose $\gamma = \gamma_{k_+}^+$. If $n - 3 + \mu_{\mathrm{CZ}}(\gamma)$ is odd, then $m := m_{k_+}^+ = 1$, and (13.3) with $[p_\gamma, q_\gamma] = \kappa_\gamma \hbar$ implies

$$\begin{aligned}
[\mathbf{H}_u, q_\gamma] &= \frac{1}{|\operatorname{Aut}^\sigma(u)|} \left[q_{\gamma_1^-}^{m_1^-} \dots q_{\gamma_{k_-}^-}^{m_{k_-}^-} p_{\gamma_1^+}^{m_1^+} \dots p_{\gamma_{k_+-1}^+}^{m_{k_+-1}^+} p_\gamma, q_\gamma \right] \\
&= \frac{\kappa_\gamma}{|\operatorname{Aut}^\sigma(u)|} \hbar q_{\gamma_1^-}^{m_1^-} \dots q_{\gamma_{k_-}^-}^{m_{k_-}^-} p_{\gamma_1^+}^{m_1^+} \dots p_{\gamma_{k_+-1}^+}^{m_{k_+-1}^+}.
\end{aligned}$$

The fraction in front of this expression is an integer since u can have only one end asymptotic to γ, and κ_γ is thus divisible by the covering multiplicity of u. If $n - 3 + \mu_{\mathrm{CZ}}(\gamma)$ is even, then we generalize this calculation by using (13.3) to write

$$[p_\gamma^m, q_\gamma] = m \kappa_\gamma \hbar p_\gamma^{m-1},$$

so then,

$$\begin{aligned}
[\mathbf{H}_u, q_\gamma] &= \frac{1}{|\operatorname{Aut}^\sigma(u)|} \left[q_{\gamma_1^-}^{m_1^-} \dots q_{\gamma_{k_-}^-}^{m_{k_-}^-} p_{\gamma_1^+}^{m_1^+} \dots p_{\gamma_{k_+-1}^+}^{m_{k_+-1}^+} p_\gamma^m, q_\gamma \right] \\
&= \frac{\kappa_\gamma m}{|\operatorname{Aut}^\sigma(u)|} \hbar q_{\gamma_1^-}^{m_1^-} \dots q_{\gamma_{k_-}^-}^{m_{k_-}^-} p_{\gamma_1^+}^{m_1^+} \dots p_{\gamma_{k_+-1}^+}^{m_{k_+-1}^+} p_\gamma^{m-1}.
\end{aligned}$$

To see that $\frac{\kappa_\gamma m}{|\operatorname{Aut}^\sigma(u)|}$ is always an integer, recall from our proof of Prop. 12.13 in the previous lecture that transformations in $\operatorname{Aut}^\sigma(u)$ permute each of the sets of punctures that are asymptotic to the same Reeb orbit. Suppose the set of positive punctures of u asymptotic to γ is partitioned by the $\operatorname{Aut}^\sigma(u)$-action into N subsets, each consisting of $\ell_1, \dots, \ell_N$ punctures, where $\ell_1 + \dots + \ell_N = m$. If z is a puncture in the ith of these subsets, then its stabilizer is a cyclic subgroup of order k_i acting on a neighborhood of z by biholomorphic rotations, where $k_i \ell_i = |\operatorname{Aut}^\sigma(u)|$. Each of these orders k_i necessarily divides the multiplicity κ_γ, so we can write $k_i a_i = \kappa_\gamma$ for some $a_i \in \mathbb{N}$. Putting all this together, we have

$$\kappa_\gamma m = \sum_{i=1}^N \kappa_\gamma \ell_i = \sum_{i=1}^N k_i a_i \ell_i = |\operatorname{Aut}^\sigma(u)| \sum_{i=1}^N a_i.$$

Following this same procedure, you should now be able to verify on your own that the coefficient appearing in $[\mathbf{H}_u, p_\gamma]$ is also always an integer. The existence of a chain complex with $\mathbb{Z}_2$ coefficients follows from this simply by projecting $\mathbb{Z}$ to $\mathbb{Z}_2$. □

13.1.2. The semiclassical limit: rational SFT. The idea of rational symplectic field theory (RSFT) is to extract as much information as possible from genus zero holomorphic curves but ignore curves of higher genus. The algebra of SFT provides a fairly obvious mechanism for this: RSFT should be what SFT becomes in the "limit as $\hbar \to 0$," i.e. the classical approximation to a quantum theory. Let

$$\mathfrak{P} := \mathfrak{W}/\hbar\mathfrak{W},$$

so $\mathfrak{P}$ is a graded unital algebra generated by the p_γ and q_γ variables and the coefficient ring R, but it does not include $\hbar$ as a generator. Since all commutators in $\mathfrak{W}$ are in $\hbar\mathfrak{W}$, the product structure of $\mathfrak{P}$ is supercommutative. Let us use the distinction between capital and lowercase letters to denote the quotient projection

$$\mathfrak{W} \to \mathfrak{P} : \mathbf{F} \mapsto \mathbf{f}.$$

We will make an exception for the letter "H": recall that $\mathbf{H}$ is not an element of $\mathfrak{W}$ since its genus zero terms have order -1 in $\hbar$, but $\hbar\mathbf{H} \in \mathfrak{W}$, so we will define

$$\mathbf{h} = \sum_u \frac{\epsilon(u)}{|\operatorname{Aut}^\sigma(u)|} e^A q^{\gamma^-} p^{\gamma^+} \in \mathfrak{P}$$

to be the image of $\hbar\mathbf{H}$ under the projection. The sum in this expression ranges over all $\mathbb{R}$-equivalence classes of index 1 curves with genus zero, so $\mathbf{h}$ will serve as the generating function of RSFT. To encode gluing of genus zero terms, note first that the commutator operation would not be appropriate since it prodcues terms for *every* possible gluing of two curves, including those which glue genus zero curves along more than one breaking orbit to produce buildings with positive arithmetic genus. We need instead to have an algebraic operation on $\mathfrak{P}$ that encodes gluing along only one breaking orbit at a time.

You already know what to expect if you've ever taken a quantum mechanics course: in the "classical limit," commutators become Poisson brackets. To express this properly, we need to make a distinction between differential operators operating from the left or the right: let

$$\frac{\overrightarrow{\partial}}{\partial q_\gamma} : \mathfrak{W} \to \mathfrak{W}$$

denote the usual operator $\frac{\partial}{\partial q_\gamma}$, which was previously defined on $\mathcal{A}[[\hbar]]$ but has an obvious extension to $\mathfrak{W}$ such that $\frac{\overrightarrow{\partial}}{\partial q_\gamma} p_{\gamma'} = 0$ for all γ'. This operator satisfies the graded Leibniz rule

$$\frac{\overrightarrow{\partial}}{\partial q_\gamma}(\mathbf{FG}) = \left(\frac{\overrightarrow{\partial}}{\partial q_\gamma}\mathbf{F}\right)\mathbf{G} + (-1)^{|q_\gamma||\mathbf{F}|}\mathbf{F}\left(\frac{\overrightarrow{\partial}}{\partial q_\gamma}\mathbf{G}\right).$$

The related operator

$$\frac{\overleftarrow{\partial}}{\partial q_\gamma} : \mathfrak{W} \to \mathfrak{W} : \mathbf{F} \mapsto \mathbf{F}\frac{\overleftarrow{\partial}}{\partial q_\gamma}$$

is defined exactly the same way on individual variables p_γ and q_γ, but satisfies a slightly different Leibniz rule,

$$(\mathbf{F}\mathbf{G})\frac{\overleftarrow{\partial}}{\partial q_\gamma} = \mathbf{F}\left(\mathbf{G}\frac{\overleftarrow{\partial}}{\partial q_\gamma}\right) + (-1)^{|q_\gamma||\mathbf{G}|}\left(\mathbf{F}\frac{\overleftarrow{\partial}}{\partial q_\gamma}\right)\mathbf{G}.$$

The point of writing $\frac{\overleftarrow{\partial}}{\partial q_\gamma}$ so that it acts from the right is to obey the usual conventions of superalgebra: signs change whenever the order of two odd elements (or operators) is interchanged. Partial derivatives with respect to p_γ can be defined analogously on $\mathfrak{W}$. With this notation in hand, the **graded Poisson bracket** on $\mathfrak{W}$ is defined by

$$\{\mathbf{F},\mathbf{G}\} = \sum_\gamma \kappa_\gamma \left(\mathbf{F}\frac{\overleftarrow{\partial}}{\partial p_\gamma}\frac{\overrightarrow{\partial}}{\partial q_\gamma}\mathbf{G} - (-1)^{|\mathbf{F}||\mathbf{G}|}\mathbf{G}\frac{\overleftarrow{\partial}}{\partial p_\gamma}\frac{\overrightarrow{\partial}}{\partial q_\gamma}\mathbf{F}\right), \tag{13.4}$$

where the sum ranges over all good Reeb orbits. In the same manner, the differential operators and the bracket $\{\ ,\ \}$ can also be defined on $\mathfrak{P}$.

It is easy to check that $\{\ ,\ \}$ on $\mathfrak{W}$ *almost* satisfies a version of (13.3): we have

$$\begin{aligned}\{\mathbf{F},\mathbf{G}\mathbf{K}\} &= \{\mathbf{F},\mathbf{G}\}\mathbf{K} + (-1)^{|\mathbf{F}||\mathbf{G}|}\mathbf{G}\{\mathbf{F},\mathbf{K}\} + \mathcal{O}(\hbar),\\ \{\mathbf{F}\mathbf{G},\mathbf{K}\} &= \mathbf{F}\{\mathbf{G},\mathbf{K}\} + (-1)^{|\mathbf{G}||\mathbf{K}|}\{\mathbf{F},\mathbf{K}\}\mathbf{G} + \mathcal{O}(\hbar)\end{aligned} \tag{13.5}$$

for all $\mathbf{F},\mathbf{G},\mathbf{K} \in \mathfrak{W}$. The extra terms denoted by $\mathcal{O}(\hbar)$ arise from the fact that in proving (13.5), we must sometimes reorder products $\mathbf{F}\mathbf{G}$ by writing them as $(-1)^{|\mathbf{F}||\mathbf{G}|}\mathbf{G}\mathbf{F} + [\mathbf{F},\mathbf{G}]$, where $[\mathbf{F},\mathbf{G}] = \mathcal{O}(\hbar)$. Since the terms with $\hbar$ disappear in $\mathfrak{P}$, the relations become exact in $\mathfrak{P}$:

$$\begin{aligned}\{\mathbf{f},\mathbf{g}\mathbf{k}\} &= \{\mathbf{f},\mathbf{g}\}\mathbf{k} + (-1)^{|\mathbf{f}||\mathbf{g}|}\mathbf{g}\{\mathbf{f},\mathbf{k}\},\\ \{\mathbf{f}\mathbf{g},\mathbf{k}\} &= \mathbf{f}\{\mathbf{g},\mathbf{k}\} + (-1)^{|\mathbf{g}||\mathbf{k}|}\{\mathbf{f},\mathbf{k}\}\mathbf{g}\end{aligned} \tag{13.6}$$

for all $\mathbf{f},\mathbf{g},\mathbf{k} \in \mathfrak{P}$.

PROPOSITION 13.5. *For all* $\mathbf{F},\mathbf{G} \in \mathfrak{W}$,

$$[\mathbf{F},\mathbf{G}] = \hbar\{\mathbf{f},\mathbf{g}\} + \mathcal{O}(\hbar^2),$$

and $\{\ ,\ \}$ *satisfies the conditions of a super Lie bracket on* $\mathfrak{P}$.

REMARK 13.6. In formulas like the one in the above proposition, we interpret $\{\mathbf{f},\mathbf{g}\} \in \mathfrak{P}$ as an element of $\mathfrak{W}$ via any choice of R-linear inclusion $\mathfrak{P} \hookrightarrow \mathfrak{W}$ that acts as the identity on the generators p_γ, q_γ. There is ambiguity in this choice due to the noncommutativity of $\mathfrak{W}$, but the ambiguity is in $\hbar\mathfrak{W}$ and thus makes no difference to the formula.

PROOF OF PROPOSITION 13.5. The formula is easily checked when $\mathbf{F}$ and $\mathbf{G}$ are individual variables of the form p_γ or q_γ; in fact the extra term $\mathcal{O}(\hbar^2)$ can be omitted in these cases. The case where $\mathbf{F}$ and $\mathbf{G}$ are general monomials follows from this via (13.3) and (13.5) using induction on the number of variables in the product. This implies the general case via bilinearity.

Given the formula, the condition $\{\mathbf{f},\mathbf{g}\} + (-1)^{|\mathbf{f}||\mathbf{g}|}\{\mathbf{g},\mathbf{f}\} = 0$ and the Poisson version of the super Jacobi identity (13.1) follow from the corresponding properties of $[\ ,\ ]$. $\square$

The proposition implies that our genus zero generating function $\mathbf{h} \in \mathfrak{P}$ satisfies $0 = \hbar^2[\mathbf{H},\mathbf{H}] = [\hbar\mathbf{H},\hbar\mathbf{H}] = \hbar\{\mathbf{h},\mathbf{h}\} + \mathcal{O}(\hbar^2)$, thus

$$\{\mathbf{h},\mathbf{h}\} = 0.$$

This relation can be interpreted as the count of boundary points of all 1-dimensional moduli spaces of genus zero curves: indeed, any pair of genus two curves $u, v \in \mathcal{M}_1^\sigma(J)/\mathbb{R}$ constributes to $\{\mathbf{h},\mathbf{h}\}$ a term of the form

$$\sum_\gamma \frac{\kappa_\gamma}{|\operatorname{Aut}^\sigma(u)||\operatorname{Aut}^\sigma(v)|} e^{A_u + A_v} q^{\gamma_u^-} \left(p^{\gamma_u^+} \overleftarrow{\frac{\partial}{\partial p_\gamma}} \right) \left(\overrightarrow{\frac{\partial}{\partial q_\gamma}} q^{\gamma_v^-} \right) p^{\gamma_v^+},$$

plus a corresponding term with the roles of u and v reversed. This sums all the monomials that one can construct by cancelling one p_γ variable from u with a matching q_γ variable from v, in other words, constructing a building by gluing v on top of u along one matching Reeb orbit.

The graded Jacobi identity will again imply that any representation of the super Lie algebra $(\mathfrak{P}, \{\ ,\ \})$ gives rise to a chain complex with $\mathbf{h}$ as its differential. For example we can take the adjoint representation,

$$\mathfrak{P} \to \operatorname{End}_R(\mathfrak{P}) : \mathbf{f} \mapsto d_\mathbf{f}, \qquad d_\mathbf{f}\mathbf{g} := \{\mathbf{f},\mathbf{g}\},$$

which satisfies $d_{\{\mathbf{f},\mathbf{g}\}} = d_\mathbf{f} d_\mathbf{g} - (-1)^{|\mathbf{f}||\mathbf{g}|} d_\mathbf{g} d_\mathbf{f}$ due to the Jacobi identity. Then $d_\mathbf{h}^2 = 0$ since $\mathbf{h}$ has odd degree and $\{\mathbf{h},\mathbf{h}\} = 0$, and the homology of **rational SFT** is defined as

$$H_*^{\mathrm{RSFT}}(M,\xi;R) := H_*(\mathfrak{P}, d_\mathbf{h}).$$

We again refer to [EGH00] for an argument that $H_*^{\mathrm{RSFT}}(M,\xi;R)$ is an invariant of the contact structure. Notice that Proposition 13.5 yields a simple relationship between the chain complexes $(\mathfrak{W}, D_\mathbf{H})$ and $(\mathfrak{P}, d_\mathbf{h})$, namely

$$D_\mathbf{H}\mathbf{F} = d_\mathbf{h}\mathbf{f} + \mathcal{O}(\hbar), \tag{13.7}$$

where $d_\mathbf{h}\mathbf{f}$ is interpreted as an element of $\mathfrak{W}$ via Remark 13.6. In other words, the projection $\mathfrak{W} \to \mathfrak{P} : \mathbf{F} \to \mathbf{f}$ is a chain map. Moreover, $d_\mathbf{H}$ is a derivation on $\mathfrak{P}$ with respect to both the product and the Poisson bracket: this follows via Proposition 13.5 and (13.7) from the fact that $D_\mathbf{H}$ satisfies the corresponding properties on $\mathfrak{W}$. We conclude that $H_*^{\mathrm{RSFT}}(M,\xi;R)$ inherits the structure of a Poisson superalgebra, and the map

$$H_*^{\mathfrak{W}}(M,\xi;R) \to H_*^{\mathrm{RSFT}}(M,\xi;R)$$

induced by the chain map $(\mathfrak{W}, D_\mathbf{H}) \to (\mathfrak{P}, d_\mathbf{h})$ is both an algebra homomorphism and a homomorphism of graded super Lie algebras.

13.1.3. The contact homology algebra. Contact homology is the most popular tool in the SFT package and was probably the first to be understood beyond the more straightforward cylindrical theory. In situations where cylindrical contact homology cannot be defined due to bubbling of holomorphic planes, the next simplest thing one can do is to define a theory that counts genus zero curves with one positive end but *arbitrary* numbers of negative ends (cf. Exercise 10.14 in Lecture 10).

The proper algebraic setting for such a theory turns out to be the algebra $\mathcal{A}$ generated by the q_γ variables, and it can be derived from RSFT by setting all p_γ variables to zero. Using the obvious inclusion $\mathcal{A} \hookrightarrow \mathfrak{P}$, define $\partial_{\mathrm{CH}} : \mathcal{A} \to \mathcal{A}$ by

$$\partial_{\mathrm{CH}}\mathbf{f} = d_{\mathbf{h}}\mathbf{f}|_{p=0}.$$

We can thus write $d_{\mathbf{h}}\mathbf{f} = \partial_{\mathrm{CH}}\mathbf{f} + \mathcal{O}(p)$, where

$$\mathcal{O}(p^k)$$

will be used generally to denote any formal sum consisting exclusively of terms of the form $p_{\gamma_1} \dots p_{\gamma_k}\mathbf{f}$ for $\mathbf{f} \in \mathfrak{P}$. Now observe that for any good orbit γ,

$$d_{\mathbf{h}}p_\gamma = \{\mathbf{h}, p_\gamma\} = -(-1)^{|p_\gamma|}\sum_{\gamma'}\left(p_\gamma \frac{\overleftarrow{\partial}}{\partial p_{\gamma'}}\right)\left(\frac{\overrightarrow{\partial}}{\partial q_{\gamma'}}\mathbf{h}\right) = -(-1)^{|p_\gamma|}\frac{\partial \mathbf{h}}{\partial q_\gamma} = \mathcal{O}(p)$$

since every term in $\mathbf{h}$ has at least one p variable. It follows that $d_{\mathbf{h}}(\mathcal{O}(p)) = \mathcal{O}(p)$, so the fact that $d_{\mathbf{h}}^2 = 0$ implies $\partial_{\mathrm{CH}}^2 = 0$, and **contact homology** is defined as

$$HC_*(M, \xi; R) := H_*(\mathcal{A}, \partial_{\mathrm{CH}}).$$

Since $d_{\mathbf{h}}$ is a derivation on $\mathfrak{P}$, the formula $d_{\mathbf{h}}\mathbf{f} = \partial_{\mathrm{CH}}\mathbf{f} + \mathcal{O}(p)$ implies that ∂_{CH} is likewise a derivation on $\mathcal{A}$, so $HC_*(M, \xi; R)$ has the structure of a graded supercommutative algebra with unit. Moreover, the projection $\mathfrak{P} \to \mathcal{A} : \mathbf{f} \mapsto \mathbf{f}|_{p=0}$ is a chain map, giving rise to an algebra homomorphism

$$H_*^{\mathrm{RSFT}}(M, \xi; R) \to HC_*(M, \xi; R).$$

The invariance of $HC_*(M, \xi; R)$ will follow from the invariance of $H_*^{\mathrm{SFT}}(M, \xi; R)$, to be discussed in §13.3.1 below.

To interpret ∂_{CH}, we can separate the part of $\mathbf{h}$ that is linear in p variables, writing

$$\mathbf{h} = \sum_\gamma \mathbf{h}_\gamma(q)p_\gamma + \mathcal{O}(p^2),$$

where for each good Reeb orbit γ, $\mathbf{h}_\gamma(q)$ denotes a polynomial in q variables with coefficients in R. Since elements $\mathbf{f} \in \mathcal{A}$ have no dependence on p variables, we then have

$$d_{\mathbf{h}}\mathbf{f} = \{\mathbf{h}, \mathbf{f}\} = \sum_\gamma \kappa_\gamma \left(\mathbf{h}\frac{\overleftarrow{\partial}}{\partial p_\gamma}\right)\left(\frac{\overrightarrow{\partial}}{\partial q_\gamma}\mathbf{f}\right) = \sum_\gamma \kappa_\gamma \mathbf{h}_\gamma \frac{\partial \mathbf{f}}{\partial q_\gamma} + \mathcal{O}(p),$$

hence

$$\partial_{\mathrm{CH}}\mathbf{f} = \sum_\gamma \kappa_\gamma \mathbf{h}_\gamma \frac{\partial \mathbf{f}}{\partial q_\gamma}.$$

In particular, ∂_{CH} acts on each generator $q_\gamma \in \mathcal{A}$ as

$$\partial_{\mathrm{CH}} q_\gamma = \kappa_\gamma \mathbf{h}_\gamma = \sum_u \frac{\epsilon(u)\kappa_\gamma}{\mathrm{Aut}^\sigma(u)} e^A q^{\boldsymbol{\gamma}^-},$$

where the sum is over all $\mathbb{R}$-equivalence classes of index 1 J-holomorphic curves u with genus zero, one positive end at γ, and negative ends $\boldsymbol{\gamma}^-$, and homology class $A \in H_2(M)/G$.

13.1.4. Algebraic overtwistedness. Even the simplest of the three differential graded algebras described above is too large to compute in most cases. The major exception is the case of overtwisted contact manifolds.

THEOREM 13.7. *If (M, ξ) is overtwisted, then $HC_*(M, \xi; R) = 0$ for all choices of the coefficient ring R.*

REMARK 13.8. If X is an algebra with unit, then saying $X = 0$ is equivalent to saying that $1 = 0$ in X.

The notion of overtwisted contact structures in dimension three was introduced by Eliashberg in [Eli89], who proved that they are *flexible* in the sense that their classification up to isotopy reduces to the purely obstruction-theoretic classification of almost contact structures up to homotopy. This means in effect that an overtwisted contact structure carries no distinctly contact geometric information, so it should not be surprising when "interesting" contact invariants such as $HC_*(M, \xi)$ vanish. The three-dimensional case of Theorem 13.7 seems to have been among the earliest insights about SFT: its first appearance in the literature was in [Eli98], and a proof later appeared in a paper by Mei-Lin Yau [Yau06], which includes a brief appendix sketching Eliashberg's original proof. We will discuss Eliashberg's proof in detail in Lecture 16.

The definitive higher-dimensional notion of overtwistedness was introduced a few years ago by Borman-Eliashberg-Murphy [BEM15], following earlier steps in this direction by Niederkrüger [Nie06] and others. There are now two known proofs of Theorem 13.7 in higher dimensions: the first uses the fact that since overtwisted contact manifolds are flexible, they always admit an embedding of a *plastikstufe*, which implies vanishing of contact homology by an unpublished result of Bourgeois and Niederkrüger (see [Bou09, Theorem 4.10] for a sketch of the argument). The second argument appeals to an even more recent result of Casals-Murphy-Presas [CMP] showing that (M, ξ) is overtwisted if and only if it is supported by a negatively stabilized open book, in which case $HC_*(M, \xi) = 0$ was proven by Bourgeois and van Koert [BvK10].

It is not known whether the vanishing of contact homology *characterizes* overtwistedness, i.e. there are not yet any examples of tight contact manifolds with $HC_*(M, \xi) = 0$. I will go out on a limb and say that such examples seem unlikely to exist in dimension three but are much more likely in higher dimensions; in fact various candidates are known [MNW13, CDvK], but we do not yet have adequate methods to prove that any of them are tight. The analogous question about Legendrian submanifolds and relative contact homology was recently answered by Ekholm

[**Ekh**], giving examples of Legendrians that are not *loose* in the sense of Murphy [**Mur**] but have vanishing Legendrian contact homology.

Nevertheless, the lack of known counterexamples has given rise to the following definition.

DEFINITION 13.9. A closed contact manifold (M, ξ) is **algebraically overtwisted** if $HC_*(M, \xi; R) = 0$ for every choice of the coefficient ring R.

REMARK 13.10. The coefficient ring is not always mentioned in statements of the above definition, but it should be. We will see in §13.3.2 below that this detail makes a difference to issues like symplectic filling obstructions. Note that for any nested pair of subgroups $G \subset G' \subset H_2(M)$, the natural projection $H_2(M)/G' \to H_2(M)/G$ induces an algebra homomorphism

$$HC_*(M, \xi; \mathbb{Q}[H_2(M)/G']) \to HC_*(M, \xi; \mathbb{Q}[H_2(M)/G]).$$

Since algebra homomorphisms necessarily map $1 \mapsto 1$ and $0 \mapsto 0$, the target of this map must vanish whenever its domain does, so for checking Definition 13.9, it suffices to check the case $R = \mathbb{Q}[H_2(M)]$.

We've seen above that there exist algebra homomorphisms

$$H_*^{\mathfrak{W}}(M, \xi; R) \to H_*^{\mathrm{RSFT}}(M, \xi; R) \to HC_*(M, \xi; R), \tag{13.8}$$

thus the vanishing of either of the algebras $H_*^{\mathfrak{W}}(M, \xi; R)$ or $H_*^{\mathrm{RSFT}}(M, \xi; R)$ with all coefficient rings R is another sufficient condition for algebraic overtwistedness. Bourgeois and Niederkrüger observed that, in fact, these conditions are also necessary:

THEOREM 13.11 ([**BN10**]). *For any coefficient ring R, the following conditions are equivalent:*

(1) $HC_*(M, \xi; R) = 0$,
(2) $H_*^{\mathrm{RSFT}}(M, \xi; R) = 0$,
(3) $H_*^{\mathrm{SFT}}(M, \xi; R) = 0$.

PROOF. The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are immediate from the algebra homomorphisms (13.8), thus it will suffice to prove (1) $\Rightarrow$ (3). Suppose $1 = 0 \in HC_*(M, \xi; R)$, which means $\partial_{\mathrm{CH}}\mathbf{f} = 1$ for some $\mathbf{f} \in \mathcal{A}$. Using the obvious inclusion $\mathcal{A} \hookrightarrow \mathfrak{W}$, this means

$$D_{\mathbf{H}}\mathbf{f} = 1 - \mathbf{G},$$

where $\mathbf{G} = \mathcal{O}(p, \hbar)$, i.e. $\mathbf{G}$ is a sum of terms that all contain at least one p_γ variable or a power of $\hbar$. It follows that $\mathbf{G}^k = \mathcal{O}(p^k, \hbar^k)$ for all $k \in \mathbb{N}$, and the infinite sum

$$\sum_{k=0}^{\infty} \mathbf{G}^k$$

is therefore an element of $\mathfrak{W}$, as the coefficient in front of any fixed monomial $\hbar^k p^\gamma$ in this sum is a polynomial function of the q variables. This sum is then a multiplicative inverse of $1 - \mathbf{G}$, and since

$$0 = D_{\mathbf{H}}^2\mathbf{f} = 0 = -D_{\mathbf{H}}\mathbf{G},$$

it also satisfies $D_{\mathbf{H}}\left((1-\mathbf{G})^{-1}\right) = 0$. Using the fact that $D_{\mathbf{H}}$ is a derivation, we therefore have

$$D_{\mathbf{H}}\left((1-\mathbf{G})^{-1}\mathbf{f}\right) = (1-\mathbf{G})^{-1}(1-\mathbf{G}) = 1,$$

implying $1 = 0 \in H_*^{\mathrm{SFT}}(M, \xi; R)$. □

13.2. SFT generating functions for cobordisms

All invariance proofs in SFT are based on a generating function analogous to $\mathbf{H}$ that counts index 0 holomorphic curves in symplectic cobordisms. The basic definition is a straightforward extension of what we saw in Lecture 12, but there is an added wrinkle due to the fact that, in general, one must include *disconnected* curves in the count.

13.2.1. Weak, strong and stable cobordisms. First some remarks about the category we are working in. Since the stated purpose of SFT is to define invariants of contact structures, we have been working since Lecture 12 with symplectizations of contact manifolds rather than more general stable Hamiltonian structures. We've made use of this restriction on several occasions, namely so that we can assume:

(1) All nontrivial holomorphic curves in $\mathbb{R} \times M$ have at least one positive puncture;
(2) The energy of a holomorphic curve in $\mathbb{R} \times M$ can be bounded in terms of its positive asymptotic orbits.

It will be useful however for certain applications to permit a slightly wider class of stable Hamiltonian structure. Recall that a hypersurface V in an almost complex manifold (W, J) is called **pseudoconvex** if the maximal complex subbundle

$$\xi := TV \cap J(TV) \subset TV$$

defines a contact structure on V whose canonical conformal symplectic bundle structure tames $J|_\xi$. For example, if α is a contact form on M and $J \in \mathcal{J}(\alpha)$, then each of the hypersurfaces $\{\text{const}\} \times M$ is pseudoconvex in $(\mathbb{R} \times M, J)$. The contact structure ξ induces an orientation on the hypersurface V; if V comes with its own orientation (e.g. as a boundary component of W), then we call it *pseudoconvex* if ξ is a positive contact structure with respect to this orientation, and **pseudoconcave** otherwise. For example, if (W, ω) is a symplectic cobordism from (M_-, ξ_-) to (M_+, ξ_+) and $J \in \mathcal{J}(W, \omega, \alpha_+, \alpha_-)$, then M_+ is pseudoconvex and M_- is pseudoconcave.

DEFINITION 13.12. Given an odd-dimensional manifold M, we will say that an almost complex structure J on $\mathbb{R} \times M$ is **pseudoconvex** if $\{r\} \times M$ is a pseudoconvex hypersurface in $(\mathbb{R} \times M, J)$ for every $r \in \mathbb{R}$, with the induced orientation such that ∂_r and $\{r\} \times M$ are positively transverse.

If $\mathcal{H} = (\omega, \lambda)$ is a stable Hamiltonian structure on M, then pseudoconvexity of $J \in \mathcal{J}(\mathcal{H})$ imposes conditions on $\mathcal{H}$, in particular λ must be a contact form. It also requires $J|_\xi$ to be tamed by $d\lambda|_\xi$, but unlike the case when $J \in \mathcal{J}(\lambda)$, $J|_\xi$ need not be *compatible* with it, i.e. the positive bilinear form $d\lambda(\cdot, J\cdot)|_\xi$ need not be symmetric. As always, $J|_\xi$ must be compatible with $\omega|_\xi$, but ω need not be an *exact* form for this to hold—the freedom to change $[\omega] \in H^2_{\mathrm{dR}}(M)$ will be the main benefit of this generalization, particularly when we discuss weak symplectic fillings below.

PROPOSITION 13.13. *Suppose $\mathcal{H} = (\omega, \lambda)$ is a stable Hamiltonian structure on a closed manifold M and $J \in \mathcal{J}(\mathcal{H})$ is pseudoconvex. Then all nonconstant finite-energy J-holomorphic curves in $\mathbb{R} \times M$ have at least one positive puncture, and their energies satisfy a uniform upper bound in terms of the periods of their positive asymptotic orbits.*

PROOF. It is straightforward to check that either of the two proofs of Proposition 10.9 given in Lecture 10 generalizes to any J on $\mathbb{R} \times M$ that is pseudoconvex. In particular, pseudoconvexity implies that if $u : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$ is a J-holomorphic curve, then $u^* d\lambda \ge 0$, with equality only at points where u is tangent to ∂_r and the Reeb vector field. Stokes' theorem thus gives

$$0 \le \int_{\dot{\Sigma}} u^* d\lambda = \sum_{z \in \Gamma^+} T_z - \sum_{z \in \Gamma^-} T_z, \tag{13.9}$$

where $T_z > 0$ denotes the period of the asymptotic orbit at each positive/negative puncture $z \in \Gamma^{\pm}$. Since $J|_{\xi}$ is also tamed by $\omega|_{\xi}$ and ω annihilates the Reeb vector field, we similarly have $u^*\omega \ge 0$, with the same condition for equality, and the compactness of M then implies an estimate of the form

$$0 \le u^*\omega \le c u^* d\lambda$$

for every J-holomorphic curve $u : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$, with a constant $c > 0$ that depends only on M, $\mathcal{H}$ and J. In light of (13.9), this implies an upper bound on $\int_{\dot{\Sigma}} u^*\omega$ in terms of the periods T_z for $z \in \Gamma^+$. Writing $\omega_\varphi = \omega + d(\varphi(r)\lambda)$ for suitable C^0-small increasing functions $\varphi : \mathbb{R} \to \mathbb{R}$, we can then apply Stokes' theorem to the second term in

$$E(u) = \sup_{\varphi} \int_{\dot{\Sigma}} u^*\omega_\varphi = \int_{\dot{\Sigma}} u^*\omega + \sup_{\varphi} \int_{\dot{\Sigma}} u^* d(\varphi(r)\lambda),$$

implying a similar upper bound for $E(u)$. □

COROLLARY 13.14. *For any stable Hamiltonian structure $\mathcal{H} = (\omega, \lambda)$ with a nondegenerate Reeb vector field $R_{\mathcal{H}}$ and a pseudoconvex $J \in \mathcal{J}(\mathcal{H})$, one can use closed $R_{\mathcal{H}}$-orbits and count J-holomorphic curves in $\mathbb{R} \times M$ to define the chain complexes $(\mathcal{A}[[\hbar]], \mathbf{D}_{\mathrm{SFT}})$, $(\mathfrak{W}, D_{\mathbf{H}})$, $(\mathfrak{P}, d_{\mathbf{h}})$ and $(\mathcal{A}, \partial_{\mathrm{CH}})$.*

We shall denote the homologies of the above chain complexes with coefficients in $R = \mathbb{Q}[H_2(M)/G]$ by

$$H_*^{\mathrm{SFT}}(M, \mathcal{H}, J; R), \quad H_*^{\mathfrak{W}}(M, \mathcal{H}, J; R), \quad H_*^{\mathrm{RSFT}}(M, \mathcal{H}, J; R), \quad HC_*(M, \mathcal{H}, J; R).$$

We make no claim at this point about these homologies being invariant. For the examples that we actually care about, this will turn out to be an irrelevant question due to Proposition 13.16 and Exercise 13.32 below.

EXAMPLE 13.15. Suppose α is a contact form on (M, ξ) and $\mathcal{H} = (\Omega, \alpha)$ is a stable Hamiltonian structure. Then for all constants $c > 0$ sufficiently large, $\mathcal{H}_c := (\Omega + c\, d\alpha, \alpha)$ is also a stable Hamiltonian structure and there exists a pseudoconvex

$J_c \in \mathcal{J}(\mathcal{H}_c)$. To see the latter, notice that $\mathcal{H}'_c := \left(\frac{1}{c}\Omega + d\alpha, \alpha\right)$ is another family of stable Hamiltonian structures, with $\mathcal{J}(\mathcal{H}'_c) = \mathcal{J}(\mathcal{H}_c)$ for all c, and $\mathcal{H}'_c \to (d\alpha, \alpha)$ as $c \to \infty$. Thus one can select $J_c \in \mathcal{J}(\mathcal{H}_c)$ converging to some $J_\infty \in \mathcal{J}(\alpha)$ as $c \to \infty$, and these are pseudoconvex for $c > 0$ sufficiently large since J_∞ is.

PROPOSITION 13.16. *In the setting of Example 13.15, assume α is nondegenerate and $J_\infty \in \mathcal{J}(\alpha)$ is generic. If $HC_*(M, \xi; R) = 0$, then $HC_*(M, \mathcal{H}_c, J_c; R)$ also vanishes for all $c > 0$ sufficiently large.*

PROOF. We will assume in the following that the usual (unrealistic) transversality assumptions hold, but the essential idea of the argument would not change in the presence of abstract perturbations.

Let $(\mathcal{A}, \partial^\infty_{\mathrm{CH}})$ denote the contact homology chain complex generated by closed R_α-orbits, with $\partial^\infty_{\mathrm{CH}}$ counting J_∞-holomorphic curves in $\mathbb{R} \times M$. The assumption $HC_*(M, \xi; R) = 0$ means there exists an element $\mathbf{f} \in \mathcal{A}$ with $\partial^\infty_{\mathrm{CH}}\mathbf{f} = 1$. Here $\mathbf{f}$ is a polynomial function of the q_γ variables, and $\partial^\infty_{\mathrm{CH}}\mathbf{f}$ counts a specific finite set of Fredholm regular index 1 curves in $(\mathbb{R} \times M, J_\infty)$. Now let $(\mathcal{A}, \partial^c_{\mathrm{CH}})$ denote the chain complex for $HC_*(M, \mathcal{H}_c, J_c; R)$, and notice that since the stable Hamiltonian structures $(d\alpha, \alpha)$ and $\mathcal{H}_c$ define matching Reeb vector fields, the set of generators is unchanged. There is also no change to this complex if we replace $\mathcal{H}_c = (\Omega + c\, d\alpha, \alpha)$ by $\mathcal{H}'_c = \left(\frac{1}{c}\Omega + d\alpha, \alpha\right)$: this changes the energies of individual J_c-holomorphic curves, but the sets of finite-energy curves are still the same in both cases. We can assume $J_c \to J_\infty$ in C^∞ as $c \to \infty$. The implicit function theorem then extends each of the finitely many J_∞-holomorphic curves counted by $\partial^\infty\mathbf{f}$ uniquely to a smooth 1-parameter family of J_c-holomorphic curves for $c > 0$ sufficiently large.[2] We claim that these are the only curves counted by $\partial^c_{\mathrm{CH}}\mathbf{f}$ when $c > 0$ is large. Indeed, there would otherwise exist a sequence $c_k \to \infty$ for which additional J_{c_k}-holomorphic index 1 curves u_k contribute to $\partial^{c_k}_{\mathrm{CH}}\mathbf{f}$, and since $\mathbf{f}$ has only finitely many terms representing possible positive asymptotic orbits, we can find a subsequence for which all the u_k have the same positive asymptotic orbits. A further subsequence then has all the same negative asymptotic orbits as well since the Reeb flow is nondegenerate and the total period of the negative orbits is bounded by the total period of the positive orbits. Finally, since the sequence of stable Hamiltonian structures $\mathcal{H}'_{c_k}$ converges to $(d\alpha, \alpha)$, the curves u_k have uniformly bounded energy with respect to $\mathcal{H}'_{c_k}$, so that SFT compactness yields a subsequence converging to a J_∞-holomorphic building of index 1, which can only be one of the curves counted by $\partial^\infty_{\mathrm{CH}}\mathbf{f}$. This contradicts the uniqueness in the implicit function theorem and thus proves the claim. We conclude that for all $c > 0$ sufficiently large, $\partial^c_{\mathrm{CH}}\mathbf{f} = 1$. □

DEFINITION 13.17. Assume (W, ω) is a symplectic cobordism with stable boundary $\partial W = -M_- \sqcup M_+$, with induced stable Hamiltonian structures $\mathcal{H}_\pm = (\omega_\pm, \lambda_\pm)$ at $M_\pm$, and suppose J is an almost complex structure on the completion $\widehat{W}$ that is ω-tame on W and belongs to $\mathcal{J}(\mathcal{H}_\pm)$ on the cylindrical ends. We will say that J

[2] In case you are concerned about the parametric moduli space being an orbifold instead of a manifold, just add asymptotic markers so that there is no isotropy, and divide by the appropriate combinatorial factors to count.

is **pseudoconvex near infinity**[3] if the $\mathbb{R}$-invariant almost complex structures $J_\pm$ defined by restricting J to $[0,\infty) \times M_+$ and $(-\infty, 0] \times M_-$ are both pseudoconvex.

Note that the condition on J in the above definition can only be satisfied if $\lambda_\pm$ are both positive contact forms on $M_\pm$, but the 2-forms $\omega_\pm$ need not be exact.

Proving contact invariance of SFT requires counting curves in trivial exact symplectic cobordisms, but it is also natural to try to say things about non-exact **strong** symplectic cobordisms using SFT.[4] These fit naturally into our previously established picture since every strong cobordism has collar neighborhoods near the boundary in which it matches the symplectization of a contact manifold. The following more general notion of cobordism is also natural from a contact topological perspective, but fits less easily into the SFT picture.

DEFINITION 13.18 ([MNW13]). Given closed contact manifolds (M_+, ξ_+) and (M_-, ξ_-) of dimension $2n-1$, a **weak symplectic cobordism** from (M_-, ξ_-) to (M_+, ξ_+) is a compact symplectic manifold (W, ω) with $\partial W = -M_- \sqcup M_+$ admitting an ω-tame almost complex structure J for which the almost complex manifold (W, J) is pseudoconvex at M_+ and pseudoconcave at M_-, with

$$\xi_\pm = TM_\pm \cap J(TM_\pm).$$

Weak cobordisms are characterized by the existence of a tame almost complex structure J whose restriction to $\xi_\pm$ is tamed by *two* symplectic bundle structures, $\omega|_{\xi_\pm}$ and $d\alpha_\pm|_{\xi_\pm}$ (for any choices of contact forms $\alpha_\pm$ defining $\xi_\pm$). Notice that in dimension 4, the second condition is mostly vacuous, and the weak cobordism condition just reduces to

$$\omega|_{\xi_\pm} > 0.$$

In this form, the low-dimensional case of Definition 13.18 has been around since the late 1980's, and there are many interesting results about it, e.g. examples of contact 3-manifolds that are weakly but not strongly fillable [Gir94, Eli96]. We will see in §13.3.2 that this distinction is detectable via SFT. Higher-dimensional examples of this phenomenon were found in [MNW13].

One major difference between weak and strong cobordisms is that the latter are always exact near the boundary, as the Liouville vector field is dual to a primitive of ω. It turns out that up to deformation, weak fillings that are exact at the boundary are the same thing as strong fillings—this was first observed by Eliashberg in dimension three [Eli91, Prop. 3.1], and was extended to higher dimensions in [MNW13]:

PROPOSITION 13.19. *Suppose (W, ω) is a weak filling of a $(2n-1)$-dimensional contact manifold (M, ξ) such that $\omega|_{TM}$ is exact. Then after a homotopy of ω through*

[3] If I were being hypercorrect about use of language, I might insist on saying that J is "pseudoconvex near $+\infty$ and *pseudoconcave* near $-\infty$," as the orientation reversal at the negative boundary makes M_- technically a pseudoconcave hypersurface in $(\widehat{W}, J)$, not pseudoconvex. But this definition will only be useful to us in cases where $M_- = \emptyset$, so my linguistic guilt is limited.

[4] By *strong cobordism*, we mean the usual notion of a compact symplectic manifold with convex and/or concave boundary components (see §1.4). The word "strong" is included in order to contrast this notion with its weaker cousin described in Definition 13.18.

a family of symplectic forms that vary only in a collar neighborhood of ∂W *and define weak fillings of* (M, ξ), (W, ω) *is a strong filling of* (M, ξ).

PROOF. Choose any contact form α for ξ, denote its Reeb vector field by R_α, and let $\Omega = \omega|_{TM}$. Identify a collar neighborhood of ∂W in W smoothly with $(-\epsilon, 0] \times M$, with the coordinate on $(-\epsilon, 0]$ denoted by r, such that ∂_r and R_α span the symplectic complement of ξ at ∂W and satisfy $\omega(\partial_r, R_\alpha) = 1$. Then ω and $\Omega + d(r\alpha)$ are cohomologous symplectic forms on $(-\epsilon, 0] \times M$ that match at $r = 0$, hence a Moser deformation argument implies they are isotopic. We can therefore assume without loss of generality that $\omega = \Omega + d(r\alpha)$ on the collar near ∂W.

By assumption, $\Omega = d\eta$ for some 1-form η on M, and since (W, ω) is a weak filling of $(M, \xi = \ker \alpha)$, we can choose a complex structure J_ξ on ξ that is tamed by both $d\alpha|_\xi$ and $d\eta|_\xi$. Now choose a smooth cutoff function $\beta : [0, \infty) \to [0, 1]$ that has compact support and equals 1 near 0. We claim that

$$\omega := d(\beta(r)\eta) + d(r\alpha)$$

is a symplectic form on $[0, \infty) \times M$ if $|\beta'|$ is sufficiently small. Indeed, writing $\omega = dr \wedge (\alpha + \beta'(r)\,\eta) + [\beta(r)\, d\eta + r\, d\alpha]$, we have

$$\omega^n = n\, dr \wedge \alpha \wedge [\beta(r)\, d\eta + r\, d\alpha]^{n-1} + n\beta'(r)\, dr \wedge \eta \wedge [\beta(r)\, d\eta + r\, d\alpha]^{n-1}.$$

The first term is positive and bounded away from zero since $d\eta|_\xi$ and $d\alpha|_\xi$ both tame J_ξ, hence do does $\beta\, d\eta + r\, d\alpha|_\xi$. The second term is then harmless if $|\beta'|$ is sufficiently small, proving $\omega^n > 0$.

This defines an extension of the original weak filling to a symplectic completion $\widehat{W} = W \cup_M ([0, \infty) \times M)$, and for each $r_0 \geq 0$, the compact subdomains defined by $r \leq r_0$ define weak fillings of $(\{r_0\} \times M, \xi)$ since $\omega|_\xi = (\beta(r_0)\, d\eta + r_0\, d\alpha)|_\xi$ also tames J_ξ. Notice that for r_0 sufficiently large, the $d\eta$ term disappears, so ω has a primitive that restricts to $\{r_0\} \times M$ as a contact form for ξ, meaning we have a *strong* filling of this hypersurface. The desired deformation of ω can therefore be defined by pulling back via a smooth family of diffeomorphisms $(-\epsilon, 0] \to (-\epsilon, r_0]$, where r_0 varies from 0 to a sufficiently large constant. □

Unlike strong cobordisms, being a weak cobordism is an open condition: if (W, ω) is a weak cobordism, then so is $(W, \omega + \epsilon\sigma)$ for any $\epsilon > 0$ sufficiently small and a closed 2-form σ, which need not be exact at ∂W. As a consequence, the cylindrical ends of a completed weak cobordism cannot always be deformed to look like the symplectization of a contact manifold. This is where Definition 13.17 comes in useful. The proof of the next lemma is very much analogous to Proposition 13.19.

LEMMA 13.20 ([MNW13, Lemma 2.10]). *Suppose* (W, ω) *is a weak filling of a* $(2n-1)$*-dimensional contact manifold* (M, ξ), α *is a contact form for* ξ *and* Ω *is a closed* 2*-form on* M *with* $[\Omega] = [\omega|_{TM}] \in H^2_{\mathrm{dR}}(M)$. *Then for any constant* $c > 0$ *sufficiently large, after a homotopy of* ω *through a family of symplectic forms that vary only in a collar neighborhood of* ∂W *and define weak fillings of* (M, ξ), $\omega|_{TM} = \Omega + c\, d\alpha$. □

The following result then provides a suitable model that can be used as Ω in the above lemma when $\omega|_{TM}$ is nonexact. The statement below is restricted to the case where $[\omega|_{TM}]$ is a rational cohomology class; the reason for this is that it relies on a Donaldson-type existence result for contact submanifolds obtained as zero-sets of approximately holomorphic sections, due to Ibort, Martiínez-Torres and Presas [IMTP00]. It seems likely that the rationality condition could be lifted with more work, and in dimension three this is known to be true; see [NW11, Prop. 2.6].

LEMMA 13.21 ([CV15, Prop. 2.18]). *For any rational cohomology class $\eta \in H^2(M;\mathbb{Q})$ on a closed $(2n-1)$-dimensional contact manifold (M,ξ), there exists a closed 2-form Ω and a nondegenerate contact form α for ξ such that (Ω,α) is a stable Hamiltonian structure.* □

Combining all of the above results (including Example 13.15) proves:

PROPOSITION 13.22. *Suppose (W,ω) is a weak filling of a $(2n-1)$-dimensional contact manifold (M,ξ) such that $[\omega|_{TM}] \in H^2_{\mathrm{dR}}(M)$ is rational or $n = 2$. Fix a nondegenerate contact form α for ξ. Then there exists a closed 2-form Ω cohomologous to $\omega|_{TM}$ such that $\mathcal{H} := (\Omega,\alpha)$ is a stable Hamiltonian structure, and for all $c > 0$ sufficiently large, ω can be deformed in a collar neighborhood of ∂W, through a family of symplectic forms defining weak fillings of (M,ξ), to a new weak filling for which ∂W is also stable and inherits the stable Hamiltonian structure $\mathcal{H}_c := (\Omega + c\,d\alpha,\alpha)$. In particular, after this deformation, the completed stable filling admits a tame almost complex structure that is pseudoconvex near infinity and may be assumed C^∞-close to any given $J \in \mathcal{J}(\alpha)$.* □

We will use this in §13.3.2 to define obstructions to weak fillability via SFT.

REMARK 13.23. There is apparently no analogue of Propositions 13.19 and 13.22 for negative boundary components of weak cobordisms, and this is one of a few reasons why they are not often discussed. For example, if L is a Lagrangian torus in the standard symplectic 4-ball $\mathbb{D}^4$, then the complement of a neighborhood of L in B^4 defines a strong cobordism from the standard contact $\mathbb{T}^3$ to S^3. The symplectic form on this cobordism is obviously exact, but if any result analogous to Proposition 13.19 were to hold at the concave boundary, then we could deform it to a Liouville cobordism. No such Liouville cobordism exists—it would imply that the Lagrangian $L \subset B^4$ is exact, thus violating Gromov's famous theorem [Gro85] on exact Lagrangians.

13.2.2. Counting disconnected index 0 curves. Fix a symplectic cobordism (W,ω) with stable boundary $\partial W = -M_- \sqcup M_+$ carrying stable Hamiltonian structures $\mathcal{H}_\pm = (\omega_\pm,\lambda_\pm)$, along with a generic almost complex structure J that is ω-tame on W, belongs to $\mathcal{J}(\mathcal{H}_\pm)$ on the cylindrical ends, and is pseudoconvex near infinity. This implies that the stabilizing 1-forms $\lambda_\pm$ are both contact forms. Let us also assume that the $\lambda_\pm$ are both nondegenerate, and that the induced $\mathbb{R}$-invariant almost complex structures $J_\pm \in \mathcal{J}(\mathcal{H}_\pm)$ are sufficiently generic to achieve regularity for all holomorphic curves under consideration. In particular, these assumptions mean that all the usual SFT chain complexes are well defined for $(M_\pm,\mathcal{H}_\pm,J_\pm;R_\pm)$ with any choice of coefficient ring $R_\pm = \mathbb{Q}[H_2(M_\pm)/G_\pm]$. Denote the corresponding SFT generating functions by $\mathbf{H}_\pm$.

Recall from Lecture 12 that the auxiliary data on M_+ and M_- includes a choice of capping surface C_γ for each closed Reeb orbit γ (or a capping *chain* with rational coefficients if $H_1(M_\pm)$ has torsion). These surfaces satisfy

$$\partial C_\gamma = \sum_i m_i [C_i^\pm] - [\gamma],$$

where the m_i are integers and $C_i^\pm \subset M_\pm$ are fixed curves forming a basis of $H_1(M_\pm)$. Assume $H_1(W)$ is torsion free, in which case the same is true of $H_1(M_+)$ and $H_1(M_-)$. (Only minor modifications are needed if this assumption fails to hold, see Remark 13.1.) We can then fix the following additional auxiliary data:

(1) A collection of **reference curves**

$$S^1 \cong C_1, \dots, C_r \subset W$$

whose homology classes from a basis of $H_1(W)$.

(2) A unitary trivialization of TW along each of the reference curves $C_1, \dots, C_r$, denoted collectively by τ.

(3) A **spanning surface** $S_i^\pm$ for each of the positive/negative reference curves $C_i^\pm \subset M_\pm$, i.e. a smooth map of a compact and oriented surface with boundary into W such that

$$\partial S_i^\pm = \sum_j m_{ji} [C_j] - [C_i^\pm]$$

in the sense of singular 2-chains, where $m_{ji} \in \mathbb{Z}$ are the unique coefficients with $[C_i^\pm] = \sum_j m_{ji}[C_j] \in H_1(W)$.

Now to any collections of orbits $\boldsymbol{\gamma}^\pm = (\gamma_1^\pm, \dots, \gamma_{k_\pm}^\pm)$ in $M_\pm$ and a relative homology class $A \in H_2(W, \bar{\boldsymbol{\gamma}}^+ \cup \bar{\boldsymbol{\gamma}}^-)$ with $\partial A = \sum_i [\gamma_i^+] - \sum_j [\gamma_j^-]$, we can associate an absolute homology class in two steps: first add A to suitable sums of the capping surfaces $C_{\gamma_i^\pm}$ producing a 2-chain whose boundary is a linear combination of positive and negative reference curves, then add a suitable linear combination of the $S_i^\pm$ so that the boundary becomes the *trivial* linear combination of $C_1, \dots, C_r$. With this understood, we can now associate an absolute homology class

$$[u] \in H_2(W)$$

to any asymptotically cylindrical J-holomorphic curve $u : (\dot{\Sigma}, j) \to (\widehat{W}, J)$, and this defines the notation $\mathcal{M}_{g,m}(J, A, \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-)$ with $A \in H_2(W)$. We now require the trivializations of $\xi_\pm$ along each $C_i^\pm$ to be compatible with τ in the sense that they extend to trivializations of TW along the capping surfaces $S_i^\pm$. With this convention, the Fredholm index formula takes the expected form

$$\operatorname{ind}(u) = (n-3)\chi(\dot{\Sigma}) + 2c_1([u]) + \sum_{i=1}^{k_+} \mu_{\mathrm{CZ}}(\gamma_i) - \sum_{j=1}^{k_-} \mu_{\mathrm{CZ}}(\gamma_j).$$

If $H_1(W)$ has torsion, then this whole discussion can be adapted as in §12.7.1 by replacing integral homology with rational homology and capping surfaces with capping chains, and the Conley-Zehnder indices can be defined modulo 2.

We will also need to impose a compatibility condition relating the coefficient rings $R_{\pm} = \mathbb{Q}[H_2(M_{\pm})/G_{\pm}]$ to a corresponding choice on the cobordism W. Choose a subgroup $G \subset H_2(W)$ such that

$$\langle [\omega], A \rangle = 0 \quad \text{for all} \quad A \in G, \tag{13.10}$$

and such that the maps $H_2(M_{\pm}) \to H_2(W)$ induced by the inclusions $M_{\pm} \hookrightarrow W$ send $G_{\pm}$ into G. If $[\omega] \neq 0 \in H^2_{\text{dR}}(W)$, then we will have to deal with noncompact sequences of J-holomorphic curves that have unbounded energy, so it becomes necessary to "complete" R to a **Novikov ring** $\overline{R}$, which contains R but also includes infinite formal sums

$$\sum_{i=1}^{\infty} c_i e^{A_i} \quad \text{such that} \quad \langle [\omega], A_i \rangle \to +\infty \text{ as } i \to \infty.$$

Note that the evaluation $\langle [\omega], A \rangle \in \mathbb{R}$ is well defined for $A \in H_2(W)/G$ due to (13.10).

Analogously to our definition of $\mathbf{H}$ in Lecture 12, the generating function for index 0 curves in $\widehat{W}$ is defined as a formal power series in the variables $\hbar$, q_γ (for orbits in M_-), and p_γ (for orbits in M_+), with coefficients in $\overline{R}$:

$$\mathbf{F} = \sum_{u \in \mathcal{M}_0^\sigma(J)} \frac{\epsilon(u)}{|\operatorname{Aut}^\sigma(u)|} \hbar^{g-1} e^A q^{\gamma^-} p^{\gamma^+}, \tag{13.11}$$

where $\mathcal{M}_0^\sigma(J)$ denotes the moduli space of connected J-holomorphic curves u in $\widehat{W}$ with $\operatorname{ind}(u) = 0$ and only good asymptotic orbits, modulo permutations of the punctures, and for each u:

- g is the genus of u;
- A is the equivalence class of $[u] \in H_2(W)$ in $H_2(W)/G$;
- $\boldsymbol{\gamma}^{\pm} = (\gamma_1^{\pm}, \dots, \gamma_{k_{\pm}}^{\pm})$ are the asymptotic orbits of u after arbitrarily fixing orderings of its positive and negative punctures;
- $\epsilon(u) \in \{1, -1\}$ is the sign of u as a point in the 0-dimensional component of $\mathcal{M}^\$(J)$ (after choosing an ordering of the punctures and asymptotic markers), relative to a choice of coherent orientations on $\mathcal{M}^\$(J)$.

As usual, the product $\epsilon(u) q^{\gamma^-} p^{\gamma^+}$ is independent of choices. We shall regard $\mathbf{F}$ as an element in an enlarged operator algebra that includes q and p variables for good orbits in both M_+ and M_-, related to each other by the supercommutation relations

$$[p_{\gamma_-}, q_{\gamma_+}] = [p_{\gamma_+}, q_{\gamma_-}] = [q_{\gamma_-}, q_{\gamma_+}] = [p_{\gamma_-}, p_{\gamma_+}] = 0$$

whenever γ_- is an orbit in M_- and γ_+ is an orbit in M_+. Since all curves counted by $\mathbf{F}$ have index 0, $\mathbf{F}$ is homogeneous with degree

$$|\mathbf{F}| = 0.$$

Notice that for any fixed monomial $q^{\gamma^-} p^{\gamma^+}$, the corresponding set of curves in $\mathcal{M}_0^\sigma(J)$ may be infinite if ω is nonexact, but SFT compactness implies that the set of such curves with any given bound on $\int_{\dot{\Sigma}} u^*\omega$ is bounded. As a consequence, the coefficient of $q^{\gamma^-} p^{\gamma^+}$ in $\mathbf{F}$ belongs to the Novikov ring $\overline{R}$.

Consider next the series

$$\exp(\mathbf{F}) := \sum_{k=0}^{\infty} \frac{1}{k!} \mathbf{F}^k.$$

We will be able to view this as a formal power series in q and p variables and a formal Laurent series in $\hbar$ with coefficients in $\overline{R}$, though it is not obvious at first glance whether its coefficients are in any sense finite. We will deduce this after interpreting it as a count of *disconnected* index 0 curves: first, write

$$\exp(\mathbf{F}) = \sum_{k=0}^{\infty} \frac{1}{k!} \Bigg(\sum_{(u_1,\dots,u_k)\in(\mathcal{M}_0^\sigma(J))^k} \frac{\epsilon(u_1)\dots\epsilon(u_k)}{|\operatorname{Aut}^\sigma(u_1)|\dots|\operatorname{Aut}^\sigma(u_k)|} \hbar^{g_1+\dots+g_k-k} e^{A_1+\dots A_k} \cdot q^{\gamma_1^-} p^{\gamma_1^+} \dots q^{\gamma_k^-} p^{\gamma_k^+} \Bigg).$$

Observe that since each of the curves $u_i \in \mathcal{M}_0^\sigma(J)$ in this expansion has index 0, the monomials $q^{\gamma_i^-} p^{\gamma_i^+}$ all have even degree and thus the order in which they are written does not matter. Now for a given collection of distinct curves $v_1, \dots, v_N$ and integers $k_1, \dots, k_N \in \mathbb{N}$ with $k_1 + \dots + k_N = k$, the various permutations of

$$(u_1, \dots, u_k) := (\underbrace{v_1, \dots, v_1}_{k_1}, \dots, \underbrace{v_N, \dots, v_N}_{k_N}) \in (\mathcal{M}_0^\sigma(J))^k$$

occur $\frac{k!}{k_1!\dots k_N!}$ times in the above sum, so if we forget the ordering, then the contribution of this particular k-tuple of curves to $\exp(\mathbf{F})$ is

$$\frac{\epsilon(u_1)\dots\epsilon(u_k)}{k_1!\dots k_N!|\operatorname{Aut}^\sigma(u_1)|\dots|\operatorname{Aut}^\sigma(u_k)|} \hbar^{g_1+\dots+g_k-k} e^{A_1+\dots+A_k} q^{\gamma_1^-} p^{\gamma_1^+} \dots q^{\gamma_k^-} p^{\gamma_k^+}.$$

Notice next that the denominator $k_1!\dots k_N!|\operatorname{Aut}^\sigma(u_1)|\dots|\operatorname{Aut}^\sigma(u_k)|$ is the order of the automorphism group of the *disconnected* curve formed by the disjoint union of $u_1, \dots, u_k$: the extra factors $k_i!$ come from automorphisms that permute connected components of the domain. Thus $\exp(\mathbf{F})$ can also be written as in (13.11), but with $\mathcal{M}_0^\sigma(J)$ replaced by the moduli space of *potentially disconnected* index 0 curves with unordered punctures, and $g-1$ generalized to $g_1+\dots+g_k-k$ for any curve that has k connected components of genera $g_1, \dots, g_k$. One subtlety that was glossed over in the above discussion: the sum also includes the unique curve with *zero* components, i.e. the "empty" J-holomorphic curve, which appears as the initial 1 in the series expansion of $\exp(\mathbf{F})$.

With this interpretation of $\exp(\mathbf{F})$ understood, we can now address the possibility that the infinite sum defining $\exp(\mathbf{F})$ might include infinitely many terms for a given monomial $\hbar^m q^{\gamma^-} p^{\gamma^+}$, i.e. that there are infinitely many disconnected index 0 curves with fixed asymptotic orbits and a fixed sum of the genera minus the number of connected components. We claim that this can indeed, happen, but only if the curves belong to a sequence of homology classes $A_i \in H_2(M)/G$ with $\langle[\omega], A_i\rangle \to \infty$, hence the coefficient of $\hbar^m q^{\gamma^-} p^{\gamma^+}$ in $\exp(\mathbf{F})$ belongs to the Novikov ring $\overline{R}$. The danger here comes only from *closed* curves, since a disjoint union of two curves with punctures always has strictly more punctures. Notice also that for any given tuples of orbits $\boldsymbol{\gamma}^\pm$, there exists a number $c \in \mathbb{R}$ depending only on these orbits and the chosen capping surfaces such that every (possibly disconnected) J-holomorphic curve $u : \dot{\Sigma} \to \widehat{W}$ asymptotic to $\boldsymbol{\gamma}^\pm$ satisfies

$$\langle [\omega], [u] \rangle \geq c.$$

This follows from the fact that the integral of ω over the relative homology class of u always has a nonnegative integrand.

LEMMA 13.24. *Given constants $C \in \mathbb{R}$ and $k \in \mathbb{Z}$, there exists a number $N \in \mathbb{N}$ such that if $u : (\Sigma, j) \to (\widehat{W}, J)$ is a closed J-holomorphic curve satisfying $\int_\Sigma u^*\omega \leq C$, with m connected components of genera $g_1, \ldots, g_m$ satisfying $g_1 + \ldots + g_m - m = k$, then $m \leq N$.*

PROOF. Note first that for each integer $g \geq 0$, there is an **energy threshold**, i.e. a constant $c_g > 0$ such that every nonconstant closed and connected J-holomorphic curve $u : \Sigma \to \widehat{W}$ of genus g has

$$\int_\Sigma u^*\omega \geq c_g.$$

This is an easy consequence of SFT compactness: indeed, if there were no such constant, then we would find a sequence $u_k : \Sigma \to \widehat{W}$ of connected closed curves with genus g such that

$$E(u_k) = \int_\Sigma u^*\omega \to 0;$$

here we have used the fact that Σ is closed and $\int_\Sigma u^*\omega_\varphi$ depends only on the homology class of u in order to simplify the usual definition of energy for asymptotically cylindrical curves. SFT compactness then gives a subsequence of u_k that converges to a stable holomorphic building in which every component has zero energy and is therefore constant. Since there are no marked points in the picture, no such building exists, so this is a contradiction.

Now if u is a disconnected curve satisfying the stated conditions, the bound on $\int_\Sigma u^*\omega$ combines with the energy threshold to give a bound for each $g \geq 0$ on the number of connected components of u with genus g. In particular, there is a bound on the number of components with genus 0 or 1. All other components contribute positively to the left hand side of the relation $\sum_{i=1}^m (g_i - 1) = k$, so this implies a universal bound on m. □

COROLLARY 13.25. *Fix constants $C \in \mathbb{R}$ and $k \in \mathbb{Z}$, and tuples of Reeb orbits $\boldsymbol{\gamma}^\pm$, and assume that the usual transversality conditions hold. Then there exist at most finitely many potentially disconnected J-holomorphic curves $u : \dot{\Sigma} \to \widehat{W}$ with index 0 such that the number of connected components m and the genera $g_1, \ldots, g_m$ of its components satisfy $g_1 + \ldots + g_m - m = k$.*

COROLLARY 13.26. *The expression* $\exp(\mathbf{F})$ *is a formal power series in q and p variables and a formal Laurent series in $\hbar$, with coefficients in the Novikov ring $\overline{R}$.*

The necessity of considering disconnected curves becomes clear when one tries to translate the compactness and gluing theory of J-holomorphic curves in $\widehat{W}$ into algebraic relations. In particular, consider the 1-dimensional moduli space of connected index 1 curves in $\widehat{W}$ with genus g. The boundary points of the compactification of this space consist of two types of buildings:

TYPE 1 : A main level of index 0 and an upper level of index 1;
TYPE 2 : A main level of index 0 and a lower level of index 1.

This is clear under the usual transversality assumptions since regular curves in $\widehat{W}$ must have index at least 0, while regular curves in the symplectizations $\mathbb{R} \times M_\pm$ have index at least 1 unless they are trivial cylinders. The building must also be connected and have arithmetic genus g, but there is nothing to guarantee that each individual level is connected. In fact, we already saw this issue in Lecture 12 when proving $\mathbf{H}^2 = 0$, but it was simpler to deal with there, because disconnected regular curves of index 1 in a symplectization always have a unique nontrivial component, while the rest are trivial cylinders. In the cobordism $\widehat{W}$, on the other hand, a disconnected index 0 curve can be formed by any disjoint union of index 0 curves, all of which are nontrivial. Exponentiation provides a convenient way to encode all data about disconnected curves in terms of connected curves.

Since the union of all buildings of types 1 and 2 described above forms the boundary of a compact oriented 1-manifold, the count of these buildings is zero, and this fact is encoded in the so-called **master equation**

$$\mathbf{H}_- \exp(\mathbf{F})|_{p_-=0} - \exp(\mathbf{F})\mathbf{H}_+|_{q_+=0} = 0, \tag{13.12}$$

where the expressions "$p_- = 0$" and "$q_+ = 0$" mean that we discard all terms in $\mathbf{H}_- \exp(\mathbf{F}) - \exp(\mathbf{F})\mathbf{H}_+$ containing any variables p_γ for orbits in M_- or q_γ for orbits in M_+. The resulting expression is therefore a formal power series in q variables for orbits in M_- and p variables for orbits in M_+, representing a count of generally disconnected index 1 holomorphic buildings in $\widehat{W}$ with the specified asymptotics. The various ways to form such buildings by choices of gluings is again encoded by the commutator algebra. The master equation (13.12) can be used to prove the chain map property for counts of curves in cobordisms, thus it is an essential piece of the invariance proof for each of the homology theories introduced above.

EXERCISE 13.27. Fill in the details of the proof of (13.12).

13.3. Full SFT as a BV_∞-algebra

In this section we discuss the specific theory $H_*^{\mathrm{SFT}}(M, \xi; R)$, defined as the homology of the chain complex $(\mathcal{A}[[\hbar]], \mathbf{D}_{\mathrm{SFT}})$. The case $G = H_2(M)$ with trivial group ring coefficients $\mathbb{Q}[H_2(M)/G] = \mathbb{Q}$ will be abbreviated as

$$H_*^{\mathrm{SFT}}(M, \xi) := H_*^{\mathrm{SFT}}(M, \xi; \mathbb{Q}).$$

As we defined it, $\mathbf{D}_{\mathrm{SFT}}$ acts on $\mathcal{A}[[\hbar]]$ by treating the generating function $\mathbf{H}$ as a differential operator via the substitution

$$p_\gamma = \kappa_\gamma \hbar \frac{\partial}{\partial q_\gamma}. \tag{13.13}$$

According to [**CL09**], this makes $(\mathcal{A}[[\hbar]], \mathbf{D}_{\mathrm{SFT}})$ into a BV_∞-algebra; we'll have no particular need to discuss here what that means, but one convenient feature is the expansion

$$\mathbf{D}_{\mathrm{SFT}} = \frac{1}{\hbar} \sum_{k=1}^{\infty} \mathbf{D}_{\mathrm{SFT}}^{(k)} \hbar^k, \tag{13.14}$$

in which each $\mathbf{D}_{\mathrm{SFT}}^{(k)} : \mathcal{A} \to \mathcal{A}$ is a differential operator *of order* $\le k$ (see [**CL09**, §5]). For each $k \in \mathbb{N}$, $\mathbf{D}_{\mathrm{SFT}}^{(k)}$ is a count of all index 1 holomorphic curves that have genus $g \ge 0$ and $m \ge 1$ positive punctures such that $g + m = k$. In particular, $\mathbf{D}_{\mathrm{SFT}}^{(1)}$ is simply the contact homology differential ∂_{CH}, and the expansion (13.14) implies together with $\mathbf{D}_{\mathrm{SFT}}^2 = 0$ that $(\mathbf{D}_{\mathrm{SFT}}^{(1)})^2 = 0$, hence we again see the chain complex for contact homology hidden inside a version of the "full" SFT complex.

13.3.1. Cobordism maps and invariance. One can use the master equation (13.12) to prove invariance of $H_*^{\mathrm{SFT}}(M, \xi; R)$ by a straightforward generalization of the usual Floer-theoretic argument. Suppose $(W, d\lambda)$ is an exact symplectic cobordism from (M_-, ξ_-) to (M_+, ξ_+) with $\lambda|_{TM_\pm} = \alpha_\pm$, and choose a generic almost complex structure J on $\widehat{W}$ that is $d\lambda$-compatible on W and restricts to the cylindrical ends as generic elements $J_\pm \in \mathcal{J}(\alpha_\pm)$. Let $(\mathcal{A}^\pm[[\hbar]], \mathbf{D}_{\mathrm{SFT}}^\pm)$ denote the chain complexes associated to the data $(\alpha_\pm, J_\pm)$, and for simplicity in this initial discussion, choose the trivial coefficient ring $R = \mathbb{Q}$ for both. We then define a map

$$\mathbf{\Phi} : \mathcal{A}^+[[\hbar]] \to \mathcal{A}^-[[\hbar]] : \mathbf{f} \mapsto \exp(\mathbf{F})\mathbf{f}|_{q_+=0},$$

where the generating function $\exp(\mathbf{F})$ is regarded as a differential operator via the substitution (13.13), with $e^A := 1$ for all $A \in H_2(W)$ since we are using trivial coefficients, and "$q_+ = 0$" means that after applying $\exp(\mathbf{F})$ to change $\mathbf{f}$ into a function of q variables for orbits in both M_+ and M_-, we discard all terms that involve orbits in M_+. The exactness of the cobordism implies that negative powers of $\hbar$ do not appear in $\mathbf{\Phi f}$, thus producing an element of $\mathcal{A}^-[[\hbar]]$: indeed, since there are no holomorphic curves in $\widehat{W}$ without positive punctures, every term in $\mathbf{F}$ contains at least one p variable, so that negative powers of $\hbar$ do not appear in $\exp(\mathbf{F})$ after applying (13.13).

The master equation for $\mathbf{F}$ now translates into the fact that $\mathbf{\Phi}$ is a chain map,

$$\mathbf{D}_{\mathrm{SFT}}^- \circ \mathbf{\Phi} - \mathbf{\Phi} \circ \mathbf{D}_{\mathrm{SFT}}^+,$$

thus it descends to homology. The geometric meaning of $\mathbf{\Phi}$ is straightforward to describe: analogous to (12.11) in Lecture 12, we can write

$$\mathbf{\Phi} q^{\boldsymbol{\gamma}} = \sum_{g=0}^{\infty} \sum_{\boldsymbol{\gamma}'} \hbar^{g+k-1} n_g(\boldsymbol{\gamma}, \boldsymbol{\gamma}', k) q^{\boldsymbol{\gamma}'}, \tag{13.15}$$

where $n_g(\boldsymbol{\gamma}, \boldsymbol{\gamma}', k)$ is a product of some combinatorial factors with a signed count of disconnected index 0 holomorphic curves with connected components of genus $g_1, \ldots, g_m$ satisfying $g_1 + \ldots + g_m - m = g - 1$, and with positive ends at $\boldsymbol{\gamma}$ and negative ends at $\boldsymbol{\gamma}'$, where k is the number of positive ends.

Let's discuss two applications of the cobordism map $\mathbf{\Phi}$. First, note that if W is a *trivial* symplectic cobordism $[0,1] \times M$, then the above discussion can easily be generalized with $(\mathcal{A}^{\pm}, \mathbf{D}^{\pm}_{\mathrm{SFT}})$ both defined over the same group ring $R = \mathbb{Q}[H_2(M)/G]$ for any choice of $G \subset H_2(M)$. There is no need to consider a Novikov ring in defining $\mathbf{F}$ here since the cobordism is exact. We therefore obtain a chain map with arbitrary group ring coefficients, and extending this discussion along standard Floer-theoretic principles will imply that the chain map is an isomorphism: this can be used in particular to prove that $H^{\mathrm{SFT}}_*(M, \xi; R)$ does not depend on the choices of contact form and almost complex structure. There are two additional steps involved in this argument: first, one needs to use a chain homotopy to prove that $\mathbf{\Phi}$ does not depend on the choice of almost complex structure J on $\widehat{W}$. Given a generic homotopy $\{J_s\}_{s\in[0,1]}$, the chain homotopy map

$$\mathbf{\Psi} : \mathcal{A}^+[[\hbar]] \to \mathcal{A}^-[[\hbar]]$$

is defined as a differential operator in the same manner as $\mathbf{\Phi}$, but counting pairs (s, u) where $s \in [0,1]$ is a parameter value for which J_s is nongeneric and u is a disconnected J_s-holomorphic curve in $\widehat{W}$ with index -1. We saw how this works for cylindrical contact homology in Lecture 10, but there is a new subtlety now that should be mentioned: in principle, a *disconnected* index -1 curve in $\widehat{W}$ could have arbitrarily many components, including perhaps many with index -1 and others with arbitrarily large index. Even worse, the compactified 1-dimensional space of pairs (s, u) for J_s-holomorphic curves u of index 0 may include buildings that have symplectization levels of index greater than 1, balanced by disjoint unions of many index -1 curves in the main level. This sounds horrible, but it can actually be ignored, for the following reason: first, since there are only finitely many pairs (s, u) where u is a *connected* J_s-holomorphic curve with index -1, one can (if transversality is achievable at all) use a genericity argument to assume without loss of generality that for any given $s \in [0,1]$, at most *one* connected index -1 curve exists. This means that in any building that has multiple index -1 components, those components are just multiple copies of the same curve. Now, since that curve has odd index, it is represented by a monomial $q^{\gamma^-}p^{\gamma^+}$ that contains an odd number of odd generators, and any nontrivial product of such generators therefore *disappears* in $\mathcal{A}$ since odd generators anticommute with themselves. This algebraic miracle encodes a convenient fact about coherent orientations: whenever one of the horrible buildings described above appears, one can reorder two of the index -1 components to produce from it a different building that lives in a moduli space with the opposite orientation. Gluing this building back together then produces a continuation of the 1-dimensional moduli space, so that the horrible building can actually be interpreted as an "interior" point of the 1-dimensional space, rather than boundary. The actual count of boundary points is then exactly what we want it to be: it is represented algebraically by the chain homotopy relation!

Finally, compositions of cobordism maps can be understood via a stretching argument that is not substantially different from the case of cylindrical contact homology. Since the trivial cobordism with $\mathbb{R}$-invariant data gives a cobordism map that just counts trivial cylinders and is therefore the identity, it follows that cobordism maps relating different pairs of data $(\alpha_\pm, J_\pm)$ are always invertible, and this proves the invariance of $H_*^{\mathrm{SFT}}(M, \xi; R)$.

The second application concerns nontrivial exact cobordisms, and it is immediate from the fact that $\mathbf{\Phi}$ is a chain map:

Theorem 13.28. *Any exact cobordism $(W, d\lambda)$ from (M_-, ξ_-) to (M_+, ξ_+) gives rise to a $\mathbb{Q}[[\hbar]]$-linear map*

$$H_*^{\mathrm{SFT}}(M_+, \xi_+) \to H_*^{\mathrm{SFT}}(M_-, \xi_-).$$

□

It is much more complicated to say what happens in the event of a nonexact cobordism, but slightly easier if we restrict our attention to fillings, i.e. the case with $M_- = \emptyset$. Assume (W, ω) is a compact symplectic manifold with stable boundary M, inheriting a stable Hamiltonian structure $\mathcal{H} = (\Omega, \alpha)$ for which α is a nondegenerate contact form, and assume also that the completion $\widehat{W}$ admits an almost complex structure J that is ω-tame on W and has a pseudoconvex restriction $J_+ \in \mathcal{J}(\mathcal{H})$ to the cylindrical end. We saw in Proposition 13.22 that these conditions can always be achieved for a weak filling after deforming the symplectic structure. Let

$$G := \ker[\omega] := \{A \in H_2(W) \mid \langle [\omega], A \rangle = 0\},$$

and choose $G_+ \subset H_2(M)$ to be any subgroup such that the map $H_2(M) \to H_2(W)$ induced by the inclusion $M \hookrightarrow W$ sends G_+ into G. In other words, G_+ can be any subgroup of $\ker[\Omega] \subset H_2(M)$. Define the group rings

$$R_+ = \mathbb{Q}[H_2(M)/G_+], \qquad R = \mathbb{Q}[H_2(W)/\ker[\omega]],$$

with the Novikov completion of R denoted by $\overline{R}$. The map $H_2(M)/G_+ \to H_2(W)/G$ induced by $M \hookrightarrow W$ then gives a natural ring homomorphism

$$R_+ \to \overline{R}. \tag{13.16}$$

If ω is not exact, then it may no longer be true that every term in $\mathbf{F}$ has at least one p variable. Let us write

$$\mathbf{F} = \mathbf{F}_0 + \mathbf{F}_1,$$

where $\mathbf{F}_0$ contains no p variables and $\mathbf{F}_1 = \mathcal{O}(p)$, i.e. $\mathbf{F}_0$ counts all closed curves in $\widehat{W}$, and $\mathbf{F}_1$ counts everything else. Since $\mathbf{F}_0$ and $\mathbf{F}_1$ have even degree, they commute, and thus

$$\exp(\mathbf{F}) = \exp(\mathbf{F}_0)\exp(\mathbf{F}_1).$$

where $\exp(\mathbf{F}_0)$ is an invertible element of $\overline{R}[[\hbar, \hbar^{-1}]]$ since $\exp(-\mathbf{F}_0)\exp(\mathbf{F}_0) = 1$. By the master equation,

$$\exp(\mathbf{F}_0)\exp(\mathbf{F}_1)\mathbf{H} = \mathcal{O}(q),$$

hence $\exp(\mathbf{F}_1)\mathbf{H} = \exp(-\mathbf{F}_0)\mathcal{O}(q) = \mathcal{O}(q)$ since $\exp(-\mathbf{F}_0)$ contains no p variables. Using the substitution (13.13), and using (13.16) to map coefficients in R_+ to $\overline{R}$, it follows that $\exp(\mathbf{F}_1)$ gives rise to a differential operator

$$\mathbf{\Phi} : \mathcal{A}[[\hbar]] \to \overline{R}[[\hbar]] : \mathbf{f} \mapsto \exp(\mathbf{F}_1)\mathbf{f}|_{q=0},$$

which is a chain map to the SFT of the empty set with Novikov coefficients, meaning

$$\mathbf{\Phi} \circ \mathbf{D}_{\mathrm{SFT}} = 0.$$

This chain map counts the disconnected index 0 curves in $\widehat{W}$ whose connected components all have at least one positive puncture.

THEOREM 13.29. *Suppose (W, ω) is a compact symplectic manifold with stable boundary $(M, \mathcal{H} = (\Omega, \alpha))$, where α is a nondegenerate contact form, and its completion $\widehat{W}$ admits an almost complex structure that is ω-tame on W and has a generic and pseudoconvex restriction $J_+ \in \mathcal{J}(\mathcal{H})$ to the cylindrical end. Let $\overline{R}$ denote the Novikov completion of $\mathbb{Q}[H_2(W)/\ker[\omega]]$, and let $R_+ = \mathbb{Q}[H_2(M)/G_+]$, where $G_+ \subset H_2(M)$ is any subgroup on which the evaluation of $[\Omega] \in H^2_{\mathrm{dR}}(M)$ vanishes. Then there exists an $\overline{R}[[\hbar]]$-linear map $H_*^{\mathrm{SFT}}(M, \mathcal{H}, J_+; R_+) \to \overline{R}[[\hbar]]$.* □

13.3.2. Algebraic torsion. We can now generalize the notion of algebraic overtwistedness. Notice that since every term in $\mathbf{D}_{\mathrm{SFT}}$ is a differential operator of order at least 1,

$$\mathbf{D}_{\mathrm{SFT}}\mathbf{f} = 0 \quad \text{for all} \quad \mathbf{f} \in R[[\hbar]],$$

hence every element of the extended coefficient ring $R[[\hbar]]$ represents an element of $H_*^{\mathrm{SFT}}(M, \xi; R)$ that may or may not be trivial. Since $\mathbf{D}_{\mathrm{SFT}}$ commutes with all elements of $R[[\hbar]]$, the subset consisting of elements that are trivial in homology forms an ideal. The following definition originates in [LW11].

DEFINITION 13.30. We say that a closed contact manifold (M, ξ) has **algebraic torsion of order** k (or k-*torsion* for short) with coefficients in R if

$$[\hbar^k] = 0 \in H_*^{\mathrm{SFT}}(M, \xi; R).$$

The numerical invariant

$$\mathrm{AT}(M, \xi; R) \in \mathbb{N} \cup \{0, \infty\}$$

is defined to be the smallest integer k such that (M, ξ) has algebraic k-torsion but no $(k-1)$-torsion, or ∞ if there is no algebraic torsion of any order.

Several consequences of algebraic torsion can be read off quickly from the properties of SFT cobordism maps. Consider first the case of trivial coefficients $R = \mathbb{Q}$, which we shall refer to as **untwisted** algebraic torsion and abbreviate

$$\mathrm{AT}(M, \xi) := \mathrm{AT}(M, \xi; \mathbb{Q}).$$

If (W, ω) is a strong filling of (M, ξ), then the hypotheses of Theorem 13.29 are fulfilled even with $G_+ = H_2(M)$ since ω is exact at the boundary, thus we obtain a $\mathbb{Q}[[\hbar]]$-linear map $H_*^{\mathrm{SFT}}(M, \xi) \to \overline{R}[[\hbar]]$, with $\overline{R}$ denoting the Novikov completion of $\mathbb{Q}[H_2(W)/\ker[\omega]]$. If $[\hbar^k] = 0 \in H_*^{\mathrm{SFT}}(M, \xi)$, then the cobordism map implies a contradiction since $\hbar^k$ does not equal 0 in $\overline{R}[[\hbar]]$. Similarly, if $(W, d\lambda)$ is an exact cobordism from (M_-, ξ_-) to (M_+, ξ_+), then the cobordism map $H_*^{\mathrm{SFT}}(M_+, \xi_+) \to H_*^{\mathrm{SFT}}(M_-, \xi_-)$ of Theorem 13.28 is also $\mathbb{Q}[[\hbar]]$-linear, and thus any algebraic k-torsion in (M_+, ξ_+) is inherited by (M_-, ξ_-). This proves:

THEOREM 13.31. *Contact manifolds with* $\mathrm{AT}(M,\xi) < \infty$ *are not strongly fillable. Moreover, if there exists an exact symplectic cobordism from* (M_-,ξ_-) *to* (M_+,ξ_+)*, then* $\mathrm{AT}(M_-,\xi_-) \le \mathrm{AT}(M_+,\xi_+)$. □

It is known (see [**Wen13**]) that the second part of the above theorem does not hold for strong symplectic cobordisms in general, so exactness of cobordisms is a meaningful symplectic topological condition, not just a technical hypothesis. It is also known thanks to a construction of Ghiggini [**Ghi05**] that strong and exact fillability are not equivalent conditions, but Ghiggini's proof of this uses Heegaard Floer homology; thus far it is not known whether this phenomenon can be detected via SFT or other holomorphic curve techniques.

There are also many known examples of contact manifolds that have untwisted algebraic torsion but are weakly fillable. The simplest are the tight tori $(\mathbb{T}^3,\xi_k)$ for $k \ge 2$, for which weak fillings were first constructed by Giroux [**Gir94**], but Eliashberg [**Eli96**] showed that strong fillings do not exist, and we will see in Lecture 16 that $\mathrm{AT}(\mathbb{T}^3,\xi_k) = 1$. The weak/strong distinction can often be detected via the choice of coefficients in SFT. We saw in §13.2.1 that a weak filling of a contact manifold (M,ξ) can always be deformed so as to have stable boundary with data $(\mathcal{H} = (\Omega,\alpha), J_+)$ for which α is a nondegenerate contact form and J_+ is C^∞-close to any given element of $\mathcal{J}(\alpha)$. Proposition 13.16 showed that if (M,ξ) is algebraically overtwisted, then the contact homology for the stable Hamiltonian data $(\mathcal{H}, J_+)$ can also be made to vanish.

EXERCISE 13.32. Generalize the proof of Prop. 13.16 to show that if (M,ξ) has algebraic k-torsion with coefficients in R, then also $[\hbar^k] = 0 \in H_*^{\mathrm{SFT}}(M,\mathcal{H}_c,J_c;R)$ for sufficiently large $c > 0$.

It then follows using Theorem 13.29 that algebraic torsion with suitably twisted coefficients also gives an obstruction to weak filling. Let us say that (M,ξ) has **fully twisted** algebraic k-torsion whenever $[\hbar^k] = 0 \in H_*^{\mathrm{SFT}}(M,\xi;\mathbb{Q}[H_2(M)])$. Note that in parallel with Remark 13.10, any nested pair of subgroups $G \subset G' \subset H_2(M)$ gives rise to a map

$$H_*^{\mathrm{SFT}}(M,\xi;\mathbb{Q}[H_2(M)/G']) \to H_*^{\mathrm{SFT}}(M,\xi;\mathbb{Q}[H_2(M)/G]),$$

which is a morphism in the sense that it maps the unit and all powers of $\hbar$ to themselves. This implies that (M,ξ) has fully twisted k-torsion if and only if it has k-torsion for every choice of coefficients.

THEOREM 13.33. *If* (M,ξ) *is a closed contact manifold with a finite order of algebraic torsion with coefficients in* $R = \mathbb{Q}[H_2(M)/G]$ *for some subgroup* G*, then* (M,ξ) *does not admit any weak symplectic filling* (W,ω) *for which* $[\omega|_{TM}] \in H^2_{\mathrm{dR}}(M)$ *is rational and annihilates all elements of* G*. In particular, if* (M,ξ) *has fully twisted algebraic torsion of some finite order, then it is not weakly fillable.*

REMARK 13.34. The rationality condition in Theorem 13.33 can probably be lifted, and is known to be unnecessary at least in dimension three. It is clear in any case that if (M,ξ) admits a weak filling (W,ω), then one can always make a small perturbation of ω to produce a weak filling for which $[\omega|_{TM}] \in H^2(M;\mathbb{Q})$.

We will see some concrete examples of algebraic torsion computations in Lecture 16. Let us conclude this discussion for now with the observation that algebraic torsion of order *zero* is a notion we've seen before:

PROPOSITION 13.35. *For any closed contact manifold (M, ξ) and group ring $R = \mathbb{Q}[H_2(M)/G]$, the following conditions are equivalent:*

(1) (M, ξ) has algebraic 0-torsion (with coefficients in R);
(2) (M, ξ) is algebraically overtwisted (with coefficients in R);
(3) $H_*^{\mathrm{SFT}}(M, \xi; R) = 0$.

PROOF. It is obvious that (3) implies (1). Since $\mathbf{D}_{\mathrm{SFT}}\mathbf{f} = \partial_{\mathrm{CH}}\mathbf{f} + \mathcal{O}(\hbar)$ for $\mathbf{f} \in \mathcal{A}$, the $R[[\hbar]]$-linear map

$$\mathcal{A}[[\hbar]] \to \mathcal{A} : \mathbf{F} \mapsto \mathbf{F}|_{\hbar=0}$$

defines a chain map $(\mathcal{A}[[\hbar]], \mathbf{D}) \to (\mathcal{A}, \partial_{\mathrm{CH}})$ and thus descends to an $R[[\hbar]]$-linear map $H_*^{\mathrm{SFT}}(M, \xi; R) \to HC_*(M, \xi; R)$. The existence of this map proves that (1) implies (2).

To prove that (2) implies (3), recall first that if there exists $\mathbf{f} \in \mathcal{A}$ with $\partial_{\mathrm{CH}}\mathbf{f} = 1$, then the fact that $HC_*(M, \xi; R) = 0$ follows easily since for any $\mathbf{g} \in \mathcal{A}$ with $\partial_{\mathrm{CH}}\mathbf{g} = 0$, the graded Leibniz rule implies $\partial_{\mathrm{CH}}(\mathbf{fg}) = (\partial_{\mathrm{CH}}\mathbf{f})\mathbf{g} - \mathbf{f}(\partial_{\mathrm{CH}}\mathbf{g}) = \mathbf{g}$. This works because ∂_{CH} is a derivation—but $\mathbf{D}_{\mathrm{SFT}}$ is not one, so the same trick will not quite work for $\mathbf{D}_{\mathrm{SFT}}$. The trick in proving $H_*^{\mathrm{SFT}}(M, \xi; R) = 0$ will be to quantify the failure of $\mathbf{D}_{\mathrm{SFT}}$ to be a derivation. For our purposes, it suffices to know that

$$\mathbf{D}_{\mathrm{SFT}}(\mathbf{FG}) = (\mathbf{D}_{\mathrm{SFT}}\mathbf{F})\mathbf{G} + (-1)^{|\mathbf{F}|}\mathbf{F}(\mathbf{D}_{\mathrm{SFT}}\mathbf{G}) + \mathcal{O}(\hbar) \tag{13.17}$$

holds for all $\mathbf{F}, \mathbf{G} \in \mathcal{A}[[\hbar]]$, which follows from the fact that ∂_{CH} is a derivation.

With this remark out of the way, suppose $\mathbf{f} \in \mathcal{A}$ satisfies $\partial_{\mathrm{CH}}\mathbf{f} = 1$, in which case

$$\mathbf{D}_{\mathrm{SFT}}\mathbf{f} = 1 + \hbar\mathbf{G} \tag{13.18}$$

for some $\mathbf{G} \in \mathcal{A}[[\hbar]]$. We claim then that for any $\mathbf{Q} \in \mathcal{A}[[\hbar]]$ with $\mathbf{D}_{\mathrm{SFT}}\mathbf{Q} = 0$, there exists $\mathbf{Q}_1 \in \mathcal{A}[[\hbar]]$ with

$$\mathbf{D}_{\mathrm{SFT}}(\mathbf{fQ}) = \mathbf{Q} + \hbar\mathbf{Q}_1 \tag{13.19}$$

and $\mathbf{D}_{\mathrm{SFT}}\mathbf{Q}_1 = 0$. Indeed, (13.19) follows from (13.17) and (13.18) since $\mathbf{D}_{\mathrm{SFT}}\mathbf{Q} = 0$, and $\mathbf{D}_{\mathrm{SFT}}\mathbf{Q}_1 = 0$ then follows by applying $\mathbf{D}_{\mathrm{SFT}}$ to (13.19) and using $\mathbf{D}_{\mathrm{SFT}}^2 = 0$. Fixing $\mathbf{Q}_0 := \mathbf{Q} \in \mathcal{A}[[\hbar]]$, we can now define a sequence $\mathbf{Q}_k \in \mathcal{A}[[\hbar]]$ satisfying $\mathbf{D}_{\mathrm{SFT}}\mathbf{Q}_k = 0$ for all integers $k \ge 0$ via the inductive condition

$$\mathbf{D}_{\mathrm{SFT}}(\mathbf{fQ}_k) = \mathbf{Q}_k + \hbar\mathbf{Q}_{k+1}.$$

Then $\sum_{k=0}^{\infty}(-1)^k\hbar^k\mathbf{Q}_k \in \mathcal{A}[[\hbar]]$, and

$$\mathbf{D}_{\mathrm{SFT}}\left(\mathbf{f}\sum_{k=0}^{\infty}(-1)^k\hbar^k\mathbf{Q}_k\right) = \mathbf{Q}.$$

□

LECTURE 14

Transversality and embedding controls in dimension four

The final three lectures will be included in the published version of this book. For updates on publication, see the author's website

https://www.mathematik.hu-berlin.de/~wendl/publications.html#notes

LECTURE 15

Intersection theory for punctured holomorphic curves

The final three lectures will be included in the published version of this book. For updates on publication, see the author's website

https://www.mathematik.hu-berlin.de/~wendl/publications.html#notes

LECTURE 16

Torsion computations and applications

The final three lectures will be included in the published version of this book. For updates on publication, see the author's website

https://www.mathematik.hu-berlin.de/~wendl/publications.html#notes

APPENDIX A

Sobolev spaces

Contents

In this appendix, we review some of the standard properties of Sobolev spaces, in particular using them to prove Propositions 2.7, 2.8 and 2.10 from §2.2, and elucidating the construction of Sobolev spaces of sections on vector bundles. A good reference for the necessary background material is [**AF03**].

A.1. Approximation, extension and embedding theorems

Unless otherwise noted, all functions in the following are assumed to be defined on a nonempty open subset

$$\mathcal{U} \subset \mathbb{R}^n$$

with its standard Lebesgue measure, and taking values in a finite-dimensional normed vector space that will usually not need to be specified, though occasionally we will assume it is $\mathbb{R}$ or $\mathbb{C}$ so that one can define products of functions. The domain $\mathcal{U}$ will also sometimes have additional conditions specified such as boundedness or regularity at the boundary, though we will try not to add too many more restrictions than are really needed. The most useful assumption to impose on $\mathcal{U}$ is known as the **strong local Lipschitz condition**: if $\mathcal{U}$ is bounded, then it means simply that near every boundary point of $\mathcal{U}$, one can find smooth local coordinates in which $\mathcal{U}$ looks like the region bounded by the graph of a Lipschitz-continuous function, and in this case we call $\mathcal{U}$ a **bounded Lipschitz domain**. If $\mathcal{U}$ is unbounded, then one needs to impose extra conditions guaranteeing e.g. uniformity of Lipschitz constants, and the precise definition becomes a bit lengthy (see [**AF03**, §4.9]). For our purposes, all we really need to know about the strong local Lipschitz condition is that that it is satisfied both by bounded Lipschitz domains and by relatively tame unbounded domains such as $(0, 1) \times (0, \infty) \subset \mathbb{R}^2$ which have smooth boundary with finitely many corners. We will repeatedly need to use the generalized version of **Hölder's inequality**, which states that for any finite collection of measurable functions $f_1, \dots, f_m$,

$$\left\| \prod_{i=1}^m |f_i| \right\|_{L^p} \le \prod_{i=1}^m \|f_i\|_{L^{p_i}} \quad \text{for } 1 \le p \le p_1, \dots, p_m \le \infty \text{ with } \frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}. \tag{A.1}$$

This is an easy corollary of the standard version,

$$\big\| |f| \cdot |g| \big\|_{L^1} \le \|f\|_{L^p} \cdot \|g\|_{L^q} \quad \text{whenever } 1 \le p, q \le \infty \text{ and } 1 = \frac{1}{p} + \frac{1}{q}.$$

For an integer $k \ge 0$ and real number $p \in [1, \infty]$ we define $W^{k,p}(\mathcal{U})$ as in §2.2 to be the Banach space of all $f \in L^p(\mathcal{U})$ which have weak partial derivatives $\partial^\alpha f \in L^p(\mathcal{U})$ for all $|\alpha| \le k$. For $p = 2$, these spaces are also often denoted by

$$H^k(\mathcal{U}) := W^{k,2}(\mathcal{U}),$$

and they admit Hilbert space structures with inner product

$$\langle f, g \rangle_{H^k} = \sum_{|\beta| \le k} \langle \partial^\alpha f, \partial^\alpha g \rangle_{L^2}.$$

We denote by

$$W_0^{k,p}(\mathcal{U}) \subset W^{k,p}(\mathcal{U}), \qquad H_0^k(\mathcal{U}) \subset H^k(\mathcal{U})$$

the closed subspaces defined as the closures of $C_0^\infty(\mathcal{U})$ with respect to the relevant norms. Since $C_0^\infty(\mathcal{U})$ is dense in $L^p(\mathcal{U})$ for $1 \le p < \infty$ (see e.g. [**LL01**, §2.19]), there is no difference between $W^{0,p}(\mathcal{U})$ and $W_0^{0,p}(\mathcal{U})$ for $p < \infty$, but in general $W_0^{k,p}(\mathcal{U}) \ne W^{k,p}(\mathcal{U})$ for $k \ge 1$, with a few notable exceptions such as the case $\mathcal{U} = \mathbb{R}^n$ (cf. Corollary A.2 below). Let

$$W^{k,p}_{\text{loc}}(\mathcal{U}) := \Big\{ \text{functions } f \text{ on } \mathcal{U} \ \Big| \ f \in W^{k,p}(\mathcal{V}) \text{ for all open subsets } \mathcal{V} \subset \mathcal{U} \text{ with compact closure } \overline{\mathcal{V}} \subset \mathcal{U} \Big\},$$

and we say that a sequence $f_j \in W^{k,p}_{\text{loc}}(\mathcal{U})$ converges in $W^{k,p}_{\text{loc}}$ to $f \in W^{k,p}_{\text{loc}}(\mathcal{U})$ if the restrictions to all precompact open subsets $\mathcal{V} \subset \overline{\mathcal{V}} \subset \mathcal{U}$ converge in $W^{k,p}(\mathcal{V})$. Recall that for $k \in \{0, 1, 2, \dots, \infty\}$, $C^k(\mathcal{U})$ denotes the space of functions on $\mathcal{U}$ with continuous derivatives up to order k, while

$$C^k(\overline{\mathcal{U}}) \subset C^k(\mathcal{U})$$

is the space of $f \in C^k(\mathcal{U})$ such that for all $|\alpha| \le k$, $\partial^\alpha f$ is bounded and uniformly continuous.

THEOREM A.1 ([**AF03**, §3.17, 3.22]). *For any open subset $\mathcal{U} \subset \mathbb{R}^n$, and any $k \ge 0$, $1 \le p < \infty$, the subspace*

$$C^\infty(\mathcal{U}) \cap W^{k,p}(\mathcal{U}) \subset W^{k,p}(\mathcal{U})$$

is dense. Moreover, if $\mathcal{U} \subset \mathbb{R}^n$ satisfies the strong local Lipschitz condition, then the space

$$\Big\{ f \in C^\infty(\mathcal{U}) \ \Big| \ f = \tilde{f}|_{\mathcal{U}} \text{ for some } \tilde{f} \in C_0^\infty(\mathbb{R}^n) \Big\}$$

is also dense in $W^{k,p}(\mathcal{U})$, so in particular,

$$C^\infty(\overline{\mathcal{U}}) \cap W^{k,p}(\mathcal{U}) \subset W^{k,p}(\mathcal{U})$$

is dense. □

COROLLARY A.2. *The space* $C_0^\infty(\mathbb{R}^n)$ *is dense in* $W^{k,p}(\mathbb{R}^n)$ *for every* $k \ge 0$ *and* $p \in [1,\infty)$. □

Here is another useful characterization of $W_0^{k,p}(\mathcal{U})$:

THEOREM A.3 ([**AF03**, §5.29]). *Assume* $\mathcal{U} \subset \mathbb{R}^n$ *is an open subset satisfying the strong local Lipschitz condition. Then a function* $f \in W^{k,p}(\mathcal{U})$ *belongs to* $W_0^{k,p}(\mathcal{U})$ *if and only if the function* $\tilde{f}$ *on* $\mathbb{R}^n$ *defined to match* f *on* $\mathcal{U}$ *and* 0 *everywhere else belongs to* $W^{k,p}(\mathbb{R}^n)$. □

While it is obvious from the definitions that functions in $W_0^{k,p}(\mathcal{U})$ always admit extensions of class $W^{k,p}$ over $\mathbb{R}^n$, this is much less obvious for functions in $W^{k,p}(\mathcal{U})$ in general, and it is not true without sufficient assumptions about the regularity of $\partial\mathcal{U}$. For our purposes it suffices to consider the following case.

THEOREM A.4 ([**AF03**, §5.22]). *Assume* $\mathcal{U} \subset \mathbb{R}^n$ *is a bounded open subset such that* $\partial\overline{\mathcal{U}}$ *is a submanifold of class* C^m *for some* $m \in \{1,2,3,\dots,\infty\}$. *Then there exists a linear operator* E *that maps functions defined almost everywhere on* $\mathcal{U}$ *to functions defined almost everywhere on* $\mathbb{R}^n$ *and has the following properties:*

- *For every function* f *on* $\mathcal{U}$, $Ef|_{\mathcal{U}} \equiv f$ *almost everywhere;*
- *For every nonnegative integer* $k \le m$ *and every* $p \in [1,\infty)$, E *defines a bounded linear operator* $W^{k,p}(\mathcal{U}) \to W^{k,p}(\mathbb{R}^n)$.

□

COROLLARY A.5. *Suppose* $\mathcal{U},\mathcal{U}' \subset \mathbb{R}^n$ *are open subsets such that* $\mathcal{U}$ *has compact closure contained in* $\mathcal{U}'$. *If* $\mathcal{U}$ *satisfies the hypothesis of Theorem A.4, then the resulting extension operator* E *can be chosen such that it maps each* $W^{k,p}(\mathcal{U})$ *for* $k \le m$ *and* $1 \le p < \infty$ *into* $W_0^{k,p}(\mathcal{U}')$.

PROOF. Choose a smooth function $\rho : \mathcal{U}' \to [0,1]$ that has compact support and equals 1 on $\overline{\mathcal{U}}$, then replace the operator E given by Theorem A.4 with the operator $f \mapsto \rho \cdot Ef$. □

To state the Sobolev embedding theorem in its proper generality, recall that for $0 < \alpha \le 1$, the **Hölder seminorm** of a function f on $\mathcal{U}$ is defined by

$$|f|_{C^\alpha} := |f|_{C^\alpha(\mathcal{U})} := \sup_{x \ne y \in \mathcal{U}} \frac{|f(x) - f(y)|}{|x-y|^\alpha},$$

and $C^{k,\alpha}(\mathcal{U})$ is then defined as the Banach space of functions $f \in C^k(\overline{\mathcal{U}})$ for which the norm

$$\|f\|_{C^{k,\alpha}} := \|f\|_{C^k} + \max_{|\beta|=k} |\partial^\beta f|_{C^\alpha}$$

is finite. In reading the following statement, it is important to remember that elements of $W^{k,p}(\mathcal{U})$ are technically not functions, but rather *equivalence classes* of functions defined almost everywhere. Thus when we say e.g. that there is an inclusion $W^{k,p}(\mathcal{U}) \hookrightarrow C^{m,\alpha}(\mathcal{U})$, the literal meaning is that for every function f representing an element of $W^{k,p}(\mathcal{U})$, one can change the values of f in a unique way on some set of measure zero in $\mathcal{U}$ so that after this change, $f \in C^{m,\alpha}(\mathcal{U})$. Continuity of the inclusion means that there is a bound of the form

$$\|f\|_{C^{m,\alpha}} \le c\|f\|_{W^{k,p}}$$

for all $f \in W^{k,p}(\mathcal{U})$, where $c > 0$ is a constant which may in general depend on m, α, k, p and $\mathcal{U}$, but not on f.

THEOREM A.6 ([**AF03**, §4.12]). *Assume $\mathcal{U} \subset \mathbb{R}^n$ is an open subset satisfying the strong local Lipschitz condition, $k \ge 1$ is an integer and $1 \le p < \infty$.*

(1) If $kp > n$ and $k - n/p < 1$, then there exist continuous inclusions

$$W^{k,p}(\mathcal{U}) \hookrightarrow C^{0,\alpha}(\mathcal{U}) \quad \text{for each } \alpha \in (0, k - n/p],$$
$$W^{k,p}(\mathcal{U}) \hookrightarrow L^q(\mathcal{U}) \quad \text{for each } q \in [p, \infty].$$

(2) If $kp < n$ and $p^* > p$ is defined by the condition

$$\frac{1}{p^*} = \frac{1}{p} - \frac{k}{n},$$

then there exist continuous inclusions

$$W^{k,p}(\mathcal{U}) \hookrightarrow L^q(\mathcal{U}), \qquad \text{for each } q \in [p, p^*].$$

(3) If $kp = n$, then there exist continuous inclusions

$$W^{k,p}(\mathcal{U}) \hookrightarrow L^q(\mathcal{U}), \qquad \text{for each } q \in [p, \infty).$$

Moreover, the spaces $W_0^{k,p}(\mathcal{U})$ admit similar inclusions under no assumption on the open subset $\mathcal{U} \subset \mathbb{R}^n$. □

Under the same assumption on the domain $\mathcal{U}$, one can apply Theorem A.6 to successive derivatives of functions in $W^{k,p}(\mathcal{U})$ and thus obtain the following inclusions for any integer $d \ge 0$:

$$W^{k+d,p}(\mathcal{U}) \hookrightarrow C^{d,\alpha}(\mathcal{U}) \quad \text{if } kp > n \text{ and } 0 < \alpha \le k - n/p < 1, \tag{A.2}$$

$$W^{k+d,p}(\mathcal{U}) \hookrightarrow W^{d,q}(\mathcal{U}) \quad \text{if } kp > n \text{ and } p \le q \le \infty, \tag{A.3}$$

$$W^{k+d,p}(\mathcal{U}) \hookrightarrow W^{d,q}(\mathcal{U}) \quad \text{if } kp < n \text{ and } p \le q \le p^*, \text{ with } \frac{1}{p^*} = \frac{1}{p} - \frac{k}{n}, \tag{A.4}$$

$$W^{k+d,p}(\mathcal{U}) \hookrightarrow W^{d,q}(\mathcal{U}) \quad \text{if } kp = n \text{ and } p \le q < \infty. \tag{A.5}$$

This last inclusion can then be composed with (A.2) for an arbitrarily large choice of q, giving another inclusion

$$W^{k+d,p}(\mathcal{U}) \hookrightarrow C^{d-1,\alpha}(\mathcal{U}) \quad \text{if } kp = n \text{ and } 0 < \alpha < 1. \tag{A.6}$$

REMARK A.7. The embedding theorem suggests that one should intuitively think of $W^{k,p}(\mathcal{U})$ as consisting of functions with "$k - n/p$ continuous derivatives," where the number $k - n/p$ may in general be a non-integer and/or negative. This provides a useful mnemonic for results about embeddings of one Sobolev space into another, such as the following.

Corollary A.8. *Assume $\mathcal{U} \subset \mathbb{R}^n$ is an open subset satisfying the strong local Lipschitz condition, $1 \le p, q < \infty$, and $k, m \ge 0$ are integers satisfying*

$$k \ge m, \qquad p \le q, \qquad \text{and} \qquad k - \frac{n}{p} \ge m - \frac{n}{q}.$$

Then there exists a continuous inclusion $W^{k,p}(\mathcal{U}) \hookrightarrow W^{m,q}(\mathcal{U})$. □

By the Arzelà-Ascoli theorem, the natural inclusion

$$C^{k,\alpha'}(\mathcal{U}) \hookrightarrow C^{k,\alpha}(\mathcal{U})$$

for $\alpha < \alpha'$ is a compact operator whenever $\mathcal{U} \subset \mathbb{R}^n$ is bounded. It follows that if $\mathcal{U} \subset \mathbb{R}^n$ in (A.2) is bounded and α is *strictly* less than the extremal value $k - n/p$, then the inclusion (A.2) is also compact. A similar statement holds for the inclusion (A.4) when $p \le q < p^*$, and this is known as the **Rellich-Kondrachov compactness theorem**. We summarize these as follows:

Theorem A.9 ([**AF03**, §6.3]). *Assume $\mathcal{U} \subset \mathbb{R}^n$ is a bounded Lipschitz domain, $k \ge 1$ and $d \ge 0$ are integers and $1 \le p < \infty$.*

(1) If $kp > n$ and $k - n/p < 1$, then the inclusions

$$W^{k+d,p}(\mathcal{U}) \hookrightarrow C^{d,\alpha}(\mathcal{U}) \quad \text{for } \alpha \in (0, k - n/p),$$
$$W^{k+d,p}(\mathcal{U}) \hookrightarrow W^{d,q}(\mathcal{U}) \quad \text{for } q \in [p, \infty)$$

are compact.

(2) If $kp \le n$ and $p^* \in (p, \infty]$ is defined by the condition $1/p^* = 1/p - k/n$, then the inclusions

$$W^{k+d,p}(\mathcal{U}) \hookrightarrow W^{d,q}(\mathcal{U}) \quad \text{for } q \in [p, p^*)$$

are compact.

In particular, the continuous inclusion $W^{k,p}(\mathcal{U}) \hookrightarrow W^{m,q}(\mathcal{U})$ in Corollary A.8 is compact whenever the inequality $k - n/p \ge m - n/q$ is strict. □

A.2. Products, compositions, and rescaling

We now restate and prove Propositions 2.7, 2.8 and 2.10 from §2.2. These are all corollaries of the Sobolev embedding theorem, so in particular they hold for the same class of domains $\mathcal{U} \subset \mathbb{R}^n$, and the restrictions on $\mathcal{U}$ can be dropped at the cost of replacing each space $W^{k,p}$ by $W_0^{k,p}$.

We begin by generalizing Prop. 2.7, hence we consider Sobolev spaces of functions valued in $\mathbb{R}$ or $\mathbb{C}$ so that pointwise products of functions are well defined almost everywhere. We say that there is a **continuous product map**,

$$W^{k_1,p_1}(\mathcal{U}) \times \ldots \times W^{k_m,p_m}(\mathcal{U}) \to W^{k,p}(\mathcal{U}),$$

or a continuous product **pairing** in the case $m = 2$, if for every set of functions $f_i \in W^{k_i,p_i}(\mathcal{U})$ with $i = 1, \ldots, m$, the pointwise product function $f_1 \cdot \ldots \cdot f_m$ is in $W^{k,p}(\mathcal{U})$ and there is an estimate of the form

$$\|f_1 \cdot \ldots \cdot f_m\|_{W^{k,p}} \le c \|f_1\|_{W^{k_1,p_1}} \cdot \ldots \cdot \|f_m\|_{W^{k_m,p_m}}$$

for some constant $c > 0$ not depending on $f_1, \dots, f_m$. The case $m = 2$, $k_1 = k_2 = k$ and $p_1 = p_2 = p$ is especially interesting, as the space $W^{k,p}(\mathcal{U})$ is then a **Banach algebra**. More generally, one can ask under what circumstances multiplication by functions of class $W^{k,p}$ defines a bounded linear operator on functions of class $W^{m,q}$. A hint about this comes from the world of classically differentiable functions: multiplication by C^k-smooth functions defines a continuous map $C^m \to C^m$ if and only if $k \ge m$. The corresponding answer in Sobolev spaces turns out to be that functions of class $W^{k,p}$ need to have strictly more than zero derivatives in the sense of Remark A.7, and at least as many derivatives as functions of class $W^{m,q}$.

THEOREM A.10. *Assume $\mathcal{U} \subset \mathbb{R}^n$ is an open subset satisfying the strong local Lipschitz condition, k, p, m and q satisfy the same numerical hypotheses as in Corollary A.8 (so in particular $W^{k,p}(\mathcal{U})$ embeds continuously into $W^{m,q}(\mathcal{U})$), and $kp > n$. Then there exists a continuous product pairing*

$$W^{k,p}(\mathcal{U}, \mathbb{C}) \times W^{m,q}(\mathcal{U}, \mathbb{C}) \to W^{m,q}(\mathcal{U}, \mathbb{C}) : (f, g) \mapsto fg.$$

The following preparatory lemma will be useful both for proving the product estimate and for further results below. It is an easy consequence of Theorem A.6 and Hölder's inequality.

LEMMA A.11. *Assume $\mathcal{U} \subset \mathbb{R}^n$ is an open subset satisfying the strong local Lipschitz condition, $m \ge 2$ is an integer, and we are given positive numbers $p_1, \dots, p_m \ge 1$ and integers $k_1, \dots, k_m \ge 0$. Let $I := \left\{ i \in \{1, \dots, m\} \mid k_i p_i \le n \right\}$. Then for any $q \ge 1$ satisfying*

$$\sum_{i \in I} \left(\frac{1}{p_i} - \frac{k_i}{n} \right) < \frac{1}{q} \le \sum_{i=1}^{m} \frac{1}{p_i},$$

there is a continuous product map

$$W^{k_1,p_1}(\mathcal{U}) \times \dots \times W^{k_m,p_m}(\mathcal{U}) \to L^q(\mathcal{U}).$$

PROOF. By the generalized Hölder inequality (A.1), it suffices to show that for any $q \ge 1$ in the stated range, one can find numbers $q_1, \dots, q_m \in [q, \infty]$ satisfying $1/q = 1/q_1 + \dots + 1/q_m$ for which Theorem A.6 provides continuous inclusions

$$W^{k_i,p_i}(\mathcal{U}) \hookrightarrow L^{q_i}(\mathcal{U})$$

for each $i = 1, \dots, m$. Whenever $k_i p_i > n$, this inclusion is valid with q_i chosen freely from the interval $[p_i, \infty]$, so $1/q_i$ can then take any value subject to the constraint

$$0 \le \frac{1}{q_i} \le \frac{1}{p_i}.$$

If on the other hand $k_i p_i \le n$, then we can arrange $1/q_i$ to take any value in the range

$$\frac{1}{p_i} - \frac{k_i}{n} < \frac{1}{q_i} \le \frac{1}{p_i}.$$

Adding these up, the range of values for $\sum_i \frac{1}{q_i}$ that we can achieve in this way covers the stated interval. ∎

PROOF OF THEOREM A.10. By density of smooth functions, it suffices to prove that an estimate of the form

$$\|fg\|_{W^{m,q}} \le c\|f\|_{W^{k,p}}\|g\|_{W^{m,q}}$$

holds for all $f \in C^\infty(\mathcal{U}) \cap W^{k,p}(\mathcal{U})$ and $g \in C^\infty(\mathcal{U}) \cap W^{m,q}(\mathcal{U})$. Equivalently, we need to show that for all f and g of this type and every multiindex α of degree $|\alpha| \le m$, there is a constant $c > 0$ independent of f and g such that

$$\|\partial^\alpha(fg)\|_{L^q} \le c\|f\|_{W^{k,p}}\|g\|_{W^{m,q}}.$$

Since f and g are smooth, we are free to use the product rule in computing $\partial^\alpha(fg)$, which will then be a linear combination of terms of the form $\partial^\beta f \cdot \partial^\gamma g$ where $|\alpha| = |\beta| + |\gamma|$, hence we have reduced the problem to proving a bound

$$\|\partial^\beta f \cdot \partial^\gamma g\|_{L^q} \le c\|f\|_{W^{k,p}}\|g\|_{W^{m,q}}$$

for every pair of multiindices β, γ with $|\beta| + |\gamma| \le m$. Since $\partial^\beta f \in W^{k-|\beta|,p}(\mathcal{U})$ and $\partial^\gamma f \in W^{m-|\gamma|,q}(\mathcal{U})$, the result follows if we can assume that for every pair of integers $a, b \ge 0$ satisfying $a + b \le m$, there exists a continuous product pairing

$$W^{k-a,p}(\mathcal{U}) \times W^{m-b,q}(\mathcal{U}) \to L^q(\mathcal{U}). \tag{A.7}$$

If $(k-a)p > n$, then $W^{k-a,p} \hookrightarrow L^\infty$ and (A.7) is immediate since $W^{m-b,q} \hookrightarrow L^q(\mathcal{U})$. For the remaining cases, we shall apply Lemma A.11, noting that the condition $1/q \le 1/p + 1/q$ is trivially satisfied.

If $(m-b)q > n$ but $(k-a)p \le n$, then the hypotheses of the lemma are satisfied if and only if

$$\frac{1}{p} - \frac{k-a}{n} < \frac{1}{q}.$$

Since $\frac{1}{p} - \frac{k}{n} \le \frac{1}{q} - \frac{m}{n}$ by assumption, we have

$$\frac{1}{p} - \frac{k-a}{n} = \frac{1}{p} - \frac{k}{n} + \frac{a}{n} \le \frac{1}{q} - \frac{m}{n} + \frac{a}{n} \le \frac{1}{q}$$

since $a \le m$, and equality holds only if $a = m$, $b = 0$ and $k - n/p = m - n/q$, which implies $mq > n$. In this case $W^{m-b,q} = W^{m,q} \hookrightarrow L^\infty$, and the pairing (A.7) follows because $W^{k-a,p} = W^{k-m,p}$ embeds continuously into L^q: the latter follows from Theorem A.6 since $\frac{1}{p} - \frac{k-m}{n} = \frac{1}{q}$.

Finally, when $(k-a)p \le n$ and $(m-b)q \le n$, the hypotheses of the lemma are satisfied since

$$\left(\frac{1}{p} - \frac{k-a}{n}\right) + \left(\frac{1}{q} - \frac{m-b}{n}\right) \le \frac{1}{p} - \frac{k}{n} + \frac{1}{q} - \frac{m}{n} + \frac{m}{n} = \left(\frac{1}{p} - \frac{k}{n}\right) + \frac{1}{q} < \frac{1}{q},$$

where we've used the assumption $kp > n$ and the fact that $a + b \le m$. □

The next result generalizes Proposition 2.8 and concerns the following question: if $f : \mathcal{U} \to \mathbb{R}^m$ is a function of class $W^{k,p}$ whose graph lies in some open subset $\mathcal{V} \subset \mathcal{U} \times \mathbb{R}^m$, and $\Psi : \mathcal{V} \to \mathbb{R}^N$ is another function, under what conditions can we conclude that the function

$$\mathcal{U} \to \mathbb{R}^N : x \mapsto \Psi(x, f(x))$$

is in $W^{k,p}(\mathcal{U}, \mathbb{R}^N)$? We will abbreviate this function in the following by $\Psi \circ (\mathrm{Id} \times f)$, and we would also like to know whether it depends continuously (in the $W^{k,p}$-topology) on f and Ψ. The following theorem is stated rather generally, but on first reading you may prefer to assume $\mathcal{U} \subset \mathbb{R}^n$ is bounded, in which case some of the hypotheses become vacuous. We will say that an open subset $\mathcal{V} \subset \mathcal{U} \times \mathbb{R}^m$ is a **star-shaped neighborhood of** $f : \mathcal{U} \to \mathbb{R}^m$ if it contains the graph of f_0 and

$$(x, v) \in \mathcal{V} \quad \Rightarrow \quad (x, tv + (1-t) f_0(x)) \in \mathcal{V} \text{ for all } t \in [0, 1].$$

THEOREM A.12. *Assume $\mathcal{U} \subset \mathbb{R}^n$ is an open subset satisfying the strong local Lipschitz condition, $p \in [1, \infty)$ and $k \in \mathbb{N}$ satisfy $kp > n$, and $\mathcal{V} \subset \mathcal{U} \times \mathbb{R}^m$ is a star-shaped neighborhood of some function $f_0 \in W^{k,p}(\mathcal{U}, \mathbb{R}^m)$. Assume also $\mathcal{O}^{k,p}(\mathcal{U}; \mathcal{V}) \subset W^{k,p}(\mathcal{U}, \mathbb{R}^m)$ is an open neighborhood of f_0 such that*

$$(x, f(x)) \in \mathcal{V} \quad \text{for all } x \in \mathcal{U} \text{ and } f \in \mathcal{O}^{k,p}(\mathcal{U}; \mathcal{V}),$$

and $\mathcal{O}^k(\overline{\mathcal{V}}, \mathbb{R}^N) \subset C^k(\overline{\mathcal{V}}, \mathbb{R}^N)$ is a subset such that all $\Psi \in \mathcal{O}^k(\overline{\mathcal{V}}, \mathbb{R}^n)$ have the following properties:[1]

(1) There exists a bounded subset $\mathcal{K} \subset \mathcal{U}$ such that $\Psi(x, v)$ is independent of x for all $x \in \mathcal{U} \setminus \mathcal{K}$;

(2) $\Psi \circ (\mathrm{Id} \times f_0) \in L^p(\mathcal{U}, \mathbb{R}^N)$.

Then there is a well-defined and continuous map

$$\mathcal{O}^k(\overline{\mathcal{V}}, \mathbb{R}^N) \times \mathcal{O}^{k,p}(\mathcal{U}; \mathcal{V}) \to W^{k,p}(\mathcal{U}, \mathbb{R}^N) : (\Psi, f) \mapsto \Psi \circ (\mathrm{Id} \times f).$$

PROOF. We will show first that if $f \in \mathcal{O}^{k,p}(\mathcal{U}; \mathcal{V})$ is smooth, then $\Psi \circ (\mathrm{Id} \times f)$ belongs to $W^{k,p}(\mathcal{U}, \mathbb{R}^N)$ for every $\Psi \in \mathcal{O}^k(\overline{\mathcal{V}}, \mathbb{R}^N)$. Since $\mathcal{V}$ is a star-shaped neighborhood of f_0, we have

$$\begin{aligned}
|\Psi(x, f(x)) - \Psi(x, f_0(x))| &= \left| \int_0^1 \frac{d}{dt} \Psi\big(x, tf(x) + (1-t) f_0(x)\big)\, dt \right| \\
&\le \left(\int_0^1 \left| D_2 \Psi\big(x, tf(x) + (1-t) f_0(x)\big) \right| dt \right) \cdot |f(x) - f_0(x)| \\
&\le \|\Psi\|_{C^1(\mathcal{V})} \cdot |f(x) - f_0(x)|
\end{aligned}$$

for all $x \in \mathcal{U}$, implying

$$\|\Psi \circ (\mathrm{Id} \times f) - \Psi \circ (\mathrm{Id} \times f_0)\|_{L^p} \le \|\Psi\|_{C^1(\mathcal{V})} \cdot \|f - f_0\|_{L^p},$$

hence $\Psi \circ (\mathrm{Id} \times f) \in L^p(\mathcal{U}, \mathbb{R}^N)$.

For $\ell = 1, \ldots, k$, we can regard the ℓth derivative of Ψ with respect to variables in $\mathbb{R}^m$ as a bounded and uniformly continuous map from $\mathcal{V}$ into the vector space of symmetric ℓ-multilinear maps from $\mathbb{R}^m$ to $\mathbb{R}^N$, denoting this by

$$D_2^\ell \Psi : \mathcal{V} \to \mathrm{Hom}((\mathbb{R}^m)^{\otimes \ell}, \mathbb{R}^N).$$

Denote the partial derivatives with respect to variables in $\mathcal{U} \subset \mathbb{R}^n$ by

$$D_1^\beta \Psi : \mathcal{V} \to \mathbb{R}^N,$$

[1]Both of the conditions on $\Psi \in \mathcal{O}^k(\overline{\mathcal{V}}, \mathbb{R}^n)$ are vacuous if $\mathcal{U} \subset \mathbb{R}^n$ is bounded.

where β is a multiindex in n variables. Now for any multiindex α with $|\alpha| \le k$, the derivative $\partial^\alpha(\Psi \circ (\mathrm{Id} \times f))$ is a linear combination of product functions of the form

$$(D_1^\gamma D_2^\ell \Psi \circ (\mathrm{Id} \times f))(\partial^{\beta_1} f, \dots, \partial^{\beta_\ell} f) : \mathcal{U} \to \mathbb{R}^N, \tag{A.8}$$

where $\ell + |\gamma| \in \{1, \dots, |\alpha|\}$ and $|\beta_1| + \dots + |\beta_\ell| = |\alpha| - |\gamma|$. If $\ell = 0$ but $|\gamma| > 0$, then this expression is clearly in $L^p(\mathcal{U}, \mathbb{R}^N)$ since it is continuous and $D_1^\gamma \Psi(x, v) = 0$ for $x \in \mathcal{U} \setminus \mathcal{K}$, where $\mathcal{K}$ is bounded. For $\ell \ge 1$, it satisfies

$$\left\| (D_1^\gamma D_2^\ell \Psi \circ (\mathrm{Id} \times f))(\partial^{\beta_1} f, \dots, \partial^{\beta_\ell} f) \right\|_{L^p(\mathcal{U})} \le \| D_1^\gamma D_2^\ell \Psi \|_{C^0(\mathcal{V})} \cdot \left\| \prod_{j=1}^{\ell} |\partial^{\beta_j} f| \right\|_{L^p(\mathcal{U})}$$

if the product on the right hand side has finite L^p-norm. The latter is trivially true if $\ell = 1$. To deal with the $\ell \ge 2$ case, note that $\partial^{\beta_j} f \in W^{k-|\beta_j|,p}(\mathcal{U})$ for each $j = 1, \dots, \ell$, so the necessary bound will follow from the existence of a continuous product map

$$W^{k-m_1,p}(\mathcal{U}) \times \dots \times W^{k-m_\ell,p}(\mathcal{U}) \to L^p(\mathcal{U})$$

for $m_j := |\beta_j|$, and we claim that such a product map does exist whenever $kp > n$ and $m_1, \dots, m_\ell \ge 0$ are integers satisfying $m_1 + \dots + m_\ell \le k$. To see this, note first that since $W^{k-m_j,p} \hookrightarrow L^\infty$ whenever $(k - m_j)p > n$, it suffices to prove the claim under the assumption that $(k - m_j)p \le n$ for every $j = 1, \dots, \ell$. In this case, Lemma A.11 provides the desired product map if the condition

$$\sum_{j=1}^{\ell} \left(\frac{1}{p} - \frac{k - m_j}{n} \right) < \frac{1}{p} \le \sum_{j=1}^{\ell} \frac{1}{p}$$

is satisfied. And it is: using $kp > n$, $\ell \ge 2$ and $m_1 + \dots + m_\ell \le k$, we find

$$\begin{aligned} \sum_{j=1}^{\ell} \left(\frac{1}{p} - \frac{k - m_j}{n} \right) &= \ell \left(\frac{1}{p} - \frac{k}{n} \right) + \frac{m_1 + \dots + m_\ell}{n} \\ &\le \frac{1}{p} + (\ell - 1) \left(\frac{1}{p} - \frac{k}{n} \right) < \frac{1}{p}. \end{aligned}$$

This proves that $\Psi \circ (\mathrm{Id} \times f) \in W^{k,p}(\mathcal{U}, \mathbb{R}^N)$.

Next, suppose $f \in \mathcal{O}^{k,p}(\mathcal{U}; \mathcal{V})$ is not necessarily smooth but $f_i \in \mathcal{O}^{k,p}(\mathcal{U}; \mathcal{V})$ is a sequence of smooth functions converging to f in $W^{k,p}$, while $\Psi_i \in \mathcal{O}^k(\overline{\mathcal{V}}, \mathbb{R}^N)$ converges to $\Psi \in \mathcal{O}^k(\overline{\mathcal{V}}, \mathbb{R}^N)$ in C^k. Then the same argument we used to estimate $\|\Psi \circ (\mathrm{Id} \times f) - \Psi \circ (\mathrm{Id} \times f_0)\|_{L^p}$ shows that $\Psi \circ (\mathrm{Id} \times f_i) \to \Psi \circ (\mathrm{Id} \times f)$ in L^p, and since f_i is also C^0-convergent, the compactly supported functions $D_1^\gamma \Psi \circ (\mathrm{Id} \times f_i)$ converge to $D_1^\gamma \Psi \circ (\mathrm{Id} \times f)$ in L^p for each multiindex with $1 \le |\gamma| \le k$. For $\ell \ge 1$ and $|\gamma| + \ell \le k$, $D_1^\gamma D_2^\ell \Psi_i \circ (\mathrm{Id} \times f_i)$ converges to $D_1^\gamma D_2^\ell \Psi \circ (\mathrm{Id} \times f)$ in $C^0(\overline{\mathcal{U}}, \mathbb{R}^N)$, and each of the derivatives $\partial^{\beta_j} f_i$ appearing in (A.8) also converges in $L^p(\mathcal{U})$. In light of the continuous product maps discussed above, it follows that each derivative $\partial^\alpha(\Psi_i \circ (\mathrm{Id} \times f_i))$ for $|\alpha| \le k$ is L^p-convergent, and its limit is necessarily the corresponding weak derivative $\partial^\alpha(\Psi \circ (\mathrm{Id} \times f))$, hence (see Exercise A.13 below) $\Psi \circ (\mathrm{Id} \times f) \in W^{k,p}(\mathcal{U}, \mathbb{R}^N)$ and $\Psi_i \circ (\mathrm{Id} \times f_i) \xrightarrow{W^{k,p}} \Psi \circ (\mathrm{Id} \times f)$. □

EXERCISE A.13. Show that if f_i is a sequence of smooth functions on an open set $\mathcal{U} \subset \mathbb{R}^n$ with $f_i \stackrel{L^p}{\to} f$ and $\partial^\alpha f_i \stackrel{L^p}{\to} g$ for some multiindex α and functions $f, g \in L^p(\mathcal{U})$, then $\partial^\alpha f = g$ in the sense of distributions.

The following result on coordinate transformations of the domain can be proved in an analogous way to Theorem A.12, though it is considerably easier since there is no need to worry about Sobolev product maps (and thus no need to assume $kp > n$ or impose regularity conditions on the domain).

THEOREM A.14 ([**AF03**, §3.41]). *Assume $k \in \mathbb{N}$, $1 \le p \le \infty$, and $\mathcal{U}, \mathcal{U}' \subset \mathbb{R}^n$ are open subsets with a C^k-smooth diffeomorphism $\varphi : \mathcal{U} \to \mathcal{U}'$ such that all derivatives of φ and φ^{-1} up to order k are bounded and uniformly continuous. Then there is a well-defined Banach space isomorphism*

$$W^{k,p}(\mathcal{U}') \to W^{k,p}(\mathcal{U}) : f \mapsto f \circ \varphi.$$

□

We now restate and prove Proposition 2.10. We denote by $\mathring{\mathbb{D}}^n, \mathring{\mathbb{D}}^n_\epsilon \subset \mathbb{R}^n$ the open balls of radius 1 and ϵ respectively about the origin.

THEOREM A.15. *Assume $p \in [1, \infty)$ and $k \in \mathbb{N}$ satisfy $kp > n$, and for each $f \in W^{k,p}(\mathring{\mathbb{D}}^n)$ and $\epsilon \in (0, 1]$, define $f_\epsilon \in W^{k,p}(\mathring{\mathbb{D}}^n)$ by*

$$f_\epsilon(x) := f(\epsilon x).$$

Then there exist constants $C > 0$ and $r > 0$ such that for every $f \in W^{k,p}(\mathring{\mathbb{D}}^n)$,

$$\|f_\epsilon - f(0)\|_{W^{k,p}(\mathring{\mathbb{D}}^n)} \le C\epsilon^r \|f - f(0)\|_{W^{k,p}(\mathring{\mathbb{D}}^n)} \quad \text{for all } \epsilon \in (0, 1].$$

PROOF. Let β denote a multiindex of order $|\beta| = k$. Then using a change of variables, we have

$$\begin{aligned} \|\partial^\beta(f_\epsilon - f(0))\|^p_{L^p(\mathring{\mathbb{D}}^n)} &= \epsilon^{kp} \int_{\mathbb{D}^n} |\partial^\beta f(\epsilon x)|^p = \epsilon^{kp-n} \int_{\mathbb{D}^n_\epsilon} |\partial^\beta f(x)|^p \\ &\le \epsilon^{kp-n} \|\partial^\beta f\|^p_{L^p(\mathring{\mathbb{D}}^n)} \le \epsilon^{kp-n} \|f - f(0)\|^p_{W^{k,p}(\mathring{\mathbb{D}}^n)}, \end{aligned}$$

and $\epsilon^{kp-n} \to 0$ as $\epsilon \to 0$ since $kp - n > 0$.

Next, suppose $|\beta| = m \in \{1, \dots, k-1\}$. Then $\partial^\beta f$ and $\partial^\beta f_\epsilon$ are in $W^{k-m,p}(\mathring{\mathbb{D}}^n)$, and if $(k - m)p < n$, Theorem A.6 gives a continuous inclusion

$$W^{k-m,p}(\mathring{\mathbb{D}}^n) \hookrightarrow L^q(\mathring{\mathbb{D}}^n) \tag{A.9}$$

with $q > p$ satisfying $1/q + (k - m)/n = 1/p$. Likewise, if $(k - m)p \ge n$, then (A.9) is a continuous inclusion for arbitrarily large choices of $q \ge p$. We will therefore assume in general that (A.9) holds with $q \in (p, \infty)$ satisfying

$$\frac{1}{q} + \frac{1}{r} = \frac{1}{p},$$

where $r = \frac{n}{k-m}$ if $(k-m)p < n$ and otherwise $r = p + \delta$ for some $\delta > 0$ which may be chosen arbitrarily small. Given this, we use Hölder's inequality and find

$$\begin{aligned}
\|\partial^\beta(f_\epsilon - f(0))\|^p_{L^p(\mathring{\mathbb{D}}^n)} &= \epsilon^{mp} \int_{\mathbb{D}^n} |\partial^\beta f(\epsilon x)|^p = \epsilon^{mp-n} \int_{\mathbb{D}^n_\epsilon} |\partial^\beta f(x)|^p \\
&\le \epsilon^{mp-n} \|\partial^\beta f\|^p_{L^q(\mathring{\mathbb{D}}^n_\epsilon)} \|1\|^p_{L^r(\mathring{\mathbb{D}}^n_\epsilon)} \\
&\le \epsilon^{mp-n} \left[\mathrm{Vol}(\mathbb{D}^n_\epsilon)\right]^{p/r} \|\partial^\beta f\|^p_{L^q(\mathring{\mathbb{D}}^n)} \\
&\le c\epsilon^{mp-n} \left[\mathrm{Vol}(\mathbb{D}^n_\epsilon)\right]^{p/r} \|\partial^\beta f\|^p_{W^{k-m,p}(\mathring{\mathbb{D}}^n)} \\
&\le c\epsilon^{mp-n} \left[\mathrm{Vol}(\mathbb{D}^n_\epsilon)\right]^{p/r} \|f - f(0)\|^p_{W^{k,p}(\mathring{\mathbb{D}}^n)}
\end{aligned}$$

for some constant $c > 0$. Writing $\mathrm{Vol}(\mathbb{D}^n_\epsilon) = C\epsilon^n$ for a suitable constant $C > 0$, the exponent on ϵ in this expression becomes

$$mp - n + \frac{np}{r},$$

which is positive whenever $r = p + \delta$ with $\delta > 0$ sufficiently small since $m \ge 1$, and in the case $r = n/(k-m)$, it becomes simply $kp - n > 0$.

Finally, to bound the L^p-norm of $f_\epsilon - f(0)$ itself, we can use the fact that $f \in W^{k,p}$ is Hölder continuous, i.e. it satisfies

$$|f(x) - f(0)| \le c\|f - f(0)\|_{W^{k,p}(\mathring{\mathbb{D}}^n)} |x|^\alpha \quad \text{for all } x \in \mathring{\mathbb{D}}^n$$

for suitable constants $c > 0$ and $\alpha \in (0, 1)$. Thus

$$\begin{aligned}
\|f_\epsilon - f(0)\|^p_{L^p(\mathring{\mathbb{D}}^n)} &= \int_{\mathbb{D}^n} |f(\epsilon x) - f(0)|^p \le c^p \|f - f(0)\|^p_{W^{k,p}} \int_{\mathbb{D}^n} |\epsilon x|^{\alpha p} \\
&= c^p \|f - f(0)\|^p_{W^{k,p}} \epsilon^{\alpha p} \int_{\mathbb{D}^n} |x|^{\alpha p} \\
&= \epsilon^{\alpha p} \frac{c^p \,\mathrm{Vol}(S^{n-1})}{\alpha p + n} \|f - f(0)\|^p_{W^{k,p}}.
\end{aligned}$$

□

A.3. Spaces of sections of vector bundles

In this section, fix a field

$$\mathbb{F} := \mathbb{R} \text{ or } \mathbb{C},$$

assume M is a smooth n-dimensional manifold, possibly with boundary, and $\pi : E \to M$ is a smooth vector bundle of rank m over $\mathbb{F}$. This comes with a "bundle atlas" $\mathcal{A}(\pi)$, a set whose elements $\alpha \in \mathcal{A}(\pi)$ each consist of the following data:

(1) An open subset $\mathcal{U}_\alpha \subset M$;
(2) A smooth local coordinate chart $\varphi_\alpha : \mathcal{U}_\alpha \xrightarrow{\cong} \Omega_\alpha$, where Ω_α is an open subset of $\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \ge 0\}$;
(3) A smooth local trivialization $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \xrightarrow{\cong} \mathcal{U}_\alpha \times \mathbb{F}^m$.

Smoothness of φ_α and Φ_α means as usual that for every pair $\alpha, \beta \in \mathcal{A}(\pi)$, the coordinate transformations

$$\varphi_{\beta\alpha} := \varphi_\beta^{-1} \circ \varphi_\alpha : \Omega_{\alpha\beta} \xrightarrow{\cong} \Omega_{\beta\alpha}, \qquad \Omega_{\alpha\beta} := \varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$$

and transition maps

$$g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{GL}(m, \mathbb{F}) \quad \text{such that} \quad \begin{array}{l} \Phi_\beta \circ \Phi_\alpha^{-1}(x, v) = (x, g_{\beta\alpha}(x)v) \\ \text{for } x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta,\ v \in \mathbb{F}^m \end{array}$$

are smooth, and we shall assume the bundle atlas is maximal in the sense that any triple $(\mathcal{U}, \varphi, \Phi)$ that is smoothly compatible with every $\alpha \in \mathcal{A}(\pi)$ also belongs to $\mathcal{A}(\pi)$.

Any $\alpha \in \mathcal{A}(\pi)$ now associates to sections $\eta : M \to E$ their local coordinate representatives

$$\eta^\alpha := \mathrm{pr}_2 \circ \Phi_\alpha \circ \eta \circ \varphi_\alpha^{-1} : \Omega_\alpha \to \mathbb{F}^m,$$

where $\mathrm{pr}_2 : \mathcal{U}_\alpha \times \mathbb{F}^m \to \mathbb{F}^m$ is the projection, and the representatives with respect to two distinct $\alpha, \beta \in \mathcal{A}(\pi)$ are related by

$$\eta^\beta = (g_{\beta\alpha} \circ \varphi_\beta^{-1})(\eta^\alpha \circ \varphi_{\alpha\beta}) \quad \text{on } \Omega_{\beta\alpha} \subset \Omega_\beta.$$

For $p \in [1, \infty]$ and each integer $k \geq 0$, we then define the topological vector space of sections of class $W^{k,p}_{\mathrm{loc}}$ by

$$W^{k,p}_{\mathrm{loc}}(E) := \big\{ \eta : M \to E \bigm| \text{sections such that } \eta^\alpha \in W^{k,p}_{\mathrm{loc}}(\mathring{\Omega}_\alpha, \mathbb{F}^m) \text{ for all } \alpha \in \mathcal{A}(\pi) \big\},$$

where convergence $\eta_j \to \eta$ in $W^{k,p}_{\mathrm{loc}}(E)$ means that $\eta_j^\alpha \to \eta^\alpha$ in $W^{k,p}_{\mathrm{loc}}(\mathring{\Omega}_\alpha, \mathbb{F}^m)$ for all $\alpha \in \mathcal{A}(\pi)$. Note that Ω_α is not necessarily an open subset of $\mathbb{R}^n$ since it may contain points in $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \{0\}$, but its interior $\mathring{\Omega}_\alpha$ is open in $\mathbb{R}^n$, and $W^{k,p}_{\mathrm{loc}}(\mathring{\Omega}_\alpha)$ is thus defined as in §A.1. Strictly speaking, elements of $\eta \in W^{k,p}_{\mathrm{loc}}(E)$ are not sections but *equivalence classes* of sections defined almost everywhere—the latter notion is defined with respect to any measure arising from a smooth volume element on M, and it does not depend on this choice.

It turns out that $W^{k,p}_{\mathrm{loc}}(E)$ can be given the structure of a Banach space if M is compact. This follows from the fact that M can then be covered by a finite subset of the atlas $\mathcal{A}(\pi)$, but we must be a little bit careful: not all charts in $\mathcal{A}(\pi)$ are equally suitable for defining $W^{k,p}$-norms on sections, because e.g. even a nice smooth section $\eta \in \Gamma(E)$ may have $\|\eta^\alpha\|_{W^{k,p}(\mathring{\Omega}_\alpha)} = \infty$ if $\Omega_\alpha \subset \mathbb{R}^n_+$ is unbounded. One way to deal with this is as follows: we will say that $\alpha \in \mathcal{A}(\pi)$ is a **precompact chart** if there exists $\alpha' \in \mathcal{A}(\pi)$ and a compact subset $\mathcal{K} \subset M$ such that

$$\mathcal{U}_\alpha \subset \mathcal{K} \subset \mathcal{U}_{\alpha'}.$$

When this is the case, $\Omega_\alpha \subset \mathbb{R}^n_+$ is necessarily bounded, and the transition maps between two precompact charts necessarily have bounded derivatives of all orders, as they are restrictions to precompact subsets of maps that are smooth on larger domains. If M is compact, then one can always find a finite subset $I \subset \mathcal{A}(\pi)$ consisting of precompact charts such that $M = \bigcup_{\alpha \in I} \mathcal{U}_\alpha$.

DEFINITION A.16. Suppose $E \to M$ is a smooth vector bundle over a compact manifold M, and $I \subset \mathcal{A}(\pi)$ is a finite set of precompact charts such that $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ is an open cover of M. We then define $W^{k,p}(E)$ as the vector space of all sections $\eta : M \to E$ for which the norm

$$\|\eta\|_{W^{k,p}} := \|\eta\|_{W^{k,p}(E)} := \sum_{\alpha \in I} \|\eta^\alpha\|_{W^{k,p}(\mathring{\Omega}_\alpha)}$$

is finite.

The norm in the above definition depends on auxiliary choices, but it is easy to see that the resulting definition of the space $W^{k,p}(E)$ and its topology do not. In fact:

PROPOSITION A.17. *If M is compact, then $W^{k,p}(E) = W^{k,p}_{\mathrm{loc}}(E)$, and a sequence η_j converges to η in $W^{k,p}_{\mathrm{loc}}(E)$ if and only if the norm given in Definition A.16 satisfies $\|\eta_j - \eta\|_{W^{k,p}(E)} \to 0$.*

The proposition is an immediate consequence of the following.

LEMMA A.18. *Suppose M is a smooth manifold, $\pi : E \to M$ is a smooth vector bundle, $\{\beta\} \cup J \subset \mathcal{A}(\pi)$ is a finite collection of charts such that $M = \bigcup_{\alpha \in J} \mathcal{U}_\alpha$ and all coordinate transformations and transition maps relating any two charts in the collection $\{\beta\} \cup J$ have bounded derivatives of all orders (e.g. it suffices to assume all are precompact). Then there exists a constant $c > 0$ such that*

$$\|\eta^\beta\|_{W^{k,p}(\mathring{\Omega}_\beta)} \le c \sum_{\alpha \in J} \|\eta^\alpha\|_{W^{k,p}(\mathring{\Omega}_\alpha)}$$

for all sections $\eta : M \to E$ with $\eta^\alpha \in W^{k,p}(\mathring{\Omega}_\alpha)$ for every $\alpha \in J$.

PROOF. Choose a partition of unity $\{\rho_\alpha : M \to [0,1]\}_{\alpha \in J}$ subordinate to the finite open cover $\{\mathcal{U}_\alpha\}_{\alpha \in J}$. Now $\eta = \sum_{\alpha \in J} \rho_\alpha \eta$, and each $\rho_\alpha \eta$ is supported in $\mathcal{U}_\alpha$, so $(\rho_\alpha \eta)^\beta$ has support in $\Omega_{\beta\alpha} = \varphi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$. Thus using Theorem A.14 with the fact that $g_{\beta\alpha}$, φ_β^{-1}, $\varphi_{\alpha\beta}$ and $\varphi_{\beta\alpha} = \varphi_{\alpha\beta}^{-1}$ are all smooth functions with bounded derivatives of all orders on the domains in question, we find

$$\begin{aligned}
\|\eta^\beta\|_{W^{k,p}(\mathring{\Omega}_\beta)} &= \left\| \sum_{\alpha \in J} (\rho_\alpha \eta)^\beta \right\|_{W^{k,p}(\mathring{\Omega}_\beta)} \le \sum_{\alpha \in J} \|(\rho_\alpha \eta)^\beta\|_{W^{k,p}(\mathring{\Omega}_{\beta\alpha})} \\
&= \sum_{\alpha \in J} \|(\rho_\alpha \circ \varphi_\beta^{-1})(g_{\beta\alpha} \circ \varphi_\beta^{-1})(\eta^\alpha \circ \varphi_{\alpha\beta})\|_{W^{k,p}(\mathring{\Omega}_{\beta\alpha})} \\
&\le c \sum_{\alpha \in J} \|\eta^\alpha\|_{W^{k,p}(\mathring{\Omega}_{\alpha\beta})} \le c \sum_{\alpha \in J} \|\eta^\alpha\|_{W^{k,p}(\mathring{\Omega}_\alpha)}.
\end{aligned}$$

□

COROLLARY A.19. *If M is compact, then the norm on $W^{k,p}(E)$ given by Definition A.16 is independent of all auxiliary choices up to equivalence of norms.* □

THEOREM A.20. *For any smooth vector bundle $\pi : E \to M$ over a compact manifold M, $W^{k,p}(E)$ is a Banach space.*

PROOF. If $\eta_j \in W^{k,p}(E)$ is a Cauchy sequence, then for some chosen finite collection $I \subset \mathcal{A}(\pi)$ of precompact charts covering M, the sequences η_j^α for $\alpha \in I$ are Cauchy in $W^{k,p}(\mathring{\Omega}_\alpha)$ and thus have limits $\xi^{(\alpha)} \in W^{k,p}(\mathring{\Omega}_\alpha, \mathbb{F}^m)$. Choosing a partition of unity $\{\rho_\alpha : M \to [0,1]\}_{\alpha \in I}$ subordinate to $\{\mathcal{U}_\alpha\}_{\alpha \in I}$, we can now associate to each $\alpha \in I$ a section $\eta_{\infty,\alpha} \in W^{k,p}(E)$ characterized uniquely by the condition that it vanishes outside of $\mathcal{U}_\alpha$ and is represented in the trivialization on $\mathcal{U}_\alpha$ by

$$\eta_{\infty,\alpha}^\alpha = (\rho_\alpha \circ \varphi_\alpha^{-1})\xi^{(\alpha)}.$$

We claim that $\rho_\alpha \eta_j \to \eta_{\infty,\alpha}$ in $W^{k,p}(E)$ for each $\alpha \in I$. Indeed, we have

$$(\rho_\alpha \eta_j)^\alpha = (\rho_\alpha \circ \varphi_\alpha^{-1})\eta_j^\alpha \to (\rho_\alpha \circ \varphi_\alpha^{-1})\xi^{(\alpha)} = \eta_{\infty,\alpha}^\alpha \quad \text{in } W^{k,p}(\mathring{\Omega}_\alpha)$$

since $\eta_j^\alpha \to \xi^{(\alpha)}$. For all other $\beta \in I$ not equal to α, $(\rho_\alpha \eta_j)^\beta - \eta_{\infty,\alpha}^\beta \in W^{k,p}(\mathring{\Omega}_\beta, \mathbb{F}^m)$ has support in $\Omega_{\beta\alpha} = \varphi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$, thus

$$\|(\rho_\alpha \eta_j)^\beta - \eta_{\infty,\alpha}^\beta\|_{W^{k,p}(\mathring{\Omega}_\beta)} = \|(\rho_\alpha \eta_j)^\beta - \eta_{\infty,\alpha}^\beta\|_{W^{k,p}(\mathring{\Omega}_{\beta\alpha})} \le c\|(\rho_\alpha \eta_j)^\alpha - \eta_{\infty,\alpha}^\alpha\|_{W^{k,p}(\mathring{\Omega}_\alpha)},$$

where the inequality comes from Lemma A.18 after replacing M with $\mathcal{U}_\alpha$, and $\mathcal{U}_\beta$ with $\mathcal{U}_\beta \cap \mathcal{U}_\alpha$ (note that the lemma does not require M to be compact). With the claim established, we have

$$\eta_j = \sum_{\alpha \in I} \rho_\alpha \eta_j \to \sum_{\alpha \in I} \eta_{\infty,\alpha} \quad \text{in } W^{k,p}(E).$$

□

EXERCISE A.21. For $\mathcal{U} \subset \mathbb{R}^n$ an open subset, the space $W^{k,p}_{\text{loc}}(\mathcal{U})$ was defined in §A.1, but one can give it an alternative definition in the present context by viewing functions on $\mathcal{U}$ as sections of a trivial vector bundle over $\mathcal{U}$, with the latter viewed as a noncompact smooth n-manifold. Show that these two definitions of $W^{k,p}_{\text{loc}}(\mathcal{U})$ are equivalent.

EXERCISE A.22. Suppose $\mathcal{U} \subset \mathbb{R}^n$ is a bounded open subset with smooth boundary, so its closure $\overline{\mathcal{U}} \subset \mathbb{R}^n$ is a smooth compact submanifold with boundary, and let $E \to \overline{\mathcal{U}}$ be a trivial vector bundle. Show that there is a canonical Banach space isomorphism between $W^{k,p}(\mathcal{U})$ as defined in §A.1 and $W^{k,p}(E)$ as defined in the present section. *Hint: Recall that sections in $W^{k,p}(E)$ are only required to be defined almost everywhere, so in particular if the domain M is a manifold with boundary, they need not be well defined on ∂M.*

In light of Exercise A.22, the natural generalization of $W^{k,p}_0(\mathcal{U})$ in the present setting is

$$W^{k,p}_0(E) := \overline{C^\infty_0(E|_{M\setminus\partial M})},$$

i.e. it is the closure in the $W^{k,p}$-norm of the space of smooth sections that vanish near the boundary. Density of smooth sections will imply that this is the same as $W^{k,p}(E)$ if M is closed, but in general $W^{k,p}_0(E)$ is a closed subspace of $W^{k,p}(E)$.

The partition of unity argument in Theorem A.20 contains all the essential ideas needed to generalize results about Sobolev spaces on domains in $\mathbb{R}^n$ to compact manifolds. We now state the essential results, leaving the proofs as exercises.

THEOREM A.23. *Assume M is a smooth compact n-manifold, possibly with boundary, $\pi : E \to M$ is a smooth vector bundle of finite rank, $k \geq 0$ is an integer and $1 \leq p < \infty$. Then the Banach space $W^{k,p}(E)$ has the following properties.*

(1) *The space $\Gamma(E)$ of smooth sections is dense in $W^{k,p}(E)$.*
(2) *If $N \subset M$ is a smooth compact n-dimensional submanifold with boundary, then there exists a bounded linear operator*

$$E : W^{k,p}(E|_N) \to W_0^{k,p}(E)$$

which is an extension *operator in the sense that $E\eta|_N = \eta$ for all $\eta \in W^{k,p}(E|_N)$. Moreover, a section $\eta \in W^{k,p}(E|_N)$ belongs to $W_0^{k,p}(E|_N)$ if and only if the section $\tilde{\eta}$ defined to match η on N and to vanish on $M \setminus N$ belongs to $W^{k,p}(E)$.*
(3) *If $kp > n$, then for each integer $d \geq 0$, there exists a continuous and compact inclusion*

$$W^{k+d,p}(E) \hookrightarrow C^d(E).$$

(4) *The natural inclusion*

$$W^{k+1,p}(E) \hookrightarrow W^{k,p}(E)$$

is compact.
(5) *Suppose $F, G \to M$ are smooth vector bundles such that there exists a smooth bundle map*

$$E \otimes F \to G : \eta \otimes \xi \mapsto \eta \cdot \xi.$$

Then if $kp > n$ and $0 \leq m \leq k$, there exists a continuous product pairing

$$W^{k,p}(E) \times W^{m,p}(F) \to W^{m,p}(G) : (\eta, \xi) \mapsto \eta \cdot \xi.$$

In particular, products of $W^{k,p}$ sections give $W^{k,p}$ sections whenever $kp > n$.
(6) *Suppose $F \to M$ is another smooth vector bundle, $\mathcal{V} \subset E$ is an open subset that intersects every fiber of E, and we consider the spaces*

$$W^{k,p}(\mathcal{V}) := \left\{ \eta \in W^{k,p}(E) \;\middle|\; \eta(M) \subset \mathcal{V} \right\}$$

and

$$C^k_M(\mathcal{V}, F) := \left\{ \Phi : \mathcal{V} \to F \mid \text{fiber-preserving maps of class } C^k \right\},$$

where the latter is assigned the topology of C^k-convergence on compact subsets. If $kp > n$, then $W^{k,p}(\mathcal{V})$ is an open subset of $W^{k,p}(E)$, and the map

$$C^k_M(\mathcal{V}, F) \times W^{k,p}(\mathcal{V}) \to W^{k,p}(F) : (\Phi, \eta) \mapsto \Phi \circ \eta$$

is well defined and continuous.
(7) *If N is another smooth compact manifold and $\varphi : N \to M$ is a smooth diffeomorphism, then there is a Banach space isomorphism*

$$W^{k,p}(E) \to W^{k,p}(\varphi^* E) : \eta \mapsto \eta \circ \varphi.$$

□

A.4. Some remarks on domains with cylindrical ends

For bundles $\pi : E \to M$ with M noncompact, $W^{k,p}(E)$ is not generally well defined without making additional choices. When $M = \dot{\Sigma} = \Sigma \setminus \Gamma$ is a punctured Riemann surface and $\pi : E \to \dot{\Sigma}$ is equipped with an asymptotically Hermitian structure $\{(E_z, J_z, \omega_z)\}_{z\in\Gamma}$, one nice way to define $W^{k,p}(E)$ was introduced in §4.1: one takes it to be the space of sections in $W^{k,p}_{\text{loc}}(E)$ whose $W^{k,p}$-norms on each cylindrical end are finite with respect to a choice of asymptotic trivialization. This definition requires the convenient fact that complex vector bundles over S^1 are always trivial, though one can also do without this by using the ideas in the previous section. Indeed, any collection of local trivializations on the asymptotic bundle $E_z \to S^1$ covering S^1 gives rise via the asymptotically Hermitian structure to a collection of trivializations on E covering the corresponding cylindrical end $\dot{\mathcal{U}}_z$. The key fact is then that S^1 is compact, hence one can always choose such a covering to be finite: combining this with a finite covering of $\dot{\Sigma}$ in the complement of its cylindrical ends by precompact charts, we obtain a covering of $\dot{\Sigma}$ by a finite collection of bundle charts that are not all precompact, but nonetheless have the property that all transition maps have bounded derivatives of all orders. This is enough to define a $W^{k,p}$-norm for sections of $E \to \dot{\Sigma}$ as in Definition A.16 and to prove that it does not depend on the choices of charts or local trivializations, though it does depend on the asymptotically Hermitian structure.

With this definition understood, one can easily generalize the Sobolev embedding theorem and other important statements in Theorem A.23 to the setting of an asymptotically Hermitian bundle over a punctured Riemann surface. We shall leave the details of this generalization as an exercise, but take the opportunity to point out a few important differences from the compact case.

First, since $\dot{\Sigma}$ is not compact, neither are the inclusions

$$W^{k+d,p}(E) \hookrightarrow C^d(E), \qquad W^{k+1,p}(E) \hookrightarrow W^{k,p}(E).$$

The proof of compactness fails due to the fact that cylindrical ends require local trivializations over unbounded domains of the form $(0,\infty) \times (0,1) \subset \mathbb{R}^2$, for which Theorem A.9 does not hold. And indeed, considering unbounded shifts on the infinite cylinder $\dot{\Sigma} = \mathbb{R} \times S^1$, it is easy to find a sequence of $W^{k,p}$-bounded functions with $kp > 2$ that do not have a C^0-convergent subsequence. That is the bad news.

The good news is that if $\eta \in W^{k+d,p}(E)$ for $kp > 2$, then one can say considerably more about η than just that it is C^d-smooth. Indeed, restricting to one of the cylindrical ends $[0,\infty) \times S^1 \subset \dot{\Sigma}$, notice that finiteness of the $W^{k+d,p}$-norm over $\dot{\Sigma}$ implies

$$\|\eta\|_{W^{k+d,p}((R,\infty)\times S^1)} \to 0 \quad \text{as} \quad R \to \infty.$$

Since these domains are all naturally diffeomorphic for different values of R, the C^d-norm of η over $(R,\infty) \times S^1$ is bounded by the $W^{k+d,p}$-norm via a constant that does not depend on R, so this implies an asymptotic decay condition

$$\|\eta\|_{C^d([R,\infty)\times S^1)} \to 0 \quad \text{as} \quad R \to \infty$$

for every $\eta \in W^{k+d,p}(E)$.

Here is another useful piece of good news: since $\dot{\Sigma}$ does not have boundary, $W^{k,p}(E) = W_0^{k,p}(E)$.

THEOREM A.24. *Given an asymptotically Hermitian bundle E over a punctured Riemann surface $\dot{\Sigma}$, the space $C_0^\infty(E)$ of smooth sections with compact support is dense in $W^{k,p}(E)$ for all $k \geq 0$ and $1 \leq p < \infty$.*

PROOF. We can assume as in Definition A.16 that the $W^{k,p}$-norm for sections η of E is given by

$$\|\eta\|_{W^{k,p}} = \sum_{\alpha \in I} \|\eta^\alpha\|_{W^{k,p}(\Omega_\alpha)},$$

where $I \subset \mathcal{A}(\pi)$ is a finite collection of bundle charts

$$\alpha = \left(\varphi_\alpha : \mathcal{U}_\alpha \xrightarrow{\cong} \Omega_\alpha\,,\ \Phi_\alpha : E|_{\mathcal{U}_\alpha} \xrightarrow{\cong} \mathcal{U}_\alpha \times \mathbb{C}^n\right)$$

such that each of the open sets $\Omega_\alpha \subset \mathbb{C}$ is either bounded or (for charts over the cylindrical ends) of the form

$$\Omega_\alpha = (0, \infty) \times \omega_\alpha \subset \mathbb{R}^2 = \mathbb{C}$$

for some bounded open subset $\omega_\alpha \subset \mathbb{R}$. Now given $\eta \in W^{k,p}(E)$, Theorem A.1 provides for each $\alpha \in I$ a sequence $\eta_j^\alpha \in W^{k,p}(\Omega_\alpha)$ of smooth functions with bounded support such that $\eta_j^\alpha \to \eta^\alpha$ in $W^{k,p}(\Omega_\alpha)$. Choose a partition of unity $\{\rho_\alpha : \dot{\Sigma} \to [0,1]\}_{\alpha \in I}$ subordinate to the open cover $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ and let

$$\eta_j := \sum_{\alpha \in I} \rho_\alpha(\eta_j^\alpha \circ \varphi_\alpha) \in W^{k,p}(E).$$

These sections are smooth and have compact support since the η_j^α have bounded support in Ω_α, and they converge in $W^{k,p}$ to η. □

APPENDIX B

The Floer C_ε space

The C_ε-topology for functions was introduced by Floer [**Flo88b**] to provide a Banach manifold of perturbed geometric structures without departing from the smooth category: it is a way to circumvent the annoying fact that spaces of smooth functions which arise naturally in geometric settings are not Banach spaces. The construction of C_ε spaces generally depends on several arbitrary choices and is thus far from canonical, but this detail is unimportant since the C_ε space itself is never the main object of interest. What is important is merely the properties that it has, namely that it not only embeds continuously into C^∞ and contains an abundance of non-trivial functions, but also is a separable Banach space and can therefore be used in the Sard-Smale theorem for genericity arguments. We shall prove these facts in this appendix.

Fix a smooth finite-rank vector bundle $\pi : E \to M$ over a finite-dimensional compact manifold M, possibly with boundary. For each integer $k \geq 0$, we denote by $C^k(E)$ the Banach space of C^k-smooth sections of E; note that the norm on $C^k(E)$ depends on various auxiliary choices but is well defined up to equivalence of norms since M is compact. Now if $(\varepsilon_k)_{k=0}^\infty$ is a sequence of positive numbers with $\varepsilon_k \to 0$, set

$$C_\varepsilon(E) = \left\{ \eta \in \Gamma(E) \;\middle|\; \|\eta\|_{C_\varepsilon} < \infty \right\},$$

where the C_ε-norm is defined by

$$\|\eta\|_{C_\varepsilon} = \sum_{k=0}^{\infty} \varepsilon_k \|\eta\|_{C^k}. \tag{B.1}$$

The norm for $C_\varepsilon(E)$ is somewhat more delicate than for $C^k(E)$, e.g. its equivalence class is not obviously independent of auxiliary choices. This remark is meant as a sanity check, but it should not cause extra concern since, in practice, the space $C_\varepsilon(E)$ is typically regarded as an auxiliary choice in itself. In many applications, one fixes an open subset $\mathcal{U} \subset M$ and considers the closed subspace

$$C_\varepsilon(E;\mathcal{U}) = \left\{ \eta \in C_\varepsilon(E) \;\middle|\; \eta|_{M \setminus \mathcal{U}} \equiv 0 \right\}.$$

Remark B.1. The requirement for M to be compact can be relaxed as long as $\mathcal{U} \subset M$ has compact closure: e.g. in one situation of frequent interest in this book, we take M to be the noncompact completion of a symplectic cobordism. In this case $C_\varepsilon(E;\mathcal{U})$ can be defined as a closed subspace of $C_\varepsilon(E|_{M_0})$ where $M_0 \subset M$ is any compact manifold with boundary that contains the closure of $\mathcal{U}$. For this reason, we lose no generality in continuing under the assumption that M is compact.

In order to prove things about $C_\varepsilon(E)$, we will need to specify a more precise definition of the C^k-norms. To this end, define a sequence of vector bundles $E^{(k)} \to M$ for integers $k \ge 0$ inductively by

$$E^{(0)} := E, \qquad E^{(k+1)} := \mathrm{Hom}(TM, E^{(k)}).$$

Choose connections and bundle metrics on both TM and E; these induce connections and bundle metrics on each of the $E^{(k)}$, so that for any section $\xi \in \Gamma(E^{(k)})$, the covariant derivative $\nabla\xi$ is now a section of $E^{(k+1)}$. In particular for $\eta \in \Gamma(E)$, we can define the "kth covariant derivative" of η as a section

$$\nabla^k \eta \in \Gamma(E^{(k)}).$$

Using the bundle metrics to define C^0-norms for sections of $E^{(k)}$, we can then define

$$\|\eta\|_{C^k(E)} = \sum_{m=0}^{k} \|\nabla^m \eta\|_{C^0(E^{(m)})},$$

where by convention $\nabla^0 \eta := \eta$. We will assume throughout the following that the C^k-norms appearing in (B.1) are defined in this way.

THEOREM B.2. *$C_\varepsilon(E)$ is a Banach space.*

PROOF. We need to show that C_ε-Cauchy sequences converge in the C_ε-norm. It is clear from the definitions that if $\eta_j \in C_\varepsilon(E)$ is Cauchy, then η_j is also C^k-Cauchy for every $k \ge 0$, hence its derivatives $\nabla^k \eta_j$ for every k are C^0-convergent to continuous sections ξ^k of $E^{(k)}$. This convergence implies that $\xi^{k+1} = \nabla \xi^k$ in the sense of distributions, hence by the equivalence of classical and distributional derivatives (see e.g. [**LL01**, §6.10]), $\eta_\infty := \xi^0$ is smooth with $\nabla^k \eta_\infty = \xi^k$, so that $\nabla^k \eta_j \to \nabla^k \eta_\infty$ in $C^0(E^{(k)})$ for all k.

We claim $\eta_\infty \in C_\varepsilon(E)$. Choose $N > 0$ such that $\|\eta_i - \eta_j\|_{C_\varepsilon} < 1$ for all $i, j \ge N$. Then for every $m \in \mathbb{N}$ and every $i \ge N$,

$$\begin{aligned}\sum_{k=0}^{m} \varepsilon_k \|\eta_i\|_{C^k} &\le \sum_{k=0}^{m} \varepsilon_k \|\eta_i - \eta_N\|_{C^k} + \sum_{k=0}^{m} \varepsilon_k \|\eta_N\|_{C^k} \\ &\le \|\eta_i - \eta_N\|_{C_\varepsilon} + \|\eta_N\|_{C_\varepsilon} < 1 + \|\eta_N\|_{C_\varepsilon}.\end{aligned}$$

Fixing m and letting $i \to \infty$, we then have

$$\sum_{k=0}^{m} \varepsilon_k \|\eta_\infty\|_{C^k} \le 1 + \|\eta_N\|_{C_\varepsilon}$$

for all m, so we can now let $m \to \infty$ and conclude $\|\eta_\infty\|_{C_\varepsilon} \le 1 + \|\eta_N\|_{C_\varepsilon} < \infty$.

The argument that $\|\eta_j \to \eta_\infty\|_{C_\varepsilon} \to 0$ as $j \to \infty$ is similar: pick $\epsilon > 0$ and N such that $\|\eta_i - \eta_j\|_{C_\varepsilon} < \epsilon$ for all $i, j \ge N$. Then for a fixed $m \in \mathbb{N}$, we can let $i \to \infty$ in the expression $\sum_{k=0}^{m} \varepsilon_k \|\eta_i - \eta_j\|_{C^k} < \epsilon$, giving

$$\sum_{k=0}^{m} \varepsilon_k \|\eta_\infty - \eta_j\|_{C^k} \le \epsilon.$$

This is true for every m, so we can take $m \to \infty$ and conclude $\|\eta_\infty - \eta_j\|_{C_\varepsilon} \le \epsilon$ for all $j \ge N$. □

To show that $C_\varepsilon(E)$ is also separable, we will follow a hint[1] from [HS95] and embed it isometrically into another Banach space that can be more easily shown to be separable. For each integer $k \ge 0$, define the vector bundle

$$F^{(k)} = E^{(0)} \oplus \ldots \oplus E^{(k)},$$

and let X_ε denote the vector space of all sequences

$$\xi := (\xi^0, \xi^1, \xi^2, \ldots) \in \prod_{k=0}^{\infty} C^0(F^{(k)})$$

such that

$$\|\xi\|_{X_\varepsilon} := \sum_{k=0}^{\infty} \varepsilon_k \|\xi^k\|_{C^0} < \infty.$$

EXERCISE B.3. Adapt the proof of Theorem B.2 to show that X_ε is also a Banach space.

LEMMA B.4. *X_ε is separable.*

PROOF. Since $C^0(F^{(k)})$ is separable for each $k \ge 0$, we can fix countable dense subsets $P^k \subset C^0(F^{(k)})$. The set

$$P := \left\{ (\xi^0, \ldots, \xi^N, 0, 0, \ldots) \in X_\varepsilon \;\middle|\; N \ge 0 \text{ and } \xi^k \in P^k \text{ for all } k = 0, \ldots, N \right\}$$

is then countable and dense in X_ε. □

THEOREM B.5. *$C_\varepsilon(E)$ is separable.*

PROOF. Consider the injective linear map

$$C_\varepsilon(E) \hookrightarrow X_\varepsilon : \eta \mapsto \big(\eta, (\eta, \nabla\eta), (\eta, \nabla\eta, \nabla^2\eta), \ldots\big).$$

This is an isometric embedding and thus presents $C_\varepsilon(E)$ as a closed linear subspace of X_ε, hence the theorem follows from Lemma B.4 and the fact that subspaces of separable metric spaces are always separable. □

Note that given any open subset $\mathcal{U} \subset M$, Theorems B.2 and B.5 also hold for $C_\varepsilon(E;\mathcal{U})$, as a closed subspace of $C_\varepsilon(E)$. So far in this discussion, however, there has been no guarantee that $C_\varepsilon(E)$ or $C_\varepsilon(E;\mathcal{U})$ contains anything other than the zero-section, though it is clear that in theory, one should always be able to enlarge the space by choosing new sequences ε_k that converge to zero faster. The following result says that $C_\varepsilon(E;\mathcal{U})$ can always be made large enough to be useful in applications.

THEOREM B.6. *Given an open subset $\mathcal{U} \subset M$, the sequence ε_k can be chosen to have the following properties:*

(1) $C_\varepsilon(E;\mathcal{U})$ is dense in the space of continuous sections vanishing outside $\mathcal{U}$.
(2) Given any point $p \in \mathcal{U}$, a neighborhood $\mathcal{N}_p \subset \mathcal{U}$ of p, a number $\delta > 0$ and a continuous section η_0 of E, there exists a section $\eta \in \Gamma(E)$ and a smooth compactly supported function $\beta : \mathcal{N}_p \to [0,1]$ such that

$$\beta\eta \in C_\varepsilon(E;\mathcal{U}), \qquad \beta(p)\eta(p) = \eta_0(p), \qquad \textit{and} \qquad \|\eta - \eta_0\|_{C^0} < \delta.$$

[1] Thanks to Sam Lisi for explaining to me what the hint in [HS95] was referring to.

PROOF. Note first that it suffices to find two separate sequences ε_k and ε'_k that have the first and second property respectively, as the sequence of minima $\min(\varepsilon_k, \varepsilon'_k)$ will then have both properties.

The following construction for the first property is based on a suggestion by Barney Bramham. Observe first that the space $C^0(E;\mathcal{U})$ of continuous sections vanishing outside $\mathcal{U}$ is a closed subspace of $C^0(E)$ and is thus separable, so we can choose a countable C^0-dense subset $P \subset C^0(E;\mathcal{U})$. Moreover, the space of *smooth* sections vanishing outside $\mathcal{U}$ is dense in $C^0(E;\mathcal{U})$, hence we can assume without loss of generality that the sections in P are smooth. Now write $P = \{\eta_1, \eta_2, \eta_3, \dots\}$ and define $\varepsilon_k > 0$ for every integer $k \ge 0$ to have the property

$$\varepsilon_k < \frac{1}{2^k} \min\left\{ \frac{1}{\|\eta_1\|_{C^k}}, \dots, \frac{1}{\|\eta_k\|_{C^k}} \right\}.$$

Then every η_j is in $C_\varepsilon(E;\mathcal{U})$, as

$$\|\eta_j\|_{C_\varepsilon} < \sum_{k=0}^{j-1} \varepsilon_k \|\eta_j\|_{C^k} + \sum_{k=j}^{\infty} \frac{1}{2^k} < \infty.$$

The second property is essentially local, so it can be deduced from Lemma B.7 below. ◻

LEMMA B.7. *Suppose $\beta : \mathring{\mathbb{D}}^n \to [0,1]$ is a smooth function with compact support on the open unit ball $\mathring{\mathbb{D}}^n \subset \mathbb{R}^n$ and $\beta(0) = 1$. One can choose a sequence of positive numbers $\varepsilon_k \to 0$ such that for every $\eta_0 \in \mathbb{R}^m$ and $r > 0$, the function $\eta : \mathbb{R}^n \to \mathbb{R}^m$ defined by*

$$\eta(p) := \beta(p/r)\eta_0$$

satisfies $\sum_{k=0}^\infty \varepsilon_k \|\eta\|_{C^k} < \infty$.

PROOF. Define $\varepsilon_k > 0$ so that for $k \ge 1$,

$$\varepsilon_k = \frac{1}{k^k \|\beta\|_{C^k}}.$$

Then

$$\sum_{k=1}^{\infty} \varepsilon_k \|\eta\|_{C^k} \le \sum_{k=1}^{\infty} \frac{1}{k^k \|\beta\|_{C^k}} \frac{\|\beta\|_{C^k}}{r^k} = \sum_{k=1}^{\infty} \left(\frac{1/r}{k}\right)^k < \infty.$$

◻

APPENDIX C

Genericity in the space of asymptotic operators

The purpose of this appendix is to prove Lemma 3.17, which was needed for our definition of spectral flow in §3.2. The proof combines some ideas from that section with the technique used in Lecture 7 to prove generic transversality of moduli spaces via the Sard-Smale theorem. Some knowledge of that technique should thus be considered a prerequisite for this appendix; if you have never seen it before and were directed here after reading the statement of Lemma 3.17, you might want to skip this for now and come back after you've read as far as Lecture 7.

Recalling the notation from Lecture 3, we fix the real Hilbert spaces

$$\mathcal{H} = L^2(S^1, \mathbb{R}^{2n}), \qquad \mathcal{D} = H^1(S^1, \mathbb{R}^{2n}),$$

the symmetric index 0 Fredholm operator

$$\mathbf{T}_{\mathrm{ref}} = -J_0\,\partial_t : \mathcal{D} \to \mathcal{H}$$

and, given a smooth loop of symmetric matrices $S : S^1 \to \mathrm{End}_{\mathbb{R}}^{\mathrm{sym}}(\mathbb{R}^{2n})$, refer to any operator of the form

$$\mathbf{A} = -J_0\,\partial_t - S : \mathcal{D} \to \mathcal{H}$$

as an **asymptotic operator**. Such operators belong to the space of symmetric compact perturbations of $\mathbf{T}_{\mathrm{ref}}$,

$$\mathrm{Fred}_{\mathbb{R}}^{\mathrm{sym}}(\mathcal{D}, \mathcal{H}, \mathbf{T}_{\mathrm{ref}}) = \left\{ \mathbf{T}_{\mathrm{ref}} + \mathbf{K} : \mathcal{D} \to \mathcal{H} \;\middle|\; \mathbf{K} \in \mathscr{L}_{\mathbb{R}}^{\mathrm{sym}}(\mathcal{H}) \right\},$$

which we regard as a smooth Banach manifold via its obvious identification with the space $\mathscr{L}_{\mathbb{R}}^{\mathrm{sym}}(\mathcal{H})$ of symmetric bounded linear operators on $\mathcal{H}$. For $k \in \mathbb{N}$, we denote by

$$\mathrm{Fred}_{\mathbb{R}}^{\mathrm{sym},k}(\mathcal{D}, \mathcal{H}, \mathbf{T}_{\mathrm{ref}}) \subset \mathrm{Fred}_{\mathbb{R}}^{\mathrm{sym}}(\mathcal{D}, \mathcal{H}, \mathbf{T}_{\mathrm{ref}})$$

the finite-codimensional submanifold determined by the condition $\dim_{\mathbb{R}} \ker \mathbf{A} = \dim_{\mathbb{R}} \operatorname{coker} \mathbf{A} = k$.

Here is the statement of Lemma 3.17 again.

Lemma. *Fix a smooth map $S : [-1, 1] \times S^1 \to \mathrm{End}_{\mathbb{R}}^{\mathrm{sym}}(\mathbb{R}^{2n})$ and consider the 1-parameter family of operators*

$$\mathbf{A}_s := -J_0\,\partial_t - S(s, \cdot) \in \mathrm{Fred}_{\mathbb{R}}^{\mathrm{sym}}(\mathcal{D}, \mathcal{H}, \mathbf{T}_{\mathrm{ref}})$$

for $s \in [-1, 1]$. Then after a C^∞-small perturbation of S fixed at $s = \pm 1$, one can assume the following:

(1) *For every $s \in (-1, 1)$, all eigenvalues of $\mathbf{A}_s$ (regarded as an unbounded operator on $\mathcal{H}$) are simple.*

(2) All intersections of the path

$$(-1,1) \to \mathrm{Fred}^{\mathrm{sym}}_{\mathbb{R}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}}) : s \mapsto \mathbf{A}_s$$

with $\mathrm{Fred}^{\mathrm{sym},1}_{\mathbb{R}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$ *are transverse.*

We shall now prove this by constructing a Floer-type space of C_ε-smooth (see Appendix B) perturbed families of asymptotic operators, and using the Sard-Smale theorem to find a countable collection of comeager subsets whose intersection contains perturbations achieving the desired conditions.

Choose a sequence of positive numbers $(\varepsilon)_{k=0}^{\infty}$ with $\varepsilon_k \to 0$ to define a separable Banach space

$$\mathcal{A}_\varepsilon := \left\{ B \in C^\infty([-1,1]\times S^1, \mathrm{End}^{\mathrm{sym}}_{\mathbb{R}}(\mathbb{R}^{2n})) \;\middle|\; \|B\|_{C_\varepsilon} < \infty \text{ and } B(\pm 1,\cdot) \equiv 0 \right\},$$

and assume via Theorem B.6 that $\mathcal{A}_\varepsilon$ is dense in the Banach space of continuous functions $[-1,1]\times S^1 \to \mathrm{End}^{\mathrm{sym}}_{\mathbb{R}}(\mathbb{R}^{2n})$ vanishing at $\{\pm 1\}\times S^1$. We then consider perturbed 1-parameter families of asymptotic operators of the form

$$\mathbf{A}^B_s := \mathbf{A}_s + B(s,\cdot) : \mathcal{D} \to \mathcal{H}$$

for $B \in \mathcal{A}_\varepsilon$, $s \in [-1,1]$. For each $k \in \mathbb{N}$ and $B \in \mathcal{A}_\varepsilon$, define the set

$$\mathcal{V}^k(B) = \left\{ (s,\lambda) \in (-1,1)\times\mathbb{R} \;\middle|\; \dim_{\mathbb{R}} \ker\left(\mathbf{A}^B_s - \lambda\right) = k \right\}.$$

To show that eigenvalues are generically simple, we need to show that for a comeager set of choices of $B \in \mathcal{A}_\varepsilon$, $\mathcal{V}^k(B)$ is empty for all $k \geq 2$. Given $(s_0,\lambda_0) \in \mathcal{V}^k(B)$, recall from §3.2 that there exist decompositions

$$\mathcal{D} = V \oplus K, \qquad \mathcal{H} = W \oplus K$$

where $K = \ker\left(\mathbf{A}^B_{s_0} - \lambda_0\right)$, $W = \mathrm{im}\left(\mathbf{A}^B_{s_0} - \lambda_0\right)$ is the L^2-orthogonal complement of K, and $V = W \cap \mathcal{D}$, so that any symmetric bounded linear operator $\mathbf{T}$ in a sufficiently small neighborhood $\mathcal{O} \subset \mathscr{L}^{\mathrm{sym}}_{\mathbb{R}}(\mathcal{D},\mathcal{H})$ of $\mathbf{A}^B_{s_0} - \lambda_0$ can be written in block form

$$\mathbf{T} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}$$

with $\mathbf{A} : V \to W$ invertible, giving rise to a smooth map

$$\Phi : \mathcal{O} \to \mathrm{End}^{\mathrm{sym}}_{\mathbb{R}}(K) : \mathbf{T} \mapsto \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$$

whose zero-set is precisely the set of nearby symmetric operators with k-dimensional kernel. A neighborhood of (s_0,λ_0) in $\mathcal{V}^k(B)$ can thus be identified with the zero-set of the map

$$\Psi_B(s,\lambda) := \Phi(\mathbf{A}^B_s - \lambda) \in \mathrm{End}^{\mathrm{sym}}_{\mathbb{R}}(K),$$

defined for $(s,\lambda) \in (-1,1)\times\mathbb{R}$ sufficiently close to (s_0,λ_0). Notice that the derivative $d\Psi_B(s,\lambda) : \mathbb{R}\oplus\mathbb{R} \to \mathrm{End}^{\mathrm{sym}}_{\mathbb{R}}(K)$ is Fredholm since its domain and target are both finite dimensional, and it can only ever be surjective when $k = \dim_{\mathbb{R}} K = 1$.

The following space will now play the role of a "universal moduli space" as in Lecture 7: let

$$\mathcal{V}^k = \left\{ (s,\lambda,B) \in (-1,1)\times\mathbb{R}\times\mathcal{A}_\varepsilon \;\middle|\; (s,\lambda) \in \mathcal{V}^k(B) \right\}.$$

The proof that this is a smooth Banach manifold depends on the following algebraic lemma.

LEMMA C.1. *Fix an asymptotic operator* $\mathbf{A} = -J_0\,\partial_t - S$ *and a linear transformation*

$$\Upsilon : \ker \mathbf{A} \to \ker \mathbf{A}$$

that is symmetric with respect to the L^2*-product. Then there exists a smooth loop* $B : S^1 \to \mathrm{End}^{\mathrm{sym}}(\mathbb{R}^{2n})$ *such that*

$$\langle \eta, B\xi \rangle_{L^2} = \langle \eta, \Upsilon\xi \rangle_{L^2}$$

for all $\eta, \xi \in \ker \mathbf{A}$.

PROOF. Note first that every nontrivial loop $\eta \in \ker \mathbf{A} \subset H^1(S^1, \mathbb{R}^{2n})$ is smooth and nowhere zero since it satisfies a linear first-order ODE with smooth coefficients. It follows that if we fix a basis $(\eta_1, \ldots, \eta_k)$ for $\ker \mathbf{A}$, then the vectors $\eta_1(t), \ldots, \eta_k(t) \in \mathbb{R}^{2n}$ are also linearly independent for all $t \in S^1$ and thus span a smooth S^1-family of k-dimensional subspaces $V_t \subset \mathbb{R}^{2n}$, each equipped with a distinguished basis. It follows that there exists a unique smooth S^1-family of linear transformations $\widehat{B}(t) : V_t \to V_t$ such that for every $\eta \in \ker \mathbf{A}$, $\widehat{B}(t)\eta(t) = (\Upsilon\eta)(t)$ for all t. Extend $\widehat{B}(t)$ arbitrarily to a smooth family of linear maps on $\mathbb{R}^{2n}$.

The matrices $\widehat{B}(t) \in \mathrm{End}_{\mathbb{R}}(\mathbb{R}^{2n})$ need not be symmetric, but they do satisfy

$$\langle \eta, \widehat{B}\xi \rangle_{L^2} = \langle \eta, \Upsilon\xi \rangle_{L^2} \quad \text{for all } \eta, \xi \in \ker \mathbf{A}.$$

Since Υ is symmetric, this implies moreover that for all $\eta, \xi \in \ker \mathbf{A}$,

$$\langle \eta, \Upsilon\xi \rangle_{L^2} = \langle \xi, \Upsilon\eta \rangle_{L^2} = \langle \xi, \widehat{B}\eta \rangle_{L^2} = \langle \eta, \widehat{B}^{\mathrm{T}}\xi \rangle_{L^2}.$$

The loop $B := \frac{1}{2}(\widehat{B} + \widehat{B}^{\mathrm{T}})$ thus has the desired properties. □

Now using the previously described construction in the space of symmetric Fredholm operators, a neighborhood of any point (s_0, λ_0, B_0) in $\mathcal{V}^k$ can be identified with the zero-set of a smooth map of the form

$$\Psi(s, \lambda, B) := \Psi_B(s, \lambda) \in \mathrm{End}^{\mathrm{sym}}_{\mathbb{F}}(K),$$

defined for all (s, λ, B) sufficiently close to (s_0, λ_0, B_0) in $(-1, 1) \times \mathbb{R} \times \mathcal{A}_\varepsilon$, where $K = \ker\left(\mathbf{A}^{B_0}_{s_0} - \lambda_0\right)$. The partial derivative of Ψ with respect to the third variable at (s_0, λ_0, B_0) is then a linear map

$$\mathbf{L} := D_3\Psi(s_0, \lambda_0, B_0) : \mathcal{A}_\varepsilon \to \mathrm{End}^{\mathrm{sym}}_{\mathbb{R}}(K)$$

of the form

$$\mathbf{L}B : K \to K : \eta \mapsto \pi_K(B(s_0, \cdot)\eta), \tag{C.1}$$

where $\pi_K : W \oplus K \to K$ is the orthogonal projection. We claim that $\mathbf{L}$ is surjective. Indeed, for any $\Upsilon \in \mathrm{End}^{\mathrm{sym}}_{\mathbb{R}}(K)$, Lemma C.1 provides a smooth loop $C_0 : S^1 \to \mathrm{End}^{\mathrm{sym}}_{\mathbb{R}}(K)$ such that

$$\pi_K(C_0\eta) = \Upsilon\eta \quad \text{for all } \eta \in K,$$

and this can be extended to a smooth function $C : [-1, 1] \times S^1 \to \mathrm{End}^{\mathrm{sym}}_{\mathbb{R}}(K)$ satisfying $C(s_0, \cdot) \equiv C_0$ and $C(\pm 1, \cdot) \equiv 0$ since $s_0 \neq \pm 1$. The function C might fail to be of class C_ε, but since it can be approximated arbitrarily well in the C^0-norm by functions in $\mathcal{A}_\varepsilon$, we conclude that the image of $\mathbf{L}$ is dense in $\mathrm{End}^{\mathrm{sym}}_{\mathbb{R}}(K)$. Since the latter is finite dimensional, the claim follows.

The implicit function theorem now gives $\mathcal{V}^k$ the structure of a smooth Banach submanifold of $(-1,1)\times\mathbb{R}\times\mathcal{A}_\varepsilon$, and it is separable since the latter is also separable. Consider the projection

$$\pi : \mathcal{V}^k \to \mathcal{A}_\varepsilon : (s,\lambda,B) \mapsto B, \tag{C.2}$$

which is a smooth map of separable Banach manifolds whose fibers $\pi^{-1}(B)$ are the spaces $\mathcal{V}^k(B)$. Using Lemma 7.18, the fact that each map Ψ_B is Fredholm implies that π is also a Fredholm map, so the Sard-Smale theorem implies that the regular values of π form a comeager subset

$$\mathcal{A}^{\mathrm{reg},k}_\varepsilon \subset \mathcal{A}_\varepsilon.$$

The intersection

$$\mathcal{A}^{\mathrm{reg}}_\varepsilon := \bigcap_{k\in\mathbb{N}} \mathcal{A}^{\mathrm{reg},k}_\varepsilon$$

is then another comeager subset of $\mathcal{A}_\varepsilon$, with the property that for each $B \in \mathcal{A}^{\mathrm{reg}}_\varepsilon$ and every $(s,\lambda) \in \mathcal{V}^k(B)$, $d\Psi_B(s,\lambda)$ is (by Lemma 7.18) surjective. As was observed previously, this is impossible for dimensional reasons if $k \ge 2$, implying that $\mathcal{V}^k(B)$ is then empty.

To find perturbations that also achieve the transversality condition, we use a similar argument: define for each $B \in \mathcal{A}_\varepsilon$ the subset

$$\mathcal{V}^0(B) = \left\{ s \in (-1,1) \;\middle|\; \dim_{\mathbb{R}} \ker \mathbf{A}^B_s = 1 \right\},$$

along with the corresponding universal set

$$\mathcal{V}^0 = \left\{ (s,B) \in (-1,1)\times\mathcal{A}_\varepsilon \;\middle|\; s \in \mathcal{V}^0(B) \right\}.$$

A neighborhood of any (s_0,B_0) in $\mathcal{V}^0$ is then the zero-set of a smooth map of the form

$$\Psi(s,B) = \Phi(\mathbf{A}^B_s) \in \mathrm{End}^{\mathrm{sym}}_{\mathbb{R}}(\ker \mathbf{A}^{B_0}_{s_0}),$$

defined for all $(s,B) \in (-1,1)\times\mathcal{A}_\varepsilon$ close enough to (s_0,B_0). For a fixed $B \in \mathcal{A}_\varepsilon$ near B_0 and $s_1 \in \mathcal{V}^0(B)$ near s_0, a neighborhood of s_1 in $\mathcal{V}^0(B)$ is then the zero-set of $\Psi_B(s) := \Psi(s,B)$, and the intersection of the path $s \mapsto \mathbf{A}_s \in \mathrm{Fred}^{\mathrm{sym}}_{\mathbb{R}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$ with $\mathrm{Fred}^{\mathrm{sym},1}_{\mathbb{R}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$ at $s = s_1$ is transverse if and only if

$$d\Psi_B(s_1) : \mathbb{R} \to \mathrm{End}^{\mathrm{sym}}_{\mathbb{R}}(\ker \mathbf{A}^{B_0}_{s_0})$$

is surjective. At (s_0,B_0), the partial derivative of Ψ with respect to B is again the same operator

$$\mathbf{L} = D_2\Psi(s_0,B_0) : \mathcal{A}_\varepsilon \to \mathrm{End}^{\mathrm{sym}}_{\mathbb{R}}(\ker \mathbf{A}^{B_0}_{s_0})$$

as in (C.1), which we've already seen is surjective due to Lemma C.1. Thus one can apply the Sard-Smale theorem to the projection

$$\mathcal{V}^0 \to \mathcal{A}_\varepsilon : (s,B) \mapsto B,$$

obtaining a comeager subset $\mathcal{A}^{\mathrm{reg},0}_\varepsilon \subset \mathcal{A}_\varepsilon$ such that all paths $\mathbf{A}_s + B(s,\cdot)$ for $B \in \mathcal{A}^{\mathrm{reg},0}_\varepsilon$ satisfy the required transversality condition. The comeager subset $\mathcal{A}^{\mathrm{reg},0}_\varepsilon \cap$

$\mathcal{A}_\varepsilon^{\text{reg}} \subset \mathcal{A}_\varepsilon$ thus consists of perturbed families of operators for which all desired conditions are satisfied, and it contains a sequence converging in the C^∞-topology to 0. This concludes the proof of Lemma 3.17.

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