TRANSVERSALITY AND SUPER-RIGIDITY FOR MULTIPLY COVERED HOLOMORPHIC CURVES

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Abstract. We develop new techniques to study regularity questions for moduli spaces of pseudoholomorphic curves that are multiply covered. Among the main results, we show that unbranched multiple covers of closed holomorphic curves are generically regular, and simple index 0 curves in dimensions greater than four are generically super-rigid, implying e.g. that the Gromov-Witten invariants of Calabi-Yau 3-folds reduce to sums of local invariants for finite sets of embedded curves. We also establish partial results on super-rigidity in dimension four and regularity of branched covers, and briefly discuss the outlook for bifurcation analysis. The proofs are based on a general stratification result for moduli spaces of multiple covers, framed in terms of a representation-theoretic splitting of Cauchy-Riemann operators with symmetries.

Contents

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1. Introduction

Motivation. The issue of transversality in Gromov's theory of pseudoholomorphic curves [Gro85] has always been problematic, and has attracted renewed interest in recent years. While many powerful symplectic invariants such as Gromov-Witten theory, Hamiltonian Floer homology and symplectic field theory are based on holomorphic curves, most of them run into severe technical complications unless multiply covered curves can be excluded, thus necessitating rather sophisticated techniques that typically replace the standard nonlinear Cauchy-Riemann equation by an abstract perturbation, see e.g. [LT98b, FO99, Rua99, Sie, CM07, HWZ17, Par16]. Aside from the technical challenges that these methods pose, they are non-ideal for many applications: for instance abstract perturbations destroy intersection theory in symplectic 4-manifolds, and in Calabi-Yau 3-folds they obscure information that one might hope to find in the geometric relationship between simple curves and their multiple covers, as exemplified by the Gopakumar-Vafa formula [GV, BP01, PT14, IP18, DIW].

The motivating principle of this paper is in some sense orthogonal to that of abstract perturbations: our aim will be to extend the transversality theory for the *standard* pseudoholomorphic curve equation as far as it can reasonably be pushed, i.e. to prove transversality when it is possible, and in other cases to isolate the precise phenomena which make it impossible and explain what is true instead. Let us start by singling out two situations in which this program is not obviously hopeless.

Example 1.1. If $u : (\Sigma, j) \to (M, J)$ is a closed J-holomorphic curve and $\varphi : (\widetilde{\Sigma}, \tilde{j}) \to (\Sigma, j)$ is an *unbranched* cover of closed connected Riemann surfaces with degree $d \in \mathbb{N}$, then the virtual dimensions of the moduli spaces containing u and $u \circ \varphi : (\widetilde{\Sigma}, \tilde{j}) \to (M, J)$, also known as the *indices* of these two curves, are related by

$$\operatorname{ind}(u \circ \varphi) = d \cdot \operatorname{ind}(u).$$

Since $\operatorname{ind}(u \circ \varphi)$ is then nonnegative whenever $\operatorname{ind}(u) \geqslant 0$, there is no obvious reason why $u \circ \varphi$ could not achieve transversality generically, but traditional methods in the theory of J-holomorphic curves do not prove this except when $u \circ \varphi$ is simply covered, or in certain 4-dimensional cases [HLS97], or more recently, when $\operatorname{ind}(u) = 0$ if a sufficiently large space of perturbed almost complex structures is allowed [GW17].

Example 1.2. Suppose $u : (\Sigma, j) \to (M, J)$ is a closed simply covered curve with index 0 and $\varphi : (\widetilde{\Sigma}, \tilde{j}) \to (\Sigma, j)$ is a branched cover of closed connected Riemann surfaces with degree $d \in \mathbb{N}$ and $Z(d\varphi) \geqslant 0$ as the algebraic count of branch points. Then combining the Riemann-Hurwitz formula

$$-\chi(\widetilde{\Sigma}) + d \cdot \chi(\Sigma) = Z(d\varphi) \tag{1.1}$$

with the standard index formula for closed holomorphic curves gives the relation

$$\operatorname{ind}(u \circ \varphi) = d \cdot \operatorname{ind}(u) - (n-3)Z(d\varphi) = -(n-3)Z(d\varphi), \tag{1.2}$$

where $\dim_{\mathbb{R}} M = 2n$. This shows that $u \circ \varphi$ lives in a space of nonpositive virtual dimension when $\dim M \geqslant 6$ and thus cannot achieve transversality if φ has branch points, as the space of holomorphic branched covers then has dimension $2Z(d\varphi) > 0$. It is interesting however to observe that u must be immersed if J is generic, so it has a well-defined normal bundle $N_u \to \Sigma$, and restricting the linearized Cauchy-Riemann operators for u and $u \circ \varphi$ to the normal bundle and its pullback gives operators $\mathbf{D}_u^N$ and $\mathbf{D}_{u\circ\varphi}^N$ with indices related by

$$\operatorname{ind}(\mathbf{D}_{u\circ\varphi}^N) = d \cdot \operatorname{ind}(\mathbf{D}_u^N) - (n-1)Z(d\varphi) = -(n-1)Z(d\varphi).$$

The latter is always nonpositive, so $\mathbf{D}^N_{u\circ\varphi}$ can be injective, and this condition has a geometric meaning: it implies that $u \circ \varphi$ can never be the limit of a sequence of somewhere injective curves (see Proposition B.1). In fact, the only other curves near $u \circ \varphi$ are other branched covers of the form $u \circ \varphi'$ for φ' near φ, and the cokernels of the operators $\mathbf{D}^N_{u\circ\varphi}$ define an obstruction bundle over the space of branched covers which can be used to compute Gromov-Witten invariants. This phenomenon is known as *super-rigidity*, see Definition 2.3.

Considerable interest in super-rigidity has been motivated by the study of Gromov-Witten invariants in Calabi-Yau 3-folds, where all moduli spaces of holomorphic curves without marked points have virtual dimension zero. In this case it can be interpreted as a Morse-Bott condition for families of "degenerate" (i.e. multiply covered) curves, so that the Gromov-Witten counts of these curves are expressed by integrating Euler classes of obstruction bundles over finitely many such families—these integrals define the so-called "multiple cover contributions," also known as the *local Gromov-Witten invariants* of the underlying embedded curves. A substantial body of results has emerged during the past two decades on local Gromov-Witten invariants and their consequences for Calabi-Yau 3-folds in the presence of the super-rigidity hypothesis, using both algebro-geometric [Pan99, BKL01, BP01, BP05, BP08] and symplectic methods [LZ07, Zin11, DWa]. In spite of these developments, a general result establishing the super-rigidity hypothesis itself has thus far been unavailable. In the algebraic category it is known to hold in some cases and not in others [BP06], and while it was conjectured in [BP01] to hold generically in symplectic manifolds, proofs have been found only in very special settings (e.g. [LP07, LP12] for certain Kähler surfaces), and a strategy was even outlined in [LZ07] to *disprove* the conjecture for higher genus curves.

Results. The first of the main results stated in §1.1 below settles the super-rigidity question for symplectic manifolds of dimension at least six: by Theorem A, super-rigidity does hold in this setting for all simple closed J-holomorphic curves of index 0 if J is generic, and it also holds in dimension four for curves of low genus. Complementary to this, we will see in Theorem B that transversality holds for the unbranched multiple covers in Example 1.1, and we will also be able to prove some transversality results for branched covers (Theorem C). The actual main result of this paper is Theorem D, which implies the aforementioned results by stratifying the space of all multiply covered J-holomorphic curves into smooth submanifolds, with precise formulas for their dimensions. The dimensions are determined by a general picture of Cauchy-Riemann type operators with symmetries described in §2.2, which has its origins in Taubes's work on the Gromov invariant of symplectic 4-manifolds [Tau96a]. As in Taubes's paper, the approach adopted here also lends itself to the study of bifurcations and wall crossing for multiple covers, on which we will make some brief remarks in §2.4 but save the detailed examination for future work.

The difficulty. As with any transversality result, the proof of our main theorem boils down to establishing that a certain bounded linear operator is surjective. The type of operator that arises has appeared before, e.g. in the context of wall-crossing arguments [Tau96a, IP18] (see also [Eft16]), and it has previously been dealt with by various ad hoc methods that suffice for certain specific applications, but would not be general enough for the problems studied here. The solution to this difficulty is probably the most technically novel element in the present paper: it is reduced to a local property of Cauchy-Riemann type operators known as *Petri's condition*, which involves a "decoupling" between the pointwise linear dependence relations for local solutions of a linear Cauchy-Riemann type equation and of its formal adjoint equation. Section 5 of this paper proves that Petri's condition holds generically for Cauchy-Riemann type operators, and this should be regarded as the main step that makes all of our other results possible.

Outlook. While the results in this paper focus specifically on closed holomorphic curves, there is no obvious obstruction to applying the same techniques to study punctured curves in symplectic cobordisms. As with [Tau96a] and the Gromov invariant, this can be expected to have important applications to the foundations of Embedded Contact Homology [Hut14], e.g. for defining cobordism maps and proving invariance without reliance on Seiberg-Witten theory. It also raises the intriguing possibility of localizing (in the sense of Corollary 1.6 below) and/or proving integrality results for invariants in symplectic field theory [EGH00]. A few special cases of super-rigidity in the punctured case have previously been observed in [Wen10, Fab13]; those examples were restricted to dimension four, but the results of the present article suggest that super-rigidity is likely to be a considerably more general phenomenon.

Since the first version of this paper appeared, A. Doan and T. Walpuski have initiated a program extending the equivariant transversality methods introduced here to more general classes of elliptic problems; see [DWb]. More recently, Bai and Swaminathan [BS] have also carried out the first step in the bifurcation analysis proposed in §2.4, and applied it toward defining an extension of Taubes's Gromov invariant to Calabi-Yau 3-folds.

1.1. **Super-rigidity and transversality theorems.** To state the main results, assume (M, ω) is a symplectic manifold with

$$\dim M = 2n \geqslant 4,$$

and J_{fix} is a smooth almost complex structure that is **compatible** with ω, meaning that $\omega(\cdot, J_{\text{fix}} \cdot)$ defines a Riemannian metric on M. We fix also an open subset $\mathcal{U} \subset M$ with compact closure, and consider the space

$$\mathcal{J}(M, \omega\,;\, \mathcal{U}, J_{\text{fix}})$$

of smooth ω-compatible almost complex structures on M that match J_{fix} outside of $\mathcal{U}$, with its natural C^∞-topology.

Remark 1.3. The existence of a symplectic form on M is not required for any of the arguments in this paper, but we are including it in the setup since it is important in applications—all results could alternatively be stated and proved for the larger space of ω-tame almost complex structures, or for arbitrary almost complex structures on a smooth (not necessarily symplectic) manifold.

Following the usual convention among symplectic topologists, we will say that a subset of a topological space is a **Baire subset** if it is comeager, i.e. it is a countable intersection of open and dense subsets. The intersection of a countable sequence of Baire subsets is again a Baire subset, and by the Baire category theorem, any Baire subset of a complete metric space is dense. We will say that a given property is true **generically** (e.g. for generic J) whenever there exists a Baire subset of the space of all admissible data (e.g. in $\mathcal{J}(M, \omega\,;\, \mathcal{U}, J_{\text{fix}})$) such that the property holds for all choices of data in that subset.

Given $J \in \mathcal{J}(M, \omega\,;\, \mathcal{U}, J_{\text{fix}})$, a closed connected Riemann surface (Σ, j) and a J-holomorphic curve $u : (\Sigma, j) \to (M, J)$, the **index** of u is the integer

$$\operatorname{ind}(u) = (n-3)\chi(\Sigma) + 2c_1(u), \tag{1.3}$$

where we abbreviate $c_1(u) := \langle c_1(TM, J), [u] \rangle$, $[u] := u_*[\Sigma] \in H_2(M)$. A closed and connected J-holomorphic curve $\tilde{u} : (\widetilde{\Sigma}, \tilde{\jmath}) \to (M, J)$ is said to be a (d-fold) **multiple cover** of u if $\tilde{u} = u \circ \varphi$ for some holomorphic map $\varphi : (\widetilde{\Sigma}, \tilde{\jmath}) \to (\Sigma, j)$ of degree $d \geqslant 2$, and u is called **simple** if it is nonconstant and is not a multiple cover of any other curve.

The notion of super-rigidity was outlined already in Example 1.2; see Definition 2.3 for a more precise formulation. We will also use the term *Fredholm regular* to refer to the standard notion of transversality for moduli spaces of unparametrized J-holomorphic curves, cf. Proposition 2.2 below. In each of the following theorems, (M, ω) is a symplectic manifold of dimension $2n$ with a compatible almost complex structure J_{fix}, and $\mathcal{U} \subset M$ is an open subset with compact closure.

Theorem A (super-rigidity)**.** *If* $\dim M \geqslant 6$*, then there exists a Baire subset* $\mathcal{J}^{\text{reg}}$ *of the space* $\mathcal{J}(M, \omega\,; \mathcal{U}, J_{\text{fix}})$ *such that for all* $J \in \mathcal{J}^{\text{reg}}$*, every simple* J*-holomorphic curve of index* 0 *that intersects* $\mathcal{U}$ *is super-rigid. Moreover, this result also holds when* $\dim M = 4$ *for all simple index* 0 *curves of genus* 0 *or* 1*.*

Super-rigidity has a number of well-known consequences, which are especially important in the case $\dim M = 6$. These are based partly on the observation that the space of all covers of super-rigid curves is an open and closed subset of the ambient moduli space of J-holomorphic curves, see Proposition B.1 in Appendix B. Applying Gromov compactness and the standard implicit function theorem for simple curves, plus the fact that simple J-holomorphic curves of index 0 are generically embedded and disjoint from each other in dimensions greater than four, this implies:

Corollary 1.4. *For generic compatible* J *in a closed symplectic* 6*-manifold* (M, ω)*, there exist for each integer* $g \geqslant 0$ *and real number* $E > 0$ *at most finitely many distinct simple* J*-holomorphic curves* u *of genus* g *in homology classes* $[u] = A \in H_2(M)$ *with* $c_1(A) = 0$ *and* $\omega(A) \leqslant E$*. Moreover, these curves are embedded and pairwise disjoint.* ☐

Remark 1.5. Doan and Walpuski [DWa] have recently shown that if one fixes the class $A \in H_2(M)$ in Corollary 1.4, then it is not actually necessary to fix the genus g, i.e. for generic J, there exist at most finitely-many simple curves *of any genus* homologous to A. Their proof uses techniques from geometric measure theory.

Using results of Zinger [Zin11] (see also Lee-Parker [LP12]), Theorem A also implies that for generic J, the space of branched covers of an embedded index 0 curve admits a well-defined obstruction bundle which can be used to compute Gromov-Witten invariants. In particular, if $\dim M \geqslant 6$ and $u : (\Sigma, j) \to (M, J)$ is an embedded J-holomorphic curve of genus g with $c_1(u) = 0$, one can apply [Zin11, Theorem 1.2] with no marked point constraints to study the space of J-holomorphic curves with image in $u(\Sigma)$, so that Theorem A establishes hypothesis (b) in Zinger's result, implying that the cokernels of the normal operators $\mathbf{D}^N_{u\circ\varphi}$ for φ varying in the space $\overline{\mathcal{M}}_h(d[\Sigma], j)$ of degree d nodal holomorphic curves in (Σ, j) with arithmetic genus h form a well-defined and oriented orbibundle

$$\mathcal{O}b^u \to \overline{\mathcal{M}}_h(d[\Sigma], j)$$

with $\operatorname{rank}_{\mathbb{R}} \mathcal{O}b^u = (n-1)(2h - 2 + d(2 - 2g))$. Note that by the Riemann-Hurwitz formula, the term $2h - 2 + d(2 - 2g)$ is simply the algebraic count of branch points $Z(d\varphi)$ for any map φ in the non-nodal stratum of $\overline{\mathcal{M}}_h(d[\Sigma], j)$.[1] The obstruction bundle is interesting mainly in the 6-dimensional case, since $n = 3$ means that $\operatorname{rank}_{\mathbb{R}} \mathcal{O}b^u$ matches the real virtual dimension of $\overline{\mathcal{M}}_h(d[\Sigma], j)$, and the count of solutions to an abstract perturbation of the holomorphic curve equation can then be computed by integrating the Euler class $e(\mathcal{O}b^u)$ over the virtual fundamental cycle of $\overline{\mathcal{M}}_h(d[\Sigma], j)$ in the sense of [LT98a, LT98b, FO99]. This produces a formula for the *local* Gromov-Witten invariants of the curve u,

$$N^h_d(u) = \int_{[\overline{\mathcal{M}}_h(d[\Sigma], j)]^{\text{vir}}} e(\mathcal{O}b^u) \in \mathbb{Q},$$

defined for every $d \in \mathbb{N}$ and $h \geqslant g$. These numbers depend only on the germ of the almost complex manifold (M, J) at $u(\Sigma)$. Note that $N^g_1(u) = \pm 1$, with the sign depending on the canonically oriented determinant line of $\mathbf{D}^N_u$.

Combining the obstruction bundle discussion with Corollary 1.4, let

$$N^g_A(M, \omega) \in \mathbb{Q}$$

denote the 0-point Gromov-Witten invariant of (M, ω) for genus g curves in a class $A \in H_2(M)$ with $c_1(A) = 0$.

[1] One must keep in mind however that the non-nodal stratum of $\overline{\mathcal{M}}_h(d[\Sigma])$ may be empty even if $\overline{\mathcal{M}}_h(d[\Sigma])$ itself is not, e.g. this is the case whenever $d = 1$ and $h > g$.

Corollary 1.6 (via [Zin11, Theorem 1.2]). *Suppose (M, ω) is a closed symplectic 6-manifold, $g \geqslant 0$ is an integer and $A \in H_2(M)$ satisfies $c_1(A) = 0$. Then for generic ω-compatible almost complex structures J,*

$$N_A^g(M, \omega) = \sum_{i=1}^N N_{d_i}^g(u_i),$$

where the sum ranges over the (by Corollary 1.4) finite set of pairwise disjoint embedded J-holomorphic curves $u_1, \ldots, u_N$ that have genera at most g and homology classes satisfying $d_i[u_i] = A$ for some $d_1, \ldots, d_N \in \mathbb{N}$. □

In particular in the Calabi-Yau case, with $c_1(TM, \omega) = 0$, this corollary localizes all of the Gromov-Witten invariants of (M, ω).

We next state two results on transversality for multiple covers.

Theorem B (transversality, unbranched). *There exists a Baire subset $\mathcal{J}^{\mathrm{reg}} \subset \mathcal{J}(M, \omega\,; \mathcal{U}, J_{\mathrm{fix}})$ such that for all $J \in \mathcal{J}^{\mathrm{reg}}$, for every simple J-holomorphic curve $u : (\Sigma, j) \to (M, J)$ intersecting $\mathcal{U}$ and every unbranched holomorphic cover $\varphi : (\widetilde{\Sigma}, \tilde{j}) \to (\Sigma, j)$ of closed Riemann surfaces, the J-holomorphic curve $u \circ \varphi : (\widetilde{\Sigma}, \tilde{j}) \to (M, J)$ is Fredholm regular.*

Remark 1.7. The case $\mathrm{ind}(u) = 0$ of Theorem B has been proved previously in [GW17], though with stronger assumptions: for technical reasons, it was necessary in that paper to assume that $u(\Sigma)$ is *contained entirely* in $\mathcal{U}$, and in dimension four also to allow perturbations of J that are ω-tame but not necessarily ω-compatible. The present paper uses a completely different approach to the transversality problem and is thus able to remove these restrictions. As explained in [GW17], the theorem implies an integrality result for the Gromov-Witten invariants in dimension four.

It is generally harder to achieve transversality for covers $u \circ \varphi$ with branch points, e.g. the index relation (1.2) shows that $\mathrm{ind}(u \circ \varphi)$ can easily become negative in dimensions greater than six. More seriously, if u is Fredholm regular, then one can always find a smooth family of other multiple covers near $u \circ \varphi$ obtained by varying both u and φ in their respective moduli spaces; since the latter lives in a space of real dimension $2Z(d\varphi)$, the condition

$$\mathrm{ind}(u \circ \varphi) \geqslant \mathrm{ind}(u) + 2Z(d\varphi)$$

is evidently necessary in order for $u \circ \varphi$ to be Fredholm regular. Observe that if φ has $r \geqslant 0$ critical values, then this condition is satisfied whenever $\mathrm{ind}(u) \geqslant (n-1)r$: indeed, each critical value is the image of at most $d-1$ branch points (counted algebraically), so we have $Z(d\varphi) \leqslant (d-1)r$ and (1.2) implies

$$\begin{aligned} \mathrm{ind}(u \circ \varphi) &= \mathrm{ind}(u) + (d-1)\,\mathrm{ind}(u) - (n-3)Z(d\varphi) \\ &\geqslant \mathrm{ind}(u) + (n-1)Z(d\varphi) - (n-3)Z(d\varphi) = \mathrm{ind}(u) + 2Z(d\varphi). \end{aligned}$$

The next result states that the condition $\mathrm{ind}(u) \geqslant (n-1)r$ is also, in some sense, sufficient.

Theorem C (transversality, branched). *There exists a Baire subset $\mathcal{J}^{\mathrm{reg}} \subset \mathcal{J}(M, \omega\,; \mathcal{U}, J_{\mathrm{fix}})$ such that the following holds for all $J \in \mathcal{J}^{\mathrm{reg}}$. Suppose $u : (\Sigma, j) \to (M, J)$ is a simple J-holomorphic curve intersecting $\mathcal{U}$ and satisfying*

$$\mathrm{ind}(u) \geqslant (n-1)r$$

for some integer $r \geqslant 0$, and $\varphi : (\widetilde{\Sigma}, \tilde{j}) \to (\Sigma, j)$ is a holomorphic branched cover of closed connected Riemann surfaces with r distinct critical values. Then there exists a J-holomorphic curve and a holomorphic branched cover

$$u_\epsilon : (\Sigma, j_\epsilon) \to (M, J) \quad \text{and} \quad \varphi_\epsilon : (\widetilde{\Sigma}, \tilde{j}_\epsilon) \to (\Sigma, j_\epsilon)$$

such that u_ϵ, φ_ϵ, j_ϵ and $\tilde{j}_\epsilon$ are arbitrarily C^∞-close to u, φ, j and $\tilde{j}$ respectively, and $u_\epsilon \circ \varphi_\epsilon : (\widetilde{\Sigma}, \tilde{j}_\epsilon) \to (M, J)$ is Fredholm regular.

Note that whenever $\operatorname{ind}(u \circ \varphi)$ is also *strictly* greater than $\operatorname{ind}(u) + 2Z(d\varphi)$, one can combine this result with the implicit function theorem to deduce the existence of simple J-holomorphic curves that are small perturbations of multiple covers of u.

The proofs of these theorems are inspired by the work of Taubes [Tau96a], whose definition of the Gromov invariant for symplectic 4-manifolds required a special case of Theorem B along with related bifurcation-theoretic results (cf. §2.4) for multiply covered holomorphic tori. Roughly speaking, the idea is to study the local structure of spaces of the form

$$\mathcal{M}(k,c) := \left\{ \tilde{u} = u \circ \varphi \;\middle|\; \dim \ker \mathbf{D}^N_{\tilde{u}} = k \text{ and } \dim \operatorname{coker} \mathbf{D}^N_{\tilde{u}} = c \right\}, \tag{1.4}$$

where $k, c \geqslant 0$ are fixed integers, u varies in the moduli space of simple J-holomorphic curves and φ varies in the moduli space of holomorphic branched covers. Ideally, one would like to show that these spaces are smooth manifolds for generic J, and to compute their codimensions in the space of pairs (u, φ). This turns Theorems A and B into "dimension counting" problems, as whenever one can show that the codimension of $\mathcal{M}(k,c)$ is larger than the dimension of the ambient space for suitable values of k and c, one may conclude that either $\ker \mathbf{D}^N_{\tilde{u}}$ or $\operatorname{coker} \mathbf{D}^N_{\tilde{u}}$ must be trivial. This discussion is oversimplified in at least three respects: first, we will not be able to find any nice structure on $\mathcal{M}(k,c)$ if φ varies in the space of *all* branched covers, but it will help to confine it to certain substrata of that space in which all branch points have prescribed branching orders. For similar reasons, it will also help to confine u to substrata in which its number of critical points and their orders are constrained, and this is easily done. More seriously, the space $\mathcal{M}(k,c)$ as sketched above can have different codimensions on different components, as its codimension depends intricately on symmetry information which is ignored in (1.4). We will therefore need to define a more elaborate version of $\mathcal{M}(k,c)$ which depends on a splitting of the operator $\mathbf{D}^N_{\tilde{u}}$ into summands corresponding to irreducible representations of the (generalized) symmetry group of the cover. This idea is borrowed directly from [Tau96a], though the details are somewhat more involved since, in contrast to the case of unbranched covers of tori, we cannot assume that all covers are regular or that their symmetry groups are abelian. We will see that once the formalism is developed in sufficient generality, it "breaks the symmetry" of $\mathbf{D}^N_{\tilde{u}}$ enough to make dimension counting arguments much more effective.

Remark 1.8. A slightly different variation on the ideas in [Tau96a] has been implemented by Eftekhary to prove a partial result toward super-rigidity in dimension six, see [Eft16].

Here is an outline of the rest of the paper.

After establishing some standard definitions and notation, §2 will further elucidate the ideas sketched above and formulate a precise version of the statement that $\mathcal{M}(k,c)$ from (1.4) is a smooth submanifold, Theorem D. This will then be used as a black box to prove Theorems A, B and C in §2.3, followed in §2.4 by a brief informal discussion of bifurcation theory. The remainder of the paper is then devoted to the proof of Theorem D. In §3, we explain the splitting construction for Cauchy-Riemann operators with symmetries and prove some lemmas based on a mixture of elliptic regularity for punctured Cauchy-Riemann operators, topology of covering spaces, and representation theory of finite groups. The summands in the splitting are also Cauchy-Riemann operators, whose indices are a somewhat delicate computation, carried out in §4. In §5 we prove a local genericity result for Cauchy-Riemann operators that takes on the role usually played by unique continuation in applications of the Sard-Smale theorem, and the latter will be used in §6 to complete the proof of Theorem D. Finally, §7 deals with super-rigidity in the four-dimensional case, which is something of an anomaly and requires different techniques based on intersection theory. The appendices provide various results that may be considered "standard" and yet, in this author's experience, seem to cause sufficient confusion among experts to warrant some discussion; their proofs require a few ideas that will in any case be useful elsewhere in the paper.

1.2. **Apologies and acknowledgements.** The super-rigidity problem has a slightly troubled history, and as the author of a new paper on the subject, it would behoove me at this point to apologize for having caused some of that trouble: I am aware of three previous attempts to prove some version of Theorem A which were later either withdrawn or revised to prove much weaker statements, and I was an author of one of them (the original version of [GW17]). To make matters worse, earlier versions of the present paper also contained a major error in §5 on which the main results were crucially dependent, causing the paper to be withdrawn for several months while the offending section underwent an extensive rewrite. (For more on the history of failed super-rigidity proofs, see Appendix D.) With all this in mind, I would sympathize with any reader's inclination to greet this paper with a dose of skepticism, though it seems worth pointing out that rather than being an attempt to rescue the (probably unrescuable) proof originally attempted in [GW17], the approach taken here has almost nothing in common with the previous one, other than the considerable debt that both of them owe to the ideas of Taubes [Tau96b, Tau96a].

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2. The main idea

2.1. Some definitions.

Let us now fix some notation and definitions that will be essential in the rest of the paper.

Given integers $g, m \geqslant 0$ and a class $A \in H_2(M)$, the moduli space of **unparametrized J-holomorphic curves** $\mathcal{M}_{g,m}(A, J)$ can be defined as the set of equivalence classes of tuples (Σ, j, Θ, u) where (Σ, j) is a closed connected Riemann surface of genus g, $\Theta \subset \Sigma$ is an ordered set of m distinct points (the **marked points**), and $u : (\Sigma, j) \to (M, J)$ is a J-holomorphic map satisfying $[u] := u_*[\Sigma] = A$, with equivalence defined by $(\Sigma, j, \Theta, u) \sim (\Sigma', \psi^* j, \psi^{-1}(\Theta), u \circ \psi)$ for diffeomorphisms $\psi : \Sigma' \to \Sigma$. The **Gromov compactification** of $\mathcal{M}_{g,m}(A, J)$ is the space $\overline{\mathcal{M}}_{g,m}(A, J)$ of (equivalence classes of) **stable nodal curves** $(S, j, \Theta, \Delta, u)$, where now S may be disconnected, and the original data are augmented by an unordered set of distinct points in $S \backslash \Theta$, arranged into unordered pairs

$$\Delta = \{\{\widehat{z}_1, \check{z}_1\}, \ldots, \{\widehat{z}_r, \check{z}_r\}\},$$

such that $u(\widehat{z}_i) = u(\check{z}_i)$ for each $i = 1, \ldots, r$. We call the pairs $\{\widehat{z}_i, \check{z}_i\}$ **nodes**, and each individual $\widehat{z}_i$ or $\check{z}_i \in S$ a **nodal point**. The curves in $\overline{\mathcal{M}}_{g,m}(A, J)$ are required to have **arithmetic genus** g, which means that the surface obtained from S by performing connected sums at all matched pairs of nodal points is a closed connected surface of genus g. The stability condition requires that any component of $S \backslash (\Theta \cup \Delta)$ on which u is constant should have negative Euler characteristic. With this condition, $\overline{\mathcal{M}}_{g,m}(A, J)$ can be given a natural topology as a metrizable Hausdorff space, and it is compact whenever J is tamed by a symplectic form. A definition of the topology may be found e.g. in [BEH$^+$03]; for convergent sequences in $\mathcal{M}_{g,m}(A, J)$, it amounts to the notion of C^∞-convergence for j and u after a choice of parametrization for which all domains and marked point sets are identified. Curves $[(S, j, \Theta, \Delta, u)] \in \overline{\mathcal{M}}_{g,m}(A, J)$ with $\Delta = \varnothing$ can equivalently be regarded as elements of $\mathcal{M}_{g,m}(A, J)$, and are thus called **smooth** curves to distinguish them from nodal curves.

Remark 2.1. In this paper, the word "curve" always means "smooth curve" (i.e. without nodes) unless the word "nodal" is explicitly included. Similarly, all dimensions and Fredholm indices in this paper are *real* (not complex) unless otherwise specified. This usage differs somewhat from the algebraic geometry literature.

When there is no danger of confusion, we shall sometimes abuse notation by writing equivalence classes $[(\Sigma, j, \Theta, u)] \in \mathcal{M}_{g,m}(A, J)$ or $[(S, j, \Theta, \Delta, u)] \in \overline{\mathcal{M}}_{g,m}(A, J)$ via the abbreviations $u \in \mathcal{M}_{g,m}(A, J)$ or $u \in \overline{\mathcal{M}}_{g,m}(A, J)$ respectively, and we will refer to the restriction of a nodal curve $[(S, j, \Theta, \Delta, u)]$ to any connected component of its domain S as a **smooth component** of u. We shall also abbreviate

$$\mathcal{M}_g(A, J) := \mathcal{M}_{g,0}(A, J), \quad \text{and} \quad \overline{\mathcal{M}}_g(A, J) := \overline{\mathcal{M}}_{g,0}(A, J).$$

Recall that $\mathcal{M}_g(A, J)$ has **virtual dimension** equal to the index of any curve $u \in \mathcal{M}_g(A, J)$ as written in (1.3), while the virtual dimension of the moduli space with marked points is

$$\text{vir-dim}\, \mathcal{M}_{g,m}(A, J) = \text{vir-dim}\, \mathcal{M}_g(A, J) + 2m.$$

The multiply covered curves form a distinguished closed subset of $\overline{\mathcal{M}}_g(A, J)$. Given any $u \in \mathcal{M}_g(A, J)$ with domain (Σ, j), and integers $h \geqslant 0$, $d \geqslant 1$, define the space of stable **nodal d-fold covers** of u,

$$\overline{\mathcal{M}}_h(d; u) = \left\{ [(S, \tilde{j}, \Delta, u \circ \varphi)] \in \overline{\mathcal{M}}_h(dA, J) \;\middle|\; [(S, \tilde{j}, \Delta, \varphi)] \in \overline{\mathcal{M}}_h(d[\Sigma], j) \right\},$$

so in particular, each smooth component $\tilde{u}_i$ of $\tilde{u} \in \overline{\mathcal{M}}_h(d; u)$ belongs to a space $\mathcal{M}_{g_i}(d_i; u)$ of smooth branched covers $u \circ \varphi_i$ of some degreee $d_i \geqslant 0$, such that $\sum_i d_i = d$. Note that $\overline{\mathcal{M}}_h(d; u)$ may in general be strictly larger than the closure of $\mathcal{M}_h(d; u)$ in the Gromov topology—to cite one well-known example, the space $\mathcal{M}_1([S^2], i)$ of smooth degree 1 holomorphic tori in (S^2, i) is empty, but $\overline{\mathcal{M}}_1([S^2], i)$ contains a nodal curve with a constant component of genus 1.

Recall next that every J-holomorphic curve $u : (\Sigma, j) \to (M, J)$ gives rise to a **linearized Cauchy-Riemann operator**

$$\mathbf{D}_u : \Gamma(u^*TM) \to \Omega^{0,1}(\Sigma, u^*TM),$$

i.e. the linearization at u of the nonlinear Cauchy-Riemann operator $\bar{\partial}_J(u) := Tu + J \circ Tu \circ j \in \Omega^{0,1}(\Sigma, u^*TM)$, whose zero-set is the space of all J-holomorphic maps with domain (Σ, j). The operator $\mathbf{D}_u$ takes vector fields along u to $(0,1)$-forms valued in the complex vector bundle (u^*TM, J), and can be written explicitly as

$$\mathbf{D}_u \eta = \nabla \eta + J(u) \circ \nabla \eta \circ j + (\nabla_\eta J) \circ Tu \circ j$$

for any choice of symmetric connection ∇ (cf. [Wena, §2.4]). Recall moreover that whenever u is nonconstant, its critical points are isolated and one can find a smooth splitting of complex vector bundles

$$u^*TM = T_u \oplus N_u \tag{2.1}$$

such that T_u matches the image of du at regular points; see e.g. [Wen10, §3.3] for details. We shall refer to N_u as the **generalized normal bundle** of u. In many cases of interest in this paper, u will be a cover of an immersed J-holomorphic curve v, so N_u is then simply the pullback of the normal bundle of v via the cover. We define the **normal Cauchy-Riemann operator** at u as the restriction of $\mathbf{D}_u$ to sections of N_u, composed with the projection $\pi_N : u^*TM \to N_u$ along T_u, hence

$$\mathbf{D}_u^N = \pi_N \circ \mathbf{D}_u|_{\Gamma(N_u)} : \Gamma(N_u) \to \Omega^{0,1}(\Sigma, N_u).$$

In general, a neighborhood of any element in $\mathcal{M}_{g,m}(A, J)$ can be identified with the zero-set of a smooth Fredholm section of a Banach space bundle, modulo a finite group action if there are nontrivial automorphisms. We say that $u \in \mathcal{M}_g(A, J)$ is **Fredholm regular** whenever it is a transverse intersection of this section with the zero-section. Note that whenever this condition holds, it automatically also holds after adding any finite collection of marked points and viewing u as an element of $\mathcal{M}_{g,m}(A, J)$. The implicit function theorem gives the open set of regular curves in $\mathcal{M}_{g,m}(A, J)$ the structure of a smooth orbifold with dimension equal to its virtual dimension, and local isotropy groups determined by the automorphism groups of the curves—in particular, the set of regular simple curves forms a manifold, though orbifold singularities can appear when multiple covers are included. The following convenient repackaging of the regularity condition comes from [Wen10, Corollary 3.13].

Proposition 2.2. *A closed and connected J-holomorphic curve $u : (\Sigma, j) \to (M, J)$ is Fredholm regular if and only if its normal operator $\mathbf{D}_u^N : W^{k,p}(N_u) \to W^{k-1,p}(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, N_u))$ is surjective for some (and therefore all) $k \in \mathbb{N}$ and $p \in (1, \infty)$.* $\square$

Definition 2.3. A closed, connected, simple J-holomorphic curve $u : (\Sigma, j) \to (M, J)$ is called **super-rigid** if it satisfies the following:

(1) $\mathrm{ind}(u) = 0$;
(2) $u : \Sigma \to M$ is an immersion;
(3) For all closed connected Riemann surfaces $(\tilde{\Sigma}, \tilde{j})$ and holomorphic maps $\varphi : (\tilde{\Sigma}, \tilde{j}) \to (\Sigma, j)$ of positive degree, the curve $\tilde{u} := u \circ \varphi : (\tilde{\Sigma}, \tilde{j}) \to (M, J)$ admits no nontrivial solutions to the normal linearized equation $\mathbf{D}_{\tilde{u}}^N \eta = 0$.

Proposition B.1 in Appendix B proves that if u is a super-rigid curve, then the only possible sequences that converge to a nodal branched cover of u consist of other covers of u. In the language of the present section, this means:

Corollary 2.4 (of Proposition B.1)**.** *Suppose (M, J) is an almost complex manifold and $u \in \mathcal{M}_g(A, J)$ is a super-rigid curve in M. Then for every $h \geqslant 0$ and $d \geqslant 1$, $\overline{\mathcal{M}}_h(d; u)$ is an open and closed subset of $\overline{\mathcal{M}}_h(dA, J)$.* $\square$

2.2. **A stratification theorem.** We now explain in precise terms the stratification result that underlies the main theorems of §1.1.

2.2.1. *Splitting the linearization at a doubly covered curve.* Suppose $v : (\Sigma, j) \to (M, J)$ is a simple J-holomorphic curve with genus $g \geqslant 0$, and $\varphi : (\Sigma', j') \to (\Sigma, j)$ is a holomorphic branched cover with degree $d \geqslant 1$, giving rise to the multiply covered curve $u = v \circ \varphi : (\Sigma', j') \to (M, J)$ of genus $h \geqslant 0$. We assume as always that Σ and Σ' are both closed and connected, and for the sake of intuition, we begin in this subsection with the special case $d = 2$. The automorphism group

$$\mathrm{Aut}(u) = \mathrm{Aut}(\varphi) := \left\{ \psi : (\Sigma', j') \xrightarrow{\cong} (\Sigma', j') \;\middle|\; \varphi = \varphi \circ \psi \right\}$$

then contains a unique nontrivial element ψ, and the space of sections $\Gamma(N_u)$ has a natural splitting

$$\Gamma(N_u) = \Gamma_+(N_u) \oplus \Gamma_-(N_u)$$

where $\Gamma_\pm(N_u) := \{\eta \in \Gamma(N_u) \mid \eta = \pm\eta \circ \psi\}$. Splitting $\Omega^{0,1}(\Sigma', N_u) = \Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma', N_u))$ in the same way, one obtains a splitting of the normal Cauchy-Riemann operator

$$\mathbf{D}_u^N = \mathbf{D}_{u,+}^N \oplus \mathbf{D}_{u,-}^N \tag{2.2}$$

into two operators $\mathbf{D}_{u,\pm}^N : \Gamma_\pm(N_u) \to \Gamma_\pm(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma', N_u))$. It is not hard to see that $\mathbf{D}_{u,+}^N$ is in some sense equivalent to $\mathbf{D}_v^N$, as its domain and target both consist of sections that are pullbacks via φ of sections over Σ. The operators $\mathbf{D}_{u,+}^N$ and $\mathbf{D}_{u,-}^N$ have unique extensions over the spaces of symmetric/antisymmetric sections of Sobolev class $W^{k,p}$ for $k \in \mathbb{N}$ and $p \in (1, \infty)$, giving bounded linear operators

$$\mathbf{D}_{u,\pm}^N : W_\pm^{k,p}(N_u) \to W_\pm^{k,p}(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma', N_u)),$$

and the standard transversality theory for simple curves then implies that $\mathbf{D}_{u,+}^N$ can be assumed surjective (and also injective if v is immersed with index 0) if J is chosen generically. We will see that the problem of proving surjectivity or injectivity for $\mathbf{D}_u^N$ becomes more tractable when viewed as two independent problems for the operators $\mathbf{D}_{u,+}^N$ and $\mathbf{D}_{u,-}^N$.

In order to generalize this discussion beyond the degree 2 case, it helps to adopt an alternative perspective based on representation theory. Let $\Theta \subset \Sigma$ denote a finite subset that contains all critical values of φ, and set

$$\Theta' := \varphi^{-1}(\Theta), \qquad \dot{\Sigma} := \Sigma \backslash \Theta, \qquad \dot{\Sigma}' := \Sigma' \backslash \Theta', \tag{2.3}$$

so that $\dot{\Sigma}' \xrightarrow{\varphi} \dot{\Sigma}$ is a smooth covering map with $G := \mathrm{Aut}(\varphi) \cong \mathbb{Z}_2$ as its group of deck transformations. Define

$$\rho : G \to S_2 : g \mapsto \rho_g$$

as the isomorphism to the symmetric group on $\{1,2\}$. We can then identify the covering map $\dot{\Sigma}' \xrightarrow{\varphi} \dot{\Sigma}$ with

$$\left(\dot{\Sigma}' \times \{1,2\}\right) \Big/ G \to \dot{\Sigma} : [(z,i)] \mapsto \varphi(z),$$

where G acts on $\dot{\Sigma}'$ by deck transformations and on $\{1,2\}$ via ρ. Now if (e_1, e_2) denotes the standard basis of $\mathbb{R}^2$, then ρ also gives rise to a real permutation representation

$$\boldsymbol{\rho} : G \to \mathrm{GL}(2,\mathbb{R}), \qquad \boldsymbol{\rho}(g)e_i := e_{\rho_g(i)},$$

and a corresponding real vector bundle $V^{\boldsymbol{\rho}} \to \dot{\Sigma}$ defined as the $\mathbb{Z}_2$-quotient of a trivial bundle over $\dot{\Sigma}'$,

$$V^{\boldsymbol{\rho}} := \left(\dot{\Sigma}' \times \mathbb{R}^2\right) \Big/ G.$$

The space of sections of the twisted normal bundle

$$N_v^{\boldsymbol{\rho}} := N_v \otimes_{\mathbb{R}} V^{\boldsymbol{\rho}} \to \dot{\Sigma}$$

then has a natural identification with the space of sections of $N_u = \varphi^* N_v$: indeed, we can represent sections of $N_v^{\boldsymbol{\rho}}$ as $\mathbb{Z}_2$-equivariant sections $\eta = \sum_{i=1}^2 \eta^i \otimes e_i$ of $\varphi^* N_v \otimes_{\mathbb{R}} \mathbb{R}^2$, which satisfy the relation $\eta^i \circ \psi = \eta^{\rho_\psi(i)}$, thus a corresponding section $\widehat{\eta} \in \Gamma(\varphi^* N_v)$ can be defined under the identification of $\dot{\Sigma}'$ with $(\dot{\Sigma}' \times \{1,2\})/G$ by

$$\widehat{\eta}([(z,i)]) = \eta^i(z).$$

Under this identification, $\mathbf{D}_u^N$ becomes a Cauchy-Riemann type operator on the twisted bundle $N_v^{\boldsymbol{\rho}}$, defined locally by $\mathbf{D}_u^N(\eta \otimes s) = (\mathbf{D}_v^N \eta) \otimes s$ whenever s is a local section of $V^{\boldsymbol{\rho}}$ that has a constant lift to the trivial bundle $\dot{\Sigma}' \times \mathbb{R}^2$.

The above construction appears cumbersome at first glance, but it has the following advantage: the decomposition $\Gamma(N_u) = \Gamma_+(N_u) \oplus \Gamma_-(N_u)$ now corresponds to a splitting of the twisted bundle $N_v^{\boldsymbol{\rho}}$ into subbundles

$$N_v^{\boldsymbol{\rho}} = N_v^{\boldsymbol{\theta}_+} \oplus N_v^{\boldsymbol{\theta}_-} := (N_v \otimes_{\mathbb{R}} V^{\boldsymbol{\theta}_+}) \oplus (N_v \otimes_{\mathbb{R}} V^{\boldsymbol{\theta}_-})$$

where $V^{\boldsymbol{\theta}_\pm} := (\dot{\Sigma}' \times W_\pm)/G$ are defined in terms of the natural splitting of $\mathbb{R}^2 = W_+ \oplus W_-$ into irreducible G-invariant subspaces

$$W_\pm = \mathbb{R} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \subset \mathbb{R}^2.$$

This is the simplest nontrivial example of what turns out to be a general principle: splittings of Cauchy-Riemann operators for multiply covered curves arise from decompositions of permutation representations into irreducible summands. To turn $\boldsymbol{\rho} = \boldsymbol{\theta}_+ \oplus \boldsymbol{\theta}_-$ into a splitting of Cauchy-Riemann operators, we still have a small analytical issue to cope with since the bundles $N_v^{\boldsymbol{\theta}_\pm}$ are defined over $\dot{\Sigma}$ and do not both extend over the punctures. In place of (2.2), we therefore obtain a splitting

$$\dot{\mathbf{D}}_u^N = \dot{\mathbf{D}}_{u,\boldsymbol{\theta}_+}^N \oplus \dot{\mathbf{D}}_{u,\boldsymbol{\theta}_-}^N,$$

where the dots over the operators indicate that we are restricting them to the punctured domain $\dot{\Sigma}'$. We will see in §3.2 how to define suitable weighted Sobolev spaces over $\dot{\Sigma}$ and $\dot{\Sigma}'$ so that the punctured operators have the same indices, kernels and cokernels as their unpunctured counterparts.

Remark 2.5. A slightly different approach to defining twisted Cauchy-Riemann operators is taken by Doan and Walpuski [DWb], who express it in the elegant language of local systems.

2.2.2. *The codimension of a multiply covered curve.* We return now to the general case of a closed connected J-holomorphic curve $u = v \circ \varphi : (\Sigma', j') \to (M, J)$ of genus h, where $v : (\Sigma, j) \to (M, J)$ is simple with genus g and $\varphi : (\Sigma', j') \to (\Sigma, j)$ has degree $d \in \mathbb{N}$. We continue using the notation $\dot{\Sigma}' \xrightarrow{\varphi} \dot{\Sigma}$ for the d-fold covering map obtained by deleting some finite subsets that include the critical values and their preimages. Recall that φ is called **regular** if $|\operatorname{Aut}(\varphi)| = \deg(\varphi) = d$. This condition was secretly important in the above discussion of the $d = 2$ case, as the definition of the twisted bundle N_v^{ρ} required identifying $\dot{\Sigma}$ with the quotient of $\dot{\Sigma}'$ by deck transformations. In general, $\operatorname{Aut}(\varphi)$ can have order smaller than d and may even be trivial, but we can use some notions from elementary covering space theory to get around this.

Definition 2.6. The **generalized automorphism group** of a d-fold branched cover $\varphi : \Sigma' \to \Sigma$ is the quotient $G := \pi_1(\dot{\Sigma})/H$, where H is the normal core[2] of $\varphi_*(\pi_1(\dot{\Sigma}'))$, and $\dot{\Sigma}$ and $\dot{\Sigma}'$ are defined by (2.3) with Θ as the set of critical values of φ.

Remark 2.7. Like fundamental groups, the generalized automorphism group G of $\varphi : \Sigma' \to \Sigma$ depends on choices of base points in $\dot{\Sigma}$ and $\dot{\Sigma}'$, but its isomorphism class is independent of these choices. We will see below that G is a finite group of order at most $d!$ that is isomorphic to $\operatorname{Aut}(\varphi)$ if and only if $\varphi : \Sigma' \to \Sigma$ is regular, and more generally, G has a natural identification with the automorphism group of a certain regular branched cover of Σ that is determined by φ and a choice of base points, and factors through φ.

Definition 2.8. A **regular presentation** of the holomorphic d-fold branched cover $\varphi : (\Sigma', j') \to (\Sigma, j)$ is a tuple $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$ consisting of:

- A finite subset $\Theta \subset \Sigma$ containing the critical values of φ and defining the punctured surfaces $\dot{\Sigma}$ and $\dot{\Sigma}'$ via (2.3);
- A connected surface $\dot{\Sigma}''$ and regular covering map $\pi : \dot{\Sigma}'' \to \dot{\Sigma}$ with finite automorphism group $G := \operatorname{Aut}(\pi)$;
- A set I with d elements;
- A transitive action of G on I, defined via a homomorphism $\rho : G \to S(I)$ from G to the symmetric group on I;
- A diffeomorphism $f : \dot{\Sigma}' \to (\dot{\Sigma}'' \times I)/G$, where G acts on $\dot{\Sigma}''$ by deck transformations and on I via ρ, such that $\varphi \circ f^{-1}$ takes the form
$$\left(\dot{\Sigma}'' \times I\right) \Big/ G \to \dot{\Sigma} : [(z, i)] \mapsto \pi(z).$$

We say that $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$ is **minimal** if $\Theta \subset \Sigma$ is the set of critical values of φ and $\rho : G \to S(I)$ is injective. Two regular presentations $(\Theta_j, \dot{\Sigma}_j'', \pi_j, G_j, \rho_j, I_j, f_j)$ of $\varphi : \Sigma' \to \Sigma$ for $j = 1, 2$ are **isomorphic** if $\Theta_1 = \Theta_2$ and there exists a diffeomorphism $\Psi : \dot{\Sigma}_1'' \to \dot{\Sigma}_2''$, a bijection $\beta : I_1 \to I_2$, and a group isomorphism $\Phi : G_1 \to G_2$ such that:

(1) $\pi_2 \circ \Psi = \pi_1$ and for all $g \in G_1$, $\Psi \circ g = \Phi(g) \circ \Psi$;
(2) For all $g \in G_1$, $\beta \circ \rho_1(g) = \rho_2(\Phi(g)) \circ \beta$;
(3) $f_2 \circ f_1^{-1}$ takes the form
$$\left(\dot{\Sigma}_1'' \times I_1\right) \Big/ G_1 \to \left(\dot{\Sigma}_2'' \times I_2\right) \Big/ G_2 : [(z, i)] \mapsto [(\Psi(z), \beta(i))].$$

Most of the regular presentations we encounter in this paper will be minimal, though an important example that is not (in particular where Θ may contain more than just the critical values) will arise in Example 3.5. Standard results about Riemann surfaces (see §3.1) imply that the regular cover $\pi : \dot{\Sigma}'' \to \dot{\Sigma}$ in any regular presentation can be extended to a holomorphic branched cover of closed connected Riemann surfaces $(\Sigma'', j'') \to (\Sigma, j)$ such that $\dot{\Sigma}'' = \Sigma'' \backslash \pi^{-1}(\Theta)$. Observe that if $i \in I$ and $G_i \subset G$ denotes the stabilizer of i under the G-action defined by ρ,

[2]Recall that the **normal core** of a subgroup H in a group Γ is the largest normal subgroup of Γ that is contained in H.

then

$$\dot{\Sigma}''/G_i \to \left(\dot{\Sigma}'' \times I\right) \Big/ G : [z] \mapsto [(z,i)]$$

is a diffeomorphism identifying $\varphi \circ f^{-1}$ with the natural projection $\dot{\Sigma}''/G_i \to \dot{\Sigma}''/G = \dot{\Sigma}$. Thus one can associate to any regular presentation a (non-unique) factorization of $\pi : \dot{\Sigma}'' \to \dot{\Sigma}$ by covering maps $\dot{\Sigma}'' \to \dot{\Sigma}' \xrightarrow{\varphi} \dot{\Sigma}$, which extends over the punctures to a factorization of $\pi : (\Sigma'', j'') \to (\Sigma, j)$ by holomorphic branched covers

$$(\Sigma'', j'') \to (\Sigma', j') \xrightarrow{\varphi} (\Sigma, j).$$

We will also show in Lemma 3.2 that $\varphi : \Sigma' \to \Sigma$ always admits a unique isomorphism class of minimal regular presentations $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$, for which G is isomorphic to the generalized automorphism group of φ, and in this case $\pi : \dot{\Sigma}'' \to \dot{\Sigma}$ is isomorphic to $\varphi : \dot{\Sigma}' \to \dot{\Sigma}$ whenever the latter happens to be already regular (cf. Example 3.4).

Given a choice of regular presentation $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$, the discussion of the degree 2 case can be generalized as follows. The transitive action $\rho : G \to S(I)$ induces a permutation representation $\boldsymbol{\rho} : G \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^I)$ on the real vector space $\mathbb{R}^I$ with basis labeled by the elements of I, and a twisted bundle $N_v^{\boldsymbol{\rho}} = N_v \otimes_{\mathbb{R}} V^{\boldsymbol{\rho}} \to \dot{\Sigma}$, where

$$V^{\boldsymbol{\rho}} := (\dot{\Sigma}'' \times \mathbb{R}^I)/G,$$

with a natural isomorphism

$$\Gamma(N_v^{\boldsymbol{\rho}}) = \Gamma(\varphi^* N_v|_{\dot{\Sigma}'}) = \Gamma(N_u|_{\dot{\Sigma}'})$$

that identifies $\mathbf{D}_u^N$ with a Cauchy-Riemann operator

$$\dot{\mathbf{D}}_{u,\boldsymbol{\rho}}^N : \Gamma(N_v^{\boldsymbol{\rho}}) \to \Omega^{0,1}(\dot{\Sigma}, N_v^{\boldsymbol{\rho}}),$$

defined on suitable exponentially weighted Sobolev spaces of sections of $N_v^{\boldsymbol{\rho}}$. (The appropriate functional-analytic setting for this operator will be specified precisely in §3.2.) Any representation $\boldsymbol{\theta} : G \to \mathrm{Aut}_{\mathbb{R}}(W)$ on a real finite-dimensional vector space W similarly gives rise to a twisted bundle $N_v^{\boldsymbol{\theta}} = N_v \otimes_{\mathbb{R}} V^{\boldsymbol{\theta}} \to \dot{\Sigma}$, with $V^{\boldsymbol{\theta}} := (\dot{\Sigma}'' \times W)/G$, and a twisted Cauchy-Riemann operator

$$\dot{\mathbf{D}}_{u,\boldsymbol{\theta}}^N : \Gamma(N_v^{\boldsymbol{\theta}}) \to \Omega^{0,1}(\dot{\Sigma}, N_v^{\boldsymbol{\theta}}),$$

which (up to conjugacy) depends only on $\dot{\mathbf{D}}_v^N$ and the isomorphism classes of the regular presentation and the representation $\boldsymbol{\theta}$. Now any representation-theoretic decomposition $\boldsymbol{\rho} = \boldsymbol{\theta}_1^{\oplus m_1} \oplus \ldots \oplus \boldsymbol{\theta}_p^{\oplus m_p}$ induces a splitting of the punctured Cauchy-Riemann operator

$$\mathbf{D}_u^N \cong \dot{\mathbf{D}}_{u,\boldsymbol{\rho}}^N = (\dot{\mathbf{D}}_{u,\boldsymbol{\theta}_1}^N)^{\oplus m_1} \oplus \ldots \oplus (\dot{\mathbf{D}}_{u,\boldsymbol{\theta}_p}^N)^{\oplus m_p}, \tag{2.4}$$

with the following useful property:

Lemma 2.9. *The normal Cauchy-Riemann operator $\mathbf{D}_u^N$ for a multiple cover is surjective or injective if and only if the same holds for all of the summands $\dot{\mathbf{D}}_{u,\boldsymbol{\theta}_j}^N$ in (2.4) with $m_j > 0$.*

Remark 2.10. We will see below that the splitting (2.4) for a multiply covered curve $u = v \circ \varphi$ can be arranged to vary smoothly as v and φ move about in their respective (suitably constrained) moduli spaces, so the indices of the summands $\dot{\mathbf{D}}_{u,\boldsymbol{\theta}_j}^N$ are constant under such variations. This immediately gives rise to "no-go" results about transversality and super-rigidity: the former is impossible on components of the moduli space where the $\dot{\mathbf{D}}_{u,\boldsymbol{\theta}_j}^N$ do not all have nonnegative index, and the latter requires them instead to have nonpositive index. Conversely, whenever either of these index conditions holds for all summands given by irreducible representations, Theorem D below will imply that the desired transversality or super-rigidity result holds for all pairs (v, φ) lying in some open and dense subset. This is the main idea behind Theorem C, and it similarly can be used to determine the feasibility of obstruction bundle arguments in general situations.

It should be emphasized that the representations of G in this discussion are *real*, not complex. We will need to use the standard fact (see §3.3) that for any finite group G, real irreducible representations $\boldsymbol{\theta} : G \to \mathrm{Aut}_{\mathbb{R}}(W)$ come in three types, characterized via the algebra $\mathbb{K} := \mathrm{End}_G(W)$ of G-equivariant real-linear maps $W \to W$:

- **Real type**: $\mathbb{K} \cong \mathbb{R}$;
- **Complex type**: $\mathbb{K} \cong \mathbb{C}$;
- **Quaternionic type**: $\mathbb{K} \cong \mathbb{H}$.

The endomorphism algebra $\mathbb{K} = \mathrm{End}_G(W)$ endows the domain and target of the operator $\dot{\mathbf{D}}^N_{u,\boldsymbol{\theta}}$ with $\mathbb{K}$-module structures, for which $\dot{\mathbf{D}}^N_{u,\boldsymbol{\theta}}$ is $\mathbb{K}$-linear.[3]

The purpose of the following definition will become clear in the statement of Theorem D below; it is independent of choices due to the uniqueness of minimal regular presentations.

Definition 2.11. The **codimension** $\mathrm{codim}(u) \geqslant 0$ of the closed, connected, d-fold covered J-holomorphic curve $u = v \circ \varphi$ is a nonnegative integer defined as follows. Choose a minimal regular presentation $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$ of φ and a complete list of pairwise non-isomorphic irreducible real representations $\{\boldsymbol{\theta}_i : G \to \mathrm{Aut}_{\mathbb{R}}(W_i)\}_{i=1,\ldots,p}$ of G, whose equivariant endomorphism algebras we denote by

$$\mathbb{K}_i := \mathrm{End}_G(W_i) \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}, \qquad i = 1, \ldots, p.$$

Then

$$\mathrm{codim}(u) := \sum_{i=1}^{p} t_i k_i c_i,$$

where $t_i := \dim_{\mathbb{R}} \mathbb{K}_i \in \{1, 2, 4\}$, $k_i := \dim_{\mathbb{K}_i} \ker \dot{\mathbf{D}}^N_{u,\boldsymbol{\theta}_i}$ and $c_i := \dim_{\mathbb{K}_i} \mathrm{coker}\, \dot{\mathbf{D}}^N_{u,\boldsymbol{\theta}_i}$ for $i = 1, \ldots, p$.

Example 2.12. When $d = 1$, u is a simple curve and its generalized automorphism group G is trivial, so there is only the trivial representation $\boldsymbol{\theta} : G \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{R})$ to consider in Definition 2.11, with $\mathrm{End}_G(\mathbb{R}) = \mathbb{R}$ and $\dot{\mathbf{D}}^N_{u,\boldsymbol{\theta}} \cong \mathbf{D}^N_u$. So in this case, $\mathrm{codim}(u) = \dim(\ker \mathbf{D}^N_u) \cdot \dim(\mathrm{coker}\, \mathbf{D}^N_u)$ can be interpreted as a measurement of the failure of transversality at u, and the standard transversality results imply that all simple curves have codimension 0 for generic J. One of the consequences of Theorem D will be that generically, this is also true for *generic* curves in the space of multiple covers, though not necessarily for all of them.

2.2.3. *Isosymmetric strata.* In order to discuss what happens to the splitting of Cauchy-Riemann operators (2.4) as v and φ move in their respective moduli spaces, we observe that the construction depends quite heavily on the branching structure of $\varphi : \Sigma' \to \Sigma$, i.e. the number of punctures $\Theta' \subset \Sigma'$ and the topological behavior of φ in their vicinity. This necessitates decomposing the space of all degree d branched covers into strata

$$\bigcup_{h \geqslant 0} \mathcal{M}_h(d[\Sigma], j) = \bigcup_{\mathbf{b}} \mathcal{M}^d_{\mathbf{b}}(j)$$

labeled by their so-called *branching data* $\mathbf{b}$. Choose an integer $r \geqslant 0$, and associate to each of the numbers $i = 1, \ldots, r$ a nonempty finite ordered set of natural numbers

$$\mathbf{b}_i = (b_i^1, \ldots, b_i^{q_i})$$

such that

$$b_i^1 + \ldots + b_i^{q_i} = d$$

and at least one of the numbers $b_i^1, \ldots, b_i^{q_i}$ is strictly greater than 1. We denote the totality of this data by $\mathbf{b} = (\mathbf{b}_1, \ldots, \mathbf{b}_r)$ and call it **branching data of degree d with r critical values**.

[3] In cases where $\mathbf{D}^N_u$ is already complex linear with respect to the natural complex structure on N_u, it is important to keep in mind that this natural complex structure has nothing to do with the one induced on $\dot{\mathbf{D}}^N_{u,\boldsymbol{\theta}}$ when $\mathbb{K} = \mathbb{C}$. In fact, these are two distinct complex structures that commute with each other, and $\dot{\mathbf{D}}^N_{u,\boldsymbol{\theta}}$ is then complex linear with respect to both of them.

Given this, let $\widetilde{\mathcal{M}}^d_{\mathbf{b}}(j)$ denote the moduli space of all closed and connected unparametrized j-holomorphic curves φ of degree d mapping into (Σ, j) with $q_1 + \ldots + q_r$ marked points

$$\zeta_1^1, \ldots, \zeta_1^{q_1}, \zeta_2^1, \ldots, \zeta_2^{q_2}, \ldots, \zeta_r^1, \ldots, \zeta_r^{q_r}$$

such that

(1) there are distinct points $w_1, \ldots, w_r \in \Sigma$ such that $\varphi^{-1}(w_i) = \{\zeta_i^1, \ldots, \zeta_i^{q_i}\}$ for each $i = 1, \ldots, r$;
(2) for each $i = 1, \ldots, r$ and $j = 1, \ldots, q_i$, φ is b_i^j-to-1 on a punctured neighborhood of ζ_i^j;
(3) φ has no critical points outside of the marked points.

Note that we do not require *every* marked point of φ to be a critical point, but we are assuming $\{w_1, \ldots, w_r\}$ is the set of critical values, whose preimages are marked points and may include both critical and regular points. For any $\varphi \in \widetilde{\mathcal{M}}_{\mathbf{b}}(j)$, we have

$$Z(d\varphi) = \sum_{i=1}^r \sum_{j=1}^{q_i} (b_i^j - 1),$$

thus d and $\mathbf{b}$ determine the genus h of φ via the Riemann-Hurwitz formula, and we shall denote by

$$\mathcal{M}^d_{\mathbf{b}}(j) \subset \mathcal{M}_h(d[\Sigma], j)$$

the image of the natural map $\widetilde{\mathcal{M}}^d_{\mathbf{b}}(j) \to \mathcal{M}_h(d[\Sigma], j)$ defined by forgetting the marked points. Note that in some cases, the Riemann-Hurwitz calculation may produce a negative genus, which just means that $\mathcal{M}^d_{\mathbf{b}}(j)$ is empty. If $\mathbf{b}$ is empty, i.e. $r = 0$, it means every $\varphi \in \mathcal{M}^d_{\mathbf{b}}(j)$ is unbranched.

It is a classical fact that $\mathcal{M}^d_{\mathbf{b}}(j)$ is a smooth manifold of real dimension $2r$, as it can be parametrized locally by the positions of the critical values $w_1, \ldots, w_r \in \Sigma$ (cf. Example 3.6). Moreover, it depends smoothly on j in the sense that if P is any smooth finite-dimensional family of complex structures on Σ, then

$$\bigcup_{j \in P} \mathcal{M}^d_{\mathbf{b}}(j) \to P$$

defines a smooth fiber bundle. We will show in §3.1 that regular presentations of $\varphi : \Sigma' \to \Sigma$ can also be arranged to vary smoothly as φ varies with fixed branching data.

Constraints must also be imposed on the simple J-holomorphic curve v so that the normal Cauchy-Riemann operators $\mathbf{D}_v^N$ and $\mathbf{D}_u^N$ vary smoothly as v moves in its moduli space. Given integers $m \geqslant 0$ and $\ell_1, \ldots, \ell_m \geqslant 1$, let

$$\mathcal{M}_{g,m}(A, J\,;\, \ell_1, \ldots, \ell_m) \subset \mathcal{M}_{g,m}(A, J)$$

denote the subset consisting of curves that have critical points of critical order ℓ_i at the ith marked point for $i = 1, \ldots, m$ and are immersed everywhere else. As explained in Appendix A, the simple curves in this space form a smooth submanifold for generic J, with codimension $2n \sum_i \ell_i$ in $\mathcal{M}_{g,m}(A, J)$. Moreover, the generalized normal bundles N_v of curves $v \in \mathcal{M}_{g,m}(A, J\,;\, \ell_1, \ldots, \ell_m)$ can be regarded as a smooth family (cf. Lemma 6.4). This is not generally true if v is allowed to move freely in $\mathcal{M}_{g,m}(A, J)$, as the topology of N_v changes when critical points of v appear, disappear or change order.

Given an integer $d \in \mathbb{N}$ and branching data $\mathbf{b}$ of degree d with $r \geqslant 0$ critical values, define

$$\mathcal{M}^d_{\mathbf{b}}(\mathcal{M}_{g,m}(A, J\,;\, \ell_1, \ldots, \ell_m)) \subset \mathcal{M}_h(dA, J)$$

to be the set of all curves admitting representatives of the form $u = v \circ \varphi : (\Sigma', j') \to (M, J)$, where $\varphi : (\Sigma', j') \to (\Sigma, j)$ parametrizes an element in $\mathcal{M}^d_{\mathbf{b}}(j)$ and $v : (\Sigma, j) \to (M, J)$ is a simple curve that intersects $\mathcal{U}$ and (after labeling its critical points as marked points in a suitable order) parametrizes an element of $\mathcal{M}_{g,m}(A, J\,;\,\ell_1, \dots, \ell_m)$. If J is generic on $\mathcal{U}$, then standard results give $\mathcal{M}^d_{\mathbf{b}}(\mathcal{M}_{g,m}(A, J\,;\,\ell_1, \dots, \ell_m))$ the structure of a smooth manifold with

$$\dim \mathcal{M}^d_{\mathbf{b}}(\mathcal{M}_{g,m}(A, J\,;\,\ell_1, \dots, \ell_m)) = 2r + (n-3)(2-2g) + 2c_1(A) - 2\sum_{i=1}^m (n\ell_i - 1).$$

Since every closed connected J-holomorphic curve belongs to such a space for a unique (up to ordering) choice of branching data $\mathbf{b}$ and critical orders $\ell_1, \dots, \ell_m$, these spaces form a smooth stratification of the moduli space of all J-holomorphic curves. They are sometimes called **isosymmetric strata**, as they have the property that all curves in the same connected component of $\mathcal{M}^d_{\mathbf{b}}(\mathcal{M}_{g,m}(A, J\,;\,\ell_1, \dots, \ell_m))$ have isomorphic generalized automorphism groups. More importantly, each isosymmetric stratum admits a smooth family of normal Cauchy-Riemann operators $\mathbf{D}^N_u$ with a smooth family of splittings as in (2.4) with respect to the irreducible representations of their generalized automorphism groups.

2.2.4. *Walls.* Here is the main stratification result.

Theorem D (stratification). *There exists a Baire subset*

$$\mathcal{J}^{\mathrm{reg}} \subset \mathcal{J}(M, \omega\,;\,\mathcal{U}, J_{\mathrm{fix}})$$

*such that the following holds for all $J \in \mathcal{J}^{\mathrm{reg}}$. For all choices of integers $g, m \geqslant 0$, $d, \ell_1, \dots, \ell_m \geqslant 1$, branching data $\mathbf{b}$ of degree d and homology classes $A \in H_2(M)$, the smooth isosymmetric stratum $\mathcal{M}^d_{\mathbf{b}}(\mathcal{M}_{g,m}(A, J\,;\,\ell_1, \dots, \ell_m))$ is a union of countably many pairwise disjoint connected smooth submanifolds, referred to in the following as **walls**, which have the following properties:*

(1) *For $u \in \mathcal{M}^d_{\mathbf{b}}(\mathcal{M}_{g,m}(A, J\,;\,\ell_1, \dots, \ell_m))$, the vector spaces* $\ker \mathbf{D}^N_u$ *and* $\operatorname{coker} \mathbf{D}^N_u$ *form the fibers of smooth vector bundles over each wall;*
(2) *The codimension in $\mathcal{M}^d_{\mathbf{b}}(\mathcal{M}_{g,m}(A, J\,;\,\ell_1, \dots, \ell_m))$ of the wall containing any given curve u is* $\operatorname{codim}(u)$.

Remark 2.13. The statement of Theorem D is specifically geared toward the applications treated in this paper, but for different purposes one could formulate various other versions, e.g. one could add more marked points to $\mathcal{M}_{g,m}(A, J\,;\,\ell_1, \dots, \ell_m)$ and impose intersection constraints on them, or one could consider generic finite-dimensional families $\{J_s\}_{s \in P}$ of almost complex structures and thus replace $\mathcal{M}_{g,m}(A, J\,;\,\ell_1, \dots, \ell_m)$ with a parametric moduli space of pairs (u, s) where $s \in P$ and u is J_s-holomorphic. Either would require no serious modifications to the proof, other than more cumbersome notation (cf. Remark 5.34).

Remark 2.14. A natural guess for the precise definition of the walls mentioned in Theorem D would be that they are maximal connected subsets of $\mathcal{M}^d_{\mathbf{b}}(\mathcal{M}_{g,m}(A, J\,;\,\ell_1, \dots, \ell_m))$ satisfying the constraint that $\dim \ker \mathbf{D}^N_u$ and $\dim \operatorname{coker} \mathbf{D}^N_u$ are constant. In fact, smooth walls can be defined in that way using the methods of [DWb], but the actual definition used in this paper is slightly more complicated: it requires a choice of a smooth family of minimal regular presentations, and the constraint to impose is then that for every finite-dimensional representation $\boldsymbol{\theta}$ of the resulting generalized automorphism group, the kernels and cokernels of the twisted Cauchy-Riemann operators $\dot{\mathbf{D}}^N_{u,\boldsymbol{\theta}}$ should have constant dimension as u varies in the wall. This would give the same result as the simpler definition if one could guarantee that every summand in the splitting (2.4) of $\mathbf{D}^N_u$ appears with positive multiplicity, i.e. that $m_i > 0$ for each of the irreducible representations $\boldsymbol{\theta}_i$, but the latter is not always true. As a consequence, a maximal connected subset on which $\ker \mathbf{D}^N_u$ and $\operatorname{coker} \mathbf{D}^N_u$ have constant dimension may in general contain multiple walls of varying codimensions, distinguished from each other by twisted Cauchy-Riemann operators corresponding to representations that play no role in the splitting of $\mathbf{D}^N_u$. This phenomenon is harmless: the important detail for our purposes is that whenever transversality or super-rigidity fails for a particular curve u, it implies that u belongs to a wall whose codimension is positive and satisfies certain estimates. The converse is neither true nor necessary.

We need two further ingredients in order to turn Theorem D into a powerful enough tool for proving the theorems of §1.1. The first is an index calculation for the twisted operators $\dot{\mathbf{D}}^N_{u,\boldsymbol{\theta}}$. The precise result is stated and proved in §4, but for the main applications we only need the following estimate, which is a corollary:

Lemma 2.15. *Given a J-holomorphic curve $v : (\Sigma, j) \to (M, J)$ with normal Cauchy-Riemann operator $\mathbf{D}^N_v$, a d-fold branched cover $\varphi : (\Sigma', j') \to (\Sigma, j)$ with $r \geqslant 0$ critical values, a regular presentation $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$ for φ and a representation $\boldsymbol{\theta} : G \to \mathrm{Aut}_{\mathbb{R}}(W)$, the resulting twisted Cauchy-Riemann operator $\dot{\mathbf{D}}^N_{u,\boldsymbol{\theta}}$ for $u = v \circ \varphi$ satisfies*

$$\dim W \cdot \left[\mathrm{ind}(\mathbf{D}^N_v) - (n-1)r\right] \leqslant \mathrm{ind}(\dot{\mathbf{D}}^N_{u,\boldsymbol{\theta}}) \leqslant \dim W \cdot \mathrm{ind}(\mathbf{D}^N_v).$$

Moreover, if the regular presentation is minimal and $\boldsymbol{\theta}$ is a faithful irreducible representation with $\mathrm{End}_G(W) \cong \mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, then the second estimate can be improved to

$$\mathrm{ind}_{\mathbb{K}}(\dot{\mathbf{D}}^N_{u,\boldsymbol{\theta}}) \leqslant \dim_{\mathbb{K}} W \cdot \mathrm{ind}_{\mathbb{R}}(\mathbf{D}^N_v) - (n-1)r,$$

and this estimate is strict in the case $\mathbb{K} = \mathbb{R}$ unless all branch points of φ have branching order 2.

For the proof of super-rigidity, we will need the next result as a means of improving the upper bound in Lemma 2.15 for representations that are not faithful.

Lemma 2.16 (see §3.4.3)**.** *Under the assumptions of Lemma 2.15, suppose the regular presentation is minimal, and the splitting (2.4) of $\mathbf{D}^N_u$ includes a summand $\dot{\mathbf{D}}^N_{u,\boldsymbol{\theta}}$ for which the representation $\boldsymbol{\theta} : G \to \mathrm{Aut}_{\mathbb{R}}(W)$ is not faithful. Then $\varphi : (\Sigma', j') \to (\Sigma, j)$ admits a factorization by holomorphic branched covers*

$$(\Sigma', j') \to (\Sigma'_0, j'_0) \xrightarrow{\varphi_0} (\Sigma, j)$$

with $\deg(\varphi_0) < d$, and $\dot{\mathbf{D}}^N_{u,\boldsymbol{\theta}}$ is conjugate to an operator $\dot{\mathbf{D}}^N_{u_0,\boldsymbol{\theta}_0}$ defined with respect to a regular presentation $(\Theta, \dot{\Sigma}''_0, \pi_0, G_0, \rho_0, I_0, f_0)$ for φ_0, where $u_0 := v \circ \varphi_0 : (\Sigma'_0, j'_0) \to (M, J)$, $G_0 := G/\ker \boldsymbol{\theta}$, and

$$\boldsymbol{\theta}_0 : G/\ker \boldsymbol{\theta} \to \mathrm{Aut}_{\mathbb{R}}(W)$$

is the faithful representation of G_0 determined by $\boldsymbol{\theta}$. Moreover, $\mathbf{D}^N_{u_0}$ also admits a splitting in the form (2.4) which has $\dot{\mathbf{D}}^N_{u_0,\boldsymbol{\theta}_0}$ as a summand.

2.3. **Proof of the main theorems modulo stratification.** Let us now take the results of the previous section as black boxes and prove the main theorems from §1.1.

Proof of Theorem A (super-rigidity) in dimension greater than four. We argue by induction on the degrees $d \in \mathbb{N}$ of branched covers. For $d = 1$, we only need to know that generic perturbations of J suffice to make all simple index 0 curves through $\mathcal{U}$ regular and immersed; this is standard (see Appendix A for the immersion property). Thus for $d \geqslant 2$, assume we have already found a Baire subset in $\mathcal{J}(M, \omega\,; \mathcal{U}, J_{\mathrm{fix}})$ for which all branched covers $u := v \circ \varphi$ with $v : (\Sigma, j) \to (M, J)$ a simple curve of index 0 and $\deg(\varphi) \leqslant d - 1$ have $\mathbf{D}^N_u$ injective. Suppose $\varphi \in \mathcal{M}^d_{\mathbf{b}}(j)$ has $r \geqslant 0$ critical values and $\deg(\varphi) = d$ and $\mathbf{D}^N_u$ is not injective for $u := v \circ \varphi$. Then picking the minimal regular presentation $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$ for φ and decomposing $\boldsymbol{\rho}$ into irreducible representations $\boldsymbol{\theta}_1^{\oplus \ell_1} \oplus \ldots \oplus \boldsymbol{\theta}_p^{\oplus \ell_p}$ of G splits $\mathbf{D}^N_u$ into twisted Cauchy-Riemann operators $\dot{\mathbf{D}}^N_{u,\boldsymbol{\theta}_i}$ for $i = 1, \ldots, p$ with

$$k_i := \dim_{\mathbb{K}_i} \ker \dot{\mathbf{D}}^N_{u,\boldsymbol{\theta}_i},$$

and at least one of the k_i must be strictly positive by Lemma 2.9. If $k_i > 0$ and $\boldsymbol{\theta}_i$ is non-faithful, then Lemma 2.16 identifies $\dot{\mathbf{D}}^N_{u,\boldsymbol{\theta}_i}$ with a summand of $\mathbf{D}^N_{u_0}$ for some other cover u_0 of v with strictly smaller degree, implying $\dim \ker \mathbf{D}^N_{u_0} > 0$ and thus violating the inductive hypothesis. We can therefore assume $k_i > 0$ for some faithful representation $\boldsymbol{\theta}_i$. But then Theorem D and

Lemma 2.15 imply that u lives in a submanifold of the $2r$-dimensional space of branched covers of v with branching data $\mathbf{b}$, having dimension at most

$$2r - t_i k_i \left[k_i - \operatorname{ind}_{\mathbb{K}_i}(\dot{\mathbf{D}}^N_{u,\boldsymbol{\theta}_i})\right] \leqslant 2r - t_i k_i[k_i + (n-1)r] = r[2 - t_i k_i(n-1)] - t_i k_i^2 < 0$$

since we are assuming $n \geqslant 3$. This gives a contradiction and thus completes the induction. □

In dimension four, the above argument fails to exclude the possibility of $\dim \ker \dot{\mathbf{D}}^N_{u,\boldsymbol{\theta}_i} = 1$ for some real-type repesentation $\boldsymbol{\theta}_i$, and this is why we do not know whether super-rigidity always holds in dimension four. We will prove in §7 that it does hold for covers of genus zero and one curves, using different techniques based on intersection theory.

Proof of Theorem B (transversality, unbranched). Suppose $v : (\Sigma, j) \to (M, J)$ is a simple curve intersecting $\mathcal{U}$ and $\varphi : (\Sigma', j') \to (\Sigma, j)$ is a d-fold unbranched cover for which $u := v \circ \varphi$ is not Fredholm regular, hence by Prop. 2.2, $\mathbf{D}^N_u$ is not surjective. Fixing the minimal regular presentation of φ and considering the splitting (2.4), we find a twisted Cauchy-Riemann operator $\dot{\mathbf{D}}^N_{u,\boldsymbol{\theta}_i}$ with

$$c_i := \dim_{\mathbb{K}_i} \operatorname{coker} \dot{\mathbf{D}}^N_{u,\boldsymbol{\theta}_i} > 0$$

for some irreducible representation $\boldsymbol{\theta}_i : G \to \operatorname{Aut}_{\mathbb{R}}(W_i)$ of the generalized automorphism group G of φ, with $\operatorname{End}_G(W_i) \cong \mathbb{K}_i \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Suppose v has exactly $m \geqslant 0$ critical points, with critical orders $\ell_1, \ldots, \ell_m$, so viewing these as marked points allows us to consider v as an element in the space $\mathcal{M}_{g,m}(A, J\,;\, \ell_1, \ldots, \ell_m)$, which has dimension

$$\dim \mathcal{M}_{g,m}(A, J\,;\, \ell_1, \ldots, \ell_m) = \operatorname{ind}(v) + 2m - 2nZ(dv) \geqslant 0.$$

The count of critical points $Z(dv)$ also appears in the relation between $\operatorname{ind}(v)$ and $\operatorname{ind} \mathbf{D}^N_v$: indeed, writing $v^*TM = T_v \oplus N_v$, we can view dv as a holomorphic section of $\operatorname{Hom}_{\mathbb{C}}(T\Sigma, T_v)$, hence

$$Z(dv) = c_1\big(\operatorname{Hom}_{\mathbb{C}}(T\Sigma, T_v)\big) = -c_1(T\Sigma) + c_1(T_v) = -\chi(\Sigma) + c_1(T_v),$$

implying $c_1(N_v) = c_1(v^*TM) - c_1(T_v) = c_1(v^*TM) - \chi(\Sigma) - Z(dv)$. Plugging in this into the Riemann-Roch formula then gives

$$\begin{aligned} \operatorname{ind} \mathbf{D}^N_v &= (n-1)\chi(\Sigma) + 2c_1(N_v) = (n-3)\chi(\Sigma) + 2c_1(v^*TM) - 2Z(dv) \\ &= \operatorname{ind}(v) - 2Z(dv). \end{aligned}$$

Meanwhile, φ lives in a discrete stratum of the space of branched covers since it has no branch points, and Lemma 2.15 reduces to an equality

$$\operatorname{ind}_{\mathbb{K}_i} \dot{\mathbf{D}}^N_{u,\boldsymbol{\theta}_i} = \dim_{\mathbb{K}_i} W_i \cdot \operatorname{ind}_{\mathbb{R}}(\mathbf{D}^N_v).$$

Now using Theorem D, we find that if J is generic, u lives in a manifold of dimension at most

$$\begin{aligned} &\dim \mathcal{M}_{g,m}(A, J\,;\, \ell_1, \ldots, \ell_m) - t_i c_i(c_i + \operatorname{ind}_{\mathbb{K}_i} \dot{\mathbf{D}}^N_{u,\boldsymbol{\theta}_i}) \\ &\qquad = \operatorname{ind}(v) + 2m - 2nZ(dv) - t_i c_i(c_i + \dim_{\mathbb{K}_i} W_i \cdot \operatorname{ind} \mathbf{D}^N_v) \\ &\qquad = \operatorname{ind}(v) + 2m - 2nZ(dv) - t_i c_i(c_i + \dim_{\mathbb{K}_i} W_i \cdot [\operatorname{ind}(v) - 2Z(dv)]) \\ &\qquad = (1 - t_i c_i \dim_{\mathbb{K}_i} W_i)\,[\operatorname{ind}(v) + 2m - 2nZ(dv)] \\ &\qquad\qquad - 2t_i c_i \dim_{\mathbb{K}_i} W_i \cdot [(n-1)Z(dv) - m] - t_i c_i^2 < 0, \end{aligned}$$

where we note that $(n-1)Z(dv) - m \geqslant 0$ since $n \geqslant 2$ and every critical point has order at least 1. □

Proof of Theorem C (transversality, branched). Assume $v : (\Sigma, j) \to (M, J)$ is simple and satisfies $\operatorname{ind}(v) \geqslant (n-1)r$, while $\varphi : (\Sigma', j') \to (\Sigma, j)$ has degree $d \in \mathbb{N}$ and r critical values. If J is generic, then by Proposition A.1 the moduli space containing v has an open and dense subset consisting of immersed curves, so we are free to assume v is immersed and thus $\operatorname{ind}(v) = \operatorname{ind} \mathbf{D}^N_v$. The key observation is then that by Lemma 2.15, the twisted operators $\dot{\mathbf{D}}^N_{u,\boldsymbol{\theta}}$ all have nonnegative index, hence Theorem D implies that all of them are surjective unless (v, φ) lies in a countable union of submanifolds with positive codimension. □

2.4. **Some remarks on wall crossing.** Part of the point of Taubes's twisted bundle setup in [Tau96a] was to understand bifurcations of isolated J-holomorphic tori under generic 1-parameter deformations in J. While bifurcation theory is not the main topic of this article, it should be clear that such a theory could be developed based on Theorem D, thus we take this opportunity to make a few observations about it.

Remark 2.17. In the time since the present article first appeared in preprint form, some interesting cases of the bifurcation analysis proposed below have been worked out in detail by Bai and Swaminathan, see [BS].

If $\{J_s\}_{s\in[0,1]}$ is a generic homotopy of compatible almost complex structures whose endpoints are generic, then as mentioned in Remark 2.13, one can modify Theorem D to the statement that the parametric moduli space

$$\mathcal{M}^d_{\mathbf{b}}(\mathcal{M}_{g,m}(A, \{J_s\}\,;\, \ell_1, \dots, \ell_m))$$

consisting of pairs (u, s) where $s \in [0, 1]$ and $u \in \mathcal{M}^d_{\mathbf{b}}(\mathcal{M}_{g,m}(A, J_s\,;\, \ell_1, \dots, \ell_m))$ is stratified by smooth submanifolds characterized by the dimensions of the kernels and cokernels of twisted Cauchy-Riemann operators, and their codimensions are given by the same formula. In this setting, suppose $\{v_\tau\}$ is a smooth 1-parameter family of simple $J_{s(\tau)}$-holomorphic curves with index 0 for some function $s(\tau) \in [0, 1]$, and $\{u_\tau = v_\tau \circ \varphi_\tau\}$ defines a corresponding 1-parameter family of unbranched covers. The latter have index 0 and will be regular for almost every τ, but a bifurcation or "wall crossing" phenomenon occurs at any parameter value τ_0 for which the family $\{u_\tau\}$ passes (necessarily transversely) through one of the codimension 1 walls given by Theorem D. When this happens, most of the twisted operators $\dot{\mathbf{D}}^N_{u_{\tau_0},\boldsymbol{\theta}}$ remain both injective and surjective, but there will be exactly one irreducible representation $\boldsymbol{\theta}$ for which

$$\dim \ker \dot{\mathbf{D}}^N_{u_{\tau_0},\boldsymbol{\theta}} = \dim \operatorname{coker} \dot{\mathbf{D}}^N_{u_{\tau_0},\boldsymbol{\theta}} = 1,$$

and $\boldsymbol{\theta}$ is necessarily of real type. Whenever $\boldsymbol{\theta}$ is not faithful, one can factor φ_τ through a cover $\widehat{\varphi}_\tau$ of smaller degree and instead examine $\widehat{u}_\tau := v_\tau \circ \widehat{\varphi}_\tau$, so that $\boldsymbol{\theta}$ becomes faithful without loss of generality (cf. Lemma 2.16). For the trivial representation, this means replacing u_τ with v_τ itself, so regularity fails for the underlying simple curve at $\tau = \tau_0$: as shown in [Tau96a], this is the case where the family $\{v_\tau\}$ undergoes a *birth-death* bifurcation. The other interesting phenomenon examined by Taubes was the *degree-doubling* bifurcation, in which v_τ remains regular but it has a double cover $u_\tau = v_\tau \circ \varphi_\tau$ which loses regularity at $\tau = \tau_0$, causing an additional 1-parameter family of simple curves $\{w_\tau\}$ to collide with $\{u_\tau\}$ at $\tau = \tau_0$. This is what happens when $\dot{\mathbf{D}}^N_{u_\tau,\boldsymbol{\theta}}$ remains an isomorphism for the trivial representation but acquires 1-dimensional kernel and cokernel for the nontrivial irreducible representation of $\mathbb{Z}_2$.

In [Tau96a], no further bifurcations beyond these two types are possible: this can be attributed to the fact that since Taubes only considers unbranched covers of tori, all covers are regular and abelian. As a consequence, all the complex irreducible representations in the picture are 1-dimensional, implying that the only faithful real-type irreducible representations one needs to consider are the trivial representation of the trivial group and the nontrivial representation of $\mathbb{Z}_2$. We should not expect this fortunate situation to hold more generally: for unbranched covers with higher genus, one certainly encounters generalized automorphism groups that are non-abelian and thus have faithful real-type representations of dimension greater than one. These should presumably give rise to bifurcation phenomena involving covers of arbitrarily high degree.

In the context of super-rigidity, it is also important to consider bifurcations that involve branched covers of index 0 curves under generic homotopies of J. Inspecting the proof of Theorem A, one should expect to see interesting phenomena whenever the dimension that was estimated at the end of the proof turns out to be at least -1, i.e.

$$2r - t_i k_i \left[k_i - \operatorname{ind}_{\mathbb{K}_i}(\dot{\mathbf{D}}^N_{u,\boldsymbol{\theta}_i}) \right] \geqslant -1.$$

Assuming we're in dimension at least six, this can only mean $t_i = k_i = 1$ and either $r = 0$ or $n = 3$. The case $r = 0$ means the cover is unbranched, so this is what we discussed in the previous paragraphs. Bifurcations involving branched covers can evidently also occur in dimension six, and in this case the improved index bound from Lemma 2.15 must be an equality. The scenario is therefore that the rank of the obstruction bundle over the space of covers $\{v_\tau \circ \varphi_\tau\}$ jumps at a particular parameter value $\tau = \tau_0$ and for some isolated element φ_{τ_0} in the space of branched covers with only simple (i.e. two-to-one) branch points: this can presumably cause both a change in the Euler class of the obstruction bundle and the breaking off of a new family of simple curves from $v_{\tau_0} \circ \varphi_{\tau_0}$. Once again the irreducible representation involved must be of real type but can have arbitrary dimension, meaning we should not expect any limitation on the degree of φ_{τ_0}, contrary to the situation in [Tau96a].

3. Splitting Cauchy-Riemann operators with symmetries

In this section we give a detailed account of the twisted bundle formalism behind Theorem D and prove several lemmas required for its proof, as well as Lemma 2.16. Instead of talking directly about J-holomorphic curves, we shall work in the context of abstract Cauchy-Riemann operators on vector bundles and their pullbacks.

3.1. **Regular presentations of branched covers.** The notion of a regular presentation was introduced in Definition 2.8. The following standard result from the theory of Riemann surfaces (see e.g. [Don11, Chapter 4, Theorem 2]) allows us to move freely back and forth between talking about holomorphic branched covers of closed Riemann surfaces and honest covering maps of punctured surfaces.

Lemma 3.1. *Suppose $(\dot{\Sigma}, j)$ is the complement of a finite set of points Θ in a closed connected Riemann surface (Σ, j), $(\dot{\Sigma}', j')$ is a connected noncompact Riemann surface, and*

$$\varphi : (\dot{\Sigma}', j') \to (\dot{\Sigma}, j)$$

is a holomorphic covering map of finite degree. Then there exists a closed connected Riemann surface (Σ', j') with a finite set of points $\Theta' \subset \Sigma'$ such that $(\dot{\Sigma}', j')$ admits a biholomorphic identification with $(\Sigma' \backslash \Theta', j')$ and φ extends over the punctures to a holomorphic branched cover $\varphi : (\Sigma', j') \to (\Sigma, j)$ with $\varphi^{-1}(\Theta) = \Theta'$. □

Assume $\varphi : (\Sigma', j') \to (\Sigma, j)$ is a d-fold holomorphic branched cover of closed connected Riemann surfaces with branching data $\mathbf{b}$ as defined in §2.2, having $r \geqslant 0$ distinct critical values. Recall from Definition 2.8 that for a regular presentation $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$ of φ, $\Theta \subset \Sigma$ is a finite set containing the critical values of φ, giving rise to the punctured surfaces

$$\dot{\Sigma} := \Sigma \backslash \Theta, \qquad \dot{\Sigma}' := \Sigma' \backslash \Theta',$$

where $\Theta' := \varphi^{-1}(\Theta)$.

Lemma 3.2. *There exists a natural bijection between the set of isomorphism classes of regular presentations of φ and the set of pairs (Θ, H) where $\Theta \subset \Sigma$ is a finite subset containing the critical values of φ and H is a finite-index normal subgroup $H \subset \pi_1(\dot{\Sigma})$ that is contained in $\varphi_*(\pi_1(\dot{\Sigma}'))$. This bijection matches any minimal regular presentation to the smallest possible choice of Θ and largest possible choice of H, i.e. the normal core of $\varphi_*(\pi_1(\dot{\Sigma}'))$. Moreover, if φ is regular and $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$ is a minimal regular presentation, then there exists a diffeomorphism $g : \dot{\Sigma}' \to \dot{\Sigma}''$ such that $\pi \circ g = \varphi$.*

Proof. Given a finite set $\Theta \subset \Sigma$ containing the critical values of φ, pick a base point $w \in \dot{\Sigma}$ and let $\tilde{\pi} : \mathscr{U} \to \dot{\Sigma}$ denote the universal cover, with $\mathscr{U}$ defined as a space of homotopy classes of paths beginning at w, so that $\pi_1(\dot{\Sigma}) := \pi_1(\dot{\Sigma}, w)$ acts naturally on $\mathscr{U}$ as the group of deck transformations for $\tilde{\pi}$. Lifting loops based at w to paths in $\dot{\Sigma}'$ then defines a homomorphism

$$\tilde{\rho} : \pi_1(\dot{\Sigma}) \to S(\varphi^{-1}(w)) : \gamma \mapsto \tilde{\rho}_\gamma$$

so that the covering map $\dot{\Sigma}' \xrightarrow{\varphi} \dot{\Sigma}$ can be identified with

$$\dot{\Sigma}' = \left(\mathscr{U} \times \varphi^{-1}(w)\right) \Big/ \pi_1(\dot{\Sigma}) \to \dot{\Sigma} : [(z, \zeta)] \mapsto \tilde{\pi}(z),$$

where $\pi_1(\dot{\Sigma})$ acts on $\mathscr{U}$ by deck transformations and on $\varphi^{-1}(w)$ via $\tilde{\rho}$. We claim that

$$\ker \tilde{\rho} \subset \pi_1(\dot{\Sigma})$$

is the normal core of $\varphi_*(\pi_1(\dot{\Sigma}'))$. Indeed, selecting a base point $w' \in \varphi^{-1}(w) \subset \dot{\Sigma}'$ to define $\pi_1(\dot{\Sigma}') := \pi_1(\dot{\Sigma}', w')$, we have

$$\varphi_*(\pi_1(\dot{\Sigma}')) = \left\{ \gamma \in \pi_1(\dot{\Sigma}) \;\middle|\; \tilde{\rho}_\gamma(w') = w' \right\},$$

which obviously contains $\ker \tilde{\rho}$. Changing the base point $w' \in \varphi^{-1}(w)$ changes the subgroup $\varphi_*(\pi_1(\dot{\Sigma}'))$ by conjugation with arbitrary elements of $\pi_1(\dot{\Sigma}')$, and the normal core is the intersection of all these conjugates, which we can now recognize as the intersection of all the stabilizers of the permutation action on $\varphi^{-1}(w)$, and that is $\ker \tilde{\rho}$.

Suppose $H \subset \pi_1(\dot{\Sigma})$ is a finite-index normal subgroup contained in $\varphi_*(\pi_1(\dot{\Sigma}'))$, and therefore also in $\ker \tilde{\rho}$. Then $\tilde{\rho}$ descends to the finite group $G := \pi_1(\dot{\Sigma})/H$, giving a homomorphism

$$\rho : G \to S(\varphi^{-1}(w)),$$

which is injective if and only if $H = \ker \tilde{\rho}$. It is now possible to define a regular presentation $(\Theta, \dot{\Sigma}'', \pi, G, \rho, \varphi^{-1}(w), f)$ of φ with π as the natural quotient projection

$$\dot{\Sigma}'' := \mathscr{U}/H \xrightarrow{\pi} \mathscr{U}/\pi_1(\dot{\Sigma}) = \dot{\Sigma}$$

and

$$\dot{\Sigma}' = \left(\mathscr{U} \times \varphi^{-1}(w)\right) \Big/ \pi_1(\dot{\Sigma}) \xrightarrow{f} \left(\dot{\Sigma}'' \times \varphi^{-1}(w)\right) \Big/ G$$

defined via the quotient projection $\mathscr{U} \to \mathscr{U}/H = \dot{\Sigma}''$. Observe that if we choose $H = \ker \tilde{\rho}$ and φ is regular, then $\varphi_*(\pi_1(\dot{\Sigma}')) \subset \pi_1(\dot{\Sigma})$ is normal and is therefore identical to H, so the natural identification of $\dot{\Sigma}'$ with $\mathscr{U}/\varphi_*(\pi_1(\dot{\Sigma}')) = \mathscr{U}/H = \dot{\Sigma}''$ gives an isomorphism between the covering maps φ and π.

Finally, suppose $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$ is a regular presentation of φ, and define the subgroup $H := \pi_*(\pi_1(\dot{\Sigma}''))$, which is normal since $\pi : \dot{\Sigma}'' \to \dot{\Sigma}$ is regular and has finite index since $\mathrm{Aut}(\pi) = G = \pi_1(\dot{\Sigma})/H$ is finite. We claim $H \subset \varphi_*(\pi_1(\dot{\Sigma}'))$: indeed, any $\gamma \in H$ is represented by a loop $\dot{\Sigma}$ based at w that lifts to a loop γ'' in $\dot{\Sigma}''$ and thus has d lifts to $\dot{\Sigma}' \cong (\dot{\Sigma}'' \times I)/G$ in the form $\gamma \times \{i\}$ for $i \in I$. We can therefore use H to define the regular presentation from the previous paragraph, with $G = \pi_1(\dot{\Sigma}'')/H$ acting on $\varphi^{-1}(w)$ via $\tilde{\rho}$, and we claim that this is isomorphic to $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$. Indeed, choosing a base point $w'' \in \pi^{-1}(w) \subset \dot{\Sigma}''$, the identification $f : \dot{\Sigma}' \to (\dot{\Sigma}'' \times I)/G$ provides a bijection

$$\beta : \varphi^{-1}(w) \to I \quad \text{such that} \quad f(w') = [(w'', \beta(w'))] \text{ for } w' \in \varphi^{-1}(w),$$

and combining this with the natural identification of $\dot{\Sigma}''$ with $\mathscr{U}/H$ gives an isomorphism of regular presentations. $\square$

Lemma 3.3. *Suppose $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$ is a minimal regular presentation of $\varphi : (\Sigma', j') \to (\Sigma, j)$, and let $\pi : (\Sigma'', j'') \to (\Sigma, j)$ denote the branched cover of closed Riemann surfaces provided by Lemma 3.1 such that $\dot{\Sigma}'' = \Sigma'' \backslash \pi^{-1}(\Theta)$. Then for each $w \in \Theta$ and $\zeta \in \pi^{-1}(w) \subset \Sigma''$,* *the branching order of π at ζ is the least common multiple of the branching orders of φ at all $z \in \varphi^{-1}(w)$. In particular, π and φ have the same sets of critical values.*

Proof. If $k \in \mathbb{N}$ is the branching order of π at ζ, we can find punctured neighborhoods $\mathcal{U}_w \subset \dot{\Sigma}$ of w and $\mathcal{U}_\zeta \subset \dot{\Sigma}''$ of ζ and identify both with the half-cylinder $[0,\infty) \times S^1$ with coordinates (s,t) such that $\pi(s,t) = (ks, kt)$. Let $G_\zeta \subset G$ denote the group of automorphisms of π that fix ζ; since $\pi : \dot{\Sigma}'' \to \dot{\Sigma}$ is a regular cover, G_ζ is necessarily a cyclic group of order k, with a generator $g \in G_\zeta$ that acts on $\mathcal{U}_\zeta \cong [0,\infty) \times S^1$ as the rotation $(s,t) \mapsto (s, t + 1/k)$. Appealing again to regularity, we can then restrict the identification $\dot{\Sigma}' = (\dot{\Sigma}'' \times I)/G$ to $\mathcal{U}_\zeta$ and obtain an identification

$$\varphi^{-1}(\mathcal{U}_w) = (\mathcal{U}_\zeta \times I) \Big/ G_\zeta.$$

The connected components of $\varphi^{-1}(\mathcal{U}_w)$ are then in bijective correspondence to the orbits of the G_ζ-action on I defined by $\rho : G \to S(I)$, with the branching order $k_z \in \mathbb{N}$ of each corresponding point $z \in \varphi^{-1}(w)$ given by the number of points in its respective orbit in I. By the orbit-stabilizer theorem, all of these numbers k_z must divide $k = |G_\zeta|$. If ℓ is their least common multiple, we conclude that $g^\ell \in G_\zeta$ acts trivially on I, which means g^ℓ is the identity since $\rho : G \to S(I)$ is injective for the minimal regular presentation, hence $\ell = k$. □

Example 3.4. If φ is regular with $\mathrm{Aut}(\varphi) = G$, then it admits a canonical minimal regular presentation $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$ where $\dot{\Sigma}'' := \dot{\Sigma}'$, $\pi := \varphi$, $I := G$, and the action $\rho : G \to S(G)$ of G on itself is defined by left multiplication

$$\rho_g(h) := gh.$$

Here the identification $\dot{\Sigma}' \xrightarrow{f} (\dot{\Sigma}'' \times G)/G$ sends $z \in \dot{\Sigma}'$ to $[(z,e)]$, where $e \in G$ is the identity element. The action of G on $\dot{\Sigma}' = (\dot{\Sigma}'' \times G)/G$ by deck transformations can now be presented as the action via right multiplication

$$G \times \dot{\Sigma}' \to \dot{\Sigma}' : (g, [(z,h)]) \mapsto [(z, hg^{-1})].$$

Notice that any regular presentation in which $\rho : G \to S(I)$ acts on I both transitively and without fixed points is isomorphic to one of this form, since for any $i \in I$, the map $G \to I : g \mapsto \rho_g(i)$ defines a bijection that transforms the action by left multiplication into ρ.

Example 3.5. The following construction underlies Lemma 2.16: any proper normal subgroup $H \subset G$ gives rise to a factorization of $\varphi : (\Sigma', j') \to (\Sigma, j)$ in the following way. Let I/H denote the set of orbits for the action $\rho|_H : H \to S(I)$. Then G/H is a finite group and ρ descends to a homomorphism

$$\rho_H : G/H \to S(I/H),$$

which acts transitively on I/H. The regular cover $\pi : \dot{\Sigma}'' \to \dot{\Sigma} = \dot{\Sigma}''/G$ now factors through the obvious projections

$$\dot{\Sigma}'' \to \dot{\Sigma}''_H := \dot{\Sigma}''/H \xrightarrow{\pi_H} \dot{\Sigma} = \dot{\Sigma}''/G,$$

and $\pi_H : \dot{\Sigma}''_H \to \dot{\Sigma}$ is a regular holomorphic cover with automorphism group G/H. We can thus define

$$\dot{\Sigma}'_H := \left(\dot{\Sigma}''_H \times (I/H)\right) \Big/ (G/H) \xrightarrow{\varphi_H} \dot{\Sigma} : [(z,i)] \mapsto \pi_H(z),$$

as well as a factorization of $\varphi : \dot{\Sigma}' \to \dot{\Sigma}$ by covering maps

$$\dot{\Sigma}' = \left(\dot{\Sigma}'' \times I\right) \Big/ G \longrightarrow \dot{\Sigma}'_H \xrightarrow{\varphi_H} \dot{\Sigma},$$

where the first map is also defined via the obvious quotient projections. It follows from Lemma 3.1 that $\dot{\Sigma}'_H$ and $\dot{\Sigma}''_H$ each arise by puncturing closed connected Riemann surfaces (Σ'_H, j'_H) and (Σ''_H, j''_H) respectively, and in particular we obtain a factorization of φ via holomorphic branched covers

$$(\Sigma', j') \to (\Sigma'_H, j'_H) \xrightarrow{\varphi_H} (\Sigma, j)$$

with $\deg(\varphi_H) \leqslant d$ equal to the number of distinct orbits of the H-action on I, hence

$$\deg(\varphi_H) < d$$

holds whenever the action of H on I is nontrivial. Note that φ_H inherits from this construction a regular presentation $(\Theta, \dot{\Sigma}''_H, \pi_H, G/H, \rho_H, I/H, f_H)$, though it need not be minimal and Θ may contain points that are not critical values of φ_H, even if $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$ is minimal. This is the main reason why non-minimal regular presentations have been included in the discussion.

It will be important to understand how the various objects constructed out of a regular presentation vary smoothly under changes in φ and j. To this end, we shall fix the following data for the remainder of §3:

- $\varphi : (\Sigma', j') \to (\Sigma, j)$ is a holomorphic branched cover of degree $d \in \mathbb{N}$ with branching data $\mathbf{b}$;
- $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$ is a regular presentation of φ;
- P is a connected smooth Banach manifold;
- $\mathcal{V} \subset \dot{\Sigma}$ is an open subset with compact closure;
- $\{j_\tau\}_{\tau \in P}$ is a smooth family of complex structures on Σ that match j outside of $\mathcal{V}$;
- $\{\psi_\tau\}_{\tau \in P}$ is a smooth family of diffeomorphisms $\psi_\tau : \Sigma \to \Sigma$ which restrict to the identity on $\mathcal{V}$ and are j-holomorphic near Θ.

We shall abbreviate the family of closed Riemann surfaces determined by j_τ as

$$\Sigma_\tau := (\Sigma, j_\tau),$$

and denote by

$$\pi : (\Sigma'', j'') \to (\Sigma, j), \qquad \Theta'' = \pi^{-1}(\Theta) \subset \Sigma''$$

the holomorphic branched cover of closed surfaces provided by Lemma 3.1 such that $\dot{\Sigma}'' = \Sigma'' \backslash \Theta''$. These choices produce a family of punctured Riemann surfaces

$$\dot{\Sigma}_\tau := (\Sigma \backslash \Theta_\tau, j_\tau) \quad \text{where} \quad \Theta_\tau := \psi_\tau(\Theta) \subset \Sigma,$$

and we define

$$\varphi_\tau := \psi_\tau \circ \varphi : \Sigma' \to \Sigma, \qquad j'_\tau := \varphi_\tau^* j_\tau \text{ on } \Sigma',$$

where we observe that j'_τ is always well defined and matches j' near Θ' since ψ_τ is holomorphic near Θ. This makes

$$\varphi_\tau : \Sigma'_\tau \to \Sigma_\tau$$

a smooth family of holomorphic branched covers, where

$$\Sigma'_\tau := (\Sigma', j'_\tau),$$

and they restrict to holomorphic covering maps of punctured surfaces $\dot{\Sigma}'_\tau \xrightarrow{\varphi} \dot{\Sigma}_\tau$, where

$$\dot{\Sigma}'_\tau := (\dot{\Sigma}', j'_\tau).$$

Example 3.6. Suppose Θ is the set of critical values of φ, $r := |\Theta|$, P is the $2r$-dimensional open ball B^{2r}, $j_\tau := j$ for all τ, and $\psi_\tau : \Sigma \to \Sigma$ is chosen to be any smooth family of diffeomorphisms supported near Θ that are holomorphic in a smaller neighborhood of Θ and such that $\psi_0 = \mathrm{Id}$ and

$$B^{2r} \to \Sigma^{\times r} : \tau \mapsto (\psi_\tau(w_1), \dots, \psi_\tau(w_r))$$

is an embedding onto an open subset, where $\Theta = \{w_1, \dots, w_r\}$. Then the branched covers $\varphi_\tau : (\Sigma', j'_\tau) \to (\Sigma, j)$ parametrize a neighborhood of φ in $\mathcal{M}^d_{\mathbf{b}}(j)$.

Example 3.7. If $v_0 : (\Sigma, j_0) \to (M, J_0)$ represents a simple element of the moduli space $\mathcal{M}_{g,m}(A, J_0\,; \ell_1, \dots, \ell_m)$ defined in Appendix A and J_0 is generic, then one can enhance the previous example as follows to parametrize a neighborhood of $u_0 := v_0 \circ \varphi$ in the space $\mathcal{M}^d_{\mathbf{b}}(\mathcal{M}_{g,m}(A, J_0\,; \ell_1, \dots, \ell_m))$. A neighborhood of v_0 in $\mathcal{M}_{g,m}(A, J_0\,; \ell_1, \dots, \ell_m)$ can be identified with a smooth submanifold X of $\bar{\partial}_{J_0}^{-1}(0)$, where $\bar{\partial}_{J_0} : \mathcal{T} \times \mathcal{B} \to \mathcal{E}$ is the nonlinear Cauchy-Riemann operator defined on the product of $\mathcal{B} := W^{k,p}(\Sigma, M)$ with a Teichmüller slice $\mathcal{T}$ through j_0, cf. Appendix A. Here $\mathcal{T}$ is a finite-dimensional smooth family of complex structures on Σ, which can all be arranged to match j_0 near Θ. A neighborhood in $\mathcal{M}^d_{\mathbf{b}}(\mathcal{M}_{g,m}(A, J_0\,;\,\ell_1, \ldots, \ell_m))$ is now parametrized by

$$P := B^{2r} \times X,$$

namely via the curves $v \circ (\psi_\sigma \circ \varphi) : (\Sigma', \varphi^*\psi_\sigma^* j) \to (M, J_0)$ for each $\tau := (\sigma, (j, v)) \in P$, and we associate to these parameters the families $j_\tau := j$ and $\psi_\tau := \psi_\sigma$.

Example 3.8. Enhancing the previous example one step further, suppose $\mathcal{J}_\varepsilon$ is an infinite-dimensional Banach manifold consisting of smooth almost complex structures and we consider a neighborhood of (v_0, J_0) in the universal moduli space

$$\mathscr{U}^*(\mathcal{J}_\varepsilon\,;\,\ell_1, \ldots, \ell_m) = \{(v, J) \mid J \in \mathcal{J}_\varepsilon,\ v \in \mathcal{M}_{g,m}(A, J\,;\,\ell_1, \ldots, \ell_m)\}\,.$$

Such a neighborhood can be identified with a finite-codimensional submanifold X in the infinite-dimensional Banach manifold $\bar\partial^{-1}(0) \subset \mathcal{T} \times \mathcal{B} \times \mathcal{J}_\varepsilon$, where $\bar\partial(j, u, J) := \bar\partial_J(j, u)$. Defining $P := B^{2r} \times X$ and the families $\{j_\tau\}$ and $\{\psi_\tau\}$ as in Example 3.7, the parameter space P is now infinite dimensional.

Observe that the branched covers in the family φ_τ all have essentially the same topological properties, e.g. their branch points and automorphism groups are identical. It is therefore trivial to extend $(\Theta, \dot\Sigma'', \pi, G, \rho, I, f)$ to a smooth family of regular presentations

$$(\Theta_\tau, \dot\Sigma'', \pi_\tau, G, \rho, I, f)$$

for φ_τ, where $\pi_\tau := \psi_\tau \circ \pi$. By the same reasoning as above, we can define on Σ'' a smooth family of complex structures $j''_\tau := \pi_\tau^* j_\tau$ such that

$$\pi_\tau : \Sigma''_\tau \to \Sigma_\tau, \qquad \Sigma''_\tau := (\Sigma'', j''_\tau)$$

becomes a smooth family of holomorphic branched covers, restricting to a smooth family of holomorphic covering maps $\dot\Sigma''_\tau \xrightarrow{\pi_\tau} \dot\Sigma_\tau$, defined on the family of punctured Riemann surfaces

$$\dot\Sigma''_\tau := (\dot\Sigma'', j''_\tau).$$

3.2. Cauchy-Riemann operators on closed and punctured domains.

Fix a complex vector bundle

$$(E, J) \to (\Sigma, j)$$

of rank $m \geqslant 1$, and define the rank m bundle of complex-antilinear maps

$$F = \overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, E) = \Lambda^{0,1}T^*\Sigma \otimes E.$$

Recall that a first-order real-linear partial differential operator $\mathbf{D} : \Gamma(E) \to \Gamma(F) = \Omega^{0,1}(\Sigma, E)$ is then called a **Cauchy-Riemann type operator** on E if it satisfies the Leibniz rule

$$\mathbf{D}(f\eta) = (\bar\partial f)\eta + f\mathbf{D}\eta$$

for all $\eta \in \Gamma(E)$ and $f \in C^\infty(\Sigma, \mathbb{R})$, where $\bar\partial f = df + i\, df \circ j \in \Omega^{0,1}(\Sigma)$. The space

$$\mathcal{CR}_{\mathbb{R}}(E)$$

of all such operators is an affine space modelled on the space of smooth real-linear bundle maps $\Gamma(\mathrm{Hom}_{\mathbb{R}}(E, F)) = \Omega^{0,1}(\Sigma, \mathrm{End}_{\mathbb{R}}(E, J))$. The **pullback** of $\mathbf{D} \in \mathcal{CR}_{\mathbb{R}}(E)$ via $\varphi : (\Sigma', j') \to (\Sigma, j)$ defines a Cauchy-Riemann operator

$$\varphi^*\mathbf{D} : \Gamma(E^\varphi) \to \Gamma(F^\varphi),$$

where we define two bundles over Σ' by

$$E^\varphi := \varphi^* E, \qquad F^\varphi := \overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma', \varphi^*E) = \Lambda^{0,1}T^*\Sigma' \otimes E^\varphi$$

and characterize $\varphi^*\mathbf{D}$ via the relation

$$(\varphi^*\mathbf{D})(\eta \circ \varphi) = \varphi^*\left(\mathbf{D}\eta\right) \quad \text{for all} \quad \eta \in \Gamma(E).$$

Example 3.9. If $v : (\Sigma, j) \to (M, J)$ is a J-holomorphic curve with generalized normal bundle $N_v \to \Sigma$, its normal Cauchy-Riemann operator $\mathbf{D}_v^N$ belongs to $\mathcal{CR}_{\mathbb{R}}(N_v)$, and if $u = v \circ \varphi : (\Sigma', j') \to (M, J)$, then $N_u = \varphi^* N_v$ and $\mathbf{D}_u^N = \varphi^* \mathbf{D}_v^N \in \mathcal{CR}_{\mathbb{R}}(N_u)$.

Remark 3.10. Note that the operator $\bar{\partial} : C^\infty(\Sigma, \mathbb{C}) \to \Omega^{0,1}(\Sigma)$ used in our definition of Cauchy-Riemann type operators makes $\bar{\partial} f$ *twice* the complex-antilinear part of the differential df. This is a common convention in J-holomorphic curve theory, but differs from the standard convention in complex analysis. We will also often use the symbol $\bar{\partial}$ to mean the coordinate-based differential operator

$$\bar{\partial} := \partial_s + i\partial_t,$$

acting on functions valued in a complex vector space and defined on open domains in $\mathbb{C}$ with complex coordinate $s + it$. The alternative convention would be to write $\bar{\partial} = \frac{1}{2}(\partial_s + i\partial_t)$.

Fixing Hermitian bundle metrics $\langle\ ,\ \rangle_E$ and $\langle\ ,\ \rangle_\Sigma$ on E and $T\Sigma$ respectively, we can integrate real parts of bundle metrics to define real-valued L^2-pairings $\langle\ ,\ \rangle_{L^2}$ on $\Gamma(E)$ and $\Gamma(F)$, which determines a **formal adjoint** operator $\mathbf{D}^* : \Gamma(F) \to \Gamma(E)$ via the relation

$$\langle \alpha, \mathbf{D}\eta \rangle_{L^2} = \langle \mathbf{D}^* \alpha, \eta \rangle_{L^2}$$

for all smooth sections $\alpha \in \Gamma(F)$ and $\eta \in \Gamma(E)$ with compact support.[4] Viewing $\mathbf{D}$ as a Fredholm operator on Sobolev spaces $W^{k,p}(E) \to W^{k-1,p}(F)$ for some $k \in \mathbb{N}$ and $p \in (1, \infty)$, we can then identify $\operatorname{coker} \mathbf{D}$ with $\ker \mathbf{D}^* \subset \Gamma(F)$, which is the L^2-orthogonal complement of $\operatorname{im} \mathbf{D} \subset W^{k-1,p}(F)$ and is a finite-dimensional space of smooth sections by elliptic regularity. Using the Riemann-Roch formula $\operatorname{ind}(\mathbf{D}) = m\chi(\Sigma) + 2c_1(E)$ and computing the algebraic count of branch points $Z(d\varphi)$ from the Riemann-Hurwitz formula, the (real) Fredholm indices of $\mathbf{D}$ and $\varphi^*\mathbf{D}$ are related by

$$\operatorname{ind}(\varphi^* \mathbf{D}) = d \cdot \operatorname{ind} \mathbf{D} - mZ(d\varphi).$$

In order to exploit the topological constructions in the previous section, we will need to work with Cauchy-Riemann type operators on punctured surfaces instead of closed surfaces. We shall now show that this can be done without loss of generality by choosing suitable weighted Sobolev spaces. Assume

$$E_\tau \to \Sigma_\tau$$

is a smooth family of rank m complex vector bundles with complex structures J_τ, equipped with a smooth family of Cauchy-Riemann operators $\mathbf{D}_\tau \in \mathcal{CR}_{\mathbb{R}}(E_\tau)$. Denote the restrictions of the bundles E_τ and

$$F_\tau := \overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma_\tau, E_\tau)$$

to the punctured surfaces $\dot{\Sigma}_\tau$ by

$$\dot{E}_\tau := E_\tau|_{\dot{\Sigma}_\tau}, \qquad \dot{F}_\tau := F_\tau|_{\dot{\Sigma}_\tau} = \overline{\operatorname{Hom}}_{\mathbb{C}}(T\dot{\Sigma}_\tau, \dot{E}_\tau).$$

Restricting $\mathbf{D}_\tau$ to $\dot{\Sigma}_\tau$ then defines a family of Cauchy-Riemann type operators

$$\dot{\mathbf{D}}_\tau \in \mathcal{CR}_{\mathbb{R}}(\dot{E}_\tau).$$

In order to understand the functional-analytic properties of $\dot{\mathbf{D}}_\tau$, we must examine its asymptotic behavior fairly carefully. Fix local holomorphic coordinate charts to identify a neighborhood of each $w \in \Theta$ in Σ with the closed unit disk $\mathbb{D} \subset \mathbb{C}$, with w corresponding to $0 \in \mathbb{D}$, and use the maps ψ_τ introduced at the end of §3.1 to produce from these a smooth family of holomorphic charts on neighborhoods of $\psi_\tau(w) \in \Theta_\tau$ for $\tau \in P$. In these coordinates, use the biholomorphic map

$$[0, \infty) \times S^1 \to \mathbb{D} \backslash \{0\} : (s, t) \mapsto e^{-2\pi(s+it)}$$

[4]The compact support condition is vacuous in the present context since Σ is compact, but the same definition is also valid on punctured domains.

to define cylindrical ends of $\dot{\Sigma}_\tau$ with holomorphic coordinates $(s,t) \in [0,\infty) \times S^1$. Choose also a smooth family of trivializations of E_τ near Θ_τ and denote the resulting trivialization of $\dot{E}_\tau$ over the cylindrical ends by Φ. The relative first Chern number[5] of $\dot{E}_\tau$ is then given by

$$c_1^\Phi(\dot{E}_\tau) = c_1(E_\tau) \in \mathbb{Z}. \tag{3.1}$$

For any tuple of real numbers

$$\boldsymbol{\delta} = \{\delta_w \in \mathbb{R}\}_{w \in \Theta},$$

we can use the chosen coordinates and trivializations over the cylindrical ends of $\dot{\Sigma}_\tau$ to define the Sobolev space with **exponential weights**

$$W^{k,p,\boldsymbol{\delta}}(\dot{E}_\tau) := \left\{ \eta \in W^{k,p}_{\text{loc}}(\dot{E}_\tau) \;\middle|\; e^{\delta_w s}\eta \in W^{k,p}([0,\infty) \times S^1) \text{ on the end near } \psi_\tau(w) \in \Theta_\tau \right\}.$$

We will also write

$$L^{p,\boldsymbol{\delta}}(\dot{E}_\tau) := W^{0,p,\boldsymbol{\delta}}(\dot{E}_\tau).$$

Note that sections $\eta \in W^{k,p,\boldsymbol{\delta}}(\dot{E}_\tau)$ have exponential decay at any end where $\delta_w > 0$, but one can also take $\delta_w < 0$, in which case η may be unbounded with exponential *growth* near w. In order to emphasize when we are using negative exponential weights, we associate to $\boldsymbol{\delta} = \{\delta_w\}_{w \in \Theta}$ the inverse set of weights

$$-\boldsymbol{\delta} := \{-\delta_w\}_{w \in \Theta}.$$

The asymptotic coordinates and trivializations also naturally give rise to asymptotic trivializations of $\dot{F}_\tau = \overline{\text{Hom}}_{\mathbb{C}}(T\dot{\Sigma}_\tau, \dot{E}_\tau)$, so we can similarly define the Banach space $W^{k-1,p,\boldsymbol{\delta}}(\dot{F}_\tau)$, which is a completion of some subset of $\Omega^{0,1}(\dot{\Sigma}_\tau, \dot{E}_\tau)$ determined by the asymptotic conditions.

Choose a smooth τ-parametrized family of Hermitian bundle metrics and connections on E_τ which match the trivial metric and connection in our chosen family of trivializations near Θ_τ. Any Cauchy-Riemann type operator on E_τ can then be written as $\mathbf{D}_\tau = \bar{\partial}_\nabla + A$ for some $A \in \Omega^{0,1}(\Sigma_\tau, \text{End}_{\mathbb{R}}(E_\tau))$, where $\bar{\partial}_\nabla := \nabla + J_\tau \circ \nabla \circ j_\tau : \Gamma(E_\tau) \to \Omega^{0,1}(\Sigma_\tau, E_\tau)$. In the chosen coordinates and trivialization near a point $w \in \Theta_\tau$, the $(0,1)$-form A can be written as

$$A = A_\tau^{(w)}(z)\, d\bar{z}$$

for some smooth function $A_\tau^{(w)} : \mathbb{D} \to \text{End}_{\mathbb{R}}(\mathbb{C}^m)$. The restriction of A to an $\text{End}_{\mathbb{R}}(\dot{E}_\tau)$-valued $(0,1)$-form $\dot{A}_\tau \in \Omega^{0,1}(\dot{\Sigma}_\tau, \text{End}_{\mathbb{R}}(\dot{E}_\tau))$ can then be written on the corresponding cylindrical end as

$$\dot{A}_\tau = \dot{A}_\tau^{(w)}(s,t)\,(-ds + i\,dt)$$

where

$$\dot{A}_\tau^{(w)}(s,t) := 2\pi e^{-2\pi(s-it)} A_\tau^{(w)}\left(e^{-2\pi(s+it)}\right), \tag{3.2}$$

and given a section $\eta \in \Gamma(\dot{E}_\tau)$ expressed as a function $\eta(s,t) \in \mathbb{C}^m$ with respect to the trivialization on the same end, $\dot{\mathbf{D}}_\tau\eta$ on this end takes the form

$$\dot{\mathbf{D}}_\tau\eta = \left(\partial_s\eta + i\partial_t\eta + \dot{A}_\tau^{(w)}\eta\right)(-ds + i\,dt) =: \left(\bar{\partial}\eta + \dot{A}_\tau^{(w)}\eta\right)(-ds + i\,dt). \tag{3.3}$$

(Here and in further local expressions below, we are using the abbreviation $\bar{\partial} := \partial_s + i\partial_t$ as mentioned in Remark 3.10.) Observe that $\dot{A}_\tau^{(w)}(s,\cdot) \to 0$ with all derivatives as $s \to \infty$. This expression shows that $\dot{\mathbf{D}}_\tau$ extends to a bounded linear operator

$$\dot{\mathbf{D}}_\tau : W^{k,p,\boldsymbol{\delta}}(\dot{E}_\tau) \to W^{k-1,p,\boldsymbol{\delta}}(\dot{F}_\tau)$$

for any choices of $k \in \mathbb{N}$, $p \in (1,\infty)$ and exponential weights $\boldsymbol{\delta} = \{\delta_w \in \mathbb{R}\}_{w \in \Theta}$. Operators of this type are standard in Floer-type theories, and especially in symplectic field theory. Appealing to the Fredholm theory on punctured surfaces developed in [Sch95], the asymptotic decay of

[5] Recall that for any complex line bundle E over a surface Σ with a trivialization Φ specified outside of some open subset in Σ with compact closure, the relative first Chern number $c_1^\Phi(E) \in \mathbb{Z}$ is defined by algebraically counting the zeroes of a generic section that is constant with respect to Φ wherever the latter is defined. This definition extends uniquely to higher rank bundles via the relation $c_1^{\Phi_1 \oplus \Phi_2}(E_1 \oplus E_2) = c_1^{\Phi_1}(E_1) + c_1^{\Phi_2}(E_2)$.

$\dot{A}_\tau^{(w)}(s,\cdot)$ means that $\dot{\mathbf{D}}_\tau : W^{k,p}(\dot{E}_\tau) \to W^{k-1,p}(\dot{F}_\tau)$ is controlled at every puncture by the so-called *trivial* asymptotic operator $-i\partial_t : H^1(S^1,\mathbb{C}^m) \to L^2(S^1,\mathbb{C}^m)$, for which 0 is an eigenvalue of maximal multiplicity. In this sense, the asymptotics are degenerate, i.e. in the SFT setting, such an operator can arise as the linearized Cauchy-Riemann operator of a holomorphic curve asymptotic to periodic orbits that live in Morse-Bott families foliating an open set. In particular, $\dot{\mathbf{D}}_\tau : W^{k,p} \to W^{k-1,p}$ is *not* Fredholm, but it becomes Fredholm when we introduce suitable weights: conjugating $\dot{\mathbf{D}}_\tau : W^{k,p,\boldsymbol{\delta}} \to W^{k-1,p,\boldsymbol{\delta}}$ with a map of the form $\Psi(\eta) = e^f\eta$ for a suitable function $f : \dot{\Sigma}_\tau \to \mathbb{R}$ (cf. [HWZ99, §6] or [Wen10, §2.1]) produces a commutative diagram

$$\begin{array}{ccc} W^{k,p,\boldsymbol{\delta}}(\dot{E}_\tau) & \xrightarrow{\dot{\mathbf{D}}_\tau} & W^{k-1,p,\boldsymbol{\delta}}(\dot{F}_\tau) \\ \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \Psi} \\ W^{k,p}(\dot{E}_\tau) & \xrightarrow{\widehat{\mathbf{D}}_\tau} & W^{k-1,p}(\dot{F}_\tau), \end{array} \tag{3.4}$$

where $\widehat{\mathbf{D}} : W^{k,p} \to W^{k-1,p}$ is another Cauchy-Riemann type operator whose asymptotic operators are offset by constants depending on the weights $\boldsymbol{\delta}$, and thus is Fredholm for suitable choices. In particular, the computation in (3.7) and (3.8) below will show that imposing the exponential growth condition $e^{-\delta s}\eta \in W^{k,p}([0,\infty) \times S^1)$ on each cylindrical end for sufficiently small $\delta > 0$ adjusts the asymptotic operators of $\widehat{\mathbf{D}}_\tau$ so that each acquires an effective Conley-Zehnder index m relative to the trivialization Φ.

We need to be a bit cautious with the weights when discussing elliptic regularity and formal adjoints: as a rule, the Sobolev constants $k \in \mathbb{N}$ and $p \in (1,\infty)$ can be changed freely, but the weights cannot. The following are immediate consequences of (3.4) after applying standard regularity arguments to $\widehat{\mathbf{D}}_\tau$, plus (in the case of Lemma 3.12) the fact that Cauchy-Riemann operators with nondegenerate asymptotics automatically impose exponential decay conditions on their kernels (cf. [Sch95, Prop. 3.1.26]):

Lemma 3.11. *Suppose $k \in \mathbb{N}$, $1 < p < \infty$, and $\boldsymbol{\delta} = \{\delta_w \in \mathbb{R}\}_{w\in\Theta}$ is any choice of exponential weights. If $\eta \in L^{p,\boldsymbol{\delta}}(\dot{E}_\tau)$ is a weak solution to $\dot{\mathbf{D}}_\tau\eta = \xi$ for $\xi \in W^{k-1,p,\boldsymbol{\delta}}(\dot{F}_\tau)$, then $\eta \in W^{k,p,\boldsymbol{\delta}}(\dot{E}_\tau)$.* □

Lemma 3.12. *Suppose $1 < p < \infty$ and the weights $\boldsymbol{\delta}$ are chosen such that $\dot{\mathbf{D}}_\tau : W^{k,p,\boldsymbol{\delta}}(\dot{E}_\tau) \to W^{k-1,p,\boldsymbol{\delta}}(\dot{F}_\tau)$ is Fredholm. If $\eta \in L^{p,\boldsymbol{\delta}}(\dot{E}_\tau)$ is a weak solution to $\dot{\mathbf{D}}_\tau\eta = 0$, then $\eta \in W^{k,q,\boldsymbol{\delta}}(\dot{E}_\tau)$ for all $k \in \mathbb{N}$ and $q \in (1,\infty)$.* □

To discuss the formal adjoint on punctured domains, one should define real L^2-products for $\Gamma(\dot{E}_\tau)$ and $\Gamma(\dot{F}_\tau)$ in terms of a family of Hermitian bundle metrics on E_τ and Riemannian metrics on $\dot{\Sigma}_\tau$ that are compatible with the conformal structure and standard on the cylindrical ends; in particular, the right metric to use on the cylindrical ends is the Euclidean metric in the coordinates $(s,t) \in [0,\infty) \times S^1$, so that ends have infinite area and the metric does not extend over the punctues. The key technical point is then the following: there are well-defined L^2-pairings

$$L^{p,\boldsymbol{\delta}} \otimes L^{q,-\boldsymbol{\delta}} \to \mathbb{R} : \eta \otimes \xi \mapsto \langle \eta, \xi \rangle_{L^2} \tag{3.5}$$

whenever $1/p + 1/q = 1$, and using the density of C_0^∞, the usual relation

$$\langle \alpha, \dot{\mathbf{D}}_\tau\eta \rangle_{L^2} = \langle \dot{\mathbf{D}}_\tau^*\alpha, \eta \rangle_{L^2} \tag{3.6}$$

for smooth compactly supported sections η and α remains valid whenever $\eta \in W^{1,p,-\boldsymbol{\delta}}(\dot{E}_\tau)$ and $\alpha \in W^{1,q,\boldsymbol{\delta}}(\dot{F}_\tau)$ for $1/p + 1/q = 1$. Using (3.4), one finds $\dot{\mathbf{D}}_\tau^* = \Psi\widehat{\mathbf{D}}_\tau^*\Psi^{-1}$, from which one can check that $\dot{\mathbf{D}}_\tau^* : W^{k,p,\boldsymbol{\delta}}(\dot{F}_\tau) \to W^{k-1,p,\boldsymbol{\delta}}(\dot{E}_\tau)$ satisfies the Fredholm property and Lemmas 3.11 and 3.12 under the same conditions on $\boldsymbol{\delta}$ as $\dot{\mathbf{D}}_\tau : W^{k,p,-\boldsymbol{\delta}}(\dot{E}_\tau) \to W^{k-1,p,-\boldsymbol{\delta}}(\dot{F}_\tau)$. The next result appears standard at first glance, but the reader should be cautioned that it depends on inclusions $W^{k,p,\boldsymbol{\delta}} \hookrightarrow W^{k,p,-\boldsymbol{\delta}}$ which hold only when all the weights are nonnegative, so e.g. one does not obtain any similar result with the roles of $\dot{\mathbf{D}}_\tau$ and $\dot{\mathbf{D}}_\tau^*$ reversed.

Proposition 3.13. *Assume $k \in \mathbb{N}$, $1 < p < \infty$, and $\boldsymbol{\delta} = \{\delta_w \geqslant 0\}_{w\in\Theta}$ is a set of nonnegative exponential weights such that*

$$\dot{\mathbf{D}}_\tau : W^{k,p,-\boldsymbol{\delta}}(\dot{E}_\tau) \to W^{k-1,p,-\boldsymbol{\delta}}(\dot{F}_\tau)$$

is Fredholm. Defining its formal adjoint as a bounded linear map

$$\dot{\mathbf{D}}_\tau^* : W^{k,p,\boldsymbol{\delta}}(\dot{F}_\tau) \to W^{k-1,p,\boldsymbol{\delta}}(\dot{E}_\tau)$$

and using the obvious inclusions $W^{k,p,\boldsymbol{\delta}}(\dot{F}_\tau) \hookrightarrow W^{k-1,p,\boldsymbol{\delta}}(\dot{F}_\tau) \hookrightarrow W^{k-1,p,-\boldsymbol{\delta}}(\dot{F}_\tau)$, we have

$$W^{k-1,p,-\boldsymbol{\delta}}(\dot{F}_\tau) = \operatorname{im}\dot{\mathbf{D}}_\tau \oplus \ker\dot{\mathbf{D}}_\tau^*.$$

In particular, $\operatorname{coker}\dot{\mathbf{D}}_\tau$ is isomorphic to the space of all sections in $L^{q,\boldsymbol{\delta}}(\dot{F}_\tau)$ for $1/p + 1/q = 1$ that are L^2-orthogonal to $\operatorname{im}\dot{\mathbf{D}}_\tau \subset L^{p,-\boldsymbol{\delta}}(\dot{F}_\tau)$ under the pairing (3.5).

Proof. If $\alpha \in \operatorname{im}\dot{\mathbf{D}}_\tau \cap \ker\dot{\mathbf{D}}_\tau^*$, then $\alpha = \dot{\mathbf{D}}_\tau\eta$ for some $\eta \in W^{k,p,-\boldsymbol{\delta}}(\dot{E}_\tau) \subset W^{1,p,-\boldsymbol{\delta}}(\dot{E}_\tau)$, while α also belongs to $W^{1,q,\boldsymbol{\delta}}(\dot{F}_\tau)$ for $1/p + 1/q = 1$ by Lemma 3.12. Thus α has a well-defined L^2-pairing with itself and (3.6) gives

$$\|\alpha\|_{L^2}^2 = \langle \alpha, \dot{\mathbf{D}}_\tau\eta\rangle_{L^2} = \langle \dot{\mathbf{D}}_\tau^*\alpha, \eta\rangle_{L^2} = 0.$$

To show that $\operatorname{im}\dot{\mathbf{D}}_\tau + \ker\dot{\mathbf{D}}_\tau^*$ is $W^{k-1,p,-\boldsymbol{\delta}}(\dot{F}_\tau)$, note first that it is a closed subspace since $\dot{\mathbf{D}}_\tau$ is Fredholm. Then in the case $k = 1$, the contrary would mean there exists a nontrivial $\lambda \in (L^{p,-\boldsymbol{\delta}}(\dot{F}_\tau))^* = L^{q,\boldsymbol{\delta}}(\dot{F}_\tau)$ for $1/p + 1/q = 1$ such that $\langle \dot{\mathbf{D}}_\tau\eta, \lambda\rangle_{L^2} = 0$ for all $\eta \in W^{1,p,-\boldsymbol{\delta}}(\dot{E}_\tau)$ and $\langle \alpha, \lambda\rangle_{L^2} = 0$ for all $\alpha \in \ker\dot{\mathbf{D}}_\tau^*$. The first condition means $\lambda \in \ker\dot{\mathbf{D}}_\tau^*$ by Lemma 3.12 and thus contradicts the second unless $\lambda = 0$. To extend this result to all $k \in \mathbb{N}$, note that if $\lambda \in W^{k-1,p,-\boldsymbol{\delta}}(\dot{F}_\tau) \subset L^{p,-\boldsymbol{\delta}}(\dot{F}_\tau)$ then the $k = 1$ case gives $\eta \in W^{1,p,-\boldsymbol{\delta}}(\dot{E}_\tau)$ and $\alpha \in \ker\dot{\mathbf{D}}_\tau^*$ such that $\dot{\mathbf{D}}_\tau\eta + \alpha = \lambda$. Then Lemma 3.12 implies $\alpha \in W^{k-1,p,\boldsymbol{\delta}}(\dot{F}_\tau) \subset W^{k-1,p,-\boldsymbol{\delta}}(\dot{F}_\tau)$, implying that $\dot{\mathbf{D}}_\tau\eta$ is also in $W^{k-1,p,-\boldsymbol{\delta}}(\dot{F}_\tau)$, so Lemma 3.11 implies $\eta \in W^{k,p,-\boldsymbol{\delta}}(\dot{E}_\tau)$ and we are done. ☐

This discussion extends easily to the pulled back operators

$$\varphi_\tau^*\mathbf{D}_\tau \in \mathcal{CR}_{\mathbb{R}}(\varphi_\tau^* E_\tau) \quad \text{and} \quad \varphi_\tau^*\dot{\mathbf{D}}_\tau \in \mathcal{CR}_{\mathbb{R}}(\varphi_\tau^* \dot{E}_\tau)$$

on bundles over Σ'_τ and $\dot{\Sigma}'_\tau$ respectively. Observe that since $\dot{\Sigma}'_\tau \xrightarrow{\varphi_\tau} \dot{\Sigma}_\tau$ has no branch points, $d\varphi_\tau$ gives a bundle isomorphism $T\dot{\Sigma}'_\tau \to \varphi_\tau^* T\dot{\Sigma}_\tau$ and we can thus identify

$$F_\tau^{\varphi_\tau}|_{\dot{\Sigma}'_\tau} = \overline{\operatorname{Hom}}_{\mathbb{C}}(T\dot{\Sigma}'_\tau, \varphi_\tau^*\dot{E}_\tau) = \overline{\operatorname{Hom}}_{\mathbb{C}}(\varphi_\tau^* T\dot{\Sigma}_\tau, \varphi_\tau^*\dot{E}_\tau) = \varphi_\tau^*\dot{F}_\tau,$$

so that $\varphi_\tau^*\dot{\mathbf{D}}_\tau$ can be viewed as a map $\Gamma(\varphi_\tau^*\dot{E}_\tau) \to \Gamma(\varphi_\tau^*\dot{F}_\tau)$. We can now define fixed holomorphic cylindrical coordinate systems $(s,t) \in [0,\infty) \times S^1$ on punctured neighborhoods of each point $\zeta \in \Theta' = \varphi_\tau^{-1}(\Theta_\tau)$ such that φ_τ takes the form

$$\dot{\Sigma}'_\tau \supset [0,\infty) \times S^1 \xrightarrow{\varphi_\tau} [0,\infty) \times S^1 \subset \dot{\Sigma}_\tau,$$
$$(s,t) \mapsto (k_\zeta s, k_\zeta t),$$

where $k_\zeta \in \mathbb{N}$ is the branching order of φ at ζ. Pulling back the trivializations Φ on E_τ near Θ_τ to define corresponding trivializations of $\varphi_\tau^* E_\tau$ near Θ', we obtain asymptotic trivializations of $\varphi_\tau^*\dot{E}_\tau$ and $\varphi_\tau^*\dot{F}_\tau$ on the cylindrical ends and can thus define weighted Sobolev norms for sections of these bundles, producing a bounded linear operator

$$\varphi_\tau^*\dot{\mathbf{D}}_\tau : W^{k,p,\boldsymbol{\delta}}(\varphi_\tau^*\dot{E}_\tau) \to W^{k-1,p,\boldsymbol{\delta}}(\varphi_\tau^*\dot{F}_\tau)$$

for all choices of $k \in \mathbb{N}$, $p \in (1,\infty)$ and exponential weights $\boldsymbol{\delta} = \{\delta_\zeta \in \mathbb{R}\}_{\zeta\in\Theta'}$. If $\boldsymbol{\delta} = \{\delta_w\}_{w\in\Theta}$ is a choice of weights for $\dot{\mathbf{D}}_\tau$, there is an induced set of weights for $\varphi_\tau^*\dot{\mathbf{D}}_\tau$ defined by

$$\varphi^*\boldsymbol{\delta} := \left\{k_\zeta \delta_{\varphi(\zeta)}\right\}_{\zeta\in\Theta'},$$

where $k_\zeta \in \{1,\ldots,d\}$ again denotes the branching order of φ at ζ.

Proposition 3.14. *Suppose $k \in \mathbb{N}$, $p \in (1,\infty)$, and the exponential weights $\boldsymbol{\delta} = \{\delta_w\}_{w\in\Theta}$ are chosen to satisfy*

$$0 < \delta_w < \frac{2\pi}{d}$$

for every $w \in \Theta$. Then for any $\mathbf{D}_\tau \in \mathcal{CR}_{\mathbb{R}}(E_\tau)$, the operators

$$\dot{\mathbf{D}}_\tau : W^{k,p,-\boldsymbol{\delta}}(\dot{E}_\tau) \to W^{k-1,p,-\boldsymbol{\delta}}(\dot{F}_\tau),$$
$$\varphi_\tau^*\dot{\mathbf{D}}_\tau : W^{k,p,-\varphi^*\boldsymbol{\delta}}(\varphi_\tau^*\dot{E}_\tau) \to W^{k-1,p,-\varphi^*\boldsymbol{\delta}}(\varphi_\tau^*\dot{F}_\tau)$$

are Fredholm and satisfy

$$\operatorname{ind}(\dot{\mathbf{D}}_\tau) = \operatorname{ind}(\mathbf{D}_\tau), \qquad \textit{and} \qquad \operatorname{ind}(\varphi_\tau^*\dot{\mathbf{D}}_\tau) = \operatorname{ind}(\varphi_\tau^*\mathbf{D}_\tau).$$

Moreover, the maps $\Gamma(E_\tau) \to \Gamma(\dot{E}_\tau)$ and $\Gamma(\varphi_\tau^*E_\tau) \to \Gamma(\varphi_\tau^*\dot{E}_\tau)$ defined by restricting smooth sections to the corresponding punctured domains define isomorphisms

$$\ker \mathbf{D}_\tau \xrightarrow{\cong} \ker \dot{\mathbf{D}}_\tau \qquad \textit{and} \qquad \ker(\varphi_\tau^*\mathbf{D}_\tau) \xrightarrow{\cong} \ker(\varphi_\tau^*\dot{\mathbf{D}}_\tau).$$

Proof. We will prove the correspondence between $\mathbf{D}_\tau$ and $\dot{\mathbf{D}}_\tau$, as the result for the pulled back operators follows by the same argument simply replacing the bundles $E_\tau \to \Sigma$ and $\dot{E}_\tau \to \dot{\Sigma}_\tau$ with $\varphi_\tau^*E_\tau \to \Sigma'$ and $\varphi_\tau\dot{E}_\tau \to \dot{\Sigma}'_\tau$ respectively.

The Fredholm property for $\dot{\mathbf{D}}_\tau$ and the index calculation follow from the usual index formula for Cauchy-Riemann operators on Riemann surfaces with cylindrical ends, proved in [Sch95] (see also [Wene, Lecture 5]), supplemented by the transformation (3.4) to handle the exponential weights (cf. [HWZ99, §6]). In particular, the condition $-2\pi < -\delta_w < 0$ for each $w \in \Theta_\tau$ guarantees that $\dot{\mathbf{D}}_\tau$ is conjugate (cf. (3.7) and (3.8) below) to a Cauchy-Riemann type operator $W^{k,p}(\dot{E}_\tau) \to W^{k-1,p}(\dot{F}_\tau)$ with nondegenerate asymptotic operators at every puncture whose Conley-Zehnder indices with respect to the trivialization Φ are $m = \operatorname{rank}_{\mathbb{C}} E_\tau$. In light of (3.1), the index formula from [Sch95] thus gives

$$\operatorname{ind}(\dot{\mathbf{D}}_\tau) = m\chi(\dot{\Sigma}_\tau) + 2c_1^\Phi(\dot{E}_\tau) + m \cdot |\Theta_\tau| = m\chi(\Sigma) + 2c_1(E_\tau) = \operatorname{ind}(\mathbf{D}_\tau).$$

Note that doing the same computation for the pulled back operators requires the stronger condition $-2\pi/d < -\delta_w < 0$ in order to ensure that all of the pulled back weights in the set $-\varphi^*\boldsymbol{\delta}$ lie in the interval $(-2\pi, 0)$.

To understand the kernels, observe that since any $\eta \in \ker \mathbf{D}_\tau$ is smooth, its restriction to $\dot{\Sigma}_\tau$ belongs to $W^{k,p,-\boldsymbol{\delta}}(\dot{E}_\tau)$ and is thus in $\ker \dot{\mathbf{D}}_\tau$.[6] Conversely, we need to show that any section $\eta \in W^{k,p,-\boldsymbol{\delta}}(\dot{E}_\tau)$ annihilated by $\dot{\mathbf{D}}_\tau$ can be extended over the punctures to a section in $W^{k,p}(E_\tau)$, which is then automatically annihilated by $\mathbf{D}_\tau$. This will follow from the asymptotic elliptic theory of the equation $\dot{\mathbf{D}}_\tau\eta = 0$. Indeed, recall from (3.3) that on the cylindrical end near any puncture $w \in \Theta_\tau$, the function $\eta(s,t) \in \mathbb{C}^m$ representing $\eta \in \ker \dot{\mathbf{D}}_\tau$ in some trivialization satisfies

$$\bar{\partial}\eta + \dot{A}_\tau^{(w)}\eta \equiv 0,$$

and

$$\eta = e^{\delta s} f \quad \text{for some } f \in W^{k,p}([0,\infty) \times S^1, \mathbb{C}^m),$$

where $\delta := \delta_w \in (0, 2\pi)$. Then $f = e^{-\delta s}\eta$ satisfies the Cauchy-Riemann type equation

$$\bar{\partial} f + (\delta + \dot{A}_\tau^{(w)}) f = \partial_s f - [-i\partial_t - (\delta + \dot{A}_\tau^{(w)})] f = 0. \tag{3.7}$$

Since $\dot{A}_\tau^{(w)}(s,\cdot) \to 0$ as $s \to \infty$, this equation is asymptotic to the equation $(\partial_s - \mathbf{A}_\delta) f = 0$ for the asymptotic operator

$$\mathbf{A}_\delta := -i\partial_t - \delta : H^1(S^1, \mathbb{C}^m) \to L^2(S^1, \mathbb{C}^m), \tag{3.8}$$

[6]Note that $\eta|_{\dot{\Sigma}_\tau}$ would not belong to $W^{k,p,-\boldsymbol{\delta}}(\dot{E}_\tau)$ in general if η were an arbitrary (not necessarily smooth) section of class $W^{k,p}$ on E_τ, nor if any of the exponential weights were nonnegative—the latter in particular permits sections in $W^{k,p,-\boldsymbol{\delta}}(\dot{E}_\tau)$ that do not decay to zero at infinity, which is crucial since arbitrary smooth sections $\eta \in \ker \mathbf{D}_\tau$ may indeed be nonzero at points in Θ_τ.

which can be regarded as a densely defined unbounded self-adjoint operator on $L^2(S^1, \mathbb{C}^m)$. The function $A_\tau^{(w)} : \mathbb{D} \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^m)$ is smooth by assumption, and (3.2) then implies that the derivatives $\partial^\alpha \dot{A}_\tau^{(w)}(s,t)$ of $\dot{A}_\tau^{(w)}$ for arbitrary multi-indices α satisfy exponential decay conditions

$$|\partial^\alpha \dot{A}_\tau^{(w)}(s,t)| \leqslant M_\alpha e^{-2\pi s}$$

for suitable constants $M_\alpha > 0$. Applying [Sie08, Theorem A.1], f therefore satisfies

$$f(s,t) = e^{\lambda s}\left[e(t) + r(s,t)\right],$$

where $e : S^1 \to \mathbb{C}^m$ is a nontrivial eigenfunction of $\mathbf{A}_\delta$ with eigenvalue $\lambda < 0$, and the remainder $r(s,t) \in \mathbb{C}^m$ decays to zero with all its derivatives uniformly in t as $s \to \infty$. The spectrum of $\mathbf{A}_\delta$ is $\{2\pi k - \delta \mid k \in \mathbb{Z}\} \subset \mathbb{R}$, hence the assumption $\delta \in (0, 2\pi)$ implies $\lambda \leqslant -\delta$, and we conclude that

$$\eta(s,t) = e^{(\delta+\lambda)s}\left[e(t) + r(s,t)\right]$$

is bounded on the cylindrical end; in fact, one can use this to show that the smooth function $\mathbb{D}\backslash\{0\} \to \mathbb{C}^m : z \mapsto \eta(z)$ defined via the transformation $z = e^{-2\pi(s+it)}$ has finite $W^{1,p}$-norm on $\mathbb{D}\backslash\{0\}$. Moreover, $\eta(z)$ has a continuous extension to $z = 0$: indeed, the extension is obviously $\eta(0) = 0$ if $\lambda < -\delta$, while in the case $\lambda = -\delta$, the eigenfunction $e(t)$ is necessarily constant, so that $\eta(s, \cdot)$ converges to this constant value as $s \to \infty$. All these conditions together imply that the continuous extension of η over the punctures is of class $W^{k,p}$, e.g. the case $k = 1$ is a standard exercise using the definition of weak derivatives (cf. [Wena, Exercise 2.118]), and the general case follows from this by elliptic regularity. □

Remark 3.15. Since sections in $W^{k,p,-\boldsymbol{\delta}}(\dot{E}_\tau)$ and its pulled back counterpart need not be bounded when the weights $-\boldsymbol{\delta}$ are negative, the punctured operators in Proposition 3.14 cannot be interpreted in any reasonable way as linearizations of nonlinear Cauchy-Riemann operators, e.g. $W^{k,p,-\boldsymbol{\delta}}(\dot{E}_\tau)$ in this case is not a subspace of a tangent space in any reasonable Banach manifold. For our purposes, the exponential growth condition is merely a technical convenience so that we can consider operators with the right index and the right kernel and cokernel while dealing with honest covering maps instead of branched covers. The geometrically meaningful operators are still $\mathbf{D}_\tau$ and $\varphi_\tau^*\mathbf{D}_\tau$, on unpunctured domains.

Remark 3.16. Suppose E_τ, Σ_τ and $\mathbf{D}_\tau$ are independent of τ but φ_τ moves in $\mathcal{M}_{\mathbf{b}}^d(j)$ as τ varies, e.g. this is the relevant situation for the proof of super-rigidity. There is then a subtle but important difference between what Proposition 3.14 says about $\dot{\mathbf{D}}_\tau$ and what it says about $\varphi_\tau^*\dot{\mathbf{D}}_\tau$. The former is a family of operators whose relationship to each other for different values of τ is not obvious from the definitions, but the proposition implies that they are all in some sense equivalent to a single operator $\mathbf{D}$ on the closed domain, so they all have isomorphic kernels. No such thing can be assumed for the pulled back operators: while $\varphi_\tau^*\dot{\mathbf{D}}_\tau$ must have the same index for all τ, there is nothing in this setup to stop the dimension of its kernel from varying wildly with τ.

3.3. **A digression on representation theory.** In preparation for the twisted bundle construction in the next section, we now collect some general facts from representation theory.

3.3.1. *Real permutation representations and subrepresentations.* Given a finite set I with $d := |I| \in \mathbb{N}$ elements and a finite group with a homomorphism

$$\rho : G \to S(I) : g \mapsto \rho_g$$

defining a transitive group action on I, we denote by $\mathbb{R}^I$ the real vector space spanned by basis vectors $\{e_i\}_{i \in I}$, with an inner product such that this basis is orthonormal. We shall use the boldface symbol $\boldsymbol{\rho}$ to denote the corresponding real d-dimensional representation of G,

$$\boldsymbol{\rho} : G \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^I) \quad \text{such that} \quad \boldsymbol{\rho}(g)e_i := e_{\rho_g(i)}. \tag{3.9}$$

We will be interested in the decomposition of $\mathbb{R}^I$ into irreducible G-invariant summands. This can be understood in terms of its complexification

$$\boldsymbol{\rho}_{\mathbb{C}} : G \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^I),$$

defined by viewing $\{e_i\}_{i\in I}$ as a complex basis of $\mathbb{C}^I$. In general, we say that a complex representation $\boldsymbol{\lambda} : G \to \mathrm{Aut}_{\mathbb{C}}(V)$ is the **complexification** of a real representation $\boldsymbol{\theta} : G \to \mathrm{Aut}_{\mathbb{R}}(W)$ if V is isomorphic to $W \oplus iW$ such that G acts on the latter by the complex-linear extension of its action on W. Recall from [Ser77, §13.2] that irreducible complex representations $\boldsymbol{\lambda} : G \to \mathrm{Aut}_{\mathbb{C}}(V)$ come in three mutually exclusive types:

- **Real type**: V admits a complex-antilinear G-invariant involution. Then $\boldsymbol{\lambda}$ is the complexification of a real irreducible representation $\boldsymbol{\theta} : G \to \mathrm{Aut}_{\mathbb{R}}(W)$. It follows that $\boldsymbol{\lambda}$ is isomorphic to its dual representation $\boldsymbol{\lambda}^* : G \to \mathrm{Aut}_{\mathbb{C}}(V^*)$, and all G-equivariant linear maps $W \to W$ are given by scalar multiplication:
$$\mathrm{End}_G(W) \cong \mathbb{R}.$$
- **Complex type**: $\boldsymbol{\lambda}$ is not isomorphic to its dual representation $\boldsymbol{\lambda}^* : G \to \mathrm{Aut}_{\mathbb{C}}(V^*)$. Then $\boldsymbol{\lambda} \oplus \boldsymbol{\lambda}^* : G \to \mathrm{Aut}_{\mathbb{C}}(V \oplus V^*)$ is the complexification of a real irreducible representation $\boldsymbol{\theta} : G \to \mathrm{Aut}_{\mathbb{R}}(W)$ obtained from $\boldsymbol{\lambda} : G \to \mathrm{Aut}_{\mathbb{C}}(V)$ by setting $W := V$ and using the obvious inclusion $\mathrm{Aut}_{\mathbb{C}}(V) \subset \mathrm{Aut}_{\mathbb{R}}(W)$. The algebra of G-equivariant real-linear maps on W is then
$$\mathrm{End}_G(W) \cong \mathbb{C}.$$
- **Quaternionic type**: $\boldsymbol{\lambda}$ is not of real type but is nonetheless isomorphic to its dual representation. Then $\boldsymbol{\lambda} \oplus \boldsymbol{\lambda} : G \to \mathrm{Aut}_{\mathbb{C}}(V \oplus V)$ is the complexification of a real irreducible representation $\boldsymbol{\theta} : G \to \mathrm{Aut}_{\mathbb{R}}(W)$ obtained from $\boldsymbol{\lambda} : G \to \mathrm{Aut}_{\mathbb{C}}(V)$ by setting $W := V$ and using the obvious inclusion $\mathrm{Aut}_{\mathbb{C}}(V) \subset \mathrm{Aut}_{\mathbb{R}}(W)$, and the algebra of G-equivariant real-linear maps on W is isomorphic to the quaternions:
$$\mathrm{End}_G(W) \cong \mathbb{H}.$$

We shall also refer to a real irreducible representation as "of **real / complex / quaternionic type**" according to which of these three constructions it comes from. With this classification in mind, we denote the various complex irreducible representations of G by

$$\boldsymbol{\lambda}_{j,\mathbb{K}} : G \to \mathrm{Aut}_{\mathbb{C}}(V_{j,\mathbb{K}}),$$

where $\mathbb{K}$ stands for $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ depending on the type, and arrange a complete list of pairwise non-isomorphic irreducible representations in the form

$$\boldsymbol{\lambda}_{1,\mathbb{R}}, \dots, \boldsymbol{\lambda}_{p,\mathbb{R}},\ \boldsymbol{\lambda}_{1,\mathbb{C}}, \boldsymbol{\lambda}^*_{1,\mathbb{C}}, \dots, \boldsymbol{\lambda}_{q,\mathbb{C}}, \boldsymbol{\lambda}^*_{q,\mathbb{C}},\ \boldsymbol{\lambda}_{1,\mathbb{H}}, \dots, \boldsymbol{\lambda}_{n,\mathbb{H}}.$$

This gives rise to a corresponding complete list

$$\boldsymbol{\theta}_{1,\mathbb{R}}, \dots, \boldsymbol{\theta}_{p,\mathbb{R}},\ \boldsymbol{\theta}_{1,\mathbb{C}}, \dots, \boldsymbol{\theta}_{q,\mathbb{C}},\ \boldsymbol{\theta}_{1,\mathbb{H}}, \dots, \boldsymbol{\theta}_{n,\mathbb{H}}$$

of pairwise non-isomorphic real irreducible representations

$$\boldsymbol{\theta}_{j,\mathbb{K}} : G \to \mathrm{Aut}_{\mathbb{R}}(W_{j,\mathbb{K}}) \quad \text{satisfying} \quad \mathrm{End}_G(W_{j,\mathbb{K}}) \cong \mathbb{K},$$

where for each j, the complexification of $\boldsymbol{\theta}_{j,\mathbb{K}}$ is $\boldsymbol{\lambda}_{j,\mathbb{R}}$ for $\mathbb{K} = \mathbb{R}$, $\boldsymbol{\lambda}_{j,\mathbb{C}} \oplus \boldsymbol{\lambda}^*_{j,\mathbb{C}}$ for $\mathbb{K} = \mathbb{C}$, and $\boldsymbol{\lambda}_{j,\mathbb{H}} \oplus \boldsymbol{\lambda}_{j,\mathbb{H}}$ for $\mathbb{K} = \mathbb{H}$. Note that the G-equivariant endomorphisms endow each $W_{j,\mathbb{K}}$ with the structure of a left $\mathbb{K}$-module such that the representation $\boldsymbol{\theta}_{j,\mathbb{K}}$ is $\mathbb{K}$-linear.

We recall a standard fact from representation theory:

Proposition 3.17. *Every finite-dimensional representation $\boldsymbol{\theta} : G \to \mathrm{Aut}(W)$ of a finite group G has a unique **isotypic decomposition**, meaning a splitting $W = W_1 \oplus \ldots \oplus W_N$ such that:*

1. *For each $i = 1, \dots, N$, $W_i \subset W$ is a G-invariant subspace on which $\boldsymbol{\theta}$ is isomorphic to a direct sum of copies of a single irreducible representation;*
2. *The irreducible representations corresponding any two distinct subspaces in the splitting are not isomorphic.*

$\square$

Since $\boldsymbol{\rho}_{\mathbb{C}}$ itself is a complexification of a real representation, every subspace in the resulting isotypic decomposition of $\mathbb{C}^I$ is either identical or orthogonal to its complex conjugate, where the conjugate always carries the dual representation. Thus we can uniquely decompose $\mathbb{C}^I$ into pairwise orthogonal G-invariant complex subspaces

$$\mathbb{C}^I = X_{1,\mathbb{R}} \oplus \ldots \oplus X_{p,\mathbb{R}} \oplus X_{1,\mathbb{C}} \oplus \bar{X}_{1,\mathbb{C}} \oplus \ldots \oplus X_{q,\mathbb{C}} \oplus \bar{X}_{q,\mathbb{C}} \oplus X_{1,\mathbb{H}} \oplus \ldots \oplus X_{n,\mathbb{H}}, \tag{3.10}$$

where each $X_{j,\mathbb{R}}$ and $X_{j,\mathbb{H}}$ is of the form $Y_{j,\mathbb{K}} \oplus iY_{j,\mathbb{K}}$ for some real subspace $Y_{j,\mathbb{K}} \subset \mathbb{R}^I$, and each $X_{j,\mathbb{C}}$ has trivial intersection with $\mathbb{R}^I$. Next, observe that every irreducible G-invariant subspace in $\mathbb{C}^I$ is either identical to its complex conjugate or intersects it trivially: indeed, any other option would produce an intersection which is a nontrivial but smaller G-invariant subspace. We can thus further decompose $X_{j,\mathbb{R}}$ and $X_{j,\mathbb{C}}$ into irreducible G-invariant subspaces

$$X_{j,\mathbb{R}} \cong V_{j,\mathbb{R}}^{\oplus k_j}, \qquad X_{j,\mathbb{C}} \cong V_{j,\mathbb{C}}^{\oplus m_j}$$

for some integers $k_j, m_j \geqslant 0$, where each $V_{j,\mathbb{R}}$ summand in $X_{j,\mathbb{R}}$ can be assumed of the form $W_{j,\mathbb{R}} \oplus iW_{j,\mathbb{R}}$ for some irreducible G-invariant real subspace $W_{j,\mathbb{R}} \subset Y_{j,\mathbb{R}}$. In $X_{j,\mathbb{H}}$, the irreducible G-invariant subspaces cannot be complexifications since the corresponding representation is not realizable over $\mathbb{R}$, thus these subspaces have trivial intersection with $\mathbb{R}^I$ and can instead be arranged in conjugate pairs:

$$X_{j,\mathbb{H}} \cong V_{j,\mathbb{H}}^{\oplus \ell_j} \oplus \overline{V_{j,\mathbb{H}}^{\oplus \ell_j}}$$

for some integers $\ell_j \geqslant 0$. From this decomposition of $\boldsymbol{\rho}_{\mathbb{C}}$ we can immediately read off a corresponding decomposition of $\boldsymbol{\rho}$: we have

$$\mathbb{R}^I = Y_{1,\mathbb{R}} \oplus \ldots \oplus Y_{p,\mathbb{R}} \oplus Y_{1,\mathbb{C}} \oplus \ldots \oplus Y_{q,\mathbb{C}} \oplus Y_{1,\mathbb{H}} \oplus \ldots \oplus Y_{n,\mathbb{H}}, \tag{3.11}$$

where the summands are all G-invariant and pairwise orthogonal, $Y_{j,\mathbb{K}} = X_{j,\mathbb{K}} \cap \mathbb{R}^I$ for $\mathbb{K} = \mathbb{R}, \mathbb{H}$, and $Y_{j,\mathbb{C}} = (X_{j,\mathbb{C}} \oplus \bar{X}_{j,\mathbb{C}}) \cap \mathbb{R}^I$, hence,

$$\dim_{\mathbb{R}} Y_{j,\mathbb{K}} = \begin{cases} \dim_{\mathbb{C}} X_{j,\mathbb{K}} & \text{if } \mathbb{K} = \mathbb{R} \text{ or } \mathbb{H}, \\ 2\dim_{\mathbb{C}} X_{j,\mathbb{K}} & \text{if } \mathbb{K} = \mathbb{C}. \end{cases}$$

These summands admit further (non-unique) decompositions into real irreducible G-invariant subspaces

$$Y_{j,\mathbb{R}} \cong W_{j,\mathbb{R}}^{\oplus k_j}, \qquad Y_{j,\mathbb{C}} \cong W_{j,\mathbb{C}}^{\oplus m_j}, \qquad Y_{j,\mathbb{K}} \cong W_{j,\mathbb{H}}^{\oplus \ell_j}.$$

3.3.2. *The regular case.* We now specialize the above discussion to the case

$$I := G, \qquad \rho_g(h) := gh,$$

in which case $\boldsymbol{\rho} : G \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^G)$ is the so-called **regular representation** of G. By a standard theorem in complex representation theory, the complexification $\boldsymbol{\rho}_{\mathbb{C}} : G \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^G)$ then contains every irreducible complex representation $\boldsymbol{\lambda}_{j,\mathbb{K}} : G \to \mathrm{Aut}_{\mathbb{C}}(V_{j,\mathbb{K}})$ as a subrepresentation with multiplicity equal to $\dim_{\mathbb{C}} V_{j,\mathbb{K}}$. This implies a similar fact about $\boldsymbol{\rho}$ that we will make use of in §6 for proving Theorem D:

Lemma 3.18. *The real regular representation $\boldsymbol{\rho} : G \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^G)$ contains every irreducible representation $\boldsymbol{\theta}_{j,\mathbb{K}} : G \to \mathrm{Aut}_{\mathbb{R}}(W_{j,\mathbb{K}})$ of G as a subrepresentation with multiplicity equal to $\dim_{\mathbb{K}} W_{j,\mathbb{K}}$.* $\square$

Next, recall that the action of G on itself by right multiplication

$$G \to S(G) : g \mapsto \rho'_g, \qquad \rho'_g h := hg^{-1}$$

commutes with ρ and thus defines a second permutation representation

$$\boldsymbol{\rho}' : G \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^G), \qquad \boldsymbol{\rho}'(g)e_h = e_{hg^{-1}}$$

which commutes with $\boldsymbol{\rho}$, giving rise to a representation

$$G \times G \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^G) : (g, h) \mapsto \boldsymbol{\rho}(g)\boldsymbol{\rho}'(h). \tag{3.12}$$

By another standard theorem of complex representation theory, the summands in the isotypic decomposition (3.10) of $\mathbb{C}^G$ are then invariant under the complexification of the $(G \times G)$-action (3.12), and they define irreducible complex representations of $G \times G$. In particular, $\boldsymbol{\rho}'$ therefore preserves each isotypic component for $\boldsymbol{\rho}$ but does not preserve any further decomposition of that component into irreducible G-invariant subspaces. For future use, we note one additional fact from complex representation theory: the action of $G \times G$ on an isotypic component in $\mathbb{C}^G$ corresponding to a given irreducible representation $\boldsymbol{\lambda} : G \to \mathrm{Aut}_{\mathbb{C}}(V)$ is isomorphic to $V \otimes V^*$, with $G \times G$ acting by

$$(G \times G) \times (V \otimes V^*) \to V \otimes V^* : ((g, h), v \otimes \alpha) \mapsto \boldsymbol{\lambda}(g)v \otimes \boldsymbol{\lambda}^*(h)\alpha,$$

cf. [Ser77, §6.2].

3.3.3. *Non-faithful representations.* An important special case of the factorization construction in Example 3.5 arises when

$$\boldsymbol{\theta} : G \to \mathrm{Aut}_{\mathbb{R}}(W)$$

is an irreducible representation that is not faithful. Choosing H to be any nontrivial normal subgroup of G contained in its kernel

$$H \subset \ker \boldsymbol{\theta} \subset G,$$

G/H then inherits an irreducible representation

$$\boldsymbol{\theta}_H : G/H \to \mathrm{Aut}_{\mathbb{R}}(W).$$

For example one can take $H = \ker \boldsymbol{\theta}$, in which case $\boldsymbol{\theta}_H$ becomes faithful. Now if $\rho : G \to S(I)$ is a transitive action on the set I of d elements, let

$$\rho_H : G/H \to S(I/H)$$

denote the induced action on the set I/H of H-orbits, and consider the corresponding permutation representations

$$\boldsymbol{\rho} : G \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^I), \qquad \boldsymbol{\rho}_H : G/H \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^{I/H}).$$

Lemma 3.19. *Under the assumptions described above, the multiplicity of* $\boldsymbol{\theta} : G \to \mathrm{Aut}_{\mathbb{R}}(W)$ *as a subrepresentation of* $\boldsymbol{\rho} : G \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^I)$ *matches the multiplicity of* $\boldsymbol{\theta}_H : G/H \to \mathrm{Aut}_{\mathbb{R}}(W)$ *as a subrepresentation of* $\boldsymbol{\rho}_H : G/H \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^{I/H})$.

Proof. Observe that in terms of the real/complex/quaternionic distinction described in §3.3.1, $\boldsymbol{\theta}$ and $\boldsymbol{\theta}_H$ are necessarily of the same type: indeed, the spaces of linear maps on W that are G-equivariant or (G/H)-equivariant are the same since H acts trivially on W. The multiplicities of both are therefore determined in the same way by the multiplicities of the corresponding *complex* irreducible representations in the complexifications of $\boldsymbol{\rho}$ and $\boldsymbol{\rho}_H$ respectively, thus it will suffice to prove a similar statement about complex representations. Namely, assume $\boldsymbol{\lambda} : G \to \mathrm{Aut}_{\mathbb{C}}(V)$ is complex irreducible, $H \subset \ker \boldsymbol{\lambda} \subset G$ is a normal subgroup and $\boldsymbol{\lambda}_H : G/H \to \mathrm{Aut}_{\mathbb{C}}(V)$ is the resulting irreducible representation of G/H. By orthonormality of characters, it will suffice to prove

$$\langle \chi_{\boldsymbol{\rho}}, \chi_{\boldsymbol{\lambda}} \rangle = \langle \chi_{\boldsymbol{\rho}_H}, \chi_{\boldsymbol{\lambda}_H} \rangle,$$

where the inner product of characters $\chi_{\boldsymbol{\lambda}} : G \to \mathbb{C}$ is given in general by

$$\langle \chi_{\boldsymbol{\lambda}}, \chi_{\boldsymbol{\lambda}'} \rangle := \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{\boldsymbol{\lambda}}(g)} \chi_{\boldsymbol{\lambda}'}(g) \in \mathbb{C}.$$

For each $i \in I$, let $G_i \subset G$ denote the stabilizer subgroup for i under the G-action on I via $\boldsymbol{\rho}$. Since the action is transitive, the orbit-stabilizer theorem implies $|G_i| = |G|/d$. The trace of a permutation matrix is the number of elements that it fixes, in other words the number of stabilizer subgroups that it belongs to, hence for each $g \in G$,

$$\chi_{\boldsymbol{\rho}}(g) = \left|\left\{i \in I \;\middle|\; g \in G_i\right\}\right|.$$

This implies

$$\langle \chi_{\boldsymbol{\rho}}, \chi_{\boldsymbol{\lambda}} \rangle = \frac{1}{|G|} \sum_{i \in I} \sum_{g \in G_i} \chi_{\boldsymbol{\lambda}}(g). \tag{3.13}$$

This can be simplified since G acts transitively on I, so the subgroups G_i for distinct $i \in I$ are all conjugate. By the conjugation-invariance of characters, this implies that all d of the sums over G_i in (3.13) are identical, so plugging in $|G_i| = |G|/d$, we have

$$\langle \chi_{\boldsymbol{\rho}}, \chi_{\boldsymbol{\lambda}} \rangle = \frac{1}{|G_i|} \sum_{g \in G_i} \chi_{\boldsymbol{\lambda}}(g),$$

where $i \in I$ in this expression can be chosen arbitrarily.

To write down a similar expression for $\langle \chi_{\boldsymbol{\rho}_H}, \chi_{\boldsymbol{\lambda}_H} \rangle$, define for each $i \in I$

$$H_i := H \cap G_i \subset G,$$

which is a subgroup of both H and G_i and is normal in the latter. There is then a natural inclusion of G_i/H_i as a subgroup of G/H, and it is the stabilizer subgroup of $[i] \in I/H$ for the permutation action of G/H on I/H. The same computation thus gives

$$\langle \chi_{\boldsymbol{\rho}_H}, \chi_{\boldsymbol{\lambda}_H} \rangle = \frac{1}{|G_i/H_i|} \sum_{[g] \in G_i/H_i} \chi_{\boldsymbol{\lambda}_H}([g]) = \frac{|H_i|}{|G_i|} \sum_{[g] \in G_i/H_i} \chi_{\boldsymbol{\lambda}_H}([g]).$$

Finally, observe that $\chi_{\boldsymbol{\lambda}}(g) = \chi_{\boldsymbol{\lambda}_H}([g])$ for each $g \in G$ since both are traces of the same linear operator acting on V, so one can replace the last expression with a sum over $g \in G_i$, giving

$$\langle \chi_{\boldsymbol{\rho}_H}, \chi_{\boldsymbol{\lambda}_H} \rangle = \frac{1}{|G_i|} \sum_{g \in G_i} \chi_{\boldsymbol{\lambda}}(g) = \langle \chi_{\boldsymbol{\rho}}, \chi_{\boldsymbol{\lambda}} \rangle.$$

□

3.4. **Twisted bundles and splittings of operators.** We can now make precise the splitting of pulled back Cauchy-Riemann type operators that was sketched in §2.2.

3.4.1. *Twisted bundles from representations.* We associate to any representation $\boldsymbol{\theta} : G \to \mathrm{Aut}_{\mathbb{R}}(W)$ the family of real vector bundles $W_\tau^{\boldsymbol{\theta}} \to \dot{\Sigma}_\tau$ defined by

$$W_\tau^{\boldsymbol{\theta}} = \left(\dot{\Sigma}_\tau'' \times W\right) \big/ G,$$

where G acts on W via $\boldsymbol{\theta}$ and on $\dot{\Sigma}_\tau''$ by deck transformations, so that $\pi_\tau : \dot{\Sigma}_\tau'' \to \dot{\Sigma}_\tau$ identifies $\dot{\Sigma}_\tau$ with $\dot{\Sigma}_\tau''/G$. This gives rise to complex vector bundles $\dot{E}_\tau^{\boldsymbol{\theta}}, \dot{F}_\tau^{\boldsymbol{\theta}} \to \dot{\Sigma}_\tau$ of rank $m \cdot \dim_{\mathbb{R}} W$, defined by

$$\dot{E}_\tau^{\boldsymbol{\theta}} = \dot{E}_\tau \otimes_{\mathbb{R}} W_\tau^{\boldsymbol{\theta}}, \qquad \dot{F}_\tau^{\boldsymbol{\theta}} = \dot{F}_\tau \otimes_{\mathbb{R}} W_\tau^{\boldsymbol{\theta}} = \overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot{\Sigma}_\tau, \dot{E}_\tau^{\boldsymbol{\theta}}).$$

Each of the bundles $W_\tau^{\boldsymbol{\theta}}$ has a canonical flat structure, i.e. it comes with a well-defined notion of constant local sections, thus $\mathbf{D}_\tau \in \mathcal{CR}_{\mathbb{R}}(E_\tau)$ determines a family of Cauchy-Riemann type operators

$$\dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}} : \Gamma(\dot{E}_\tau^{\boldsymbol{\theta}}) \to \Gamma(\dot{F}_\tau^{\boldsymbol{\theta}}) = \Omega^{0,1}(\dot{\Sigma}_\tau, \dot{E}_\tau^{\boldsymbol{\theta}})$$

such that $\dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}}(\eta \otimes v) = \dot{\mathbf{D}}_\tau \eta \otimes v$ whenever v is a constant local section of $W_\tau^{\boldsymbol{\theta}}$. Since $\dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}} \in \mathcal{CR}_{\mathbb{R}}(\dot{E}_\tau^{\boldsymbol{\theta}})$, it is Fredholm in suitable Banach space settings, in particular as a bounded linear operator

$$\dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}} : W^{k,p,-\boldsymbol{\delta}}(\dot{E}_\tau^{\boldsymbol{\theta}}) \to W^{k-1,p,-\boldsymbol{\delta}}(\dot{F}_\tau^{\boldsymbol{\theta}})$$

for any $k \in \mathbb{N}$, $p \in (1, \infty)$, and negative exponential weights $-\boldsymbol{\delta} = \{-\delta_w\}_{w \in \Theta}$ with all $\delta_w > 0$ sufficiently small. We will formulate a precise version of this statement and compute the index in

§4. Observe that aside from its obvious dependence on $\mathbf{D}_\tau$, $\dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}}$ depends on our choice of regular presentation for φ and on the representation $\boldsymbol{\theta}$, but both of them only up to isomorphism.

If $\boldsymbol{\theta}$ is irreducible with $\operatorname{End}_G(W) = \mathbb{K} \in \{\mathbb{C}, \mathbb{H}\}$, then the resulting left $\mathbb{K}$-module structure of W induces a left $\mathbb{K}$-module structure on each fiber of the twisted bundles $\dot{E}_\tau^{\boldsymbol{\theta}}$ and $\dot{F}_\tau^{\boldsymbol{\theta}}$, for which the twisted operator $\dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}}$ commutes with the action of $\mathbb{K}$, thus its kernel and cokernels are also left $\mathbb{K}$-modules. Note that if $\mathbb{K} = \mathbb{C}$, the resulting complex structure on $\dot{E}_\tau^{\boldsymbol{\theta}}$ and $\dot{F}_\tau^{\boldsymbol{\theta}}$ is different from the one defined by J; the latter does not commute with $\dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}}$ unless $\mathbf{D}_\tau$ is a J-linear operator to start with.

The most important special case of the above construction is $\dot{E}_\tau^{\boldsymbol{\rho}} \to \dot{\Sigma}_\tau$, where $\boldsymbol{\rho} : G \to \operatorname{Aut}_{\mathbb{R}}(\mathbb{R}^I)$ is the permutation representation associated to our regular presentation of φ. We define $\dot{E}_\tau^{\boldsymbol{\rho}} = \dot{E}_\tau \otimes (\mathbb{R}^I)_\tau^{\boldsymbol{\rho}} \to \dot{\Sigma}_\tau$ as above and can identify it canonically with

$$\dot{E}_\tau^{\boldsymbol{\rho}} = \left(\pi_\tau^* \dot{E}_\tau \otimes \mathbb{R}^I\right) \Big/ G,$$

so that sections of $\dot{E}_\tau^{\boldsymbol{\rho}}$ are written as G-equivariant sections of $\pi_\tau^* \dot{E}_\tau \otimes \mathbb{R}^I$, hence

$$\eta = \sum_{i \in I} \eta^i \otimes e_i$$

for $\eta^i \in \Gamma(\pi_\tau^* \dot{E}_\tau)$. Here G-equivariance means that for all $z \in \dot{\Sigma}_\tau''$ and $g \in G$,

$$\eta(gz) = (\mathbb{1} \otimes \boldsymbol{\rho}(g))\eta(z) = \sum_{i \in I} \eta^i(z) \otimes e_{\rho_g(i)},$$

hence

$$\eta^i(z) = \eta^{\rho_g(i)}(gz) \quad \text{for all} \quad z \in \dot{\Sigma}_\tau'',\ g \in G \text{ and } i \in I. \tag{3.14}$$

Writing $\dot{\Sigma}_\tau' = (\dot{\Sigma}_\tau'' \times I)/G$, this relation gives rise to a bijective correspondence

$$\begin{aligned} \Gamma(\dot{E}_\tau^{\boldsymbol{\rho}}) \to \Gamma(\varphi_\tau^* \dot{E}_\tau) : \eta \mapsto \widehat{\eta} \\ \widehat{\eta}([(z,i)]) = \eta^i(z) \end{aligned} \tag{3.15}$$

and thus natural isomorphisms

$$W^{k,p,-\boldsymbol{\delta}}(\dot{E}_\tau^{\boldsymbol{\rho}}) \to W^{k,p,-\varphi^*\boldsymbol{\delta}}(\varphi_\tau^* \dot{E}_\tau) \tag{3.16}$$

for every $k \geqslant 0$ and $p \in (1, \infty)$, where we recall from §3.2 that the pulled back exponential weights are defined by

$$\varphi^* \boldsymbol{\delta} := \left\{k_\zeta \delta_{\varphi(\zeta)}\right\}_{\zeta \in \Theta'},$$

with $k_\zeta \in \{1, \ldots, d\}$ denoting the branching order of $\varphi : \Sigma' \to \Sigma$ at $\zeta \in \Theta'$. The reason for using these particular weights in the isomorphism (3.16) is as follows. We observe first that if $\varphi : [0, \infty) \times S^1 \to [0, \infty) \times S^1$ is a holomorphic covering map of the form $(s,t) \mapsto (ms, mt)$ and $\mathbb{Z}_m$ is defined to act on $[0, \infty) \times S^1$ via the transformation $(s,t) \mapsto (s, t + 1/m)$ and its iterates, then the map $f \mapsto f \circ \varphi$ defines for each integer $k \geqslant 0$ and $p \in (1, \infty)$ an isomorphism from $W^{k,p}([0, \infty) \times S^1)$ to the closed subspace of $W^{k,p}([0, \infty) \times S^1)$ consisting of $\mathbb{Z}_m$-invariant functions. It follows that for any exponential weight δ, a function f on $[0, \infty) \times S^1$ is of class $W^{k,p,\delta}$ if and only if $f \circ \varphi$ is of class $W^{k,p,m\delta}$. The global consequence of these observations is that for $\eta \in \Gamma(\dot{E}_\tau^{\boldsymbol{\rho}})$ and the corresponding section $\widehat{\eta} \in \Gamma(\varphi_\tau^* \dot{E}_\tau)$, the $W^{k,p,-\varphi^*\boldsymbol{\delta}}$-norm of $\widehat{\eta}$ can be bounded in terms of the $W^{k,p,-\boldsymbol{\delta}}$-norm of η, and vice versa.

Observe that $(\mathbb{R}^I)_\tau^{\boldsymbol{\rho}} \to \dot{\Sigma}_\tau$ also has a well-defined real bundle metric since $\boldsymbol{\rho}$ acts on $\mathbb{R}^I$ by orthogonal transformations, so endowing E_τ with a Hermitian bundle metric induces a Hermitian bundle metric on $\dot{E}_\tau^{\boldsymbol{\rho}} = \dot{E}_\tau \otimes (\mathbb{R}^I)_\tau^{\boldsymbol{\rho}}$ such that the correspondence (3.15) also preserves L^2-products. After writing down a similar correspondence for the bundles $\dot{F}_\tau^{\boldsymbol{\rho}}$ and $\varphi_\tau^* \dot{F}_\tau$, we obtain an identification between the Cauchy-Riemann operators $\varphi_\tau \dot{\mathbf{D}}_\tau$ and $\dot{\mathbf{D}}_\tau^{\boldsymbol{\rho}}$:

$$\begin{array}{ccc} W^{k,p,-\boldsymbol{\delta}}(\dot{E}_\tau^{\boldsymbol{\rho}}) & \xrightarrow{\dot{\mathbf{D}}_\tau^{\boldsymbol{\rho}}} & W^{k-1,p,-\boldsymbol{\delta}}(\dot{F}_\tau^{\boldsymbol{\rho}}) \\ \downarrow \cong & & \downarrow \cong \\ W^{k,p,-\varphi^*\boldsymbol{\delta}}(\varphi_\tau^* \dot{E}_\tau) & \xrightarrow{\varphi_\tau^* \dot{\mathbf{D}}_\tau} & W^{k-1,p,-\varphi^*\boldsymbol{\delta}}(\varphi_\tau^* \dot{F}_\tau), \end{array} \tag{3.17}$$

3.4.2. *Splitting the twisted Cauchy-Riemann operator.* If $W \subset \mathbb{R}^I$ is any G-invariant subspace and $\boldsymbol{\theta} : G \to \mathrm{Aut}_{\mathbb{R}}(W)$ denotes the resulting subrepresentation, then we obtain corresponding subbundles

$$\dot{E}_\tau^{\boldsymbol{\theta}} \subset \dot{E}_\tau^{\boldsymbol{\rho}}, \qquad \dot{F}_\tau^{\boldsymbol{\theta}} \subset \dot{F}_\tau^{\boldsymbol{\rho}}$$

such that $\dot{\mathbf{D}}_\tau^{\boldsymbol{\rho}}$ takes sections of $\dot{E}_\tau^{\boldsymbol{\theta}}$ to sections of $\dot{F}_\tau^{\boldsymbol{\theta}}$, acting as the operator $\dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}}$. Under the correspondence (3.15), one can understand this as identifying $\Gamma(\dot{E}_\tau^{\boldsymbol{\theta}})$ and $\Gamma(\dot{F}_\tau^{\boldsymbol{\theta}})$ with closed subspaces

$$\Gamma_{\boldsymbol{\theta}}(\varphi_\tau^* \dot{E}_\tau) \subset \Gamma(\varphi_\tau^* \dot{E}_\tau), \qquad \Gamma_{\boldsymbol{\theta}}(\varphi_\tau^* \dot{F}_\tau) \subset \Gamma(\varphi_\tau^* \dot{F}_\tau),$$

with a similar definition for closed subspaces of the relevant weighted Sobolev spaces, such that $\varphi_\tau^* \dot{\mathbf{D}}_\tau$ restricts to a bounded linear operator

$$W_{\boldsymbol{\theta}}^{k,p,-\varphi^*\boldsymbol{\delta}}(\varphi_\tau^* \dot{E}_\tau) \xrightarrow{\varphi_\tau^* \dot{\mathbf{D}}_\tau} W_{\boldsymbol{\theta}}^{k-1,p,-\varphi^*\boldsymbol{\delta}}(\varphi_\tau^* \dot{F}_\tau),$$

which is conjugate to $\dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}} : W^{k,p,-\boldsymbol{\delta}}(\dot{E}_\tau^{\boldsymbol{\theta}}) \to W^{k-1,p,-\boldsymbol{\delta}}(\dot{F}_\tau^{\boldsymbol{\theta}})$ and will thus be Fredholm with any negative exponential weights that are close enough to 0. Now if

$$\mathbb{R}^I = W_1 \oplus \ldots \oplus W_N$$

is a decomposition of $\boldsymbol{\rho}$ into subrepresentations $\boldsymbol{\theta}_j : G \to \mathrm{Aut}_{\mathbb{R}}(W_j)$ for $j = 1, \ldots, N$, we obtain a direct sum decomposition

$$\dot{\mathbf{D}}_\tau^{\boldsymbol{\rho}} = \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}_1} \oplus \ldots \oplus \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}_N},$$

which is equivalent via (3.17) to a decomposition of $\varphi_\tau^* \dot{\mathbf{D}}_\tau$ over a splitting of Banach spaces

$$W^{k,p,-\varphi^*\boldsymbol{\delta}}(\varphi_\tau^* E_\tau) = \bigoplus_{j=1}^N W_{\boldsymbol{\theta}_j}^{k,p,-\varphi^*\boldsymbol{\delta}}(\varphi_\tau^* E_\tau)$$

and the corresponding decomposition of $W^{k-1,p,-\varphi^*\boldsymbol{\delta}}(\varphi_\tau^* F_\tau)$. Observe that if the subspaces $W_1, \ldots, W_N \subset \mathbb{R}^I$ are pairwise orthogonal, then the corresponding spaces of sections of $\varphi_\tau^* \dot{E}_\tau$ and $\varphi_\tau^* \dot{F}_\tau$ are L^2-orthogonal as a consequence. It is useful to note that whenever two of the representations $\boldsymbol{\theta}_i : G \to \mathrm{Aut}_{\mathbb{R}}(W_i)$ and $\boldsymbol{\theta}_j : G \to \mathrm{Aut}_{\mathbb{R}}(W_j)$ are isomorphic, the G-equivariant isomorphism $W_i \to W_j$ induces bundle isomorphisms $\dot{E}_\tau^{\boldsymbol{\theta}_i} \to \dot{E}_\tau^{\boldsymbol{\theta}_j}$ and $\dot{F}_\tau^{\boldsymbol{\theta}_i} \to \dot{F}_\tau^{\boldsymbol{\theta}_j}$ that identify $\dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}_i}$ with $\dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}_j}$, so these two operators have isomorphic kernels and cokernels. This implies:

Lemma 3.20. *Suppose $\boldsymbol{\theta}_j : G \to \mathrm{Aut}_{\mathbb{R}}(W_j)$ for $j = 1, \ldots, N$ is a collection of representations of G, and $\boldsymbol{\theta} : G \to \mathrm{Aut}_{\mathbb{R}}(W)$ is another representation such that*

$$\boldsymbol{\theta} \cong \bigoplus_{j=1}^N \boldsymbol{\theta}_j^{\oplus k_j}$$

for some integers $k_1, \ldots, k_N \geqslant 0$. Then there exist isomorphisms

$$\ker \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}} \cong \bigoplus_{j=1}^N \left(\ker \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}_j} \right)^{\oplus k_j} \quad \text{and} \quad \operatorname{coker} \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}} \cong \bigoplus_{j=1}^N \left(\operatorname{coker} \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}_j} \right)^{\oplus k_j}.$$

In particular, if $\boldsymbol{\theta}$ is the permutation representation $\boldsymbol{\rho} : G \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^I)$, this gives isomorphisms

$$\ker(\varphi_\tau^* \dot{\mathbf{D}}_\tau) \cong \bigoplus_{j=1}^N \left(\ker \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}_j} \right)^{\oplus k_j} \quad \text{and} \quad \operatorname{coker}(\varphi_\tau^* \dot{\mathbf{D}}_\tau) \cong \bigoplus_{j=1}^N \left(\operatorname{coker} \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}_j} \right)^{\oplus k_j}.$$

□

3.4.3. *Non-faithful representations revisited.* Here is a proof of Lemma 2.16. For the present discussion we drop the parameter τ from the notation since it does not play any important role.

Suppose $\boldsymbol{\theta} : G \to \mathrm{Aut}_{\mathbb{R}}(W)$ is a representation and $H \subset \ker \boldsymbol{\theta} \subset G$ is a nontrivial normal subgroup of G, giving rise to a representation

$$\boldsymbol{\theta}_H : G/H \to \mathrm{Aut}_{\mathbb{R}}(W),$$

and (following Example 3.5) a factorization of $\varphi : \Sigma' \to \Sigma$ as

$$\Sigma' \to \Sigma'_H \xrightarrow{\varphi_H} \Sigma.$$

By assumption we are using a minimal regular presentation and thus $\rho : G \to S(I)$ is injective, so H acts nontrivially on I, implying $\deg(\varphi_H) < d$. Writing $\dot{\Sigma}''_H = \dot{\Sigma}''/H$, the obvious projection map

$$\left(\dot{\Sigma}'' \times W\right) \Big/ G \to \left(\dot{\Sigma}''_H \times W\right) \Big/ (G/H)$$

is then an isomorphism of real vector bundles over $\dot{\Sigma}$ and thus gives rise to a canonical identification between the twisted bundles $\dot{E}^{\boldsymbol{\theta}}$ and $\dot{E}^{\boldsymbol{\theta}_H}$ with their Cauchy-Riemann operators $\dot{\mathbf{D}}^{\boldsymbol{\theta}}$ and $\dot{\mathbf{D}}^{\boldsymbol{\theta}_H}$. To prove the lemma, we now just need to observe that Lemma 3.19 implies $\boldsymbol{\theta}$ is a subrepresentation of $\boldsymbol{\rho}$ if and only if $\boldsymbol{\theta}_H$ is a subrepresentation of $\boldsymbol{\rho}_H$, hence the corresponding twisted operators appear simultaneously as summands in the decompositions of $\varphi^*\dot{\mathbf{D}}$ and $\varphi_H^*\dot{\mathbf{D}}$ from Lemma 3.20.

Remark 3.21. In the situation above, one should interpret $\ker \dot{\mathbf{D}}^{\boldsymbol{\theta}}$ as the set of all sections in $\ker(\varphi^*\dot{\mathbf{D}})$ that are pullbacks of sections in $\ker \dot{\mathbf{D}}^{\boldsymbol{\theta}_H}$ (interpreted as a subspace of $\ker(\varphi_H^*\dot{\mathbf{D}})$) via the branched cover $\Sigma' \to \Sigma'_H$.

3.4.4. *The regular case revisited.* Now consider the special case where $\boldsymbol{\rho}$ is the regular representation $G \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^G)$, defined via

$$\rho : G \to S(G), \qquad \rho_g(h) = gh.$$

We saw in Example 3.4 that this means $\varphi_\tau : \dot{\Sigma}'_\tau \to \dot{\Sigma}_\tau$ are all regular covers isomorphic to $\pi : \dot{\Sigma}''_\tau \to \dot{\Sigma}_\tau$, and the action of G on $\dot{\Sigma}'_\tau = (\dot{\Sigma}''_\tau \times G)/G$ by deck transformations takes the form

$$g[(z,h)] := [(z, \rho'_g(h))]$$

where $\rho' : G \to S(G)$ is the action of G on itself by right multiplication, $\rho'_g(h) = hg^{-1}$. The induced G-action on spaces of sections η of $\varphi_\tau^*\dot{E}_\tau$ is defined by

$$(g\eta)([(z,h)]) := \eta(g^{-1}[(z,h)]) = \eta([(z,hg)]).$$

Recall now from §3.3.2 that the permutation representation $\boldsymbol{\rho}' : G \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^G)$ arising from ρ' commutes with $\boldsymbol{\rho}$ and preserves the isotypic components of $\boldsymbol{\rho}$. It therefore defines an action on $\dot{E}_\tau^{\boldsymbol{\rho}}$ by fiber-preserving bundle isomorphisms, and these isomorphisms preserve each of the subbundles in the splitting

$$\dot{E}_\tau^{\boldsymbol{\rho}} = \bigoplus_{j=1}^{p} (\dot{E}_\tau^{\boldsymbol{\rho}})_{j,\mathbb{R}} \oplus \bigoplus_{j=1}^{q} (\dot{E}_\tau^{\boldsymbol{\rho}})_{j,\mathbb{C}} \oplus \bigoplus_{j=1}^{n} (\dot{E}_\tau^{\boldsymbol{\rho}})_{j,\mathbb{H}} \tag{3.18}$$

corresponding to the isotypic decomposition (3.11) of $\boldsymbol{\rho}$. In particular, this G-action by bundle isomorphisms gives a linear G-action on each of the subspaces $\Gamma((\dot{E}_\tau^{\boldsymbol{\rho}})_{j,\mathbb{K}}) \subset \Gamma(\dot{E}_\tau^{\boldsymbol{\rho}})$, and there is a similar action on sections of $\dot{F}_\tau^{\boldsymbol{\rho}}$ such that the restriction of $\dot{\mathbf{D}}_\tau^{\boldsymbol{\rho}}$ to each of these subspaces is G-equivariant. Its kernel and cokernel thus inherit natural G-actions. Under the correspondence (3.15), this action on sections of $\dot{E}_\tau^{\boldsymbol{\rho}}$ matches the action by deck transformations on $\Gamma(\varphi_\tau^*\dot{E}_\tau)$.

Lemma 3.22. *Suppose $\rho : G \to S(G)$ is defined by left multiplication, $\boldsymbol{\theta}_0 : G \to \mathrm{Aut}_{\mathbb{R}}(W)$ is an irreducible representation of G, and $\boldsymbol{\theta} : G \to \mathrm{Aut}_{\mathbb{R}}(Y)$ denotes the corresponding summand in the isotypic decomposition* (3.11) *of the regular representation $\boldsymbol{\rho} : G \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^G)$. Then every irreducible subrepresentation for the natural G-action on $\ker \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}}$ or $\operatorname{coker} \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}}$ is isomorphic to $\boldsymbol{\theta}_0$.*

Proof. Suppose first that $\boldsymbol{\theta}_0$ is of either real or quaternionic type, in which case the complexification $X := Y \oplus iY \subset \mathbb{C}^G$ of $Y \subset \mathbb{R}^G$ is also an isotypic component for the complexified regular representation $\boldsymbol{\rho}_{\mathbb{C}} : G \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^G)$. We shall denote the restriction of $\boldsymbol{\rho}_{\mathbb{C}}$ to X by

$$\boldsymbol{\lambda} : G \to \mathrm{Aut}_{\mathbb{C}}(X),$$

and let $\boldsymbol{\lambda}_0 : G \to \mathrm{Aut}_{\mathbb{C}}(V)$ denote the underlying complex irreducible representation. Regarding these complex representations as real representations on X and V respectively gives rise to corresponding twisted bundles and Cauchy-Riemann operators on them, along with a natural linear inclusion of vector bundles

$$\dot{E}_\tau^{\boldsymbol{\theta}} \hookrightarrow \dot{E}_\tau^{\boldsymbol{\lambda}} \qquad \text{such that} \qquad \ker \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}} = \ker \dot{\mathbf{D}}_\tau^{\boldsymbol{\lambda}} \cap \Gamma(\dot{E}_\tau^{\boldsymbol{\theta}}).$$

It will be useful to think of $\dot{E}_\tau^{\boldsymbol{\lambda}}$ as a *complexification* of $\dot{E}_\tau^{\boldsymbol{\theta}}$, in the following sense. While $\dot{E}_\tau^{\boldsymbol{\theta}}$ is already a complex vector bundle, $\dot{E}_\tau^{\boldsymbol{\lambda}} = \dot{E}_\tau \otimes_{\mathbb{R}} X_\tau^{\boldsymbol{\lambda}}$ naturally carries *two* complex structures J_τ and i, which commute with each other: the former acts on $\eta \otimes v \in \dot{E}_\tau \otimes_{\mathbb{R}} X_\tau^{\boldsymbol{\lambda}}$ by $J_\tau \eta \otimes v$ and the latter by $\eta \otimes iv$, using the fact that $\boldsymbol{\lambda}$ is a complex representation and $X_\tau^{\boldsymbol{\lambda}}$ is therefore naturally a complex vector bundle. From this perspective, $\dot{\mathbf{D}}_\tau^{\boldsymbol{\lambda}}$ is the natural i-complex-linear extension of $\dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}}$ to its complexified domain, and the representations defined by the G-action on $\ker \dot{\mathbf{D}}_\tau^{\boldsymbol{\lambda}}$ and $\operatorname{coker} \dot{\mathbf{D}}_\tau^{\boldsymbol{\lambda}}$ will be the complexifications of the real representations it defines on $\ker \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}}$ and $\operatorname{coker} \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}}$ respectively. In the following we shall use the symbol "$\otimes_i$" to denote complex tensor products of vector spaces and bundles with i (instead of J_τ) as the complex structure.

Recall now that as an isotypic component of the complex regular representation, X admits a complex-linear isomorphism to $V \otimes_i V^*$ such that for all $g \in G$, $\boldsymbol{\rho}(g)$ acts on $V \otimes_i V^*$ as $\boldsymbol{\lambda}_0 \otimes \mathbb{1}$, while $\boldsymbol{\rho}'(g)$ acts as $\mathbb{1} \otimes \boldsymbol{\lambda}_0^*$. The isomorphism $X \to V \otimes_i V^*$ thus gives rise to i-complex bundle isomorphisms

$$\dot{E}_\tau^{\boldsymbol{\lambda}} \to \dot{E}_\tau^{\boldsymbol{\lambda}_0} \otimes_i V^*, \qquad \dot{F}_\tau^{\boldsymbol{\lambda}} \to \dot{F}_\tau^{\boldsymbol{\lambda}_0} \otimes_i V^*,$$

where we are abusing notation to let V^* denote the trivial bundle over $\dot{\Sigma}_\tau$ with fiber V^*, and this identifies $\dot{\mathbf{D}}_\tau^{\boldsymbol{\lambda}}$ with $\dot{\mathbf{D}}_\tau^{\boldsymbol{\lambda}_0} \otimes \mathbb{1}$. We therefore have

$$\ker \dot{\mathbf{D}}_\tau^{\boldsymbol{\lambda}} \cong \ker \dot{\mathbf{D}}_\tau^{\boldsymbol{\lambda}_0} \otimes_i V^*, \qquad \operatorname{coker} \dot{\mathbf{D}}_\tau^{\boldsymbol{\lambda}} \cong \operatorname{coker} \dot{\mathbf{D}}_\tau^{\boldsymbol{\lambda}_0} \otimes_i V^*,$$

with G acting on both by $\mathbb{1} \otimes \boldsymbol{\lambda}_0^*$, hence all irreducible subrepresentations in these spaces are isomorphic to $\boldsymbol{\lambda}_0^*$, which is isomorphic to $\boldsymbol{\lambda}_0$ since the latter is not of complex type. Viewing these as complexifications of real representations on $\ker \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}}$ and $\operatorname{coker} \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}}$ as explained above, it follows via the correspondence between real and complex irreducible representations outlined in §3.3.1 that all the irreducible real subrepresentations are isomorphic to $\boldsymbol{\theta}_0$.

The main difference if $\boldsymbol{\theta}_0$ is of complex type is that $Y \oplus iY \subset \mathbb{C}^G$ is no longer an isotypic component for $\boldsymbol{\rho}_{\mathbb{C}}$, but is instead the direct sum of two isotypic components related to each other by complex conjugation

$$Y \oplus iY = X \oplus \bar{X} \subset \mathbb{C}^G,$$

corresponding to some complex irreducible representation $\boldsymbol{\lambda}_0 : G \to \mathrm{Aut}_{\mathbb{C}}(V)$ and its non-isomorphic dual $\boldsymbol{\lambda}_0^* : G \to \mathrm{Aut}_{\mathbb{C}}(V^*)$. Writing $\boldsymbol{\lambda} : G \to \mathrm{Aut}_{\mathbb{C}}(X)$ and $\bar{\boldsymbol{\lambda}} : G \to \mathrm{Aut}_{\mathbb{C}}(\bar{X})$ for the restriction of $\boldsymbol{\rho}_{\mathbb{C}}$ to these subspaces, we can then think of $\dot{\mathbf{D}}_\tau^{\boldsymbol{\lambda} \oplus \bar{\boldsymbol{\lambda}}} = \dot{\mathbf{D}}_\tau^{\boldsymbol{\lambda}} \oplus \dot{\mathbf{D}}_\tau^{\bar{\boldsymbol{\lambda}}}$ as the complexification of $\dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}}$. A repeat of the argument above using the isomorphisms $X \cong V \otimes_i V^*$ and $\bar{X} \cong V^* \otimes_i V$ then gives an i-complex-linear isomorphism

$$\ker \dot{\mathbf{D}}_\tau^{\boldsymbol{\lambda} \oplus \bar{\boldsymbol{\lambda}}} \cong (\ker \dot{\mathbf{D}}_\tau^{\boldsymbol{\lambda}_0} \otimes_i V^*) \oplus (\ker \dot{\mathbf{D}}_\tau^{\boldsymbol{\lambda}_0^*} \otimes_i V),$$

with G acting via $\mathbb{1} \otimes \boldsymbol{\lambda}_0^*$ on the first summand and $\mathbb{1} \otimes \boldsymbol{\lambda}_0$ on the second, and a similar isomorphism for cokernels. It follows that every irreducible subrepresentation in either $\ker \dot{\mathbf{D}}_\tau^{\boldsymbol{\lambda} \oplus \bar{\boldsymbol{\lambda}}}$

or coker $\dot{\mathbf{D}}_\tau^{\boldsymbol{\lambda}\oplus\bar{\boldsymbol{\lambda}}}$ is isomorphic to one of $\boldsymbol{\lambda}_0$ or $\boldsymbol{\lambda}_0^*$, and the desired result for real subrepresentations again follows via the correspondence between real and complex representations in §3.3.1. $\square$

Continuing in the setting of Lemma 3.22, let $\mathbb{K} = \mathrm{End}_G(W) \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and write $k = \dim_{\mathbb{K}} \ker \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}_0}$, $c = \dim_{\mathbb{K}} \operatorname{coker} \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}_0}$. By Lemma 3.18, $\boldsymbol{\theta} \cong \boldsymbol{\theta}_0^{\oplus m}$ with $m := \dim_{\mathbb{K}} W$, so Lemma 3.20 gives $\dim_{\mathbb{K}} \ker \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}} = km$ and $\dim_{\mathbb{K}} \operatorname{coker} \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}} = cm$. Lemma 3.22 meanwhile decomposes the representation defined by the G-action on $\ker \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}}$ as $\boldsymbol{\theta}_0^{\oplus \ell}$ for some $\ell \geqslant 0$, so $\ker \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}} \cong W^{\oplus \ell}$. Comparing dimensions, we deduce $\ell = k$, and applying the same argument to the cokernel then likewise identifies the representation defined by the G-action on $\operatorname{coker} \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}}$ with $\boldsymbol{\theta}_0^{\oplus c}$. The following consequence is the origin of the codimension formula in Theorem D (cf. 3.23).

Corollary 3.23. *In the setting of Lemma 3.22, let* $\mathbb{K} = \mathrm{End}_G(W)$. *Then the space of G-equivariant real-linear maps* $\ker \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}} \to \operatorname{coker} \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}}$ *satisfies*

$$\dim_{\mathbb{R}} \mathrm{Hom}_G\left(\ker \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}}, \operatorname{coker} \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}}\right) = \dim_{\mathbb{R}} \mathbb{K} \cdot \dim_{\mathbb{K}} \ker \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}_0} \cdot \dim_{\mathbb{K}} \operatorname{coker} \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}_0}.$$

$\square$

3.5. **Setting up the implicit function theorem.** We assume throughout this section that $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$ is the *minimal* regular presentation of $\varphi : \Sigma' \to \Sigma$. Suppose

$$\boldsymbol{\theta}_i : G \to \mathrm{Aut}_{\mathbb{R}}(W_i), \qquad i = 1, \dots, N$$

is a complete list of pairwise non-isomorphic real irreducible representations for G, with

$$\mathbb{K}_i := \mathrm{End}_G(W_i), \quad \text{and} \quad t_i := \dim_{\mathbb{R}} \mathbb{K}_i \in \{1, 2, 4\}.$$

Recall that all of the data we have been considering depends smoothly on a parameter τ, which lives in a connected Banach manifold P as described at the end of §3.1. Any N-tuples of nonnegative integers $\mathbf{k} = (k_1, \dots, k_N)$ and $\mathbf{c} = (c_1, \dots, c_N)$ now determine subsets of this parameter space

$$P(\mathbf{k}, \mathbf{c}) := \left\{ \tau \in P \;\middle|\; \dim_{\mathbb{K}_i} \ker \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}_i} = k_i \text{ and } \dim_{\mathbb{K}_i} \operatorname{coker} \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}_i} = c_i \text{ for all } i = 1, \dots, N \right\}.$$

Note that $P(\mathbf{k}, \mathbf{c})$ is automatically empty unless $k_i - c_i = \mathrm{ind}_{\mathbb{K}_i} \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}_i}$ for all $i = 1, \dots, N$, and these indices do not depend on the parameter τ. Assuming this condition holds, we would now like to present $P(\mathbf{k}, \mathbf{c})$ locally as the zero-set of a smooth map to a finite-dimensional vector space, and to compute its derivative in a special case.

We start by translating the conditions defining $P(\mathbf{k}, \mathbf{c})$ into conditions on the pulled back operators $\widehat{\varphi}_\tau^* \dot{\mathbf{D}}_\tau$ for a suitable family of *regular* covers $\widehat{\varphi}_\tau : \widehat{\Sigma}_\tau \to \dot{\Sigma}_\tau$ with $\mathrm{Aut}(\widehat{\varphi}_\tau) = G$. This can be defined by replacing the homomorphism $\rho : G \to S(I)$ with the action of G on itself by left multiplication, i.e. let

$$\widehat{\rho} : G \to S(G) : g \mapsto \widehat{\rho}_g, \qquad \widehat{\rho}_g(h) := gh,$$

so that $(\Theta_\tau, \dot{\Sigma}'', \pi_\tau, G, \widehat{\rho}, G, \mathrm{Id})$ becomes a minimal regular presentation for

$$\widehat{\Sigma}_\tau := \left(\dot{\Sigma}''_\tau \times G\right) \Big/ G \xrightarrow{\widehat{\varphi}_\tau} \dot{\Sigma}_\tau : [(z, g)] \mapsto \pi_\tau(z),$$

or rather for the extension of this map to a branched cover of closed surfaces as provided by Lemma 3.1. In keeping with our usual notational convention, $\widehat{\Sigma}_\tau$ is a fixed smooth surface $\widehat{\Sigma}$ with a fixed G-action by deck transformations but a τ-dependent family of conformal structures $\widehat{\jmath}_\tau = \widehat{\varphi}_\tau^* j_\tau$, which are fixed on the cylindrical ends.

Denote the isotypic decomposition of the regular representation $\widehat{\boldsymbol{\rho}} : G \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^G)$ by

$$\widehat{\boldsymbol{\rho}} = \bigoplus_{i=1}^N \widehat{\boldsymbol{\theta}}_i,$$

where $\widehat{\boldsymbol{\theta}}_i \cong \boldsymbol{\theta}_i^{\oplus \ell_i}$ for integers ℓ_i which are strictly positive by Lemma 3.18. Then by Lemma 3.20,

$$\ker(\widehat{\varphi}_\tau^* \dot{\mathbf{D}}_\tau) \cong \bigoplus_{i=1}^N \ker \dot{\mathbf{D}}_\tau^{\widehat{\boldsymbol{\theta}}_i} \cong \bigoplus_{i=1}^N \left(\ker \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}_i}\right)^{\oplus \ell_i},$$
$$\operatorname{coker}(\widehat{\varphi}_\tau^* \dot{\mathbf{D}}_\tau) \cong \bigoplus_{i=1}^N \operatorname{coker} \dot{\mathbf{D}}_\tau^{\widehat{\boldsymbol{\theta}}_i} \cong \bigoplus_{i=1}^N \left(\operatorname{coker} \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}_i}\right)^{\oplus \ell_i},$$

so $\tau \in P(\mathbf{k}, \mathbf{c})$ implies

$$\dim \ker(\widehat{\varphi}_\tau^* \dot{\mathbf{D}}_\tau) = \sum_{i=1}^N t_i \ell_i k_i. \tag{3.19}$$

Lemma 3.24. *Every $\sigma \in P(\mathbf{k}, \mathbf{c})$ has a neighborhood $\mathcal{U}_\sigma \subset P$ such that $\mathcal{U}_\sigma \cap P(\mathbf{k}, \mathbf{c})$ is the set of all $\tau \in \mathcal{U}_\sigma$ for which* (3.19) *holds.*

Proof. Since all the operators $\dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}_i}$ are Fredholm and they depend continuously on τ, we can assume $\dim \ker \dot{\mathbf{D}}_\tau^{\boldsymbol{\theta}_i} \leqslant \dim \ker \dot{\mathbf{D}}_\sigma^{\boldsymbol{\theta}_i}$ for all $i = 1, \dots, N$ if τ is sufficiently close to σ. Thus (3.19) can only be satisfied if none of these inequalities are strict, which means $\tau \in P(\mathbf{k}, \mathbf{c})$ since every ℓ_i is positive. □

Recall from §3.2 that the weighted Sobolev spaces $W^{k,p,-\widehat{\varphi}^* \boldsymbol{\delta}}(\widehat{\varphi}_\tau^* \dot{E}_\tau)$ and $W^{k-1,p,-\widehat{\varphi}^* \boldsymbol{\delta}}(\widehat{\varphi}_\tau^* \dot{F}_\tau)$ are defined in terms of fixed families of trivializations of E_τ near Θ_τ and holomorphic cylindrical coordinates which allow us to compute Sobolev norms on the cylindrical ends. Given $\sigma \in P(\mathbf{k}, \mathbf{c})$, choose a neighborhood $\mathcal{U}_\sigma \subset P$ that is diffeomorphic to a ball and small enough to satisfy Lemma 3.24. By assumption the bundles E_τ depend smoothly on τ, which means there is a well-defined smooth bundle $\widehat{E} \to P \times \Sigma$ with $\widehat{E}_{(\tau,z)} = (E_\tau)_z$. Choosing a suitable connection on the latter, we can use parallel transport along paths of the form $(\tau(t), \psi_{\tau(t)}(z)) \in \mathcal{U}_\sigma \times \Sigma$ with $\tau(t)$ radiating outward from σ to define a smooth family of complex bundle isomorphisms

$$\Psi_\tau : \psi_\sigma^* E_\sigma \to \psi_\tau^* E_\tau$$

which respect these fixed trivializations near Θ_τ and satisfy $\Psi_\sigma = \mathrm{Id}$. These give rise to isomorphisms $\dot{E}_\sigma \to \dot{E}_\tau$ covering the diffeomorphisms $\psi_\tau \circ \psi_\sigma^{-1} : \dot{\Sigma}_\sigma \to \dot{\Sigma}_\tau$. Notice that there are also natural real bundle isomorphisms

$$d\psi_\tau : T\Sigma \to \psi_\tau^* T\Sigma,$$

so that $d\psi_\tau \circ d\psi_\sigma^{-1}$ gives a family of isomorphisms $T\dot{\Sigma}_\sigma \to T\dot{\Sigma}_\tau$ covering $\dot{\Sigma}_\sigma \xrightarrow{\psi_\tau \circ \psi_\sigma^{-1}} \dot{\Sigma}_\tau$, and they respect the chosen holomorphic cylindrical coordinates on the ends. These then induce smooth families of isomorphisms of complex bundles over $\widehat{\Sigma}$,

$$\widehat{\varphi}_\sigma^* \dot{E}_\sigma \to \widehat{\varphi}_\tau^* \dot{E}_\tau, \qquad \widehat{\varphi}_\sigma^* \dot{F}_\sigma \to \widehat{\varphi}_\tau^* \dot{F}_\tau$$

which again are the identity for $\tau = \sigma$ and are also equivariant with respect to the natural G-action by bundle isomorphisms covering deck transformations of $\widehat{\Sigma}$. Acting with these on sections produces τ-parametrized families of G-equivariant Banach space isomorphisms which we shall also denote by Ψ_τ:

$$\begin{aligned} W^{k,p,-\widehat{\varphi}^* \boldsymbol{\delta}}(\widehat{\varphi}_\sigma^* \dot{E}_\sigma) &\xrightarrow{\Psi_\tau} W^{k,p,-\widehat{\varphi}^* \boldsymbol{\delta}}(\widehat{\varphi}_\tau^* \dot{E}_\tau), \\ W^{k-1,p,-\widehat{\varphi}^* \boldsymbol{\delta}}(\widehat{\varphi}_\sigma^* \dot{F}_\sigma) &\xrightarrow{\Psi_\tau} W^{k-1,p,-\widehat{\varphi}^* \boldsymbol{\delta}}(\widehat{\varphi}_\tau^* \dot{F}_\tau). \end{aligned} \tag{3.20}$$

Here $\Psi_\sigma = \mathrm{Id}$.

We can now use these isomorphisms to define for $\tau \in \mathcal{U}_\sigma$ a smooth family of G-equivariant Fredholm operators with fixed domain and target space,

$$\widehat{\mathbf{D}}_\tau := \Psi_\tau^{-1} \circ \widehat{\varphi}_\tau^* \dot{\mathbf{D}}_\tau \circ \Psi_\tau : W^{k,p,-\widehat{\varphi}^* \boldsymbol{\delta}}(\widehat{\varphi}_\sigma^* \dot{E}_\sigma) \to W^{k-1,p,-\widehat{\varphi}^* \boldsymbol{\delta}}(\widehat{\varphi}_\sigma^* \dot{F}_\sigma), \tag{3.21}$$

such that

$$\mathcal{U}_\sigma \cap P(\mathbf{k},\mathbf{c}) = \left\{ \tau \in \mathcal{U}_\sigma \;\middle|\; \dim\ker \widehat{\mathbf{D}}_\tau = \sum_{i=1}^N t_i \ell_i k_i \right\}.$$

In order to present the latter as the zero-set of a smooth map, let us abbreviate

$$\mathbf{X}_\sigma := W^{k,p,-\widehat{\varphi}^*\boldsymbol{\delta}}(\widehat{\varphi}_\sigma^* \dot{E}_\sigma), \qquad \mathbf{Y}_\sigma := W^{k-1,p,-\widehat{\varphi}^*\boldsymbol{\delta}}(\widehat{\varphi}_\sigma^* \dot{F}_\sigma),$$

so (3.21) defines a smooth map

$$\mathcal{U}_\sigma \to \mathscr{L}_G(\mathbf{X}_\sigma, \mathbf{Y}_\sigma) : \tau \mapsto \widehat{\mathbf{D}}_\tau,$$

where $\mathscr{L}_G(\mathbf{X}_\sigma, \mathbf{Y}_\sigma)$ denotes the Banach space of bounded real-linear maps $\mathbf{X}_\sigma \to \mathbf{Y}_\sigma$ that are G-equivariant. Since $\widehat{\mathbf{D}}_\sigma = \widehat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma$ is Fredholm, we can choose a splitting

$$\mathbf{X}_\sigma = \mathbf{V}_\sigma \oplus \ker(\widehat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma),$$

such that $\mathbf{V}_\sigma \subset \mathbf{X}_\sigma$ is a closed subspace and $\widehat{\mathbf{D}}_\sigma$ maps $\mathbf{V}_\sigma$ isomorphically to its image. By Proposition 3.13, we can similarly split

$$\mathbf{Y}_\sigma = \operatorname{im}(\widehat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma) \oplus \ker(\widehat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma^*),$$

where $\ker(\widehat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma^*)$ is equivalently the space of all sections in $W^{k-1,p,\widehat{\varphi}^*\boldsymbol{\delta}}(\widehat{\varphi}_\sigma^* \dot{F}_\sigma)$ that are L^2-orthogonal to $\operatorname{im}(\widehat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma)$. In terms of these splittings, $\widehat{\mathbf{D}}_\tau$ can be written in block form

$$\widehat{\mathbf{D}}_\tau = \begin{pmatrix} \mathbf{D}_\tau^{11} & \mathbf{D}_\tau^{12} \\ \mathbf{D}_\tau^{21} & \mathbf{D}_\tau^{22} \end{pmatrix},$$

where after shrinking $\mathcal{U}_\sigma$ if necessary, we can assume without loss of generality that $\mathbf{D}_\tau^{11} : \mathbf{V}_\sigma \to \operatorname{im}(\widehat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma)$ is invertible for all $\tau \in \mathcal{U}_\sigma$. We can therefore define a map

$$\begin{aligned} \mathbf{F}_\sigma : \mathcal{U}_\sigma &\to \operatorname{Hom}_G\left(\ker(\widehat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma), \ker(\widehat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma^*)\right) \\ \tau &\mapsto \mathbf{D}_\tau^{22} - \mathbf{D}_\tau^{21} (\mathbf{D}_\tau^{11})^{-1} \mathbf{D}_\tau^{12}. \end{aligned} \tag{3.22}$$

Lemma 3.25. *A parameter $\tau \in \mathcal{U}_\sigma$ belongs to $P(\mathbf{k},\mathbf{c})$ if and only if $\mathbf{F}_\sigma(\tau) = 0$.*

Proof. Define for each $\tau \in \mathcal{U}_\sigma$ the Banach space isomorphism

$$\mathbf{T} = \begin{pmatrix} \mathbb{1} & -(\mathbf{D}_\tau^{11})^{-1}\mathbf{D}_\tau^{12} \\ 0 & \mathbb{1} \end{pmatrix} \in \mathscr{L}(\mathbf{V}_\sigma \oplus \ker(\widehat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma)) = \mathscr{L}(\mathbf{X}_\sigma).$$

Then $\widehat{\mathbf{D}}_\tau \mathbf{T} = \begin{pmatrix} \mathbf{D}_\tau^{11} & 0 \\ \mathbf{D}_\tau^{21} & \mathbf{F}_\sigma(\tau) \end{pmatrix}$, and since $\mathbf{D}_\tau^{11}$ is invertible,

$$\ker \widehat{\mathbf{D}}_\tau \cong \ker(\widehat{\mathbf{D}}_\tau \mathbf{T}) = \{0\} \oplus \ker \mathbf{F}_\sigma(\tau) \cong \ker \mathbf{F}_\sigma(\tau).$$

The latter can only have the same dimension as $\ker(\widehat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma)$ if $\mathbf{F}_\sigma(\tau)$ vanishes. □

Observe that by Lemma 3.22, Corollary 3.23 and Schur's lemma,

$$\dim \operatorname{Hom}_G\left(\ker(\widehat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma), \ker(\widehat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma^*)\right) = \sum_{i=1}^N t_i k_i c_i. \tag{3.23}$$

The lemma implies via the implicit function theorem that a neighborhood of σ in $P(\mathbf{k},\mathbf{c})$ is a smooth submanifold with the same codimension that appears in Theorem D whenever we can show that the linearization

$$d\mathbf{F}_\sigma(\sigma) : T_\sigma P \to \operatorname{Hom}_G\left(\ker(\widehat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma), \ker(\widehat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma^*)\right)$$

is surjective.

We will need a precise formula for this linearization in the following special case. Suppose we have a smooth path

$$\gamma : (-\epsilon, \epsilon) \to P \qquad \text{with } \gamma(0) = \sigma \text{ and } \dot{\gamma}(0) = Y \in T_\sigma P$$

such that for all $\tau = \gamma(t)$:

(1) $E_\tau = E_\sigma$ (i.e. there is a canonical complex bundle isomorphism);
(2) $\psi_\tau = \mathrm{Id}$;
(3) $j_\tau = j_\sigma$.

We are then free to choose the bundle isomorphisms Ψ_τ and consequently the Banach space isomorphisms (3.20) to be the identity for all $\tau = \gamma(t)$, so $\widehat{\mathbf{D}}_{\gamma(t)} = \widehat{\varphi}_\sigma^* \dot{\mathbf{D}}_{\gamma(t)}$, where $\mathbf{D}_{\gamma(t)}$ is a smooth family of Cauchy-Riemann operators on the fixed bundle $E_\sigma \to \Sigma_\sigma$. Differentiating this family gives a real-linear bundle map

$$A_Y := \partial_t \mathbf{D}_{\gamma(t)}\big|_{t=0} \in \Gamma(\mathrm{Hom}_{\mathbb{R}}(E_\sigma, F_\sigma)),$$

and we then find that

$$\mathbf{L}(Y) := d\mathbf{F}_\sigma(\sigma)Y \in \mathrm{Hom}_G\left(\ker(\widehat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma), \ker(\widehat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma^*)\right)$$

takes the form

$$\mathbf{L}(Y)\eta = \pi\big((\widehat{\varphi}_\sigma^* A_Y)\eta\big), \tag{3.24}$$

where π is the projection

$$\mathbf{Y}_\sigma = \mathrm{im}(\widehat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma) \oplus \ker(\widehat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma^*) \xrightarrow{\pi} \ker(\widehat{\varphi}_\sigma^* \dot{\mathbf{D}}_\sigma^*).$$

The local genericity result developed in §5 below is geared toward proving that operators such as $\mathbf{L}$ are surjective.

4. Index computation

The goal of this section is to compute the Fredholm index of the twisted Cauchy-Riemann type operators introduced in §3.4. We will use the notation of §3 but dispense with the parameter τ since it is not important for the index computation, hence $\varphi : (\Sigma', j') \to (\Sigma, j)$ is a fixed branched cover, and $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$ is a fixed regular presentation. The complex vector bundles E and F with their restrictions $\dot{E}$ and $\dot{F}$ to the punctured domain $\dot{\Sigma}$ are assumed to have rank

$$m := \mathrm{rank}_{\mathbb{C}} E \in \mathbb{N},$$

and we assume

$$\boldsymbol{\theta} : G \to \mathrm{Aut}_{\mathbb{R}}(W)$$

is a (not necessarily irreducible or faithful) representation of G with

$$n := \dim W \in \mathbb{N}.$$

The resulting twisted bundles over $\dot{\Sigma}$ can be written as

$$\dot{E}^{\boldsymbol{\theta}} = \dot{E} \otimes_{\mathbb{R}} W^{\boldsymbol{\theta}}, \qquad \dot{F}^{\boldsymbol{\theta}} = \dot{F} \otimes_{\mathbb{R}} W^{\boldsymbol{\theta}},$$

in terms of the flat real vector bundle $W^{\boldsymbol{\theta}} := (\dot{\Sigma}'' \times W)/G \to \dot{\Sigma}$, and any Cauchy-Riemann type operator $\mathbf{D} \in \mathcal{CR}_{\mathbb{R}}(E)$ then gives rise to the twisted operator

$$\dot{\mathbf{D}}^{\boldsymbol{\theta}} : \Gamma(\dot{E}^{\boldsymbol{\theta}}) \to \Gamma(\dot{F}^{\boldsymbol{\theta}}).$$

We need a bit more notation in order to state a formula for $\mathrm{ind}(\dot{\mathbf{D}}^{\boldsymbol{\theta}})$. Recall that while the deck transformations $G = \mathrm{Aut}(\pi)$ act on $\dot{\Sigma}''$ without fixed points, their extensions to biholomorphic self-maps of Σ'' may fix some of the punctures, so for each $w \in \Theta$ and $\zeta \in \pi^{-1}(w) \subset \Theta'' := \pi^{-1}(\Theta)$, we can consider the stabilizer subgroup

$$G_\zeta := \{g \in G \mid g\zeta = \zeta\},$$

which is necessarily cyclic. Restricting $\boldsymbol{\theta}$ to G_ζ then defines a representation $G_\zeta \to \mathrm{Aut}_{\mathbb{R}}(W)$, which splits W into G_ζ-invariant subspaces $W = W_\zeta \oplus W_\zeta'$ such that G_ζ acts on W_ζ trivially and on W_ζ' as a direct sum of nontrivial representations. We define the number

$$n_w := \dim W_\zeta' \in \{0, \dots, n\}.$$

As implied by the notation, this depends on $w \in \Theta$ but not on the choice of preimage $\zeta \in \pi^{-1}(w)$: indeed, since G acts transitively on $\pi^{-1}(w)$, any two choices of ζ give rise to conjugate subgroups G_ζ, and using orthonormality of characters, one can compute

$$n_w = n - \dim W_\zeta = n - \frac{1}{|G_\zeta|} \sum_{g \in G_\zeta} \chi_{\boldsymbol{\theta}}(g),$$

an expression which depends only on the congugacy class of G_ζ.

Theorem 4.1. *Under the assumptions detailed above, the operator*

$$\dot{\mathbf{D}}^{\boldsymbol{\theta}} : W^{k,p,-\boldsymbol{\delta}}(\dot{E}^{\boldsymbol{\theta}}) \to W^{k-1,p,-\boldsymbol{\delta}}(\dot{F}^{\boldsymbol{\theta}})$$

is Fredholm for any $k \in \mathbb{N}$, $p \in (1,\infty)$ and negative exponential weights $-\boldsymbol{\delta} = \{-\delta_w\}_{w \in \Theta}$ satisfying $0 < \delta_w < 2\pi/|G|$ for all $w \in \Theta$. Its index is

$$\operatorname{ind}(\dot{\mathbf{D}}^{\boldsymbol{\theta}}) = n \cdot \operatorname{ind}(\mathbf{D}) - m \sum_{w \in \Theta} n_w.$$

The dimensions and indices in the above statement are all real, but note that if $\boldsymbol{\theta}$ is irreducible with $\mathbb{K} := \operatorname{End}_G(W) \in \{\mathbb{C}, \mathbb{H}\}$, then the integers n and n_w are automatically divisible by $t := \dim_{\mathbb{R}} \mathbb{K} \in \{2,4\}$, hence so is $\operatorname{ind}(\dot{\mathbf{D}}^{\boldsymbol{\theta}})$. Let us state the corollary for the faithful case in terms of the $\mathbb{K}$-linear index since it is most useful in this form.

Corollary 4.2 (cf. Lemma 2.15)**.** *Assume $(\Theta, \dot{\Sigma}'', \pi, G, \rho, I, f)$ is the mimimal regular presentation, and that $\boldsymbol{\theta}$ is faithful and irreducible with $\operatorname{End}_G(W) \cong \mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Then*

$$\operatorname{ind}_{\mathbb{K}}(\dot{\mathbf{D}}^{\boldsymbol{\theta}}) \leqslant (\dim_{\mathbb{K}} W) \cdot \operatorname{ind}_{\mathbb{R}}(\mathbf{D}) - m|\Theta|,$$

and if $\mathbb{K} = \mathbb{R}$, then the inequality is strict unless all branch points of φ have branching order 2.

Proof. By Lemma 3.3, the stabilizer subgroups G_ζ are nontrivial for all $\zeta \in \Theta''$, and the conclusion about branch points of order 2 will hold if and only if all of them are isomorphic to $\mathbb{Z}_2$. Now if $\boldsymbol{\theta}$ is faithful, it follows that all nontrivial elements $g \in G_\zeta$ for $\zeta \in \Theta''$ also act nontrivially on W, hence the decomposition of W into G_ζ-invariant subspaces contains at least a 1-dimensional $\mathbb{K}$-linear subspace on which G_ζ acts nontrivially, giving $n_w \geqslant \dim_{\mathbb{R}} \mathbb{K}$ for all $w \in \Theta$. This implies the upper bound, and in the case $\mathbb{K} = \mathbb{R}$, it is an equality if and only if $n_w = 1$ for all $w \in \Theta$, meaning each G_ζ acts on W as the $(n-1)$-fold direct sum of the trivial representation plus a real 1-dimensional nontrivial representation, which is required to be faithful. But the only nontrivial faithful real 1-dimensional representation of any finite group is the nontrivial representation of $\mathbb{Z}_2$, hence $G_\zeta \cong \mathbb{Z}_2$. □

Remark 4.3. Doan and Walpuski have recently shown that an index formula equivalent to that of Theorem 4.1 can also be derived from Kawasaki's orbifold Riemann-Roch theorem [Kaw79]. From this perspective, branch points are regarded as orbifold singularities instead of punctures; see [DWb, Appendix 2.B].

The remainder of this section is devoted to the proof of Theorem 4.1, which we shall break down into five steps.

Step 1: Some notation.
It will be convenient first to complexify the representation. We define $V := W \oplus iW$ and the complex representation

$$\boldsymbol{\lambda} : G \to \operatorname{Aut}_{\mathbb{C}}(V)$$

such that $\boldsymbol{\lambda}(g)|_W = \boldsymbol{\theta}(g)$ for all $g \in G$. Note that for $w \in \Theta$ and $\zeta \in \pi^{-1}(w) \subset \Theta''$, the trivial representation of G_ζ on V is the complexification of the trivial real representation on W, so the splitting $W = W_\zeta \oplus W'_\zeta$ explained above complexifies to a splitting $V = V_\zeta \oplus V'_\zeta$, where $V_\zeta \subset V$ is the largest complex subspace on which G_ζ acts trivially, allowing us to write

$$n_w = \dim_{\mathbb{C}} V'_\zeta = n - \dim_{\mathbb{C}} V_\zeta.$$

The complexified representation now gives rise to a complex flat bundle $V^{\boldsymbol{\lambda}} := (\dot{\Sigma}'' \times V)/G$, corresponding twisted bundles

$$\dot{E}^{\boldsymbol{\lambda}} := \dot{E} \otimes_{\mathbb{R}} V^{\boldsymbol{\lambda}}, \qquad \dot{F}^{\boldsymbol{\lambda}} := \dot{F} \otimes_{\mathbb{R}} V^{\boldsymbol{\lambda}}, \tag{4.1}$$

and a twisted Cauchy-Riemann operator

$$\dot{\mathbf{D}}^{\boldsymbol{\lambda}} : W^{k,p,-\boldsymbol{\delta}}(\dot{E}^{\boldsymbol{\lambda}}) \to W^{k-1,p,-\boldsymbol{\delta}}(\dot{F}^{\boldsymbol{\lambda}}).$$

The following point is important to understand: the tensor products in (4.1) are *real*, thus $\dot{E}^{\boldsymbol{\lambda}}$ and $\dot{F}^{\boldsymbol{\lambda}}$ each inherit two complex structures J and i, where J comes from the complex structure of E and i from that of V: they commute with each other and are defined by

$$J(\eta \otimes v) := J\eta \otimes v, \qquad i(\eta \otimes v) := \eta \otimes iv.$$

In this sense, $\dot{\mathbf{D}}^{\boldsymbol{\lambda}}$ can be regarded as the i-complex-linear extension of $\dot{\mathbf{D}}^{\boldsymbol{\theta}}$ to complexifications of the latter's domain and target space—this notion of "complexification" ignores the fact that these spaces already have native complex structures J and treats them as *real* vector spaces, which is appropriate since $\dot{\mathbf{D}}^{\boldsymbol{\theta}}$ need not be J-complex linear. We therefore obtain the relation

$$\operatorname{ind}(\dot{\mathbf{D}}^{\boldsymbol{\theta}}) = \frac{1}{2} \operatorname{ind}(\dot{\mathbf{D}}^{\boldsymbol{\lambda}}),$$

and we shall compute $\operatorname{ind}(\dot{\mathbf{D}}^{\boldsymbol{\lambda}})$ by regarding $\dot{\mathbf{D}}^{\boldsymbol{\lambda}}$ as a real-linear Cauchy-Riemann type operator on the complex vector bundle $(\dot{E}^{\boldsymbol{\lambda}}, J)$. Since $\operatorname{rank}_{\mathbb{C}} \dot{E}^{\boldsymbol{\lambda}} = \operatorname{rank}_{\mathbb{C}} E \cdot \dim_{\mathbb{R}} V = 2mn$, the punctured Riemann-Roch formula from [Sch95, §3.3] (or equivalently [Wene, Lecture 5]) gives

$$\operatorname{ind}(\dot{\mathbf{D}}^{\boldsymbol{\lambda}}) = 2mn \cdot \chi(\dot{\Sigma}) + 2c_1^{\Phi}(\dot{E}^{\boldsymbol{\lambda}}, J) + \sum_{w \in \Theta} \mu_{\mathrm{CZ}}^{\Phi}(\mathbf{A}_w^{\boldsymbol{\lambda}} - \delta_w), \tag{4.2}$$

where Φ is an arbitrary choice of asymptotic trivialization, and $\mu_{\mathrm{CZ}}^{\Phi}(\mathbf{A}_w^{\boldsymbol{\lambda}} - \delta_w) \in \mathbb{Z}$ are Conley-Zehnder indices that depend on certain asymptotic operators $\mathbf{A}_w^{\boldsymbol{\lambda}}$ to be discussed below and the exponential weight $-\delta_w \in (-2\pi/|G|, 0)$ associated to each puncture $w \in \Theta$. The main difficulty of the calculation is in choosing a suitable asymptotic trivialization in which both $c_1^{\Phi}(\dot{E}^{\boldsymbol{\lambda}}, J)$ and $\mu_{\mathrm{CZ}}^{\Phi}(\mathbf{A}_w^{\boldsymbol{\lambda}} - \delta_w)$ can be computed.

Denote

$$d' := \deg(\pi) = |G|,$$

and for each $w \in \Theta$ and $\zeta \in \pi^{-1}(w) \subset \Theta''$, let

$$k_\zeta \in \{1, \dots, d'\}$$

denote the branching order of π at ζ, meaning π is a k_ζ-to-1 map on a small punctured neighborhood of ζ. We can then choose punctured neighborhoods $\mathcal{U}_w \subset \dot{\Sigma}$ and $\mathcal{U}_\zeta \subset \dot{\Sigma}''$ of w and ζ respectively, with holomorphic cylindrical coordinates $(s,t) \in [0,\infty) \times S^1$ on each such that

$$\pi(s,t) = (k_\zeta s, k_\zeta t)$$

in coordinates on $\mathcal{U}_\zeta$. In these coordinates, any $g \in G_\zeta$ necessarily preserves the end $\mathcal{U}_\zeta$ and takes the form $g(s,t) = (s, t + j/k_\zeta)$ for some $j \in \{0, \dots, k_\zeta - 1\}$. This means that G_ζ is a cyclic group of order k_ζ, and it has a canonical generator $g_\zeta \in G_\zeta$ such that

$$g_\zeta(s,t) = (s, t + 1/k_\zeta) \quad \text{on } \mathcal{U}_\zeta.$$

In addition to the cylindrical coordinates, let us choose complex trivializations of E on each of the corresponding neighborhoods of Θ, thus giving an identification

$$\dot{E}|_{\mathcal{U}_w} = \big([0,\infty) \times S^1\big) \times E_w \tag{4.3}$$

for each $w \in \Theta$. For any choice $\zeta \in \pi^{-1}(w) \subset \Theta''$, this also gives us an identification of $\dot{E}^{\boldsymbol{\lambda}}|_{\mathcal{U}_w}$ with

$$\Big(\big([0,\infty) \times S^1\big) \times (E_w \otimes_{\mathbb{R}} V)\Big) \Big/ G_\zeta, \tag{4.4}$$

where the action of $G_\zeta = \mathbb{Z}_{k_\zeta}$ on $\big([0,\infty) \times S^1\big) \times (E_w \otimes_{\mathbb{R}} V)$ is determined by

$$g_\zeta \cdot \big((s,t), \eta \otimes v\big) = \big((s, t + 1/k_\zeta), \eta \otimes \boldsymbol{\lambda}(g_\zeta)v\big).$$

This picture can now easily be extended to the "circle compactification" of the punctured surface: let $\bar{\Sigma}$ and $\bar{\Sigma}''$ denote the compact surfaces with boundary obtained by replacing each cylindrical end $[0,\infty) \times S^1$ in $\dot{\Sigma}$ and $\dot{\Sigma}''$ respectively by the compact topological manifold $[0,\infty] \times S^1$. The connected components of $\partial\bar{\Sigma}$ and $\partial\bar{\Sigma}''$ are then in bijective correspondence with the punctures $w \in \Theta$ or $\zeta \in \Theta''$ respectively, and the choice of cylindrical coordinates identifies each of these components with S^1. We shall denote the boundary components accordingly by S^1_w, S^1_ζ for $w \in \Theta$ or $\zeta \in \Theta''$, hence

$$\partial\bar{\Sigma} = \bigsqcup_{w \in \Theta} S^1_w, \qquad \partial\bar{\Sigma}'' = \bigsqcup_{\zeta \in \Theta''} S^1_\zeta.$$

The covering map $\pi : \dot{\Sigma}'' \to \dot{\Sigma}$ now extends to a continuous covering map

$$\bar{\pi} : \bar{\Sigma}'' \to \bar{\Sigma}$$

which restricts on the boundary components to

$$\pi_\zeta := \bar{\pi}|_{S^1_\zeta} : S^1_\zeta \to S^1_{\pi(\zeta)} : t \mapsto k_\zeta t,$$

and each $g \in G$ also extends naturally to a continuous deck transformation $\bar{g} : \bar{\Sigma}'' \to \bar{\Sigma}''$ of $\bar{\pi}$, such that if $g(\zeta) = \zeta'$, then $\bar{g}$ maps $S^1_\zeta \to S^1_{\zeta'}$ via the canonical diffeomorphism composed with a translation. The identifications (4.3) and (4.4) then yield obvious extensions of $\dot{E}$ and $\dot{E}^{\boldsymbol{\lambda}}$ as topological vector bundles

$$\bar{E} \to \bar{\Sigma}, \qquad \bar{E}^{\boldsymbol{\lambda}} \to \bar{\Sigma},$$

and we have

$$\bar{E}^{\boldsymbol{\lambda}} = \big(\bar{\pi}^*\bar{E} \otimes_{\mathbb{R}} V\big) \Big/ G.$$

Step 2: Asymptotic operators on the twisted bundle.
With the essential notation in place, we can now discuss asymptotic operators. Recall that after choosing a suitable Hermitian inner product on $\dot{E}$ over the cylindrical ends, any Cauchy-Riemann type operator $\dot{\mathbf{D}}$ on $\dot{E} \to \dot{\Sigma}$ with reasonable asymptotic behavior determines real-linear operators

$$\mathbf{A}_w : \Gamma(\bar{E}|_{S^1_w}) \to \Gamma(\bar{E}|_{S^1_w}),$$

for each $w \in \Theta$, see e.g. [Wen10, §2.1]. These can be regarded as unbounded self-adjoint operators on $L^2(\bar{E}|_{S^1_w})$ with dense domain $H^1(\bar{E}|_{S^1_w})$, and we say $\mathbf{A}_w$ is **nondegenerate** whenever its kernel is trivial, in which case it determines a **Conley-Zehnder index**

$$\mu^\Phi_{\mathrm{CZ}}(\mathbf{A}_w) \in \mathbb{Z}$$

relative to any choice of complex trivialization Φ of $\bar{E}|_{S^1_w}$. In the case where $\dot{\mathbf{D}}$ is the restriction to $\dot{\Sigma}$ of some operator $\mathbf{D} \in \mathcal{CR}_{\mathbb{R}}(E)$ on Σ, the operators $\mathbf{A}_w$ are very simple and were already computed in §3.2: they are each the so-called *trivial* asymptotic operator

$$\mathbf{A}_w = -J\partial_t,$$

where ∂_t is a well-defined differential operator on $\bar{E}|_{S^1_w}$ since the fibers are all canonically identified with E_w. This operator is degenerate, but the introduction of negative exponential weights $-\delta_w < 0$ identifies $\dot{\mathbf{D}}$ with another Cauchy-Riemann type operator whose corresponding asymptotic operators are $\mathbf{A}_w - \delta_w$, which are nondegenerate for any $\delta_w > 0$ sufficiently small.

Denote by

$$\mathbf{A}^{\boldsymbol{\lambda}}_w : \Gamma(\bar{E}^{\boldsymbol{\lambda}}|_{S^1_w}) \to \Gamma(\bar{E}^{\boldsymbol{\lambda}}|_{S^1_w})$$

the asymptotic operators associated to $\dot{\mathbf{D}}^{\boldsymbol{\lambda}}$ for each $w \in \Theta$. These are easiest to understand by considering the pulled back Cauchy-Riemann operator

$$\pi^*\dot{\mathbf{D}}^{\boldsymbol{\lambda}} : W^{1,p,-\pi^*\boldsymbol{\delta}}(\pi^*\dot{E}^{\boldsymbol{\lambda}}) \to L^{p,-\pi^*\boldsymbol{\delta}}(\pi^*\dot{F}^{\boldsymbol{\lambda}}),$$

whose asymptotic operators we will denote by

$$\pi^*\mathbf{A}^{\boldsymbol{\lambda}}_\zeta : \Gamma\big((\bar\pi^*\bar E^{\boldsymbol{\lambda}})|_{S^1_\zeta}\big) \to \Gamma\big((\bar\pi^*\bar E^{\boldsymbol{\lambda}})|_{S^1_\zeta}\big)$$

for $\zeta \in \Theta''$. The relation $\left(\pi^*\dot{\mathbf{D}}^{\boldsymbol{\lambda}}\right)(\eta\circ\pi) = \pi^*\left(\dot{\mathbf{D}}^{\boldsymbol{\lambda}}\eta\right)$ for sections $\eta \in \Gamma(\dot E^{\boldsymbol{\lambda}})$ gives rise to the following relation between asymptotic operators:

$$\left(\pi^*\mathbf{A}^{\boldsymbol{\lambda}}_\zeta\right)(f\circ\pi_\zeta) = k_\zeta\cdot\left(\mathbf{A}^{\boldsymbol{\lambda}}_w f\right)\circ\pi_\zeta \quad \text{for } f \in \Gamma\big(\bar E^{\boldsymbol{\lambda}}|_{S^1_w}\big) \text{ and } \zeta \in \pi^{-1}(w). \tag{4.5}$$

This can be proved via a local computation as in §3.2: writing $\pi(s,t) = (ks,kt)$ in suitable holomorphic cylindrical coordinates and $\dot{\mathbf{D}}^{\boldsymbol{\lambda}}\eta = (\bar\partial\eta + B\eta)(-ds + i\,dt)$ for some matrix-valued function $B(s,t)$ after a choice of trivialization for $\dot E^{\boldsymbol{\lambda}}$ over the end near w, $\mathbf{A}^{\boldsymbol{\lambda}}_w$ is represented in this trivialization by the operator $-i\partial_t - B(\infty,t)$ by definition. The corresponding trivialized formula for $\pi^*\dot{\mathbf{D}}^{\boldsymbol{\lambda}}$ then comes from

$$\begin{aligned}
\pi^*\dot{\mathbf{D}}^{\boldsymbol{\lambda}}(\eta\circ\pi)(s,t) &= \pi^*\left(\dot{\mathbf{D}}^{\boldsymbol{\lambda}}\eta\right)\Big|_{(s,t)} \\
&= \big(\bar\partial\eta(ks,kt) + B(ks,kt)\eta(ks,kt)\big)\left(-d(ks) + i\,d(kt)\right) \\
&= \big(\bar\partial + k\,B(ks,kt)\big)(\eta\circ\pi)(s,t)\cdot(-ds + i\,dt),
\end{aligned}$$

hence $\pi^*\dot{\mathbf{D}}^{\boldsymbol{\lambda}}$ appears in trivialized form as the sum of $\bar\partial$ with the zeroth-order term $kB(ks,kt)$. The trivialized formula for $\pi^*\mathbf{A}^{\boldsymbol{\lambda}}_\zeta$ is thus $-i\partial_t - k\,B(\infty,kt)$, which explains the factor of k_ζ appearing in (4.5).

For the following discussion, fix $w \in \Theta$ and $\zeta \in \pi^{-1}(w)$. The definition of $\dot{\mathbf{D}}^{\boldsymbol{\lambda}}$ implies that $\pi^*\dot{\mathbf{D}}^{\boldsymbol{\lambda}}$ acts on sections $\eta\otimes v \in \Gamma(\pi^*\dot E\otimes_{\mathbb{R}} V)$ such that $(\pi^*\dot{\mathbf{D}}^{\boldsymbol{\lambda}})(\eta\otimes v) = \big[\big(\pi^*\dot{\mathbf{D}}\big)\eta\big]\otimes v$ whenever $v : \dot\Sigma'' \to V$ is constant. From this, one deduces that for any section $f\otimes v \in \Gamma\big(\bar\pi^*\bar E\otimes_{\mathbb{R}} V|_{S^1_\zeta}\big)$ where f is an arbitrary smooth map $S^1_\zeta \to E_w$ and $v : S^1_\zeta \to V$ is constant, we have

$$\pi^*\mathbf{A}^{\boldsymbol{\lambda}}_\zeta(f\otimes v) = -(J\,\partial_t f)\otimes v. \tag{4.6}$$

Now to write down a formula for $\mathbf{A}^{\boldsymbol{\lambda}}_w$, we can use the natural identification of $\Gamma\big(\bar E^{\boldsymbol{\lambda}}|_{S^1_w}\big)$ with the space of G_ζ-equivariant loops in $E_w\otimes_{\mathbb{R}} V$,

$$\Gamma\big(\bar E^{\boldsymbol{\lambda}}|_{S^1_w}\big) = \left\{F \in C^\infty(S^1_\zeta, E_w\otimes_{\mathbb{R}} V)\ \middle|\ F(t + 1/k_\zeta) = g_\zeta\cdot F(t) \text{ for all } t \in S^1_\zeta\right\}.$$

Acting on G_ζ-equivariant loops F, (4.5) and (4.6) imply

$$\mathbf{A}^{\boldsymbol{\lambda}}_w F = -\frac{1}{k_\zeta} J\,\partial_t F, \tag{4.7}$$

where it is understood that $J\partial_t$ acts on the tensor product by taking $F = f\otimes v$ to $(J\,\partial_t f)\otimes v$ whenever v is locally constant.

Step 3: Trivializations and Conley-Zehnder indices.
This is the step in which it is helpful to be working with the complexification $\dot{\mathbf{D}}^{\boldsymbol{\lambda}}$ rather than directly with $\dot{\mathbf{D}}^{\boldsymbol{\theta}}$. In order to choose a suitable trivialization Φ and compute $\mu^\Phi_{\mathrm{CZ}}(\mathbf{A}^{\boldsymbol{\lambda}}_w - \delta_w)$, we shall first split $\mathbf{A}^{\boldsymbol{\lambda}}_w$ into a direct sum of operators on J-complex line bundles. Observe that $\bar E|_{S^1_w} = S^1_w\times E_w$ is already canonically trivial, so any complex basis of E_w gives a splitting of $\mathbf{A}^{\boldsymbol{\lambda}}_w$ over an m-fold direct sum of isomorphic J-complex bundles of rank $2n$,

$$\bar E^{\boldsymbol{\lambda}}|_{S^1_w} = \left(L^{\boldsymbol{\lambda}}\right)^{\oplus m},$$

where

$$L^{\boldsymbol{\lambda}} = S^1\times(\mathbb{C}\otimes_{\mathbb{R}} V)\Big/ G_\zeta$$

and the generator of $G_\zeta = \mathbb{Z}_{k_\zeta}$ acts by $g_\zeta\cdot(t, f\otimes v) = (t + 1/k_\zeta, f\otimes\boldsymbol{\lambda}(g_\zeta)v)$. Note that $L^{\boldsymbol{\lambda}}$ carries two commuting complex structures, J and i, which act on the first and second factor of the tensor product respectively. Further: V admits a complex basis $(v_1, \dots, v_n)$ consisting of eigenvectors of $\boldsymbol{\lambda}(g_\zeta)$, and we can then define integers $p_j \in \{0, \dots, k_\zeta - 1\}$ for $j = 1, \dots, n$ by

$$\boldsymbol{\lambda}(g_\zeta) v_j = e^{2\pi i p_j / k_\zeta} v_j.$$

Here we can identify $V'_\zeta \subset V$ as the subspace spanned by all v_j such that $p_j > 0$. Identifying V with $\mathbb{C}^n$ via this eigenbasis yields a splitting

$$L^{\boldsymbol{\lambda}} = L^{\boldsymbol{\lambda}}_1 \oplus \dots \oplus L^{\boldsymbol{\lambda}}_n,$$

where for $j = 1, \dots, n$,

$$L^{\boldsymbol{\lambda}}_j := S^1 \times (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) \Big/ \mathbb{Z}_{k_\zeta},$$

with the generator $1 \in \mathbb{Z}_{k_\zeta}$ acting by $1 \cdot (t, f \otimes v) = (t + 1/k_\zeta, f \otimes e^{2\pi i p_j / k_\zeta} v)$. This bundle again carries the two commuting complex structures J and i acting on the first and second factors of the tensor product respectively; it has complex rank 2 with respect to either one. Finally, since J acts i-complex-linearly on $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, we can find eigenvectors $f_\pm \in \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ such that $J f_\pm = \pm i f_\pm$, so the splitting $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} f_+ \oplus \mathbb{C} f_-$ gives a splitting of J- and i-complex vector bundles

$$L^{\boldsymbol{\lambda}}_j = L^{\boldsymbol{\lambda}}_{j,+} \oplus L^{\boldsymbol{\lambda}}_{j,-},$$

with

$$L^{\boldsymbol{\lambda}}_{j,\pm} = (S^1 \times \mathbb{C}) \Big/ \mathbb{Z}_{k_\zeta}, \tag{4.8}$$

where the generator $1 \in \mathbb{Z}_{k_\zeta}$ acts by $1 \cdot (t, f) = (t + 1/k_\zeta, e^{2\pi i p_j / k_\zeta} f)$. Both $L^{\boldsymbol{\lambda}}_{j,+}$ and $L^{\boldsymbol{\lambda}}_{j,-}$ are complex line bundles over S^1, carrying two complex structures J and i, which satisfy $J = i$ on $L^{\boldsymbol{\lambda}}_{j,+}$ but $J = -i$ on $L^{\boldsymbol{\lambda}}_{j,-}$. This splitting of bundles gives a splitting of $\mathbf{A}^{\boldsymbol{\lambda}}_w$ in the form

$$\mathbf{A}^{\boldsymbol{\lambda}}_w = \left(\bigoplus_{j=1}^n \left(\mathbf{A}^{\boldsymbol{\lambda}}_{j,+} \oplus \mathbf{A}^{\boldsymbol{\lambda}}_{j,-} \right) \right)^{\oplus m}, \tag{4.9}$$

where for $j = 1, \dots, n$, $\mathbf{A}^{\boldsymbol{\lambda}}_{j,\pm}$ acts on

$$\Gamma(L^{\boldsymbol{\lambda}}_{j,\pm}) = \left\{ f \in C^\infty(S^1, \mathbb{C}) \;\middle|\; f(t + 1/k_\zeta) = e^{2\pi i p_j / k_\zeta} f(t) \text{ for all } t \in S^1 \right\}$$

by

$$\mathbf{A}^{\boldsymbol{\lambda}}_{j,\pm} f = \mp \frac{1}{k_\zeta} i \, \partial_t f.$$

Since $L^{\boldsymbol{\lambda}}_{j,\pm}$ are complex line bundles, $\mu^\Phi_{\mathrm{CZ}}(\mathbf{A}^{\boldsymbol{\lambda}}_{j,\pm} - \delta_w)$ can be computed in terms of winding numbers of eigenfunctions of $\mathbf{A}^{\boldsymbol{\lambda}}_{j,\pm}$, using the relation proved in [HWZ95, Theorem 3.10]. In particular, if (as will turn out to be true in our case) all eigenspaces of $\mathbf{A}^{\boldsymbol{\lambda}}_{j,\pm}$ have real dimension 2, then

$$\mu^\Phi_{\mathrm{CZ}}(\mathbf{A}^{\boldsymbol{\lambda}}_{j,\pm} - \delta_w) = 2 \operatorname{wind}^\Phi(f_{j,\pm}) + 1, \tag{4.10}$$

where $f_{j,\pm} \in \Gamma(L^{\boldsymbol{\lambda}}_{j,\pm})$ is any nontrivial eigenfunction of $\mathbf{A}^{\boldsymbol{\lambda}}_{j,\pm} - \delta_w$ with the largest possible negative eigenvalue. A $\mathbb{Z}_{k_\zeta}$-equivariant function $f : S^1 \to \mathbb{C}$ satisfies $\mathbf{A}^{\boldsymbol{\lambda}}_{j,\pm} f = \lambda f$ if and only if it is a complex multiple of

$$f_\lambda(t) := e^{\pm i k_\zeta \lambda t}, \qquad \lambda \mp \frac{2\pi p_j}{k_\zeta} \in 2\pi\mathbb{Z}. \tag{4.11}$$

Observe that since $0 < \delta_w < 2\pi/d' \leqslant 2\pi/k_\zeta$, every eigenvalue λ thus satisfies $\lambda - \delta_w \neq 0$; this proves that the perturbed asymptotic operators $\mathbf{A}^{\boldsymbol{\lambda}}_{j,\pm}$ are all nondegenerate and thus establishes the Fredholm property for $\dot{\mathbf{D}}^{\boldsymbol{\lambda}}$. Now to apply (4.10), we need to find the unique eigenvalue $\lambda = 2\pi(\ell \pm p_j / k_\zeta)$ for $\ell \in \mathbb{Z}$ such that

$$2\pi \left(\ell \pm \frac{p_j}{k_\zeta} \right) - \delta_w < 0 < 2\pi \left[(\ell + 1) \pm \frac{p_j}{k_\zeta} \right] - \delta_w.$$

Since $0 < \delta_w < 2\pi/d'$, this condition is equivalent to

$$\ell \leqslant \mp \frac{p_j}{k_\zeta} < \ell + 1,$$

so choosing the appropriate $\ell \in \mathbb{Z}$ and plugging in (4.11) leads to the formulas

$$\begin{aligned} f_{j,+}(t) &:= \begin{cases} 1 & \text{if } p_j = 0, \\ e^{-2\pi i(k_\zeta - p_j)t} & \text{if } p_j > 0, \end{cases} \\ f_{j,-}(t) &:= e^{2\pi i p_j t}. \end{aligned} \tag{4.12}$$

Let $\Phi_j^\pm$ for $j = 1, \dots, n$ denote a choice of J-complex trivializations of $L^{\boldsymbol{\lambda}}_{j,\pm}$ such that

$$\operatorname{wind}^{\Phi_j^+}(f_{j,+}) = \operatorname{wind}^{\Phi_j^-}(f_{j,-}) = 0, \qquad j = 1, \dots, n,$$

and denote by Φ_w the resulting J-complex trivialization of

$$\bar{E}^{\boldsymbol{\lambda}}\Big|_{S^1_w} = \left(\bigoplus_{j=1}^n \left(L^{\boldsymbol{\lambda}}_{j,+} \oplus L^{\boldsymbol{\lambda}}_{j,-} \right) \right)^{\oplus m}. \tag{4.13}$$

By (4.10), we now have

$$\mu_{\mathrm{CZ}}^{\Phi_j^+}(\mathbf{A}^{\boldsymbol{\lambda}}_{j,+} - \delta_w) = \mu_{\mathrm{CZ}}^{\Phi_j^-}(\mathbf{A}^{\boldsymbol{\lambda}}_{j,-} - \delta_w) = 1,$$

and thus by (4.9), $\mu_{\mathrm{CZ}}^{\Phi_w}(\mathbf{A}^{\boldsymbol{\lambda}}_w - \delta_w) = 2mn$. Note that, a priori, this construction of Φ_w depends on an arbitrary choice $\zeta \in \pi^{-1}(w)$, but the fact that $\mu_{\mathrm{CZ}}^{\Phi_w}(\mathbf{A}^{\boldsymbol{\lambda}}_w - \delta_w)$ turns out to be independent of this choice tells us that Φ_w is uniquely determined up to homotopy. Performing this construction for all punctures $w \in \Theta$, we will denote the resulting asymptotic trivialization of $\dot{E}^{\boldsymbol{\lambda}}$ simply by Φ. We've proved:

Lemma 4.4. *For the asymptotic trivialization Φ described above and each puncture $w \in \Theta$, $\mu_{\mathrm{CZ}}^{\Phi}(\mathbf{A}^{\boldsymbol{\lambda}}_w - \delta_w) = 2mn$.* □

Step 4: The relative first Chern number.
It remains to compute $c_1^\Phi(\dot{E}^{\boldsymbol{\lambda}}, J)$. Consider the pullback $\pi^*\dot{E}^{\boldsymbol{\lambda}} = \pi^*\dot{E} \otimes_{\mathbb{R}} V$. The first factor in this tensor product has a canonical homotopy class of asymptotic trivializations, which we shall denote by $\pi^*\Psi_0$, as it is the pullback of an asymptotic trivialization Ψ_0 for $\dot{E}$, satisfying $c_1^{\Psi_0}(\dot{E}) = c_1(E)$. Moreover, the second factor is globally trivial, thus $\pi^*\dot{E}^{\boldsymbol{\lambda}}$ carries a canonical asymptotic trivialization, denoted by Ψ, such that

$$c_1^\Psi(\pi^*\dot{E}^{\boldsymbol{\lambda}}) = \dim_{\mathbb{R}} V \cdot c_1^{\pi^*\Psi_0}(\pi^*\dot{E}) = 2n \cdot \deg(\pi) \cdot c_1^{\Psi_0}(\dot{E}) = 2nd' \cdot c_1(E).$$

If $\pi^*\Phi$ denotes the pullback of Φ to an asymptotic trivialization of $\pi^*\dot{E}^{\boldsymbol{\lambda}}$, we then have

$$\begin{aligned} c_1^\Phi(\dot{E}^{\boldsymbol{\lambda}}) &= \frac{1}{d'} c_1^{\pi^*\Phi}(\pi^*\dot{E}^{\boldsymbol{\lambda}}) = \frac{1}{d'}\left[c_1^\Psi(\pi^*\dot{E}^{\boldsymbol{\lambda}}) + \deg^\Psi(\pi^*\Phi) \right] \\ &= 2n \cdot c_1(E) + \frac{1}{d'} \deg^\Psi(\pi^*\Phi), \end{aligned} \tag{4.14}$$

where $\deg^\Psi(\pi^*\Phi) \in \mathbb{Z}$ denotes the sum over all punctures $\zeta \in \Theta''$ of the degrees of the transition maps $S^1 \to \mathrm{GL}(2mn, \mathbb{C})$ that change Ψ to $\pi^*\Phi$. We can compute the latter for each $w \in \Theta$ and $\zeta \in \pi^{-1}(w) \subset \Theta''$ as a sum of winding numbers over a line bundle decomposition analogous to (4.13), namely

$$\pi^*\bar{E}^{\boldsymbol{\lambda}}\Big|_{S^1_\zeta} = \pi_\zeta^*\left(\bar{E}^{\boldsymbol{\lambda}}\Big|_{S^1_w} \right) = \left(\bigoplus_{j=1}^n \left(\pi_\zeta^* L^{\boldsymbol{\lambda}}_{j,+} \oplus \pi_\zeta^* L^{\boldsymbol{\lambda}}_{j,-} \right) \right)^{\oplus m},$$

where pulling back (4.8) via the projection $\pi_\zeta : S^1 \to S^1/\mathbb{Z}_{k_\zeta}$ gives the trivial line bundle

$$\pi_\zeta^* L^{\boldsymbol{\lambda}}_{j,\pm} = S^1 \times \mathbb{C},$$

with the pulled back trivialization $\pi_\zeta^* \Phi_j^\pm$ such that the special eigenfunctions $f_{j,\pm}$ in (4.12) have zero winding as t traverses S^1. The restriction Ψ_ζ of Ψ to $\bar{\pi}^* \bar{E}^{\boldsymbol{\lambda}}|_{S^1_\zeta}$ is now the direct sum of the standard trivializations on each of the factors $\pi_\zeta^* L^{\boldsymbol{\lambda}}_{j,\pm}$, thus

$$\deg^{\Psi_\zeta}(\pi_\zeta^* \Phi_w) = m \sum_{j=1}^n \left[\operatorname{wind}_{S^1}(f_{j,+}) + \operatorname{wind}_{S^1}(f_{j,-}) \right]. \tag{4.15}$$

There is an important sublety here: recall that $J = \pm i$ on $L^{\boldsymbol{\lambda}}_{j,\pm}$, hence the orientation induced by J on $L^{\boldsymbol{\lambda}}_{j,-}$ is the *opposite* of the obvious one, and the sign of $\operatorname{wind}_{S^1}(f_{j,-})$ must be reversed accordingly, giving

$$\operatorname{wind}_{S^1}(f_{j,+}) = \begin{cases} 0 & \text{if } p_j = 0, \\ p_j - k_\zeta & \text{if } p_j > 0, \end{cases}$$
$$\operatorname{wind}_{S^1}(f_{j,-}) = -p_j.$$

Plugging this into (4.15), we have

$$\deg^{\Psi_\zeta}(\pi_\zeta^* \Phi_w) = m \sum_{j \in \{1,\dots,n\},\ p_j \neq 0} (-k_\zeta) = -m k_\zeta \dim_{\mathbb{C}} V'_\zeta.$$

Summing over all $\zeta \in \Theta''$ and plugging into (4.14) then gives

$$c_1^\Phi(\dot{E}^{\boldsymbol{\lambda}}) = 2n \cdot c_1(E) - \frac{m}{d'} \sum_{\zeta \in \Theta''} k_\zeta \dim_{\mathbb{C}} V'_\zeta.$$

Since $\dim_{\mathbb{C}} V'_\zeta = n_w$ is independent of $\zeta \in \pi^{-1}(w)$ for each $w \in \Theta$, and $\sum_{\zeta \in \pi^{-1}(w)} k_\zeta = d'$, this implies:

Lemma 4.5. $c_1^\Phi(\dot{E}^{\boldsymbol{\lambda}}) = 2n \cdot c_1(E) - m \sum_{w \in \Theta} n_w$. ☐

Step 5: Conclusion of the proof.
Finally, we combine Lemmas 4.4 and 4.5 and plug into (4.2) to obtain

$$\begin{aligned} \operatorname{ind}(\dot{\mathbf{D}}^{\boldsymbol{\lambda}}) &= 2mn \cdot \chi(\dot{\Sigma}) + 4n \cdot c_1(E) - 2m \sum_{w \in \Theta} n_w + 2mn|\Theta| \\ &= 2 \left[mn \cdot \chi(\Sigma) + 2n \cdot c_1(E) - m \sum_{w \in \Theta} n_w \right], \end{aligned}$$

and thus

$$\operatorname{ind}(\dot{\mathbf{D}}^{\boldsymbol{\theta}}) = n \left[m\chi(\Sigma) + 2c_1(E) \right] - m \sum_{w \in \Theta} n_w.$$

The expression in brackets is $\operatorname{ind}(\mathbf{D})$, so this completes the proof of Theorem 4.1.

5. Petri's condition

5.1. **The main local result.** Standard proofs of transversality results via the Sard-Smale theorem (cf. [FHS95, MS12]) typically require some kind of unique continuation lemma, which for J-holomorphic curves usually means the similarity principle. In this section we will establish a local result about Cauchy-Riemann type operators that plays this role in the proof of Theorem D. It combines the usual unique continuation property with an additional "quadratic" local condition that can be achieved under generic zeroth-order perturbations.

For any pair of smooth real vector bundles E and F over the same manifold M, one can define the **Petri map**

$$\Pi : \Gamma(E) \otimes \Gamma(F) \to \Gamma(E \otimes F), \qquad \Pi(\eta \otimes \xi)(p) := \eta(p) \otimes \xi(p).$$

Since we plan to discuss purely local conditions, let us amend this by fixing a point $p \in M$ and considering the space of *germs* of smooth sections at p,

$$\Gamma_p(E) := \Gamma(E)/\sim,$$

where $\eta, \eta' \in \Gamma(E)$ represent the same element of $\Gamma_p(E)$ if and only if they match on some neighborhood of p. The Petri map then descends to a **local Petri map** at p,

$$\Pi : \Gamma_p(E) \otimes \Gamma_p(F) \to \Gamma_p(E \otimes F).$$

It is easy to see that Π is never injective, e.g. its kernel contains $f\eta \otimes \xi - \eta \otimes f\xi$ for any two sections $\eta \in \Gamma(E)$, $\xi \in \Gamma(F)$ with a smooth function $f : M \to \mathbb{R}$. It will sometimes become injective, however, if the domain is restricted to certain spaces of solutions to linear PDEs. To express this properly, let us assume $\mathbf{D} : \Gamma(E) \to \Gamma(F)$ is a linear partial differential operator with smooth coefficients, and $\mathbf{D}^* : \Gamma(F) \to \Gamma(E)$ is its formal adjoint with respect to a choice of bundle metrics on E, F and volume form on M. For any point $p \in M$, both operators descend to linear maps on the spaces of germs of smooth sections at p, which we will denote by

$$\mathbf{D}_p : \Gamma_p(E) \to \Gamma_p(F), \qquad \mathbf{D}_p^* : \Gamma_p(F) \to \Gamma_p(E).$$

We will also assume $\mathbf{D}$ and $\mathbf{D}^*$ uniquely determine (via extension or restriction) linear maps

$$\mathbf{D} : \mathbf{X}(E) \to \mathbf{Y}(F), \qquad \mathbf{D}^* : \mathbf{X}^*(F) \to \mathbf{Y}^*(E),$$

where $\mathbf{X}(E)$, $\mathbf{Y}^*(E)$, $\mathbf{Y}(F)$ and $\mathbf{X}^*(F)$ are vector spaces of sections (or equivalence classes of sections defined almost everywhere) of the respective bundles; in typical examples, these will be Sobolev spaces, sometimes with exponential weight conditions if M is a noncompact manifold with cylindrical ends. Let us add two conditions of a local nature, both of which are satisfied for a wide class of elliptic operators, including those of Cauchy-Riemann type:

- (REGULARITY) Every section in $\ker \mathbf{D} \subset \mathbf{X}(E)$ or $\ker \mathbf{D}^* \subset \mathbf{X}^*(F)$ is smooth.
- (UNIQUE CONTINUATION AT p) The maps $\ker \mathbf{D} \to \ker \mathbf{D}_p$ and $\ker \mathbf{D}^* \to \ker \mathbf{D}_p^*$ that send each section to its germ at p are injective.

The terminology in the following definition is adapted from the work of Doan and Walpuski [DWb], who borrowed it in turn from algebraic geometry (see e.g. [ACGH85]).

Definition 5.1. Suppose $\mathbf{D} : \mathbf{X}(E) \to \mathbf{Y}(F)$ is a differential operator with formal adjoint $\mathbf{D}^* : \mathbf{X}^*(F) \to \mathbf{Y}^*(E)$ satisfying the conditions specified above, and $p \in \mathcal{U} \subset M$. We say that $\mathbf{D}$ satisfies

(1) **Petri's condition**, if the restricted Petri map $\ker \mathbf{D} \otimes \ker \mathbf{D}^* \xrightarrow{\Pi} \Gamma(E \otimes F)$ is injective;
(2) **Petri's condition over** $\mathcal{U}$ if there is no nontrivial element $t \in \ker \mathbf{D} \otimes \ker \mathbf{D}^*$ such that $\Pi(t) \in \Gamma(E \otimes F)$ vanishes identically on $\mathcal{U}$;
(3) the **local Petri condition at** p if the map $\ker \mathbf{D}_p \otimes \ker \mathbf{D}_p^* \xrightarrow{\Pi} \Gamma_p(E \otimes F)$ is injective;
(4) **Petri's condition to infinite order at** p if there is no nontrivial element $t \in \ker \mathbf{D}_p \otimes \ker \mathbf{D}_p^*$ such that $\Pi(t)$ has vanishing derivatives of all orders at p.

Every condition on the list in Definition 5.1 implies the previous one; note that the implication (3) $\Rightarrow$ (2) in particular follows from our regularity and unique continuation assumptions. The first two conditions are global in nature, as $\ker \mathbf{D}$ and $\ker \mathbf{D}^*$ depend on the global properties of $\mathbf{D}$, including the choice of domains $\mathbf{X}(E)$ and $\mathbf{X}^*(F)$. These kernels will always be finite dimensional in the cases we consider, so that it seems unsurprising (if non-obvious) that Petri's condition might hold. In contrast, the third and fourth conditions are *much* stronger and more surprising because $\ker \mathbf{D}_p$ and $\ker \mathbf{D}_p^*$ are in general infinite dimensional, but the local conditions are also more powerful, e.g. it will be extremely useful to observe that they are preserved under pullbacks via branched covers of the base.

Remark 5.2. As defined above, the global versions of Petri's condition may in general depend not only on the operator $\mathbf{D}$ but also on the auxiliary geometric data (bundle metrics and volume form) used to define $\mathbf{D}^*$, but the local conditions are independent of these choices. Indeed, whenever $\mathbf{D}_1^*$ and $\mathbf{D}_2^*$ are two operators arising as formal adjoints of $\mathbf{D}$ via different choices of the geometric data, there is a smooth bundle automorphism $\Phi : F \to F$ that maps local solutions of $\mathbf{D}_1^*\xi = 0$ to local solutions of $\mathbf{D}_2^*\xi = 0$, so that $\mathbb{1} \otimes \Phi : E \otimes F \to E \otimes F$ identifies the two different versions of $\ker \Pi \subset \ker \mathbf{D}_p \otimes \ker \mathbf{D}_p^*$.

Remark 5.3. It is clear from the definition that the set of points $p \in M$ at which the local Petri condition is *not* satisfied is open. We will see in §5.4 that Petri's condition to infinite order can sometimes be shown to hold at all points in a dense subset of some region $\mathcal{U} \subset M$, so it follows in this situation that the local Petri condition also holds at *all* points in $\mathcal{U}$.

It should be emphasized that whenever we refer to the above definition, we will be regarding all vector spaces as *real* vector spaces so that "$\otimes$" means the real tensor product, even in cases where $\mathbf{D}$ happens to be complex linear. The only exception is Example 5.5 below, which is a digression from the main topic at hand.

Example 5.4. Elliptic operators over 1-dimensional domains satisfy something much stronger than the Petri condition to infinite order, because by local uniqueness of solutions to ODEs, any linearly-independent set of local sections in $\ker \mathbf{D}$ or $\ker \mathbf{D}^*$ is also pointwise linearly independent. For similar reasons, any Cauchy-Riemann type operator $\mathbf{D} : \Gamma(E) \to \Gamma(F)$ that splits over a direct sum of complex line bundles with nonpositive first Chern numbers over a closed surface Σ must satisfy the *global* Petri condition over arbitrary subsets $\mathcal{U} \subset \Sigma$. The reason for this is that on a line bundle $E \to \Sigma$ with $c_1(E) \leqslant 0$, the similarity principle guarantees that global solutions to $\mathbf{D}\eta = 0$ are either trivial or nowhere vanishing, so that globally linearly-independent sets of solutions are also linearly independent at every point. This property might not hold for the formal adjoint $\mathbf{D}^*$, but since solutions to $\mathbf{D}^*\xi = 0$ satisfy unique continuation, any expression of the form $\sum_{ij} c^{ij}\eta_i \otimes \xi_j$ with a nontrivial set of coefficients $c^{ij} \in \mathbb{R}$ and linearly-independent sets $\{\eta_i \in \ker \mathbf{D}\}$ and $\{\xi_j \in \ker \mathbf{D}^*\}$ is still guaranteed to be nonzero at every point outside a discrete subset. Example 5.6 below shows however that the local Petri condition in this situation is not always satisfied.

Example 5.5. Complex-linear Cauchy-Riemann operators over a Riemann surface satisfy the complex version of Petri's condition to infinite order at every point, i.e. the definition above is satisfied if real tensor products are replaced by complex tensor products. One can prove this by choosing holomorphic trivializations and writing elements of $\ker \mathbf{D}$ and $\ker \mathbf{D}^*$ locally as Taylor series in z or $\bar{z}$ respectively: it then turns out that for any nontrivial $t \in \ker \mathbf{D} \otimes_{\mathbb{C}} \ker \mathbf{D}^*$, the Taylor series in z and $\bar{z}$ for the resulting section of $E \otimes_{\mathbb{C}} F$ at a given point is always nontrivial. We omit the details since we will not need this fact.

Example 5.6. If we regard the standard Cauchy-Riemann operator $\mathbf{D} = \bar{\partial}$ on a trivial line bundle and its formal adjoint $\mathbf{D}^* = -\partial$ as real-linear operators, then they do not satisfy the local Petri condition at any point. A local counterexample is given by

$$1 \otimes i\bar{z} - i \otimes \bar{z} - z \otimes i + iz \otimes 1 \in \ker \bar{\partial} \otimes_{\mathbb{R}} \ker \partial.$$

It follows that the local Petri condition is also not satisfied by any Cauchy-Riemann type operator that splits off a complex-linear summand.

Example 5.7. Here is an example of a Cauchy-Riemann type operator that does not split off any complex-linear summand but still fails to satisfy the local Petri condition: take E and F to be the trivial complex line bundle over $\mathbb{C}$, with standard bundle metrics and the standard area form, and consider $\mathbf{D} := \bar{\partial} + \kappa$, $\mathbf{D}^* = -\partial + \kappa$, where $\kappa : \mathbb{C} \to \mathbb{C}$ is complex conjugation. Using coordinates $s + it \in \mathbb{C}$, one can associate to every $\lambda \in (-1, 1)$ solutions $\eta_\lambda \in \ker \mathbf{D}$ and

$\xi_\lambda \in \ker \mathbf{D}^*$ defined by[7]

$$\eta_\lambda(s+it) := e^{\lambda s+\sqrt{1-\lambda^2}t}\left(\sqrt{1-\lambda}+i\sqrt{1+\lambda}\right),$$
$$\xi_\lambda(s+it) := e^{-\lambda s-\sqrt{1-\lambda^2}t}\left(\sqrt{1-\lambda}-i\sqrt{1+\lambda}\right).$$

Identifying the fibers $\mathbb{C}$ with $\mathbb{R}^2$ so that the fibers of $E \otimes_{\mathbb{R}} F$ become the space of real 2-by-2 matrices, the products $\Pi(\eta_\lambda \otimes \xi_\lambda)$ are now constant sections of $E \otimes_{\mathbb{R}} F$:

$$\Pi(\eta_\lambda \otimes \xi_\lambda)(s+it) = \begin{pmatrix} 1-\lambda & -\sqrt{1-\lambda^2} \\ \sqrt{1-\lambda^2} & -1-\lambda \end{pmatrix}.$$

Such products span the 3-dimensional space of real matrices of the form $\begin{pmatrix} a & b \\ -b & c \end{pmatrix}$, thus any four such products must be linearly dependent, and the dependence relation gives rise to nontrivial elements in $\ker \Pi$ by choosing four distinct values of $\lambda \in (-1,1)$.

Remark 5.8. An earlier version of this paper (see Appendix D.2) claimed that every Cauchy-Riemann type operator whose complex-antilinear part is invertible at a point p satisfies Petri's condition to infinite order at p, but Example 5.7 contradicts that.

The operators in Examples 5.6 and 5.7 are rather special, and our main objective in this section is to prove that such counterexamples cannot arise for *generic* Cauchy-Riemann type operators. To set up the result, assume now that Σ is a Riemann surface with a Hermitian bundle metric $\langle\ ,\ \rangle_\Sigma$ on $T\Sigma$. We will not require Σ to be compact since the discussion will be purely local, but fix a point $p \in \Sigma$ and an open neighborhood $\mathcal{U} \subset \Sigma$ of p with compact closure. Fix also a complex vector bundle $E \to \Sigma$ with a Hermitian bundle metric, let $F = \overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, E)$, and denote by $\mathcal{CR}_{\mathbb{R}}(E)$ the space of real-linear Cauchy-Riemann type operators $\mathbf{D} : \Gamma(E) \to \Gamma(F)$. We shall fix a specific $\mathbf{D}_{\text{fix}} \in \mathcal{CR}_{\mathbb{R}}(E)$ and define the space of all Cauchy-Riemann type operators $\mathbf{D}$ that match $\mathbf{D}_{\text{fix}}$ outside of $\mathcal{U}$:

$$\mathcal{CR}_{\mathbb{R}}(E\,;\mathcal{U},\mathbf{D}_{\text{fix}}) := \left\{\mathbf{D} \in \mathcal{CR}_{\mathbb{R}}(E) \;\middle|\; \mathbf{D}-\mathbf{D}_{\text{fix}} = 0 \text{ on } \Sigma\backslash\mathcal{U}\right\}.$$

This is an affine space over the Fréchet space of smooth sections of $\operatorname{Hom}_{\mathbb{R}}(E,F)$ that vanish outside $\mathcal{U}$, so in particular it is a complete metric space. For every $\mathbf{D} \in \mathcal{CR}_{\mathbb{R}}(E)$, $\mathbf{D}^*$ will denote the formal adjoint of $\mathbf{D}$ determined by the bundle metrics on E and Σ.

For any $\eta \in \Gamma_p(E)$, we define the **vanishing order** of η at p by

$$\operatorname{ord}(\eta;p) := \sup\left\{k \in \{0\} \cup \mathbb{N} \;\middle|\; \text{all derivatives of } \eta \text{ at } p \text{ up to order } k \text{ vanish}\right\}.$$

For $t \in \Gamma_p(E) \otimes \Gamma_p(F)$, we will then say that t **vanishes to order** k if t can be written as a finite sum $t = \sum_j \eta_j \otimes \xi_j$ such that

$$\operatorname{ord}(\eta_j;p) + \operatorname{ord}(\xi_j;p) \geqslant k \quad \text{for every } j.$$

The usual unique continuation results imply that for every $\mathbf{D} \in \mathcal{CR}_{\mathbb{R}}(E)$, nontrivial local solutions to the equations $\mathbf{D}\eta = 0$ or $\mathbf{D}^*\eta = 0$ satisfy $\operatorname{ord}(\eta;p) < \infty$ at every point. One can easily prove from this that nontrivial elements $t \in \ker \mathbf{D}_p \otimes \ker \mathbf{D}_p^*$ also cannot vanish to infinite order (see Proposition 5.12).

The machinery developed in the next two subsections will prove:

Theorem 5.9. *For every $\ell \in \mathbb{N}$, there exists an integer $k \geqslant \ell$ and a Baire subset*

$$\mathcal{CR}_{\mathbb{R}}^{\ell,\text{reg}}(E\,;\mathcal{U},\mathbf{D}_{\text{fix}}) \subset \mathcal{CR}_{\mathbb{R}}(E\,;\mathcal{U},\mathbf{D}_{\text{fix}})$$

[7] The inspiration for this example comes from the asymptotic formulas in [HWZ96, Sie08]: in particular on the cylinder $\mathbb{R} \times S^1$ with coordinates (s,t), a translation-invariant Cauchy-Riemann type equation $(\bar\partial + B(t))\eta(s,t) = 0$ always has solutions of the form $\eta(s,t) = e^{s\lambda}f(t)$, where f is an eigenfunction of the asymptotic operator $-i\partial_t - B(t)$ with eigenvalue $\lambda \in \mathbb{R}$. In the asymptotic setting one requires solutions to be periodic in t, in which case the eigenvalue λ can only take a discrete set of values, but periodicity is not necessary in Example 5.7, and λ can therefore be chosen much more freely.

with the following significance: for every $\mathbf{D} \in \mathcal{CR}^{\ell,\mathrm{reg}}_{\mathbb{R}}(E\,;\mathcal{U},\mathbf{D}_{\mathrm{fix}})$*, if* $\eta_1,\dots,\eta_\ell \in \ker \mathbf{D}_p$ *and* $\xi_1,\dots,\xi_\ell \in \ker \mathbf{D}^*_p$ *are* ℓ*-tuples of local solutions such that* $t := \sum_{j=1}^{\ell} \eta_j \otimes \xi_j \in \Gamma_p(E) \otimes \Gamma_p(F)$ *does not vanish to order* ℓ*, then* $\Pi(t) \in \Gamma_p(E \otimes F)$ *does not vanish to order* k*.*

In light of unique continuation, we now set

$$\mathcal{CR}^{\mathrm{reg}}_{\mathbb{R}}(E\,;\mathcal{U},\mathbf{D}_{\mathrm{fix}}) := \bigcap_{\ell\in\mathbb{N}} \mathcal{CR}^{\ell,\mathrm{reg}}_{\mathbb{R}}(E\,;\mathcal{U},\mathbf{D}_{\mathrm{fix}}) \subset \mathcal{CR}_{\mathbb{R}}(E\,;\mathcal{U},\mathbf{D}_{\mathrm{fix}})$$

and obtain:

Corollary 5.10. *There exists a Baire subset*

$$\mathcal{CR}^{\mathrm{reg}}_{\mathbb{R}}(E\,;\mathcal{U},\mathbf{D}_{\mathrm{fix}}) \subset \mathcal{CR}_{\mathbb{R}}(E\,;\mathcal{U},\mathbf{D}_{\mathrm{fix}})$$

such that every $\mathbf{D} \in \mathcal{CR}^{\mathrm{reg}}_{\mathbb{R}}(E\,;\mathcal{U},\mathbf{D}_{\mathrm{fix}})$ *satisfies Petri's condition to infinite order at the point* $p \in \mathcal{U}$*.*

This result can be extended in various ways. For instance, the regular set $\mathcal{CR}^{\mathrm{reg}}_{\mathbb{R}}(E\,;\mathcal{U},\mathbf{D}_{\mathrm{fix}})$ defined above depends *a priori* on the choice of a point $p \in \mathcal{U}$, but one can also find a Baire set of operators such that Petri's condition to infinite order is satisfied simultaneously at every point in $\mathcal{U}$. More generally, one can consider smooth families of operators parametrized by a finite-dimensional manifold and prove that for generic families, every operator in the family satisfies these conditions. In §5.4, we will prove that the normal Cauchy-Riemann operators of J-holomorphic curves can all be assumed to satisfy Petri's condition to infinite order in regions where J can be perturbed generically. One of the advantages of focusing on purely *local* conditions is that once we establish this result for somewhere injective curves, it carries over immediately to their multiple covers, which will be a crucial ingredient in the proof of Theorem D.

The aforementioned extensions of Corollary 5.10 are all based on the Sard-Smale theorem, but Theorem 5.9 itself requires (aside from unique continuation) only finite-dimensional analysis and linear algebra. Indeed, the conditions defining each of the spaces $\mathcal{CR}^{\ell,\mathrm{reg}}_{\mathbb{R}}(E\,;\mathcal{U},\mathbf{D}_{\mathrm{fix}})$ in the statement of the theorem depend only on the k-jet of $\mathbf{D} \in \mathcal{CR}_{\mathbb{R}}(E\,;\mathcal{U},\mathbf{D}_{\mathrm{fix}})$ at p for some finite $k \in \mathbb{N}$, and this data varies in a finite-dimensional smooth manifold. The idea behind the proof is roughly to show that the set of jets of operators not satisfying the desired conditions lives in "walls" whose codimensions can be assumed arbitrarily large by making k larger. These walls are not submanifolds in general, but are what we call "C^∞-subvarieties," whose local structure is nice enough to apply Sard's theorem as if they were manifolds. (The necessary background on C^∞-subvarieties is reviewed in Appendix C.) The main technical work behind the proof is then to estimate the ranks of certain large matrices that determine the codimensions of these subvarieties.

The rest of this section will proceed as follows. In §5.2, we introduce a general formalism for studying differential operators via jet spaces at a point, and explain how results such as Theorem 5.9 can be reduced to a specific technical lemma on estimating the ranks of certain finite-dimensional linear transformations. We will then address this problem for Cauchy-Riemann operators in §5.3, leading to the proof of Theorem 5.9. The extension to a result about normal Cauchy-Riemann operators of holomorphic curves for generic J will be stated and proved in §5.4, and §5.5 will then give an important application of Petri's condition to global transversality problems as arising in Theorem D.

5.2. **Jet space formalism.** The contents of this subsection are not specific to Cauchy-Riemann operators, but may be relevant in principle to any linear partial differential operator with smooth coefficients.

5.2.1. *Germs, jets, and the vanishing order filtration.* Fix a smooth n-dimensional manifold M with a smooth vector bundle $E \to M$ of real rank $m \in \mathbb{N}$. For a chosen point $p \in M$, we continue to denote by

$$\Gamma_p(E) := \Gamma(E)/\!\sim$$

the vector space of germs of smooth sections of E defined near p. This space has a natural filtration

$$\Gamma_p(E) = \Gamma_p(E)^0 \supset \Gamma_p(E)^1 \supset \Gamma_p(E)^2 \supset \dots, \tag{5.1}$$

where for each $k \in \mathbb{Z}$ we define $\Gamma_p(E)^k \subset \Gamma_p(E)$ as the space of germs of sections whose derivatives up to order $k-1$ at p all vanish. For $k \leqslant 0$ this is a vacuous condition, hence $\Gamma_p(E)^k = \Gamma_p(E)$. For each $k \in \mathbb{Z}$ we define the space of **k-jets of sections** at p by

$$J^k_p E := \Gamma_p(E) \big/ \Gamma_p(E)^{k+1}.$$

We will typically abuse notation by using a single symbol such as η to represent a section in $\Gamma(E)$, its germ in $\Gamma_p(E)$ and its k-jet in $J^k_p E$; when there is need for more clarity in the notation, we will sometimes write the natural quotient projections as

$$\Gamma(E) \text{ or } \Gamma_p(E) \xrightarrow{J^k_p} J^k_p E,$$

so that the k-jet of a section $\eta \in \Gamma(E)$ at p can be denoted by $J^k_p \eta \in J^k_p E$. The jet space inherits from (5.1) a finite filtration

$$J^k_p E = (J^k_p E)^0 \supset (J^k_p E)^1 \supset \dots \supset (J^k_p E)^k \supset (J^k_p E)^{k+1} = \{0\}, \tag{5.2}$$

where for each $\ell \leqslant k$, $(J^k_p E)^{\ell+1}$ is the kernel of the quotient projection $J^\ell_p : J^k_p E \to J^\ell_p E$.

There is an obvious isomorphism of $J^0_p E$ with the fiber E_p, and the spaces $J^k_p E$ for $k < 0$ are all trivial. If we choose local coordinates $(x_1, \dots, x_n)$ for M identifying p with $0 \in \mathbb{R}^n$, together with a trivialization of E near p, then $J^k_p E$ for each $k \in \mathbb{Z}$ becomes naturally identified with the vector space of $\mathbb{R}^m$-valued Taylor polynomials of degree at most k,

$$\sum_{|\alpha| \leqslant k} x^\alpha c_\alpha, \qquad c_\alpha \in \mathbb{R}^m. \tag{5.3}$$

The notation for the filtration above has been chosen so that under this identification, $(J^k_p E)^\ell$ becomes the space of Taylor polynomials of degree at most k that are also $O(|x|^\ell)$.

Given two vector spaces $V = V^0 \supset V^1 \supset V^2 \supset \dots$ and $W = W^0 \supset W^1 \supset W^2 \supset \dots$ with filtrations, we will say in general that a linear map $T : V \to W$ **preserves the filtrations** if $T(V^n) \subset W^n$ for every $n \geqslant 0$.

5.2.2. *Differential operators and formal adjoints.* Since we are mainly interested in Cauchy-Riemann type operators, for simplicity we shall only consider differential operators of order 1 in the following discussion, though the jet space formalism could easily be extended beyond this.

Given a second smooth vector bundle $F \to M$ of real rank $\ell \in \mathbb{N}$ and a first-order linear partial differential operator $\mathbf{D} : \Gamma(E) \to \Gamma(F)$ with smooth coefficients, $\mathbf{D}$ descends to a map $\Gamma_p(E) \to \Gamma_p(F)$ that sends $\ker J^k_p \subset \Gamma_p(E)$ into $\ker J^{k-1}_p \subset \Gamma_p(F)$ for each $k \in \mathbb{Z}$, thus it also descends to a linear map

$$\mathbf{D} : J^k_p E \to J^{k-1}_p F.$$

Let us denote by

$$\mathscr{D}_p(E, F) \subset \operatorname{Hom}\big(\Gamma_p(E), \Gamma_p(F)\big)$$

the vector space consisting of all germs at p of linear differential operators $\Gamma(E) \to \Gamma(F)$ of order at most 1 with smooth coefficients. The vector space of linear maps $J^k_p E \to J^{k-1}_p F$ that are induced by operators in $\mathscr{D}_p(E, F)$ will then be denoted by

$$\mathscr{D}^k_p(E, F) \subset \operatorname{Hom}\big(J^k_p E, J^{k-1}_p F\big),$$

and we will again abuse notation by using a single symbol such as $\mathbf{D}$ to denote a global differential operator $\Gamma(E) \to \Gamma(F)$, its germ in $\mathscr{D}_p(E, F)$, and the map in $\mathscr{D}^k_p(E, F)$ that it determines. Observe that $\mathscr{D}^k_p(E, F)$ is a finite-dimensional vector space isomorphic to the $(n+1)$-fold product of $J^{k-1}_p \operatorname{Hom}(E, F)$: indeed, if we fix local coordinates $(x_1, \dots, x_n)$ identifying a neighborhood of p with the n-disk $\mathbb{D}^n_\epsilon$ of some radius $\epsilon > 0$, along with local trivializations of E and F over the same neighborhood, then each $\mathbf{D} \in \mathscr{D}_p(E,F)$ is represented by an operator $C^\infty(\mathbb{D}^n_\epsilon, \mathbb{R}^m) \to C^\infty(\mathbb{D}^n_\epsilon, \mathbb{R}^\ell)$ of the form

$$\mathbf{D} = \sum_{j=1}^n a_j \partial_j + b \tag{5.4}$$

for some smooth functions $a_1, \ldots, a_n, b : \mathbb{D}^n_\epsilon \to \operatorname{Hom}(\mathbb{R}^m, \mathbb{R}^\ell)$. For a given $\eta \in \Gamma(E)$, the $(k-1)$-jet of $\mathbf{D}\eta$ at p is thus determined by the $(k-1)$-jets of the functions $a_1, \ldots, a_n, b$ at that point, and these are equivalent to bundle maps $E \to F$ defined near p.

We will also consider a subset

$$\widehat{\mathscr{D}}_p(E,F) \subset \mathscr{D}_p(E,F),$$

which is assumed to have the property that for any given $\mathbf{D} \in \widehat{\mathscr{D}}_p(E,F)$, another operator $\mathbf{D}' \in \mathscr{D}_p(E,F)$ satisfies

$$\mathbf{D}' \in \widehat{\mathscr{D}}_p(E,F) \quad \Leftrightarrow \quad \mathbf{D}' = \mathbf{D} + A \text{ for some } A \in \Gamma_p(\operatorname{Hom}(E,F)),$$

i.e. $\widehat{\mathscr{D}}_p(E,F)$ is an affine space over $\Gamma_p(\operatorname{Hom}(E,F))$. The space of maps $J^k_p E \to J^{k-1}_p F$ induced by operators $\mathbf{D} \in \widehat{\mathscr{D}}_p(E,F)$ then defines a subset

$$\widehat{\mathscr{D}}^k_p(E,F) \subset \mathscr{D}^k_p(E,F),$$

which is naturally an affine space over the finite-dimensional vector space $J^{k-1}_p \operatorname{Hom}(E,F)$.

In order to bring formal adjoints into this picture, we need to make choices of bundle metrics for E and F and a volume form on M near p; these choices will often be referred to collectively as the **geometric data**. It will be useful to fix geometric data once and for all at the point p itself, while allowing it to vary at other points near p. Concretely, fix a pair of inner products

$$g_p = \langle\ ,\ \rangle_{E_p} \text{ on } E_p, \qquad h_p = \langle\ ,\ \rangle_{F_p} \text{ on } F_p,$$

along with a nontrivial alternating multilinear n-form

$$\mu_p \in \Lambda^n T^*_p M.$$

Let us denote by $S^2 E^* \subset E^* \otimes E^* \to M$ the vector bundle of symmetric bilinear forms $E \oplus E \to \mathbb{R}$. The space of **$k$-jets of bundle metrics** on E which match g_p at p is then

$$J^k_p\big(\mathfrak{m}(E)\big) := \left\{ g \in J^k_p\big(S^2E^*\big) \;\middle|\; J^0_p g = g_p \right\},$$

and it is naturally an affine space over the finite-dimensional vector space $\ker J^0_p \subset J^k_p\big(S^2E^*\big)$. We similarly define the affine spaces

$$J^k_p\big(\mathfrak{m}(F)\big) := \left\{ h \in J^k_p\big(S^2E^*\big) \;\middle|\; J^0_p h = h_p \right\}$$

and

$$J^k_p\big(\mathfrak{v}(M)\big) := \left\{ \mu \in J^k_p\big(\Lambda^n T^*M\big) \;\middle|\; J^0_p \mu = \mu_p \right\},$$

which consist respectively of k-jets of bundle metrics on F matching h_p at p and k-jets of volume forms on M matching μ_p at p. We will again abuse notation by using a single symbol such as g or $\langle\ ,\ \rangle_E$ to denote a global bundle metric on E that matches g_p at p, or the germ of such a metric near p, or its k-jet in $J^k_p\big(\mathfrak{m}(E)\big)$; similar remarks apply to $J^k_p\big(\mathfrak{m}(F)\big)$ and $J^k_p\big(\mathfrak{v}(M)\big)$.

Any choice of smooth bundle metrics $g = \langle\ ,\ \rangle_E$ on E and $h = \langle\ ,\ \rangle_F$ on F and a volume form $\mu \in \Omega^n(M)$ assigns to each differential operator $\mathbf{D} : \Gamma(E) \to \Gamma(F)$ a **formal adjoint** $\mathbf{D}^* : \Gamma(F) \to \Gamma(E)$ satisfying the relation

$$\int_M \langle \xi, \mathbf{D}\eta \rangle_F\, \mu = \int_M \langle \mathbf{D}^*\xi, \eta \rangle_E\, \mu \quad \text{for all} \quad \eta \in C^\infty_0(E),\ \xi \in C^\infty_0(F).$$

Fix local coordinates and trivializations near p to write $\mathbf{D}$ again in the form (5.4). The chosen bundle metrics and volume form can be written in terms of the standard Euclidean inner product $\langle\ ,\ \rangle$ and volume form $dx_1 \wedge \ldots \wedge dx_n$ as

$$\langle\ ,\ \rangle_E = \langle \cdot, G\cdot \rangle, \quad \langle\ ,\ \rangle_F = \langle \cdot, H\cdot \rangle, \quad \mu = F\, dx_1 \wedge \ldots \wedge dx_n$$

for some smooth functions $F : \mathbb{D}^n_\epsilon \to \mathbb{R}$, $G : \mathbb{D}^n_\epsilon \to \operatorname{End}(\mathbb{R}^m)$ and $H : \mathbb{D}^n_\epsilon \to \operatorname{End}(\mathbb{R}^\ell)$, where F is everywhere nonzero and G and H take values in the spaces of symmetric positive-definite matrices. Note that the condition defining $\mathbf{D}^*$ does not change if the sign of μ is reversed, so without loss of generality let us assume $F > 0$. One can then compute a local formula for $\mathbf{D}^* : C^\infty(\mathbb{D}^n_\epsilon, \mathbb{R}^\ell) \to C^\infty(\mathbb{D}^n_\epsilon, \mathbb{R}^m)$ as

$$\mathbf{D}^* = -\sum_j (G^{-1} a_j^{\mathrm{T}} H)\partial_j + G^{-1}\left(b^{\mathrm{T}} H - \sum_j \left[a_j^{\mathrm{T}} H \partial_j(\ln F) + \partial_j(a_j^{\mathrm{T}} H) \right] \right). \tag{5.5}$$

We observe from this formula that the germ $\mathbf{D}^* \in \mathscr{D}_p(F, E)$ at p is determined by the corresponding germs of the geometric data g, h, μ and $\mathbf{D} \in \mathscr{D}_p(E, F)$. Moreover, if the first-order terms a_j in $\mathbf{D}$ are fixed, then for any $\xi \in \Gamma(F)$, the $(k-1)$-jet of $\mathbf{D}^*\xi$ at p is determined by the $(k-1)$-jet of g, the k-jets of μ and h, and the $(k-1)$-jet of the zeroth-order term b in $\mathbf{D}$. It follows that the correspondence assigning to each $\mathbf{D} \in \widehat{\mathscr{D}}_p(E, F)$ with germs of geometric data g, h, μ the germ of a formal adjoint $\mathbf{D}^* \in \mathscr{D}_p(F, E)$ descends to a well-defined map

$$\widehat{\mathscr{D}}^k_p(E, F) \times J^{k-1}_p\big(\mathfrak{m}(E)\big) \times J^k_p\big(\mathfrak{m}(F)\big) \times J^k_p\big(\mathfrak{v}(M)\big) \xrightarrow{\ *\ } \mathscr{D}^k_p(F, E). \tag{5.6}$$

All the spaces involved in this map are finite-dimensional manifolds, and the map is smooth.

5.2.3. *Unique continuation in tensor products.* If $V = V^0 \supset V^1 \supset V^2 \supset \ldots$ and $W = W^0 \supset W^1 \supset W^2 \supset \ldots$ are two vector spaces with filtrations, then $V \otimes W$ inherits a natural filtration

$$V \otimes W = (V \otimes W)^0 \supset (V \otimes W)^1 \supset (V \otimes W)^2 \supset \ldots,$$

where for each $n \geqslant 0$,

$$(V \otimes W)^n := (V^0 \otimes W^n) + (V^1 \otimes W^{n-1}) + \ldots + (V^n \otimes W^0).$$

Lemma 5.11. *Given two filtered vector spaces V and W, if $t \in (V \otimes W)^n$ is nontrivial, then for some $r \in \mathbb{N}$, t can be written as*

$$t = \sum_{j=1}^r v_j \otimes w_j$$

for two linearly-independent sets $v_1, \ldots, v_r \in V$ and $w_1, \ldots, w_r \in W$ such that for all $j = 1, \ldots, r$, we have

$$v_j \in V^{k_j} \text{ and } w_j \in W^{\ell_j} \quad \text{where} \quad k_j + \ell_j = n.$$

Proof. Suppose $t = \sum_{j=1}^r v_j \otimes w_j$ satisfies all of these conditions except that the set $v_1, \ldots, v_r$ is linearly dependent, so there exist constants $c_1, \ldots, c_r$ with $\sum_j c_j v_j = 0$ and not all of the c_j are zero. After reordering the set, we can assume without loss of generality that $c_1 \neq 0$ and, for every $j = 2, \ldots, r$ with $c_j \neq 0$, $k_j \geqslant k_1$. Writing $v_1 = \sum_{j=2}^r \frac{c_j}{c_1} v_j$ then gives

$$t = \sum_{j=2}^r v_j \otimes \widehat{w}_j \quad \text{where} \quad \widehat{w}_j := w_j + \frac{c_j}{c_1} w_1.$$

For each $j = 2, \ldots, r$, we now have $\ell_j = n - k_j \leqslant n - k_1 = \ell_1$, thus $w_1 \in W^{\ell_1} \subset W^{\ell_j}$ and therefore $\widehat{w}_j \in W^{\ell_j}$, hence the shortened sum also satisfies the desired conditions. One can apply a similar procedure to shorten the sum if instead $w_1, \ldots, w_r$ is linearly dependent, and repeating this enough times produces two sets that are both linearly independent. ☐

Let us say that a differential operator $\mathbf{D} \in \mathscr{D}_p(E, F)$ has the **strong unique continuation** property if there exists no nontrivial solution $\eta \in \ker \mathbf{D}$ such that $\eta \in \Gamma_p(E)^k$ for every $k \in \mathbb{N}$.

Proposition 5.12. *If $\mathbf{D} \in \mathscr{D}_p(E,F)$ and $\mathbf{D}^* \in \mathscr{D}_p(F,E)$ both have the strong unique continuation property, then there exists no nontrivial element $t \in \ker \mathbf{D} \otimes \ker \mathbf{D}^*$ such that $t \in \big(\Gamma_p(E) \otimes \Gamma_p(F)\big)^k$ for every $k \in \mathbb{N}$.*

Proof. Given $t \in \ker \mathbf{D} \otimes \ker \mathbf{D}^*$ nonzero, there are uniquely defined finite-dimensional subspaces $V \subset \ker \mathbf{D}$ and $W \subset \ker \mathbf{D}^*$ such that for any pair of linearly-independent sets $\eta_1, \dots, \eta_r \in \ker \mathbf{D}$ and $\xi_1, \dots, \xi_r \in \ker \mathbf{D}^*$ with $t = \sum_j \eta_j \otimes \xi_j$,

$$V = \operatorname{Span}\{\eta_1, \dots, \eta_r\} \quad \text{and} \quad W = \operatorname{Span}\{\xi_1, \dots, \xi_r\}.$$

We claim there exists $k \in \mathbb{N}$ such that no nontrivial $\eta \in V$ is in $\Gamma_p(E)^k$ and no nontrivial $\xi \in W$ is in $\Gamma_p(F)^k$. Indeed, if there does not exist such a number for V, then there exist sequences $\eta_j \in V$ and $k_j \in \mathbb{N}$ with $k_j \to \infty$ and $\eta_j \in \Gamma_p(E)^{k_j}$ for every j. Since V is finite dimensional, we can normalize the η_j and then find a convergent subsequence $\eta_j \to \eta_\infty \in V$ whose limit is nontrivial, but must also belong to $\bigcap_{k \in \mathbb{N}} \Gamma_p(E)^k$, giving a contradiction. The same argument works for W.

Now, fixing $k \in \mathbb{N}$ as in the previous paragraph, suppose $t \in (\Gamma_p(E) \otimes \Gamma_p(F))^{2k}$ and $t \neq 0$. Lemma 5.11 then writes t in the form $\sum_j \eta_j \otimes \xi_j$ where the η_j and ξ_j are necessarily bases of V and W respectively, but they also satisfy $\eta_j \in \Gamma_p(E)^{\ell_j}$ and $\xi_j \in \Gamma_p(F)^{m_j}$ with $\ell_j + m_j \geqslant 2k$ for each j. This implies either $\ell_j \geqslant k$ or $m_j \geqslant k$ in each case, and is thus a contradiction. □

5.2.4. *Local rescaling.* Every differential operator is locally equivalent (up to choices of coordinates and trivializations) to an arbitrarily small perturbation of an operator with constant coefficients and no lower-order terms. To make use of this observation, we shall from now on impose the following additional condition on the affine space of local operators $\widehat{\mathscr{D}}_p(E,F) \subset \mathscr{D}_p(E,F)$:

Assumption 5.13. There exists a choice of coordinates identifying a neighborhood $\mathcal{U} \subset M$ of p with $\mathbb{D}^n_\epsilon \subset \mathbb{R}^n$ and p with $0 \in \mathbb{R}^n$, along with local trivializations over $\mathcal{U}$, in which the first-order coefficients $a_j : \mathbb{D}^n_\epsilon \to \operatorname{Hom}(\mathbb{R}^m, \mathbb{R}^\ell)$ in $\mathbf{D} = \sum_j a_j \partial_j + b$ for each $\mathbf{D} \in \widehat{\mathscr{D}}_p(E,F)$ are constant.

Let us fix once and for all a neighborhood $\mathcal{U} \subset M$ of p with coordinates and trivializations for which the condition in Assumption 5.13 holds. For every $\varepsilon \in [0,1]$, we then associate to each $\mathbf{D} \in \mathscr{D}_p(E,F)$ an operator $\mathbf{D}_\varepsilon \in \mathscr{D}_p(E,F)$ such that if $\mathbf{D}$ takes the form $\mathbf{D}\eta(x) = \sum_j a_j(x)\partial_j \eta(x) + b(x)\eta(x)$ in the chosen coordinates and trivializations, then $\mathbf{D}_\varepsilon$ is given by

$$\mathbf{D}_\varepsilon \eta(x) = \sum_j a_j(\varepsilon x)\partial_j \eta(x) + \varepsilon b(\varepsilon x)\eta(x), \qquad \varepsilon \in [0,1].$$

We can similarly associate to each $\eta \in \Gamma_p(E)$ and $\xi \in \Gamma_p(F)$ germs of sections $\eta_\varepsilon \in \Gamma_p(E)$ and $\xi_\varepsilon \in \Gamma_p(F)$, which in coordinates take the form

$$\eta_\varepsilon(x) := \eta(\varepsilon x), \qquad \xi_\varepsilon(x) := \xi(\varepsilon x).$$

We then have

$$\mathbf{D}_\varepsilon \eta_\varepsilon = \varepsilon (\mathbf{D}\eta)_\varepsilon$$

for every $\mathbf{D} \in \mathscr{D}_p(E,F)$ and $\eta \in \Gamma_p(E)$. Letting these operators descend to jet spaces, we obtain for every $\mathbf{D} \in \mathscr{D}^k_p(E,F)$ a smooth 1-parameter family of operators $\big\{\mathbf{D}_\varepsilon \in \mathscr{D}^k_p(E,F)\big\}_{\varepsilon \in [0,1]}$ and linear maps

$$J^k_p E \to J^k_p E : \eta \mapsto \eta_\varepsilon,$$

which for $\varepsilon > 0$ are isomorphisms sending $\ker \mathbf{D} \subset J^k_p E$ onto $\ker \mathbf{D}_\varepsilon \subset J^k_p E$.

Next, fix geometric data consisting of bundle metrics $g = \langle\ ,\ \rangle_E$ on E and $h = \langle\ ,\ \rangle_F$ on F, and a volume form μ, such that all three match the fixed choices of data g_p, h_p and μ_p at p. Using the same coordinates and trivializations over $\mathcal{U}$, we can write $g = \langle \cdot, G\cdot \rangle$, $h = \langle \cdot, H\cdot \rangle$ and $\mu = F\, dx_1 \wedge \ldots \wedge dx_n$, and then define a smooth 1-parameter family of geometric data $g_\varepsilon, h_\varepsilon, \mu_\varepsilon$ for $\varepsilon \in [0,1]$ by replacing the functions G, H and F with

$$G_\varepsilon(x) := G(\varepsilon x), \qquad H_\varepsilon(x) := H(\varepsilon x), \qquad F_\varepsilon(x) := F(\varepsilon x).$$

Note that since p corresponds to $0 \in \mathbb{D}^n_\epsilon$ in coordinates, the modified geometric data still matches the fixed choices g_p, h_p, μ_p at p, and we can then descend to jet spaces to obtain smooth 1-parameter families

$$g_\varepsilon \in J^{k-1}_p\big(\mathfrak{m}(E)\big), \qquad h_\varepsilon \in J^k_p\big(\mathfrak{m}(F)\big), \qquad \mu_\varepsilon \in J^k_p\big(\mathfrak{v}(M)\big)$$

for $\varepsilon \in [0,1]$. Now if $\mathbf{D}^* \in \mathscr{D}_p(F,E)$ denotes the formal adjoint of $\mathbf{D} \in \widehat{\mathscr{D}}_p(E,F)$ with respect to the geometric data g,h,μ and $\mathbf{D}^*_\varepsilon \in \mathscr{D}_p(F,E)$ is defined from $\mathbf{D}^*$ via the same rescaling prescription as $\mathbf{D}_\varepsilon$ described above, then we see from (5.5) that $\mathbf{D}^*_\varepsilon$ is in fact the formal adjoint of $\mathbf{D}_\varepsilon$ with respect to the data $g_\varepsilon, h_\varepsilon, \mu_\varepsilon$. Moreover, Assumption 5.13 implies that the map $\mathscr{D}^k_p(E,F) \to \mathscr{D}^k_p(E,F)$ induced by $\mathscr{D}_p(E,F) \to \mathscr{D}_p(E,F) : \mathbf{D} \mapsto \mathbf{D}_\varepsilon$ preserves $\widehat{\mathscr{D}}^k_p(E,F)$, so we can now fit the smooth map (5.6) into the rows of a commutative diagram

$$\begin{array}{ccc}
\widehat{\mathscr{D}}^k_p(E,F) \times J^{k-1}_p\big(\mathfrak{m}(E)\big) \times J^k_p\big(\mathfrak{m}(F)\big) \times J^k_p\big(\mathfrak{v}(M)\big) & \xrightarrow{\ *\ } & \mathscr{D}^k_p(F,E) \\
\Big\downarrow{\scriptstyle \varepsilon} & & \Big\downarrow{\scriptstyle \varepsilon} \\
\widehat{\mathscr{D}}^k_p(E,F) \times J^{k-1}_p\big(\mathfrak{m}(E)\big) \times J^k_p\big(\mathfrak{m}(F)\big) \times J^k_p\big(\mathfrak{v}(M)\big) & \xrightarrow{\ *\ } & \mathscr{D}^k_p(F,E),
\end{array}$$

where the vertical maps abbreviated by "ε" are defined via the corresondences $\mathbf{D} \mapsto \mathbf{D}_\varepsilon$, $g \mapsto g_\varepsilon$, $h \mapsto h_\varepsilon$, $\mu \mapsto \mu_\varepsilon$ and $\mathbf{D}^* \mapsto \mathbf{D}^*_\varepsilon$. The case $\varepsilon = 0$ is special: since all $\mathbf{D} \in \widehat{\mathscr{D}}_p(E,F)$ have matching first-order terms and the geometric data g,h,μ all match at p, $\mathbf{D}_0$ and $\mathbf{D}^*_0$ are *uniquely-defined* operators that depend on the space $\widehat{\mathscr{D}}_p(E,F)$ and the chosen inner products g_p and h_p, but not otherwise on the specific choices of operator $\mathbf{D} \in \widehat{\mathscr{D}}_p(E,F)$ or volume form or bundle metrics. Similarly, the volume form μ_0 and bundle metrics g_0 and h_0 are fully determined by the fixed data μ_p, g_p and h_p.

5.2.5. *Right-inverses.* Henceforward we impose the following additional assumption.

Assumption 5.14. The operators $\mathbf{D}_0 : J^k_p E \to J^{k-1}_p F$ and $\mathbf{D}^*_0 : J^k_p F \to J^{k-1}_p E$ obtained by the rescaling procedure in §5.2.4 are surjective.

Remark 5.15. It is not difficult to show that Assumption 5.14 is satisfied whenever the operators in $\widehat{\mathscr{D}}_p(E,F)$ are elliptic. For Cauchy-Riemann operators in particular, this is virtually obvious, and we will write down explicit choices of right-inverses for that case in §5.3.2.

Lemma 5.16. *Under Assumption 5.14, every $\mathbf{D} \in \widehat{\mathscr{D}}^k_p(E,F)$ is surjective, and so is $\mathbf{D}^* \in \mathscr{D}^k_p(F,E)$ for every choice of geometric data $g \in J^{k-1}_p\big(\mathfrak{m}(E)\big)$, $h \in J^k_p\big(\mathfrak{m}(F)\big)$ and $\mu \in J^k_p\big(\mathfrak{v}(M)\big)$.*

Proof. Since $\mathbf{D}_\varepsilon$ converges in $\mathrm{Hom}(J^k_p E, J^{k-1}_p F)$ to $\mathbf{D}_0$ as $\varepsilon \to 0$, surjectivity of $\mathbf{D}_0$ implies for any given $\mathbf{D} \in \mathscr{D}^k_p(E,F)$ that $\mathbf{D}_\varepsilon$ is also surjective for all $\varepsilon > 0$ sufficiently small. The isomorphism $\ker \mathbf{D} \to \ker \mathbf{D}_\varepsilon$ induced by the correspondence $\eta \mapsto \eta_\varepsilon$ for all $\varepsilon > 0$ then implies that $\mathbf{D}$ is also surjective. The same argument works for the formal adjoints since $\mathbf{D}^*_\varepsilon \to \mathbf{D}^*_0$ as $\varepsilon \to 0$. $\square$

Since we are working in finite-dimensional spaces, surjectivity allows us to choose right-inverses

$$\mathbf{T}_0 : J^{k-1}_p F \to J^k_p E, \qquad \mathbf{T}^*_0 : J^{k-1}_p E \to J^k_p F$$

for $\mathbf{D}_0$ and $\mathbf{D}^*_0$ respectively. We would now like to derive from these similar right-inverses for other operators that are close to $\mathbf{D}_0$ and $\mathbf{D}^*_0$, along with explicit isomorphisms between the kernels of nearby operators. To this end, consider an open neighborhood

$$(g_0, h_0, \mu_0, \mathbf{D}_0) \in \mathcal{U} \subset J^{k-1}_p\big(\mathfrak{m}(E)\big) \times J^k_p\big(\mathfrak{m}(F)\big) \times J^k_p\big(\mathfrak{v}(M)\big) \times \widehat{\mathscr{D}}^k_p(E,F),$$

which we reserve the right to make smaller as necessary. Given $(g,h,\mu,\mathbf{D}) \in \mathcal{U}$, we will as usual denote by $\mathbf{D}^*$ the formal adjoint of $\mathbf{D}$ with respect to the geometric data (g,h,μ). Since $\mathbf{D}_0\mathbf{T}_0 = \mathbb{1}$ and $\mathbf{D}^*_0\mathbf{T}^*_0 = \mathbb{1}$, we can assume after shrinking $\mathcal{U}$ that for every $(g,h,\mu,\mathbf{D}) \in \mathcal{U}$, the operators $\mathbf{D}\mathbf{T}_0 : J_p^{k-1}F \to J_p^{k-1}F$ and $\mathbf{D}^*\mathbf{T}_0^* : J_p^{k-1}E \to J_p^{k-1}E$ are both close enough to the identity to be invertible. This gives rise to right-inverses for $\mathbf{D}$ and $\mathbf{D}^*$, defined respectively by

$$\mathbf{T} := \mathbf{T}_0(\mathbf{D}\mathbf{T}_0)^{-1} : J_p^{k-1}F \to J_p^k E, \qquad \mathbf{T}^* := \mathbf{T}_0^*(\mathbf{D}^*\mathbf{T}_0^*)^{-1} : J_p^{k-1}E \to J_p^k F.$$

Notice that $\mathbf{T}$ and $\mathbf{T}^*$ depend smoothly on $(g, h, \mu, \mathbf{D}) \in \mathcal{U}$.

For a fixed $(g, h, \mu, \mathbf{D}) \in \mathcal{U}$, arbitrary operators close to $\mathbf{D}$ in $\widehat{\mathscr{D}}_p^k(E, F)$ have the form $\widehat{\mathbf{D}} := \mathbf{D} + A$ for $A \in J_p^{k-1}\operatorname{Hom}(E, F)$ small, and the formal adjoint $\widehat{\mathbf{D}}^*$ with respect to the geometric data (g, h, μ) is then $\mathbf{D}^* + A^*$, where $A^* \in J_p^{k-1}\operatorname{Hom}(F, E)$ is the $(k-1)$-jet of the fiberwise transpose (with respect to g and h) of a smooth bundle map $E \to F$ representing A. If A is small enough,[8] then we can use the same trick again to write down right-inverses of $\widehat{\mathbf{D}}$ and $\widehat{\mathbf{D}}^*$ in the form

$$\widehat{\mathbf{T}} := \mathbf{T}(\widehat{\mathbf{D}}\mathbf{T})^{-1} = \mathbf{T}\left(\mathbb{1} + A\mathbf{T}\right)^{-1} = \mathbf{T}\sum_{j=0}^{\infty}(-1)^j(A\mathbf{T})^j,$$

$$\widehat{\mathbf{T}}^* := \mathbf{T}^*(\widehat{\mathbf{D}}^*\mathbf{T}^*)^{-1} = \mathbf{T}^*\left(\mathbb{1} + A^*\mathbf{T}^*\right)^{-1} = \mathbf{T}^*\sum_{j=0}^{\infty}(-1)^j(A^*\mathbf{T}^*)^j.$$

Shrinking the size of A further if necessary, we can then define isomorphisms

$$\Psi_{(\mathbf{D},A)} := \mathbb{1} - \widehat{\mathbf{T}}A = \sum_{j=0}^{\infty}(-1)^j(\mathbf{T}A)^j : J_p^k E \to J_p^k E,$$

$$\Psi^*_{(\mathbf{D},A)} := \mathbb{1} - \widehat{\mathbf{T}}^*A^* = \sum_{j=0}^{\infty}(-1)^j(\mathbf{T}^*A^*)^j : J_p^k F \to J_p^k F,$$

which satisfy

$$\widehat{\mathbf{D}}\Psi_{(\mathbf{D},A)} = \widehat{\mathbf{D}} - A = \mathbf{D} \quad \text{and} \quad \widehat{\mathbf{D}}^*\Psi^*_{(\mathbf{D},A)} = \widehat{\mathbf{D}}^* - A^* = \mathbf{D}^*,$$

so they restrict to isomorphisms $\ker\mathbf{D} \xrightarrow{\Psi_{(\mathbf{D},A)}} \ker\widehat{\mathbf{D}}$ and $\ker\mathbf{D}^* \xrightarrow{\Psi^*_{(\mathbf{D},A)}} \ker\widehat{\mathbf{D}}^*$ respectively. The operators $\Psi_{(\mathbf{D},A)}$ and $\Psi^*_{(\mathbf{D},A)}$ depend smoothly on both $(g, h, \mu, \mathbf{D}) \in \mathcal{U}$ and $A \in J_p^{k-1}\operatorname{Hom}(E, F)$.

5.2.6. *The universal Petri moduli space.* We now consider the subset

$$\mathcal{V}^k \subset J_p^{k-1}\big(\mathfrak{m}(E)\big) \times J_p^k\big(\mathfrak{m}(F)\big) \times J_p^k\big(\mathfrak{v}(M)\big) \times \widehat{\mathscr{D}}_p^k(E, F) \times \left(J_p^k E \otimes J_p^k F\right)$$

consisting of all tuples $(g, h, \mu, \mathbf{D}, t)$ such that

$$t \in \ker\mathbf{D} \otimes \ker\mathbf{D}^* \subset J_p^k E \otimes J_p^k F,$$

where it should be understood that $\mathbf{D}^*$ is the formal adjoint of $\mathbf{D}$ with respect to the geometric data g, h, μ. In light of Assumption 5.14 and Lemma 5.16, the obvious projection endows $\mathcal{V}^k$ with a natural vector bundle structure

$$\mathcal{V}^k \to J_p^{k-1}\big(\mathfrak{m}(E)\big) \times J_p^k\big(\mathfrak{m}(F)\big) \times J_p^k\big(\mathfrak{v}(M)\big) \times \widehat{\mathscr{D}}_p^k(E, F),$$

whose fiber over $(g, h, \mu, \mathbf{D})$ is $\ker\mathbf{D} \otimes \ker\mathbf{D}^*$. We will prefer to think of $\mathcal{V}^k$ rather as a *family* of vector bundles over the space of operators $\widehat{\mathscr{D}}_p^k(E, F)$, parametrized by the space of geometric data $(g, h, \mu) \in J_p^{k-1}\big(\mathfrak{m}(E)\big) \times J_p^k\big(\mathfrak{m}(F)\big) \times J_p^k\big(\mathfrak{v}(M)\big)$. Thus for each (g, h, μ), denote

$$\mathcal{V}^k(g, h, \mu) := \left\{(\mathbf{D}, t) \;\middle|\; (g, h, \mu, \mathbf{D}, t) \in \mathcal{V}^k\right\}.$$

[8] We will not need this detail, but it is often possible to choose $\mathbf{T}_0$ and $\mathbf{T}_0^*$ so that they have degree $+1$ with respect to the vanishing-order filtration, in which case the operators $A\mathbf{T}$, $\mathbf{T}A$, $A^*\mathbf{T}^*$ and $\mathbf{T}^*A^*$ also have this property and are therefore nilpotent. It follows in this case that all infinite series appearing in this discussion are actually finite sums, so A does not really need to be small.

It will be useful to amend these definitions in two ways. Given a pair of real vector spaces V and W, let us say that an element $t \in V \otimes W$ has **rank** r if $t = \sum_{j=1}^r v_j \otimes w_j$ for two linearly-independent sets $v_1, \dots, v_r \in V$ and $w_1, \dots, w_r \in W$. Note that if V is finite dimensional, then the rank of $t \in V \otimes W$ under the canonical isomorphism $V \otimes W \cong \mathrm{Hom}(V^*, W)$ is just the rank of the corresponding linear map $V^* \to W$. As a consequence, whenever V and W are both finite dimensional, the set of elements of rank $r \in \mathbb{N}$ in $V \otimes W$ is a smooth submanifold whose codimension is the dimension of $\mathrm{Hom}(\ker T, \operatorname{coker} T)$ for a linear map $T : V^* \to W$ of rank r, giving

$$\begin{aligned} \dim\left\{t \in V \otimes W \;\middle|\; \operatorname{rank} t = r\right\} &= \dim V \cdot \dim W - (\dim V - r) \cdot (\dim W - r) \\ &= r(\dim V + \dim W) - r^2. \end{aligned} \tag{5.7}$$

With this understood, we can define for each $r \in \mathbb{N}$ a smooth submanifold

$$\mathcal{V}^k_r := \left\{(g, h, \mu, \mathbf{D}, t) \in \mathcal{V}^k \;\middle|\; \operatorname{rank} t = r\right\},$$

which is foliated by the smooth family of smooth submanifolds

$$\mathcal{V}^k_r(g, h, \mu) := \left\{(\mathbf{D}, t) \in \mathcal{V}^k(g, h, \mu) \;\middle|\; \operatorname{rank} t = r\right\}$$

parametrized by the space of geometric data $(g, h, \mu) \in J^{k-1}_p\big(\mathfrak{m}(E)\big) \times J^k_p\big(\mathfrak{m}(F)\big) \times J^k_p\big(\mathfrak{v}(M)\big)$. Finally, recalling the filtration by vanishing orders in §5.2.1, we define for each $\ell \in \{1, \dots, k\}$ the open subset

$$\mathcal{V}^k_{r,\ell} := \left\{(g, h, \mu, \mathbf{D}, t) \in \mathcal{V}^k_r \;\middle|\; t \notin \left(J^k_p E \otimes J^k_p F\right)^\ell\right\},$$

which is likewise foliated by a smooth family of submanifolds

$$\mathcal{V}^k_{r,\ell}(g, h, \mu) := \left\{(\mathbf{D}, t) \in \mathcal{V}^k_r(g, h, \mu) \;\middle|\; t \notin \left(J^k_p E \otimes J^k_p F\right)^\ell\right\}.$$

parametrized by the geometric data (g, h, μ).

The Petri map $\Pi : \Gamma_p(E) \otimes \Gamma_p(F) \to \Gamma_p(E \otimes F)$ descends for each $k \in \mathbb{Z}$ to a linear map

$$\Pi^k : J^k_p E \otimes J^k_p F \to J^k_p(E \otimes F)$$

that preserves the filtration by vanishing orders. Since the projection map $\mathcal{V}^k_{r,\ell}(g, h, \mu) \to J^k_p E \otimes J^k_p F$ sending $(g, h, \mu, \mathbf{D}, t)$ to t is smooth and also depends smoothly on the geometric data (g, h, μ), Π^k gives rise to a smooth family of smooth maps

$$\Pi^k_{r,\ell} : \mathcal{V}^k_{r,\ell}(g, h, \mu) \to J^k_p(E \otimes F) : (\mathbf{D}, t) \mapsto \Pi^k(t), \tag{5.8}$$

whose zero-set we shall denote by

$$\mathscr{P}^k_{r,\ell}(g, h, \mu) := (\Pi^k_{r,\ell})^{-1}(0) = \left\{(\mathbf{D}, t) \in \mathcal{V}^k_{r,\ell}(g, h, \mu) \;\middle|\; \Pi^k(t) = 0\right\}.$$

This is the so-called *universal Petri moduli space*. Our main goal is to prove under suitable assumptions that it is a C^∞-subvariety in $\mathcal{V}^k_{r,\ell}(g, h, \mu)$ and to establish an effective lower bound $R \in \mathbb{N}$ on its codimension. Once this is done, Sard's theorem (see Appendix C) will imply that for almost every $\mathbf{D} \in \widehat{\mathscr{D}}^k_p(E, F)$, the space

$$\mathscr{P}^k_{r,\ell}(g, h, \mu, \mathbf{D}) := \left\{t \;\middle|\; (\mathbf{D}, t) \in \mathscr{P}^k_{r,\ell}(g, h, \mu)\right\}$$

is a C^∞-subvariety of codimension at least R in the manifold

$$\mathcal{V}^k_{r,\ell}(g, h, \mu, \mathbf{D}) := \left\{t \in \ker \mathbf{D} \otimes \ker \mathbf{D}^* \;\middle|\; \operatorname{rank} t = r,\ t \notin (J^k_p E \otimes J^k_p F)^\ell\right\}.$$

If the codimension R is large enough, this will imply that $\mathscr{P}^k_{r,\ell}(g, h, \mu, \mathbf{D})$ is empty.

Denote the linearization of the map (5.8) at the point $(\mathbf{D}, t) \in \mathscr{P}^k_{r,\ell}(g, h, \mu)$ by

$$d_2\Pi^k_{r,\ell}(g, h, \mu, \mathbf{D}, t) : T_{(\mathbf{D},t)}\mathcal{V}^k_{r,\ell}(g, h, \mu) \to J^k_p(E \otimes F),$$

where the subscript in "d_2" is meant to emphasize that this is a partial derivatve—we differentiate with respect to $(\mathbf{D}, t)$ while holding (g, h, μ) constant. Estimating the rank of $d_2\Pi^k_{r,\ell}$ requires being able to write down a sufficiently large space of tangent vectors in $T_{(\mathbf{D},t)}\mathcal{V}^k_{r,\ell}(g, h, \mu)$. Suppose that $(g, h, \mu, \mathbf{D})$ belongs to the neighborhood $\mathcal{U}$ of $(g_0, h_0, \mu_0, \mathbf{D}_0)$ chosen in §5.2.5, so we have right-inverses $\mathbf{T}, \mathbf{T}^*$ and isomorphisms $\Psi_{(\mathbf{D},A)}, \Psi^*_{(\mathbf{D},A)}$ that depend smoothly on $(g, h, \mu, \mathbf{D}) \in \mathcal{U}$ and a small zeroth-order perturbation $A \in J^{k-1}_p \operatorname{Hom}(E, F)$. We can use this to associate to every $A \in J^{k-1}_p \operatorname{Hom}(E, F)$ and $t \in \ker \mathbf{D} \otimes \ker \mathbf{D}^*$ a smooth path

$$(-\delta, \delta) \to \mathcal{V}^k(g, h, \mu) : s \mapsto (\mathbf{D} + sA, (\Psi_{(\mathbf{D},sA)} \otimes \Psi^*_{(\mathbf{D},sA)})t)$$

which passes through $(\mathbf{D}, t)$ at $s = 0$. Observe that if $t = \sum_{j=1}^r \eta_j \otimes \xi_j$ for two linearly-independent sets $\eta_1, \ldots, \eta_r \in J^k_p E$ and $\xi_1, \ldots, \xi_r \in J^k_p F$, then $\Psi_{(\mathbf{D},sA)}$ and $\Psi^*_{(\mathbf{D},sA)}$ map these to linearly-independent sets when s is close enough to 0, since both operators are then close to the identity. It follows that if $(\mathbf{D}, t) \in \mathscr{P}^k_{r,\ell}(g, h, \mu)$, then the path above is in $\mathcal{V}^k_{r,\ell}(g, h, \mu)$ for $\delta > 0$ sufficiently small. Differentiating it at $s = 0$, then feeding the resulting tangent vector into $d_2\Pi^k_{r,\ell}(g, h, \mu, \mathbf{D}, t)$ and multiplying the result by -1 for cosmetic purposes, we obtain the linear map

$$\begin{aligned}\mathbf{L}(g, h, \mu, \mathbf{D}, t) : J^{k-1}_p \operatorname{Hom}(E, F) &\to J^k_p(E \otimes F),\\ A &\mapsto \Pi^k \circ (\mathbf{T}A \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{T}^* A^*)(t).\end{aligned}$$

This depends smoothly on the data $(g, h, \mu, \mathbf{D}, t)$ and is well defined whenever $(g, h, \mu, \mathbf{D})$ is sufficiently close to $(g_0, h_0, \mu_0, \mathbf{D}_0)$. The rank of this operator is clearly less than or equal to that of $d_2\Pi^k_{r,\ell}(g, h, \mu, \mathbf{D}, t)$. We shall abbreviate the special case

$$\mathbf{L}_t := \mathbf{L}(g_0, h_0, \mu_0, \mathbf{D}_0, t) : J^{k-1}_p \operatorname{Hom}(E, F) \to J^k_p(E \otimes F) \tag{5.9}$$

for $t \in \ker \mathbf{D}_0 \otimes \ker \mathbf{D}^*_0$, as this will turn out to be the only case that matters in practice. In fact, we can now use the rescaling trick from §5.2.4 to reduce the local analysis of the space $\mathscr{P}^k_{r,\ell}(g, h, \mu)$ to the problem of estimating the rank of $\mathbf{L}_t$.

For every $\varepsilon \in (0, 1]$ and $q \in \mathbb{Z}$ and every choice of the geometric data (g, h, μ), one can define a diffeomorphism

$$\Phi_\varepsilon : \mathcal{V}^k_{r,\ell}(g, h, \mu) \xrightarrow{\cong} \mathcal{V}^k_{r,\ell}(g_\varepsilon, h_\varepsilon, \mu_\varepsilon) : (\mathbf{D}, t) \mapsto (\mathbf{D}_\varepsilon, t_\varepsilon)\,, \tag{5.10}$$

where the map $\ker \mathbf{D} \otimes \ker \mathbf{D}^* \to \ker \mathbf{D}_\varepsilon \otimes \ker \mathbf{D}^*_\varepsilon : t \mapsto t_\varepsilon$ is defined via

$$\eta \otimes \xi \mapsto \frac{1}{\varepsilon^q}\eta_\varepsilon \otimes \xi_\varepsilon. \tag{5.11}$$

The scaling factor ε^q here is not strictly necessary, but has been added for use in the proof of Lemma 5.19 below. We see that Φ_ε maps $\mathscr{P}^k_{r,\ell}(g, h, \mu)$ bijectively onto $\mathscr{P}^k_{r,\ell}(g_\varepsilon, h_\varepsilon, \mu_\varepsilon)$ for each $\varepsilon \in (0, 1]$. This map is not defined for $\varepsilon = 0$, but the data g_ε, h_ε, μ_ε, $\mathbf{D}_\varepsilon$ and $\mathbf{D}^*_\varepsilon$ do have well-defined limits as $\varepsilon \to 0$; in particular, $\mathbf{D}_0$ and $\mathbf{D}^*_0$ are both operators with constant coefficients and no zeroth-order term in our chosen local coordinates and trivializations. The following definition is highly dependent on this choice of coordinates, but so is the map Φ_ε; there will be no problem as long as the same choices are used for both.

Definition 5.17. We will say that an element of $J^k_p E$ or $J^k_p F$ is **homogeneous of degree** d if, under the natural identifications of these spaces with spaces of Taylor polynomials determined by the chosen coordinates and trivializations from Assumption 5.13, it is represented by a homogeneous polynomial of degree d. Similarly, we will call an element $t = \sum_j \eta_j \otimes \xi_j \in J^k_p E \otimes J^k_p F$ **homogeneous of degree** d if for every j, the elements $\eta_j \in J^k_p E$ and $\xi_j \in J^k_p F$ are homogeneous with degrees adding up to d.

Remark 5.18. The homogeneous elements $t \in J^k_p E \otimes J^k_p F$ of degree q are precisely those which are fixed under the map (5.11) for every $\varepsilon > 0$.

Lemma 5.19. *Suppose that for every homogeneous element $t \in \ker \Pi^k \subset J^k_p E \otimes J^k_p F$ of degree less than ℓ that also belongs to $\ker \mathbf{D}_0 \otimes \ker \mathbf{D}_0^*$, the linear map $\mathbf{L}_t : J^{k-1}_p \operatorname{Hom}(E,F) \to J^k_p(E \otimes F)$ has rank at least $R \in \mathbb{N}$. Then for every $r \in \mathbb{N}$, $\mathscr{P}^k_{r,\ell}(g,h,\mu)$ is a C^∞-subvariety of codimension at least R in $\mathcal{V}^k_{r,\ell}(g,h,\mu)$.*

Proof. Suppose $(\mathbf{D},t) \in \mathscr{P}^k_{r,\ell}(g,h,\mu)$ and let $q \in \{0,\ldots,\ell-1\}$ denote the largest integer such that $t \in (J^k_p E \otimes J^k_p F)^q$. Use this value of q to define the scaling factor in (5.11) for the definition of the diffeomorphisms Φ_ε in (5.10). Identifying k-jets with Taylor polynomials as in (5.3), we can write t as a finite sum $\sum_j \eta_j \otimes \xi_j$, where for each individual value of j, $\eta_j \in \ker \mathbf{D}$ is a polynomial of degree at most k with lowest-order term of degree $u_j \geqslant 0$, $\xi_j \in \ker \mathbf{D}^*$ is likewise a polynomial of degree at most k with lowest-order term of degree $v_j \geqslant 0$, and $u_j + v_j \geqslant q$, with equality $u_j + v_j = q$ in at least one case. It follows that $t_\varepsilon \in J^k_p E \otimes J^k_p F$ converges as $\varepsilon \to 0$ to a nontrivial homogenous element $t_0 \in \ker \mathbf{D}_0 \otimes \ker \mathbf{D}_0^* \subset J^k_p E \otimes J^k_p F$ of degree $q < \ell$, and $\Pi^k(t_0) = 0$ since $\Pi^k(t_\varepsilon) = \Pi^k(t) = 0$ for every $\varepsilon > 0$. As a consequence, $(g_\varepsilon, h_\varepsilon, \mu_\varepsilon, \mathbf{D}_\varepsilon, t_\varepsilon) \in \mathcal{V}^k$ converges as $\varepsilon \to 0$ to $(g_0, h_0, \mu_0, \mathbf{D}_0, t_0) \in \mathcal{V}^k$. Since $\mathbf{L}_{t_0}$ has rank at least R by the hypothesis of the lemma, it follows for all $\varepsilon > 0$ sufficiently small that

$$\operatorname{rank} d_2 \Pi^k_{r,\ell}(g_\varepsilon, h_\varepsilon, \mu_\varepsilon, \mathbf{D}_\varepsilon, t_\varepsilon) \geqslant \operatorname{rank} \mathbf{L}(g_\varepsilon, h_\varepsilon, \mu_\varepsilon, \mathbf{D}_\varepsilon, t_\varepsilon) \geqslant R.$$

Fix $\varepsilon > 0$ in this range. Then an arbitrary element $(\mathbf{D}',t') \in \mathcal{V}^k_{r,\ell}(g,h,\mu)$ in some small neighborhood of $(\mathbf{D},t)$ belongs to $\mathscr{P}^k_{r,\ell}(g,h,\mu)$ if and only if $\Pi^k \circ \Phi_\varepsilon(\mathbf{D}',t') = 0$. Since Φ_ε is a diffeomorphism, the linearization of $\Pi^k \circ \Phi_\varepsilon : \mathcal{V}^k_{r,\ell}(g,h,\mu) \to J^k_p(E \otimes F)$ at $(\mathbf{D},t)$ has the same image as the operator $d_2 \Pi^k_{r,\ell}(g_\varepsilon, h_\varepsilon, \mu_\varepsilon, \mathbf{D}_\varepsilon, t_\varepsilon)$, and thus has rank at least R. ☐

5.3. **Application to Cauchy-Riemann operators.** We shall now apply Lemma 5.19 for the specific case of Cauchy-Riemann type operators. For the rest of this section, assume M is a Riemann surface (Σ, j), E is a complex vector bundle of complex rank $m \in \mathbb{N}$, $F = \overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, E)$, and $\widehat{\mathscr{D}}_p(E,F)$ is the space of germs of real-linear Cauchy-Riemann type operators on E near $p \in \Sigma$. This space of operators satisfies Assumption 5.13 since one can always choose trivializations and coordinates in which every $\mathbf{D} \in \widehat{\mathscr{D}}_p(E,F)$ is a zeroth-order perturbation of $\bar\partial := \partial_s + i\partial_t$. To define formal adjoints, we assume $g = \langle\ ,\ \rangle_E$ is the real part of a Hermitian bundle metric on E, μ is the area form on Σ determined by a Hermitian bundle metric $\langle\ ,\ \rangle_\Sigma$ on $T\Sigma$, and $h = \langle\ ,\ \rangle_F$ is the real part of the Hermitian bundle metric determined on F via the natural isomorphism $F \cong T\Sigma \otimes_{\mathbb{C}} E$.

Remark 5.20. It is important to keep in mind that the operators $\mathbf{D} \in \widehat{\mathscr{D}}_p(E,F)$ are in general real- and not complex-linear, thus throughout this section, the symbols $\operatorname{Hom}(V,W)$ and $V \otimes W$ will always refer to *real*-linear maps and real tensor products unless otherwise noted, even in cases where V and W are both complex. We will use the notation $\operatorname{Hom}_{\mathbb{C}}(V,W)$ and $V \otimes_{\mathbb{C}} W$ to specify the complex analogues of these operations.

5.3.1. *A digression on real and complex tensor products.* Suppose V and W are complex vector spaces, and let $\overline{W}$ denote the complex conjugate of W, i.e. it is the same real vector space, but with a sign inserted in the definition of its complex structure. There is then a canonical complex-antilinear isomorphism $W \to \overline{W}$ defined by the identity map, and we shall denote it by

$$W \to \overline{W} : w \mapsto \bar{w}.$$

The spaces $V \otimes_{\mathbb{C}} W$ and $V \otimes_{\mathbb{C}} \overline{W}$ are both quotients of the real tensor product $V \otimes W$, e.g. we obtain $V \otimes_{\mathbb{C}} W$ from $V \otimes W$ by introducing the equivalence relation $iv \otimes w \sim v \otimes iw$, and for $V \otimes_{\mathbb{C}} \overline{W}$ the relation is instead $iv \otimes w \sim -v \otimes iw$. If the resulting quotient projections are denoted by $\pi_+ : V \otimes W \to V \otimes_{\mathbb{C}} W$ and $\pi_- : V \otimes W \to V \otimes_{\mathbb{C}} \overline{W}$, then we obtain an isomorphism

$$(\pi_+, \pi_-) : V \otimes W \xrightarrow{\cong} (V \otimes_{\mathbb{C}} W) \oplus (V \otimes_{\mathbb{C}} \overline{W}).$$

This discussion carries over verbatim to a pair of complex vector bundles E and F over Σ, giving a bundle isomorphism

$$(\pi_+, \pi_-) : E \otimes F \to (E \otimes_{\mathbb{C}} F) \oplus (E \otimes_{\mathbb{C}} \bar{F}).$$

The Petri map then fits into a commutative diagram

$$\begin{array}{ccc} \Gamma(E) \otimes \Gamma(F) & \xrightarrow{\quad\quad\quad \Pi \quad\quad\quad} & \Gamma(E \otimes F) \\ \downarrow \cong & & \downarrow \cong \\ (\Gamma(E) \otimes_{\mathbb{C}} \Gamma(F)) \oplus \big(\Gamma(E) \otimes_{\mathbb{C}} \Gamma(\bar{F})\big) & \xrightarrow{\Pi_{\mathbb{C}} \oplus \Pi_{\mathbb{C}}} & \Gamma\big((E \otimes_{\mathbb{C}} F) \oplus (E \otimes_{\mathbb{C}} \bar{F})\big), \end{array} \tag{5.12}$$

where $\Pi_{\mathbb{C}} : \Gamma(E) \otimes_{\mathbb{C}} \Gamma(F) \to \Gamma(E \otimes_{\mathbb{C}} F)$ denotes the obvious *complex-linear Petri map* that is defined for any two complex vector bundles. Suppose in particular that E and F are line bundles and we have chosen complex trivializations for both over some region $\mathcal{U}$. The bundle $\bar{F}$ inherits from this a trivialization over $\mathcal{U}$ such that the canonical map $F \to \bar{F}$ looks like complex conjugation, and $E \otimes_{\mathbb{C}} F$ and $E \otimes_{\mathbb{C}} \bar{F}$ likewise inherit natural trivializations. The diagram now allows us to identify the real Petri map with

$$\begin{gathered} C^\infty(\mathcal{U}, \mathbb{C}) \otimes C^\infty(\mathcal{U}, \mathbb{C}) \to C^\infty(\mathcal{U}, \mathbb{C}) \oplus C^\infty(\mathcal{U}, \mathbb{C}), \\ f \otimes g \mapsto (fg, f\bar{g}). \end{gathered} \tag{5.13}$$

5.3.2. *The main rank estimate.* Fix a holomorphic coordinate chart near $p \in \Sigma$ and a corresponding complex local trivialization of E such that the Hermitian bundle metrics on $T\Sigma$ and E both match the standard Hermitian inner product at p. The bundle F naturally inherits from these choices a local trivialization in which its Hermitian bundle metric also appears standard at p. These choices identify elements of $J^k_p E$ with polynomials in z and $\bar{z}$,

$$\sum_{j+\ell \leqslant k} z^j \bar{z}^\ell c_{j,\ell}, \qquad c_{j,\ell} \in \mathbb{C}^m,$$

hence

$$\dim_{\mathbb{C}} J^k_p E = \dim_{\mathbb{C}} J^k_p F = m\,(1 + 2 + \ldots + (k+1)) = \frac{m(k+1)(k+2)}{2}. \tag{5.14}$$

Every $\mathbf{D} \in \widehat{\mathscr{D}}_p(E, F)$ is now identified with an operator of the form

$$\mathbf{D} = \bar{\partial} + A : C^\infty(\mathbb{D}_\epsilon, \mathbb{C}^m) \to C^\infty(\mathbb{D}_\epsilon, \mathbb{C}^m),$$

where $\bar{\partial} = \partial_s + i\partial_t$ and $A : \mathbb{D}_\epsilon \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^m)$. The operator $\mathbf{D}_0$ obtained by rescaling as in §5.2.4 is then simply

$$\mathbf{D}_0 = \bar{\partial} := \partial_s + i\partial_t = 2\frac{\partial}{\partial \bar{z}},$$

and since the rescaled bundle metrics g_0, h_0 and area form μ_0 are all standard in these coordinates, the formal adjoint of $\mathbf{D}_0$ with respect to this geometric data is

$$\mathbf{D}_0^* = -\partial = -(\partial_s - i\partial_t) = -2\frac{\partial}{\partial z}.$$

We can therefore choose right-inverses $\mathbf{T}_0 : J^{k-1}_p F \to J^k_p E$ and $\mathbf{T}_0^* : J^{k-1}_p E \to J^k_p F$ that are uniquely determined in coordinates by the conditions

$$\mathbf{T}_0\left(z^j \bar{z}^\ell c\right) := \frac{1}{2(\ell+1)} z^j \bar{z}^{\ell+1} c, \qquad 0 \leqslant j + \ell \leqslant k-1, \quad c \in \mathbb{C}^m, \tag{5.15}$$

and

$$\mathbf{T}_0^*\left(z^j \bar{z}^\ell c\right) := -\frac{1}{2(j+1)} z^{j+1} \bar{z}^\ell c, \qquad 0 \leqslant j + \ell \leqslant k-1, \quad c \in \mathbb{C}^m. \tag{5.16}$$

These choices determine the maps $\mathbf{L}_t : J^{k-1}_p \operatorname{Hom}(E, F) \to J^k_p(E \otimes F)$ in (5.9). Observe now that the domain of this operator has a natural splitting

$$J^{k-1}_p \operatorname{Hom}(E, F) = J^{k-1}_p \operatorname{Hom}_{\mathbb{C}}(E, F) \oplus J^{k-1}_p \overline{\operatorname{Hom}}_{\mathbb{C}}(E, F).$$

If we were to restrict to complex-linear zeroth-order terms $A \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(E,F))$, then the perturbed operators $\mathbf{D} = \mathbf{D}_0 + A$ would always be equivalent to $\mathbf{D}_0$ under changes of trivialization, killing any hope that $\mathscr{P}^k_{r,\ell}(g_0,h_0,\mu_0,\mathbf{D})$ might be a smaller space than $\mathscr{P}^k_{r,\ell}(g_0,h_0,\mu_0,\mathbf{D}_0)$. For this reason, we shall restrict $\mathbf{L}_t$ to the complementary subspace consisting of $(k-1)$-jets of *antilinear* perturbations. Having done this, the following additional detail becomes relevant: for $A \in J^{k-1}_p\overline{\mathrm{Hom}}_{\mathbb{C}}(E,F)$ and $t = \sum_j \eta_j \otimes \xi_j \in \ker \mathbf{D}_0 \otimes \ker \mathbf{D}_0^*$, the commutative diagram (5.12) implies

$$
\begin{aligned}
\pi_- \circ \mathbf{L}_t(A) &= \Pi^k_{\mathbb{C}} \circ \pi_- \left(\sum_j (\mathbf{T}_0 A\eta_j \otimes \xi_j + \eta_j \otimes \mathbf{T}_0^* A^* \xi_j) \right) \\
&= \Pi^k_{\mathbb{C}} \sum_j \left(\mathbf{T}_0 A \eta_j \otimes_{\mathbb{C}} \bar{\xi}_j + \eta_j \otimes_{\mathbb{C}} \overline{\mathbf{T}_0^* A^* \xi_j} \right),
\end{aligned}
$$

where $\Pi^k_{\mathbb{C}}$ denotes the map induced on k-jets by the complex Petri map $\Pi_{\mathbb{C}}$. Since $\mathbf{T}_0$ and $\mathbf{T}_0^*$ are complex linear while A and $\xi_j \mapsto \bar{\xi}_j$ are antilinear, the expression on the right hand side is the result of applying some real-linear map to $\pi_+(t) = \sum_j \eta_j \otimes_{\mathbb{C}} \xi_j$; the point here is that real-linear operators of the form $\phi \otimes \psi$ are well defined on the complex tensor product whenever ϕ and ψ are either both complex linear or both complex antilinear. But as mentioned in Example 5.5, $\mathbf{D}_0$ satisfies the complex Petri condition, so the fact that $\Pi^k(t) = 0$ implies that $\Pi^k_{\mathbb{C}} \circ \pi_+(t) = 0$ and thus $\pi_+(t) = 0$, so that the expression vanishes automatically. We conclude from this discussion that all interesting information in $\mathbf{L}_t$ is carried by the map

$$
\widehat{\mathbf{L}}_t := \pi_+ \circ \mathbf{L}_t|_{J^{k-1}_p\overline{\mathrm{Hom}}_{\mathbb{C}}(E,F)} : J^{k-1}_p\overline{\mathrm{Hom}}_{\mathbb{C}}(E,F) \to J^k_p(E \otimes_{\mathbb{C}} F). \tag{5.17}
$$

Clearly the rank of $\widehat{\mathbf{L}}_t$ gives a lower bound for the rank of $\mathbf{L}_t$. The workhorse result behind Theorem 5.9 is now the following:

Proposition 5.21. *For every $\ell \in \mathbb{N}$, there exists a constant $C_\ell > 0$ that depends on ℓ but not on k, such that for all $t \in \ker \Pi^k \subset \ker \mathbf{D}_0 \otimes \ker \mathbf{D}_0^*$ that are homogeneous elements of degree less than ℓ in $J^k_p E \otimes J^k_p F$, the operator $\widehat{\mathbf{L}}_t : J^{k-1}_p\overline{\mathrm{Hom}}_{\mathbb{C}}(E,F) \to J^k_p(E \otimes_{\mathbb{C}} F)$ satisfies*

$$
\operatorname{rank} \widehat{\mathbf{L}}_t \geqslant C_\ell k^2.
$$

Lemma 5.22. *If Proposition 5.21 holds in the case* $\operatorname{rank}_{\mathbb{C}} E = 1$, *then it holds in general.*

Proof. For $\operatorname{rank}_{\mathbb{C}} E = m \in \mathbb{N}$, the chosen trivializations furnish local splittings $E = E_1 \oplus \ldots \oplus E_m$ and $F = F_1 \oplus \ldots \oplus F_m$ that are respected by $\mathbf{D}_0$ and $\mathbf{D}_0^*$, i.e. both are m-fold direct sums of identical operators given by $\bar{\partial}$ or $-\partial$ respectively. Their chosen right-inverses $\mathbf{T}_0$ and $\mathbf{T}_0^*$ also respect these splittings. Let us denote the resulting splittings of the kernels by $\ker \mathbf{D}_0 = K_1 \oplus \ldots \oplus K_m$ and $\ker \mathbf{D}_0^* = L_1 \oplus \ldots \oplus L_m$, so that $\ker \mathbf{D}_0 \otimes \ker \mathbf{D}_0^*$ splits into m^2 identical factors of the form $K_i \otimes L_j$. Similarly, $J^k_p(E \otimes F)$ splits into m^2 identical factors of the form $J^k_p(E_i \otimes F_j)$, and the Petri map $\Pi^k : J^k_p E \otimes J^k_p F \to J^k_p(E \otimes F)$ sends $J^k_p E_i \otimes J^k_p F_j$ to $J^k_p(E_i \otimes F_j)$ for every i and j. A homogeneous element $t \in \ker \Pi^k \subset \ker \mathbf{D}_0 \otimes \ker \mathbf{D}_0^*$ of degree $q < \ell$ is now defined by its m^2 components $t_{ij} \in \ker \Pi^k \cap (K_i \otimes L_j)$, at least one of which must be a nontrivial homogeneous element of degree q; call this component t_{uv}. Now consider the restriction of $\widehat{\mathbf{L}}_t$ to the subspace

$$
J^{k-1}_p\overline{\mathrm{Hom}}_{\mathbb{C}}(E_u, F_v) \subset J^{k-1}_p\overline{\mathrm{Hom}}_{\mathbb{C}}(E,F),
$$

defined as the $(k-1)$-jets of bundle maps $A : E \to F$ that annihilate E_i for all $i \neq u$ and have image in F_v. Since the bundle metrics g_0 and h_0 are standard in our chosen trivializations, A^* then belongs to the corresponding subspace $J^{k-1}_p\overline{\mathrm{Hom}}_{\mathbb{C}}(F_v, E_u) \subset J^{k-1}_p\overline{\mathrm{Hom}}_{\mathbb{C}}(F,E)$. Composing our restriction of $\widehat{\mathbf{L}}_t$ with the natural projection $J^k_p(E \otimes F) \to J^k_p(E_u \otimes F_v)$ then produces an operator $J^{k-1}_p\overline{\mathrm{Hom}}_{\mathbb{C}}(E_u, F_v) \to J^k_p(E_u \otimes_{\mathbb{C}} F_v)$ that matches the rank 1 case of $\widehat{\mathbf{L}}_t$, and its rank gives a lower bound for the rank of $\widehat{\mathbf{L}}_t$. □

The remainder of this subsection is devoted to proving the $\mathrm{rank}_{\mathbb{C}} E = 1$ case of Proposition 5.21.

We shall write everything in the chosen coordinates and trivializations so that elements of $J^k_p E$, $J^k_p F$ and $J^k_p(E \otimes_{\mathbb{C}} F)$ are now identified with complex-valued polynomials of degree at most k in the variables z and $\bar{z}$. The holomorphic polynomials form $\ker \mathbf{D}_0$, while the antiholomorphic polynomials form $\ker \mathbf{D}_0^*$. Using (5.13) to compute the kernel of the Petri map, it turns out that arbitrary elements of $\ker \Pi^k \subset J^k_p E \otimes J^k_p F$ now take the form[9]

$$t = \sum_{j,n=0}^{k} \left[a_{jn} \left(z^j \otimes \bar{z}^n + i z^j \otimes i \bar{z}^n \right) + b_{jn} \left(i z^j \otimes \bar{z}^n - z^j \otimes i \bar{z}^n \right) \right] + R,$$

where $a_{jn}, b_{jn} \in \mathbb{R}$ are real coefficients subject to the condition $\sum_{j+n=q} a_{jn} = \sum_{j+n=q} b_{jn} = 0$ for every $q = 0, \ldots, k$, and R is an arbitrary sum of homogeneous elements that have degrees greater than k and therefore vanish automatically under Π^k. For Proposition 5.21 we are interested only in homogeneous elements of some degree less than ℓ, so let us fix an integer $q \leqslant \ell$ and write

$$t = \sum_{j=0}^{q-1} \left[a_j \left(z^j \otimes \bar{z}^{q-1-j} + i z^j \otimes i \bar{z}^{q-1-j} \right) + b_j \left(i z^j \otimes \bar{z}^{q-1-j} - z^j \otimes i \bar{z}^{q-1-j} \right) \right],$$

where $a_j, b_j \in \mathbb{R}$ are now subject to the conditions $\sum_{j=0}^{q-1} a_j = \sum_{j=0}^{q-1} b_j = 0$ and we explicitly assume that at least one of these coefficients is nonzero. The action of an antilinear bundle map $A \in \Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(E, F))$ on a section $\eta \in \Gamma(E)$ can be written in trivializations as

$$(A\eta)(z) := \alpha(z) \overline{\eta(z)}$$

for some complex-valued function α, thus the map $A : J^k_p E \to J^{k-1}_p F$ can be written as

$$A\eta = \sum_{u+v \leqslant k-1} \alpha_{uv} z^u \bar{z}^v \bar{\eta}$$

for some coefficients $\alpha_{uv} \in \mathbb{C}$. The transpose $A^* : J^k_p F \to J^{k-1}_p E$ is given by exactly the same formula—here we are taking transposes of the 1-by-1 matrices α_{uv} and thus leaving them unchanged, as the antilinearity of A makes the transpose the appropriate transformation here instead of the Hermitian adjoint. With this data in place and the explicit formulas given in (5.15) and (5.16) for $\mathbf{T}_0$ and $\mathbf{T}_0^*$, we now obtain an explicit formula for $\widehat{\mathbf{L}}_t(A) \in J^k_p(E \otimes_{\mathbb{C}} F)$ as

$$\widehat{\mathbf{L}}_t(A) = \sum_{j=0}^{q-1} \sum_{u+v \leqslant k-q} \left(\frac{\bar{c}_j \alpha_{uv}}{v+j+1} z^u \bar{z}^{v+q} - \frac{c_j \alpha_{uv}}{u+q-j} z^{u+q} \bar{z}^v \right),$$

where we have defined

$$c_j := a_j + i b_j \in \mathbb{C} \quad \text{for} \quad j = 0, \ldots, q-1.$$

Two immediate remarks are in order: first, the second summation in this formula stops at $k-q$ instead of $k-1$ because all terms in A with degree larger than $k-q$ produce terms in $\widehat{\mathbf{L}}_t(A)$ that have degree greater than k and thus vanish in $J^k_p(E \otimes_{\mathbb{C}} F)$. Along the same lines, we notice that whenever A is given by a homogeneous polynomial of degree n, $\widehat{\mathbf{L}}_t(A)$ is likewise homogeneous with degree $n+q$, indicating a natural splitting of the map $\widehat{\mathbf{L}}_t : J^{k-1}_p \overline{\mathrm{Hom}}_{\mathbb{C}}(E, F) \to J^k_p(E \otimes_{\mathbb{C}} F)$ into factors

$$\widehat{\mathbf{L}}_t = \widehat{\mathbf{L}}_t^{(0)} \oplus \ldots \oplus \widehat{\mathbf{L}}_t^{(k-q)},$$

where for each $n = 0, \ldots, k-q$, $\widehat{\mathbf{L}}_t^{(n)}$ is defined on the space of homogeneous degree n polynomials in $J^{k-1}_p \overline{\mathrm{Hom}}_{\mathbb{C}}(E, F)$. (Strictly speaking, there are additional factors defined on homogeneous polynomials of higher degree, but we will ignore them because they are trivial.)

[9]This seems a good moment to remind the reader that all tensor products in this section are *real* tensor products unless the symbol "$\otimes_{\mathbb{C}}$" is used.

For each individual $n \in \{0, \dots, k-q\}$, the map $\widehat{\mathbf{L}}_t^{(n)}$ takes the form

$$\widehat{\mathbf{L}}_t^{(n)}(A) = \sum_{u+v=n} \alpha_{uv} \cdot \left[\left(\sum_{j=0}^{q-1} \frac{\bar{c}_j}{v+j+1} \right) z^u \bar{z}^{v+q} - \left(\sum_{j=0}^{q-1} \frac{c_j}{u+q-j} \right) z^{u+q} \bar{z}^v \right].$$

To simplify this expression, we can write $\mathbf{c} = (c_0, \dots, c_{q-1}) \in \mathbb{C}^q$ as a column vector and define for integers $u, v \geqslant 0$ the complex numbers

$$\theta_v := \begin{pmatrix} \frac{1}{v+1} & \cdots & \frac{1}{v+q} \end{pmatrix} \bar{\mathbf{c}} \quad \text{and} \quad \kappa_u := \begin{pmatrix} \frac{1}{u+q} & \cdots & \frac{1}{u+1} \end{pmatrix} \mathbf{c},$$

so that now

$$\widehat{\mathbf{L}}_t^{(n)}(A) = \sum_{u+v=n} \alpha_{uv} \cdot \left(\theta_v z^u \bar{z}^{v+q} - \kappa_u z^{u+q} \bar{z}^v \right).$$

If we now identify the homogeneous degree n part of A with the vector in $\mathbb{C}^{n+1}$ given by $(\alpha_{n,0}, \alpha_{n-1,1}, \dots, \alpha_{0,n})$, and use the monomials

$$z^{n+q}, z^{n+q-1}\bar{z}, z^{n+q-2}\bar{z}^2, \dots, z\bar{z}^{n+q-1}, \bar{z}^{n+q}$$

as a complex basis for the homogeneous degree $n+q$ part of $J_p^k(E \otimes_{\mathbb{C}} F)$, then $\widehat{\mathbf{L}}_t^{(n)}$ is represented by the $(n+q+1)$-by-$(n+1)$ complex matrix

$$\widehat{\mathbf{L}}_t^{(n)} = \begin{pmatrix} -\kappa_n & & & \\ \vdots & -\kappa_{n-1} & & \\ \theta_0 & \vdots & \ddots & \\ & \theta_1 & & -\kappa_0 \\ & & \ddots & \vdots \\ & & & \theta_n \end{pmatrix}. \tag{5.18}$$

In this matrix, all entries not written explicitly are understood to be 0.

Lemma 5.23. *For any set of distinct positive integers $i_1, \dots, i_q$, the matrix*

$$\begin{pmatrix} \dfrac{1}{i_1+q} & \cdots & \dfrac{1}{i_1+1} \\ \vdots & \ddots & \vdots \\ \dfrac{1}{i_q+q} & \cdots & \dfrac{1}{i_q+1} \end{pmatrix}$$

is invertible.

Proof. This follows from the well-known formula for so-called *Cauchy determinants*,

$$\det \begin{pmatrix} \dfrac{1}{z_1+w_1} & \cdots & \dfrac{1}{z_1+w_q} \\ \vdots & \ddots & \vdots \\ \dfrac{1}{z_q+w_1} & \cdots & \dfrac{1}{z_q+w_q} \end{pmatrix} = \frac{\prod_{i=1}^{q} \prod_{j=i}^{i-1} (z_i - z_j)(w_i - w_j)}{\prod_{i,j=1}^{q} (z_i + w_j)},$$

see e.g. [PS98, pp. 92 and 279]. □

Since at least one of the coefficients a_j or b_j is nonzero, the vector $\mathbf{c} \in \mathbb{C}^q$ cannot be annihilated by q linearly independent vectors, so we conclude:

Corollary 5.24. *In the matrix* (5.18)*, at most $q-1$ of the entries $\kappa_0, \dots, \kappa_n$ can be zero.* □

This result implies that at most $q-1$ columns of the matrix (5.18) need to be eliminated in order to produce a matrix whose columns are all linearly independent, hence if $n \geqslant q-1$, we have

$$\operatorname{rank}_{\mathbb{C}} \widehat{\mathbf{L}}_t^{(n)} \geqslant n-(q-1).$$

If $k \geqslant 2q$, then summing this estimate for $n = q, \dots, k-q$ gives

$$\operatorname{rank}_{\mathbb{C}} \widehat{\mathbf{L}}_t \geqslant 1+2+\ldots+k-2q+1 = \frac{1}{2}(k-2q+1)(k-2q+2),$$

and thus

$$\operatorname{rank} \widehat{\mathbf{L}}_t \geqslant (k-2q+1)(k-2q+2) \geqslant (k-2\ell+1)(k-2\ell+2)$$

whenever $k \geqslant 2\ell$. This estimate might not be satisfied for k underneath this threshold, but since that is only finitely many cases, we can now just choose a constant $C_\ell > 0$ small enough to achieve $C_\ell k^2 \leqslant \operatorname{rank} \widehat{\mathbf{L}}_t$ for those cases and $C_\ell k^2 \leqslant (k-2\ell+1)(k-2\ell+2)$ for all $k \geqslant 2\ell$. With this, the proof of Proposition 5.21 is complete.

5.3.3. *Proof of Theorem 5.9.* Consider the C^∞-subvarieties $\mathscr{P}^k_{r,\ell}(g,h,\mu) \subset \mathcal{V}^k_{r,\ell}(g,h,\mu)$ from §5.2.6 in the specific setting of local Cauchy-Riemann type operators $\mathbf{D} \in \widehat{\mathscr{D}}_p(E,F)$ with $\operatorname{rank}_{\mathbb{C}} E = m$. For any given operator $\mathbf{D} \in \widehat{\mathscr{D}}^k_p(E,F)$, we know from Lemma 5.16 that $\mathbf{D} : J^k_p E \to J^{k-1}_p F$ and $\mathbf{D}^* : J^k_p F \to J^{k-1}_p E$ are both surjective, thus (5.14) gives

$$\begin{aligned}\dim\ker\mathbf{D} = \dim\ker\mathbf{D}^* &= \dim J^k_p E - \dim J^{k-1}_p F \\ &= m(k+1)(k+2) - mk(k+1) = 2m(k+1),\end{aligned}$$

and plugging this into (5.7),

$$\dim \mathcal{V}^k_{r,\ell}(g,h,\mu,\mathbf{D}) = 4rm(k+1) - r^2. \tag{5.19}$$

Next, combining Proposition 5.21 with Lemma 5.19 gives:

Proposition 5.25. *For every $\ell \in \mathbb{N}$, there exists a constant $C_\ell > 0$ such that for all integers $k \geqslant \ell$ and all $r \in \mathbb{N}$, $\mathscr{P}^k_{r,\ell}(g,h,\mu) \subset \mathcal{V}^k_{r,\ell}(g,h,\mu)$ is a C^∞-subvariety of codimension at least $C_\ell k^2$.* □

Sard's theorem (see Propsition C.3) now provides a Baire subset

$$\widehat{\mathscr{D}}^{k,\mathrm{reg}}_p(E,F\,;r,\ell) \subset \widehat{\mathscr{D}}^k_p(E,F)$$

such that for all $\mathbf{D} \in \widehat{\mathscr{D}}^{k,\mathrm{reg}}_p(E,F\,;r,\ell)$, $\mathscr{P}^k_{r,\ell}(g,h,\mu,\mathbf{D})$ is a C^∞-subvariety in $\mathcal{V}^k_{r,\ell}(g,h,\mu,\mathbf{D})$ of codimension at least $C_\ell k^2$. Since this codimension grows quadratically with k while the dimension of $\mathcal{V}^k_{r,\ell}(g,h,\mu,\mathbf{D})$ grows only linearly, we conclude that for any fixed $r,\ell \in \mathbb{N}$, the space $\mathscr{P}^k_{r,\ell}(g,h,\mu,\mathbf{D})$ is empty for all k sufficiently large.

To conclude the proof of Theorem 5.9, we choose for each $\ell \in \mathbb{N}$ some $k \geqslant \ell$ large enough so that $\mathscr{P}^k_{\ell,\ell}(g,h,\mu,\mathbf{D}) = \varnothing$ for every $\mathbf{D} \in \widehat{\mathscr{D}}^{k,\mathrm{reg}}_p(E,F\,;\ell,\ell)$, and then define $\mathcal{CR}^{\ell,\mathrm{reg}}_{\mathbb{R}}(E\,;\mathcal{U},\mathbf{D}_{\mathrm{fix}})$ to be the set of all operators in $\mathcal{CR}_{\mathbb{R}}(E\,;\mathcal{U},\mathbf{D}_{\mathrm{fix}})$ whose k-jets at p belong to $\widehat{\mathscr{D}}^{k,\mathrm{reg}}_p(E,F\,;\ell,\ell)$.

5.4. **Petri's condition is satisfied for generic J.** We now return to the setting of §2 and consider the moduli space $\mathcal{M}_g(A,J)$ of unparametrized closed J-holomorphic curves $u : (\Sigma,j) \to (M,J)$ of genus $g \geqslant 0$ homologous to $A \in H_2(M)$ in a symplectic manifold (M,ω) of dimension $2n \geqslant 4$ with $J \in \mathcal{J}(M,\omega\,;\mathcal{U},J_{\mathrm{fix}})$. Here $\mathcal{U} \subset M$ is an open subset with compact closure, J_{fix} is a fixed compatible almost complex structure, and all $J \in \mathcal{J}(M,\omega\,;\mathcal{U},J_{\mathrm{fix}})$ are assumed to match J_{fix} outside of $\mathcal{U}$.

Theorem 5.26. *There exists a Baire subset $\mathcal{J}^{\mathrm{reg}} \subset \mathcal{J}(M,\omega\,;\mathcal{U},J_{\mathrm{fix}})$ such that for all $J \in \mathcal{J}^{\mathrm{reg}}$ and every $u \in \mathcal{M}_g(A,J)$ with parametrization $u : (\Sigma,j) \to (M,J)$, the normal Cauchy-Riemann operator $\mathbf{D}^N_u \in \mathcal{CR}_{\mathbb{R}}(N_u)$ satisfies Petri's condition to infinite order on an open and dense set of points in $u^{-1}(\mathcal{U})$. In particular, $\mathbf{D}^N_u$ satisfies the local Petri condition at every point in $u^{-1}(\mathcal{U})$ (cf. Remark 5.3).*

We will deduce Theorem 5.26 from the results of the previous subsection after showing essentially that the natural map from the universal moduli space of simple holomorphic curves with one marked point to the space of k-jets of normal Cauchy-Riemann operators at the marked point is always a submersion. Up to some technical details still to be addressed, the next lemma implies this. Recall that a point $z \in \Sigma$ in the domain of a smooth map $v : \Sigma \to M$ is called an **injective point** if $dv(z) : T_z\Sigma \to T_{v(z)}M$ is injective and $\{z\} = v^{-1}(v(z))$. For a simple J-holomorphic curve, the complement of the set of injective points is a discrete set.

Lemma 5.27. *Assume $J \in \mathcal{J}(M, \omega\,; \mathcal{U}, J_{\text{fix}})$, and $v : (\Sigma, j) \to (M, J)$ is a simple J-holomorphic curve with generalized normal bundle $N_v \subset v^*TM$ defined as the ω-symplectic complement of the generalized tangent bundle $T_v \subset v^*TM$. Given any $A \in \Omega^{0,1}(\Sigma, \operatorname{End}_{\mathbb{R}}(N_v))$ with support contained in the set of injective points in $v^{-1}(\mathcal{U})$, there exists a smooth family of almost complex structures*

$$\{J_\tau \in \mathcal{J}(M, \omega\,; \mathcal{U}, J_{\text{fix}})\}_{\tau \in (-\epsilon, \epsilon)}$$

such that $J_0 = J$, $J_\tau(v(z)) = J(v(z))$ for all τ and z, and the resulting family of normal Cauchy-Riemann operators $\mathbf{D}^N_{v,\tau} \in \mathcal{CR}_{\mathbb{R}}(N_v)$ for v defined with respect to J_τ satisfies

$$\partial_\tau \mathbf{D}^N_{v,\tau}\eta\big|_{\tau=0} = \pi_N \circ \nabla_\eta Y \circ Tv \circ j = A\eta$$

for $\eta \in \Gamma(N_v)$, where $Y := \partial_\tau J_\tau\big|_{\tau=0} \in \Gamma(\overline{\operatorname{End}}_{\mathbb{C}}(TM, J))$, ∇ is any connection on M, and $\pi_N : v^*TM \to N_v$ denotes the projection along T_v.

Proof. If $\{J_\tau\}$ is any smooth path in $\mathcal{J}(M, \omega\,; \mathcal{U}, J_{\text{fix}})$ with $J_0 = J$, $J_\tau(v) \equiv J(v)$ for all τ and $Y := \partial_\tau J|_{\tau=0}$, then $Y(v) \equiv 0$, hence ∇Y is well defined along v independently of any connection. For $\eta \in \Gamma(N_v)$, let us write $\nabla_\eta Y$ in block form as

$$\nabla_\eta Y = \begin{pmatrix} \nabla^T_\eta Y & \nabla^{TN}_\eta Y \\ \nabla^{NT}_\eta Y & \nabla^N_\eta Y \end{pmatrix} \in \Gamma(\overline{\operatorname{End}}_{\mathbb{C}}(v^*TM, J)) \tag{5.20}$$

with respect to the tangent-normal decomposition $v^*TM = T_v \oplus N_v$. Since N_v is the ω-symplectic orthogonal complement of T_v, the fact that J_τ is always ω-compatible then translates into conditions that constrain $\nabla^T_\eta Y$ and $\nabla^N_\eta Y$ separately and another condition that determines $\nabla^{TN}_\eta Y$ in terms of $\nabla^{NT}_\eta Y$, namely

$$\omega((\nabla^{NT}_\eta Y)v, w) + \omega(v, (\nabla^{TN}_\eta Y)w) = 0$$

for all $(v, w) \in T_v \oplus N_v$. This means that ω-compatibility does not prevent us from freely choosing $\nabla^{NT}_\eta Y$ so long as we (1) do not mind $\nabla^{TN}_\eta Y$ being determined by this choice, and (2) do this only in regions where v has no double points, so that the splitting of TM into $T_v \oplus N_v$ is unambiguous. Now using the definition of the normal Cauchy-Riemann operator, one computes that for any $\eta \in \Gamma(N_v)$,

$$\partial_\tau \mathbf{D}^N_{v,\tau}\eta\big|_{\tau=0} = \nabla^{NT}_\eta Y \circ Tv \circ j.$$

On a region where v has neither critical points nor double points and its image lies in the perturbation domain $\mathcal{U}$, we can therefore choose the normal derivatives of Y along v to make the above expression match A. $\square$

To prove Theorem 5.26, we will use the Floer C_ε-topology (cf. [Flo88, §5]) to define spaces of perturbed data. Given any $J_{\text{ref}} \in \mathcal{J}(M, \omega\,; \mathcal{U}, J_{\text{fix}})$, we define

$$T_{J_{\text{ref}}}\mathcal{J}(J, \omega\,; \mathcal{U}, J_{\text{fix}}) \subset \Gamma(\overline{\operatorname{End}}_{\mathbb{C}}(TM, J_{\text{ref}}))$$

as the space of smooth J_{ref}-antilinear bundle maps Y that vanish outside $\mathcal{U}$ and satisfy $\omega(\cdot, Y\cdot) + \omega(Y\cdot, \cdot) \equiv 0$; intuitively, this is the tangent space at J_{ref} to the smooth Fréchet manifold $\mathcal{J}(J, \omega\,; \mathcal{U}, J_{\text{fix}})$. There is a natural embedding

$$Y \mapsto J_Y := \left(\mathbb{1} + \frac{1}{2}J_{\text{ref}}Y\right) J_{\text{ref}} \left(\mathbb{1} + \frac{1}{2}J_{\text{ref}}Y\right)^{-1} \tag{5.21}$$

which takes a C^0-small neighborhood of 0 in $T_{J_{\mathrm{ref}}}\mathcal{J}(J,\omega\,;\mathcal{U},J_{\mathrm{fix}})$ homeomorphically to a neighborhood of J_{ref} in $\mathcal{J}(M,\omega\,;\mathcal{U},J_{\mathrm{fix}})$. Now choose a Riemannian metric on M in order to define the C^ν-norms on $\Gamma(\overline{\mathrm{End}}_{\mathbb{C}}(TM,J_{\mathrm{ref}}))$ for each integer $\nu \geqslant 0$, fix a sequence of positive numbers $\varepsilon_\nu \to 0$, and define the C_ε-norm

$$\|Y\|_{C_\varepsilon} := \sum_{\nu=0}^{\infty} \varepsilon_\nu \|Y\|_{C^\nu} \tag{5.22}$$

for $Y \in \Gamma(\overline{\mathrm{End}}_{\mathbb{C}}(TM,J_{\mathrm{ref}}))$. Fixing any $\delta > 0$ sufficiently small, this gives rise to a smooth, separable and metrizable Banach manifold

$$\mathcal{J}_\varepsilon := \left\{ J_Y \;\middle|\; Y \in T_{J_{\mathrm{ref}}}\mathcal{J}(J,\omega\,;\mathcal{U},J_{\mathrm{fix}}),\ \|Y\|_{C_\varepsilon} < \infty \text{ and } \|Y\|_{C^0} < \delta \right\}$$

which embeds continuously into $\mathcal{J}(J,\omega\,;\mathcal{U},J_{\mathrm{fix}})$ and contains arbitrarily C^∞-small perturbations of J_{ref}. Note that since $\mathcal{U} \subset M$ has compact closure, the equivalence classes of the individual C^ν-norms are each independent of auxiliary choices such as connections or local trivializations, but the equivalence class of the C_ε-norm may in fact depend on these choices. This is immaterial, as the choice of the sequence $\{\epsilon_\nu\}_{\nu=0}^\infty$ carries no geometric meaning in itself; what is important is rather that the space of sections of class C_ε can always be enlarged by making ε_ν converge to 0 faster. To say this more precisely, let us endow the set

$$\boldsymbol{\mathcal{E}} := \left\{ \text{sequences } \varepsilon = \{\varepsilon_\nu\}_{\nu=0}^\infty \;\middle|\; \varepsilon_\nu > 0 \text{ for all } \nu, \text{ and } \lim_{\nu\to\infty} \varepsilon_\nu = 0 \right\}$$

with a pre-order $\prec$ defined by

$$\varepsilon \prec \varepsilon' \qquad \Longleftrightarrow \qquad \limsup_{\nu\to\infty} \frac{\varepsilon_\nu}{\varepsilon'_\nu} < \infty.$$

Definition 5.28. Given a statement $S(\varepsilon)$ dependent on a choice of $\varepsilon \in \boldsymbol{\mathcal{E}}$, we will say that $S(\varepsilon)$ holds for all $\varepsilon \in \boldsymbol{\mathcal{E}}$ **with sufficiently rapid decay** if there exists $\varepsilon_0 \in \boldsymbol{\mathcal{E}}$ such that $S(\varepsilon)$ holds for all $\varepsilon \prec \varepsilon_0$.

Lemma 5.29. *The C_ε-norms on sections $Y \in T_{J_{\mathrm{ref}}}\mathcal{J}(J,\omega\,;\mathcal{U},J_{\mathrm{fix}})$ have the following properties:*

(1) *If $\varepsilon \prec \varepsilon'$ in $\boldsymbol{\mathcal{E}}$, then there exists a constant $c > 0$ such that $\|Y\|_{C_\varepsilon} \leqslant c\|Y\|_{C_{\varepsilon'}}$ for all Y.*
(2) *For any given Y, $\|Y\|_{C_\varepsilon} < \infty$ for all $\varepsilon \in \boldsymbol{\mathcal{E}}$ with sufficiently rapid decay.*
(3) *Every countable subset of $\boldsymbol{\mathcal{E}}$ has a lower bound in $\boldsymbol{\mathcal{E}}$ with respect to the pre-order $\prec$.*

Proof. Property (1) follows easily from the observation that $\varepsilon \prec \varepsilon'$ if and only if there exist constants $C > 0$ and $\nu_0 \in \mathbb{N}$ such that $\varepsilon_\nu \leqslant C\varepsilon'_\nu$ for all $\nu > \nu_0$. To prove (2), observe that any nontrivial smooth section Y vanishing outside of $\mathcal{U}$ is of class C_ε for $\varepsilon_\nu := 1/\left(2^\nu \cdot \|Y\|_{C^\nu}\right)$, then apply (1). Finally, $\varepsilon \in \boldsymbol{\mathcal{E}}$ is a lower bound for the countable subset $\{\varepsilon^{(1)}, \varepsilon^{(2)}, \varepsilon^{(3)}, \ldots\} \subset \boldsymbol{\mathcal{E}}$ whenever $\varepsilon_\nu \leqslant \min\left\{\varepsilon_\nu^{(1)}, \ldots, \varepsilon_\nu^{(\nu)}\right\}$ for every ν. □

Let us discuss the geometric data to be used in formulating the local Petri condition for a holomorphic curve. Given $J \in \mathcal{J}(M,\omega\,;\mathcal{U},J_{\mathrm{fix}})$, the complex vector bundle (TM,J) carries a natural Hermitian metric whose real part is $g_J := \omega(\cdot, J\cdot)$. If $u : (\Sigma,j) \to (M,J)$ is J-holomorphic and is immersed at the point $\zeta \in \Sigma$, then g_J can be pulled back to define a Riemannian metric on Σ near ζ in the conformal class of j, thus giving rise to an area form μ_u on Σ and compatible bundle metrics g_u on N_u and h_u on $\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma,N_u)$ near ζ, where for concreteness we are also free to assume $N_u \subset u^*TM$ is the g_J-orthogonal complement of T_u. In order to avoid ambiguity, we shall assume in the following that $\mathbf{D}_u^N$ and $(\mathbf{D}_u^N)^*$ are defined via these specific choices of geometric data near any given immersed point $\zeta \in \Sigma$; note that this would not be a valid global definition for $(\mathbf{D}_u^N)^*$ since the pulled back metric on Σ becomes singular at critical points, but this will not matter since we only intend to study finite jets of $(\mathbf{D}_u^N)^*$ at a specific immersed point. Recall from Remark 5.2 that Petri's condition does not depend on choices of geometric data. Moreover, while the global topological type of N_u may change (because the number of critical points may change) as u moves about in its moduli space, the germs of $\mathbf{D}_u^N$ and $(\mathbf{D}_u^N)^*$ at an immersed point can still be assumed to depend smoothly on u.

Let us denote by

$$\mathcal{M}^*_{g,1}(A,J) \subset \mathcal{M}_{g,1}(A,J)$$

the open subset consisting of simple curves with one marked point such that the marked point is an injective point with image in $\mathcal{U}$. We will abuse notation and write elements of $\mathcal{M}^*_{g,1}(A,J)$ as (u,ζ), where $u : (\Sigma, j) \to (M,J)$ is a specific parametrization and $\zeta \in \Sigma$ is the marked point. Using the notation of §5.2.6, we then define for each $k,r,\ell \in \mathbb{N}$ with $\ell \leqslant k$ the space

$$\widehat{\mathcal{M}}^{k,r,\ell}_{g,1}(A,J) := \left\{ (u,\zeta,t) \;\middle|\; (u,\zeta) \in \mathcal{M}^*_{g,1}(A,J),\ t \in \mathcal{V}^k_{r,\ell}(g_u,h_u,\mu_u,\mathbf{D}_u^N) \right\}$$

where g_u, h_u, μ_u are the specific choices of geometric data determined by u and g_J as described in the previous paragraph. The extra term t is an element in the tensor product of the k-jet versions of $\ker \mathbf{D}_u^N$ and $\ker(\mathbf{D}_u^N)^*$ at ζ, having rank r and not vanishing to order ℓ. We will be interested especially in the subset

$$\mathcal{M}^{k,r,\ell}_{g,1}(A,J) := \left\{ (u,\zeta,t) \in \widehat{\mathcal{M}}^{k,r,\ell}_{g,1}(A,J) \;\middle|\; \Pi^k(t) = 0 \right\}.$$

To understand the structure of these spaces, we define corresponding universal moduli spaces:

$$\begin{aligned}
\mathscr{U}^*_{g,1}(A,\mathcal{J}_\varepsilon) &:= \left\{ (u,\zeta,J) \;\middle|\; J \in \mathcal{J}_\varepsilon,\ (u,\zeta) \in \mathcal{M}^*_{g,1}(A,J) \right\},\\
\widehat{\mathscr{U}}^{k,r,\ell}_{g,1}(A,\mathcal{J}_\varepsilon) &:= \left\{ (u,\zeta,t,J) \;\middle|\; J \in \mathcal{J}_\varepsilon,\ (u,\zeta,t) \in \widehat{\mathcal{M}}^{k,r,\ell}_{g,1}(A,J) \right\},\\
\mathscr{U}^{k,r,\ell}_{g,1}(A,\mathcal{J}_\varepsilon) &:= \left\{ (u,\zeta,t,J) \;\middle|\; J \in \mathcal{J}_\varepsilon,\ (u,\zeta,t) \in \mathcal{M}^{k,r,\ell}_{g,1}(A,J) \right\}.
\end{aligned}$$

We shall always choose $\varepsilon \in \boldsymbol{\mathcal{E}}$ to have sufficiently rapid decay so that, by standard arguments as in [MS12], $\mathscr{U}^*_{g,1}(A,\mathcal{J}_\varepsilon)$ is a smooth, metrizable and separable Banach manifold such that the projection $\mathscr{U}^*_{g,1}(A,\mathcal{J}_\varepsilon) \to \mathcal{J}_\varepsilon : (u,\zeta,J) \mapsto J$ is a smooth Fredholm map whose index is the virtual dimension of $\mathcal{M}^*_{g,1}(A,J)$. It follows that the same is true for $\widehat{\mathscr{U}}^{k,r,\ell}_{g,1}(A,\mathcal{J}_\varepsilon)$, as the additional k-jet data t varies in a smooth finite-dimensional manifold that depends smoothly on the k-jet of the operator $\mathbf{D}_u^N$ at the immersed point ζ, and this in turn depends smoothly on $(u,\zeta,J) \in \mathscr{U}^*_{g,1}(A,\mathcal{J}_\varepsilon)$.

It will be convenient to impose an extra condition defining an open subset of $\mathscr{U}^{k,r,\ell}_{g,1}(A,\mathcal{J}_\varepsilon)$. For each $\ell \in \mathbb{N}$, let $C_\ell > 0$ denote the constant furnished by Proposition 5.25 in §5.3, with the roles of the bundles E, F and point p in that subsection played by N_u, $\overline{\operatorname{Hom}_{\mathbb{C}}}(T\Sigma, N_u)$ and $\zeta \in \Sigma$ respectively.

Definition 5.30. Given $J \in \mathcal{J}(M,\omega\,;\mathcal{U},J_{\text{fix}})$ and $\varepsilon \in \boldsymbol{\mathcal{E}}$, we will say that an element $(u,\zeta,t) \in \mathcal{M}^{k,r,\ell}_{g,1}(A,J)$ is **ε-regular** if $J \in \mathcal{J}_\varepsilon$ and (u,ζ,t,J) has a neighborhood $\mathcal{O} \subset \widehat{\mathscr{U}}^{k,r,\ell}_{g,1}(A,\mathcal{J}_\varepsilon)$ such that $\mathcal{O} \cap \mathscr{U}^{k,r,\ell}_{g,1}(A,\mathcal{J}_\varepsilon)$ is a C^∞-subvariety of $\widehat{\mathscr{U}}^{k,r,\ell}_{g,1}(A,\mathcal{J}_\varepsilon)$ with codimension at least $C_\ell k^2$.

Note that ε-regularity is an open condition by construction, i.e. the set of tuples $(u,\zeta,t,J) \in \mathscr{U}^{k,r,\ell}_{g,1}(A,\mathcal{J}_\varepsilon)$ such that (u,ζ,t) is ε-regular is open. The important consequence of Lemma 5.27 will be that it is generally also nonempty.

Lemma 5.31. *Any given $(u,\zeta,t) \in \mathcal{M}^{k,r,\ell}_{g,1}(A,J_{\text{ref}})$ is ε-regular for all $\varepsilon \in \boldsymbol{\mathcal{E}}$ with sufficiently rapid decay.*

Proof. Observe first that $J_{\text{ref}} \in \mathcal{J}_\varepsilon$ for every $\varepsilon \in \boldsymbol{\mathcal{E}}$. Now given $(u,\zeta,t) \in \mathcal{M}^{k,r,\ell}_{g,1}(A,J_{\text{ref}})$, define the Fréchet space

$$\mathcal{Y}_0 := \left\{ Y \in T_{J_{\text{ref}}}\mathcal{J}(M,\omega\,;\mathcal{U},J_{\text{fix}}) \;\middle|\; Y|_{u(\Sigma)} \equiv 0 \right\}$$

and for each $\varepsilon \in \boldsymbol{\mathcal{E}}$ the Banach space

$$\mathcal{Y}_\varepsilon := \left\{ Y \in T_{J_{\text{ref}}}\mathcal{J}(M,\omega\,;\mathcal{U},J_{\text{fix}}) \;\middle|\; Y|_{u(\Sigma)} \equiv 0 \text{ and } \|Y\|_{C_\varepsilon} < \infty \right\},$$

where the latter is regarded as a closed subspace of $T_{J_{\mathrm{ref}}}\mathcal{J}_\varepsilon$ with the C_ε-topology. Abbreviating $E := N_u$ and $F := \overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, N_u)$, Lemma 5.27 provides a surjective linear map

$$\Psi_0 : \mathcal{Y}_0 \to J^{k-1}_\zeta(\mathrm{Hom}_{\mathbb{R}}(E,F)) : Y \mapsto J^{k-1}_\zeta(A_Y),$$

where A_Y denotes (the germ near ζ of) the zeroth-order term determined by Y according to the formula $A_Y\eta = \pi_N \circ \nabla_\eta Y \circ Tu \circ j$. Since the target space of Ψ_0 is finite dimensional, Lemma 5.29 implies that it remains surjective when restricted to the subspace $\mathcal{Y}_\varepsilon$ for $\varepsilon \in \boldsymbol{\mathcal{E}}$ with sufficiently rapid decay. Each $Y \in \mathcal{Y}_\varepsilon$ now gives rise to a 1-parameter family of almost complex structures $J_\tau := J_{\tau Y} \in \mathcal{J}_\varepsilon$ defined via (5.21), which match J_{ref} along u and satisfy $J_0 = J_{\mathrm{ref}}$. This defines a smooth family $(u, \zeta, J_\tau) \in \mathscr{U}^*_{g,1}(A, \mathcal{J}_\varepsilon)$ that deforms the normal Cauchy-Riemann operator of u in the direction of A_Y but leaves the geometric data along u unchanged. It follows that the linearization at $(u, \zeta, t, J_{\mathrm{ref}})$ of the natural projection map[10]

$$\widehat{\mathscr{U}}^{k,r,\ell}_{g,1}(A, \mathcal{J}_\varepsilon) \to \mathcal{V}^k_{r,\ell} : (u, \zeta, t, J) \mapsto (g_u, h_u, \mu_u, \mathbf{D}^N_u, t) \tag{5.23}$$

is surjective onto $T_{(\mathbf{D}^u_N, t)}\mathcal{V}^k_{r,\ell}(g_u, h_u, \mu_u)$, and the result then follows from Proposition 5.25. ☐

Applying the Sard-Smale theorem to the projection $\widehat{\mathscr{U}}^{k,r,\ell}_{g,1}(A, \mathcal{J}_\varepsilon) \to \mathcal{J}_\varepsilon : (u, \zeta, t, J) \mapsto J$ as in Proposition C.3, we can associate to each $\varepsilon \in \boldsymbol{\mathcal{E}}$ and each set of positive integers k, r, ℓ with $k \geqslant \ell$ a Baire subset

$$\mathcal{J}^{\mathrm{reg}}_\varepsilon(k, r, \ell) \subset \mathcal{J}_\varepsilon$$

such that for all $J \in \mathcal{J}^{\mathrm{reg}}_\varepsilon(k, r, \ell)$, $\widehat{\mathcal{M}}^{k,r,\ell}_{g,1}(A, J)$ is a smooth finite-dimensional manifold and the open set of ε-regular elements in

$$\mathcal{M}^{k,r,\ell}_{g,1}(A, J) \subset \widehat{\mathcal{M}}^{k,r,\ell}_{g,1}(A, J)$$

is a C^∞-subvariety of codimension at least $C_\ell k^2$. The dimension of $\widehat{\mathcal{M}}^{k,r,\ell}_{g,1}(A, J)$ is the Fredholm index of the projection $\widehat{\mathscr{U}}^{k,r,\ell}_{g,1}(A, \mathcal{J}_\varepsilon) \to \mathcal{J}_\varepsilon$, which is larger than that of $\mathscr{U}^*_{g,1}(A, \mathcal{J}_\varepsilon) \to \mathcal{J}_\varepsilon$ by $\dim \mathcal{V}^k_{r,\ell}(g_u, h_u, \mu_u, \mathbf{D}^N_u)$. Plugging in (5.19), this gives

$$\dim \widehat{\mathcal{M}}^{k,r,\ell}_{g,1}(A, J) = \text{vir-dim}\, \mathcal{M}_{g,1}(A, J) + 4r(n-1)(k+1) - r^2.$$

This number grows linearly with k, while the codimension $C_\ell k^2$ grows quadratically, thus for any fixed r, ℓ, g, A, the integer

$$\text{vir-dim}\, \mathcal{M}^{k,r,\ell}_{g,1}(A, J) := \text{vir-dim}\, \mathcal{M}_{g,1}(A, J) + 4r(n-1)(k+1) - r^2 - C_\ell k^2 \tag{5.24}$$

becomes negative for all $k \in \mathbb{N}$ sufficiently large. Taking the countable intersection of the Baire sets $\mathcal{J}^{\mathrm{reg}}_\varepsilon(k, r, \ell)$ for all k, r, ℓ, g, A, we obtain:

Corollary 5.32. *For every $\varepsilon \in \boldsymbol{\mathcal{E}}$, there exists a Baire subset $\mathcal{J}^{\mathrm{reg}}_\varepsilon \subset \mathcal{J}_\varepsilon$ such that for all $J \in \mathcal{J}^{\mathrm{reg}}_\varepsilon$ and any given $g \geqslant 0$, $A \in H_2(M)$ and $r, \ell \in \mathbb{N}$, the set of ε-regular elements in $\mathcal{M}^{k,r,\ell}_{g,1}(A, J)$ is empty whenever k is large enough for the integer in* (5.24) *to be negative.* ☐

For the proof of Theorem 5.26, we will use a variation on a popular trick due to Taubes, presenting the desired set $\mathcal{J}^{\mathrm{reg}} \subset \mathcal{J}(M, \omega\,; \mathcal{U}, J_{\mathrm{fix}})$ as the intersection of an explicit countable collection of open and dense subsets. This depends on the ability to decompose the relevant moduli space into a countable union of compact subsets, and as preparation, the following lemma gives a way of doing this for the moduli space of complex structures. Given a smooth oriented surface Σ, we let $\mathcal{J}(\Sigma)$ denote the space of smooth complex structures on Σ compatible with the orientation, with its natural C^∞-topology. For integers $g, m \geqslant 0$, $\mathcal{M}_{g,m}$ will denote the

[10] Strictly speaking, the definition of $\mathcal{V}^k_{r,\ell}$ in this context depends on the germs near $\zeta \in \Sigma$ of the vector bundles N_u and $\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, N_u)$, which vary as (u, ζ, t, J) moves in $\widehat{\mathscr{U}}^{k,r,\ell}_{g,1}(A, \mathcal{J}_\varepsilon)$, so for the purposes of (5.23), $\mathcal{V}^k_{r,\ell}$ should be replaced with a suitable fiber bundle over $\widehat{\mathscr{U}}^{k,r,\ell}_{g,1}(A, \mathcal{J}_\varepsilon)$, of which the map in (5.23) is a section. This detail makes little difference for the present argument, however, since the family $(u, \zeta, J_\tau) \in \mathscr{U}^*_{g,1}(A, \mathcal{J}_\varepsilon)$ involves a fixed curve with a fixed marked point and $J_\tau|_{\mathrm{im}(u)}$ also fixed.

(uncompactified) moduli space of Riemann surfaces with genus g and m marked points; recall that elements of the latter are equivalence classes of tuples (Σ, j, Θ) where (Σ, j) is a Riemann surface of genus g and $\Theta \subset \Sigma$ is an ordered set of m points.

Lemma 5.33. *Given integers $g, m \geqslant 0$, fix a closed surface Σ of genus g and an ordered set of m points $\Theta = \{\zeta_1, \dots, \zeta_m\} \in \Sigma_g$. Then there exists a nested sequence of compact subsets*

$$\mathcal{J}^1(\Sigma, \Theta) \subset \mathcal{J}^2(\Sigma, \Theta) \subset \mathcal{J}^3(\Sigma, \Theta) \subset \ldots \subset \mathcal{J}(\Sigma)$$

such that every element of $\mathcal{M}_{g,m}$ has a representative (Σ, j, Θ) for some $j \in \mathcal{J}^K(\Sigma, \Theta)$, $K \in \mathbb{N}$.

Proof. Let $\pi : \mathcal{J}(\Sigma) \to \mathcal{M}_{g,m} : j \mapsto [(\Sigma, j, \Theta)]$ denote the natural projection. Choose for each $j \in \mathcal{J}(\Sigma)$ a smooth slice $\mathcal{T}_j \subset \mathcal{J}(\Sigma)$ through j for the natural action of $\operatorname{Diff}_0(\Sigma, \Theta)$ on $\mathcal{J}(\Sigma)$, i.e. $\mathcal{T}_j$ locally parametrizes the Teichmüller space of (Σ, Θ) near j. Since Teichmüller space is finite dimensional, $\mathcal{T}_j$ contains a compact neighborhood $\mathcal{V}_j \subset \mathcal{T}_j$ of j, and the image of $\mathcal{V}_j$ under π is then a neighborhood of $[(\Sigma, j, \Theta)]$ in $\mathcal{M}_{g,m}$. Since the latter is second countable, we can then find a sequence $j_1, j_2, j_3, \ldots \in \mathcal{J}(\Sigma)$ such that $\bigcup_{i \in \mathbb{N}} \pi(\mathcal{V}_{j_i}) = \mathcal{M}_{g,m}$. Set $\mathcal{J}^K(\Sigma, \Theta) := \mathcal{V}_{j_1} \cup \ldots \cup \mathcal{V}_{j_K}$. □

Proof of Theorem 5.26. For the following definition, we fix a model surface Σ_g of genus g and a point $\zeta \in \Sigma_g$, along with Riemannian metrics on Σ_g and M, denoting the various induced distance functions by $\operatorname{dist}(\cdot, \cdot)$. The Levi-Cività connection then induces connections on the bundles $E = N_u$ and $F = \overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma_g, N_u)$ appearing below, which can be used in defining metrics on the jet spaces $J^k_\zeta E$ and $J^k_\zeta F$. For each $K, \ell \in \mathbb{N}$, fix an integer $k := k(K, \ell) \geqslant \ell$ large enough so that

$$\operatorname{vir-dim} \mathcal{M}^{k,r,\ell}_{g,1}(A, J) < 0 \quad \text{for all} \quad r \in \{1, \ldots, K\}. \tag{5.25}$$

With this choice in place, we define

$$\mathcal{N}^K(J) \subset \bigcup_{r=1}^{K} \mathcal{M}^{k,r,\ell}_{g,1}(A, J)$$

as a set of elements (u, ζ, t) satisfying quantitative versions of the various conditions defining the spaces $\mathcal{M}^{k,r,\ell}_{g,1}(A, J)$. Concretely, we require every element of $\mathcal{N}^K(J)$ to be representable as a curve $u : (\Sigma_g, j) \to (M, J)$ with marked point $\zeta \in \Sigma_g$ and $t \in \mathcal{V}^k_{r,\ell}(g_u, h_u, \mu_u, \mathbf{D}^N_u)$ with $|t| = 1$ such that:

1. *Domains do not degenerate:* j belongs to the compact set $\mathcal{J}^K(\Sigma_g, \{\zeta\})$ from Lemma 5.33.
2. *Bubbles do not form:* $\sup_{z \in \Sigma_g} |du(z)| \leqslant K$.
3. *The marked point does not escape:* $\operatorname{dist}(u(\zeta), M \backslash \mathcal{U}) \geqslant 1/K$.
4. *The marked point remains an injective point:*
$$|du(\zeta)| \geqslant \frac{1}{K} \quad \text{and} \quad \inf_{z \in \Sigma_g \backslash \{\zeta\}} \frac{\operatorname{dist}(u(\zeta), u(z))}{\operatorname{dist}(\zeta, z)} \geqslant \frac{1}{K}.$$
5. *The rank of t does not blow up:* $\operatorname{rank} t \leqslant K$.
6. *The vanishing order of t does not increase:* Writing $E = N_u$ and $F = \overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma_g, E)$, the distance of $t \in J^k_\zeta E \otimes J^k_\zeta F$ from the subspace $(J^k_\zeta E \otimes J^k_\zeta F)^\ell$ is at least $1/K$.

Now let

$$\mathcal{J}^{\mathrm{reg},K} := \left\{ J \in \mathcal{J}(M, \omega\,; \mathcal{U}, J_{\mathrm{fix}}) \;\middle|\; \mathcal{N}^K(J) = \varnothing \right\}.$$

To see that $\mathcal{J}^{\mathrm{reg},K}$ is open, suppose the contrary: then there exist sequences $J_\nu \in \mathcal{J}(M, \omega\,; \mathcal{U}, J_{\mathrm{fix}})$ and $(u_\nu, \zeta, t_\nu) \in \mathcal{N}^K(J_\nu)$ with $J_\nu \to J \in \mathcal{J}^{\mathrm{reg},K}$ as $\nu \to \infty$. Assuming the parametrizations $u_\nu : (\Sigma_g, j_\nu) \to (M, J_\nu)$ satisfy all of the conditions listed above, elliptic regularity combined with the compactness of $\mathcal{J}^K(\Sigma_g, \{\zeta\})$ and the condition $|t_\nu| = 1$ then gives a subsequence converging to an element of $\mathcal{N}^K(J)$, which is a contradiction.

We claim that $\mathcal{J}^{\mathrm{reg},K}$ is also dense. To see this, recall that the reference structure J_{ref} in the definition of $\mathcal{J}_\varepsilon$ was arbitrary, so it will suffice to prove that for some $\varepsilon \in \boldsymbol{\mathcal{E}}$, $\mathcal{J}_\varepsilon$ contains arbitrarily

C_ε-small perturbations of J_{ref} that are in $\mathcal{J}^{\mathrm{reg},K}$. The argument of the previous paragraph shows that $\mathcal{N}^K(J_{\mathrm{ref}})$ is compact, so since ε-regularity is an open condition, Lemma 5.31 implies after taking a lower bound for finitely many choices of $\varepsilon \in \boldsymbol{\mathcal{E}}$ that every element of $\mathcal{N}^K(J_{\mathrm{ref}})$ is ε-regular, and so therefore is everything in some open neighborhood of $\mathcal{N}^K(J_{\mathrm{ref}}) \times \{J_{\mathrm{ref}}\}$ in $\mathscr{U}^{k,r,\ell}_{g,1}(A, \mathcal{J}_\varepsilon)$. Since $\mathcal{J}^{\mathrm{reg}}_\varepsilon \subset \mathcal{J}_\varepsilon$ is a Baire subset, we can choose a sequence $J_\nu \in \mathcal{J}^{\mathrm{reg}}_\varepsilon$ with $J_\nu \to J_{\mathrm{ref}}$, and we claim that $J_\nu \in \mathcal{J}^{\mathrm{reg},K}$ for all ν sufficiently large. If not, then after restricting to a subsequence, there exists a sequence $(u_\nu, \zeta_\nu, t_\nu) \in \mathcal{N}^K(J_\nu)$ which converges by the compactness argument in the previous paragraph to an element of $\mathcal{N}^K(J_{\mathrm{ref}})$, implying that $(u_\nu, \zeta_\nu, t_\nu)$ is ε-regular for ν large. In light of the assumption $\operatorname{vir-dim} \mathcal{M}^{k,r,\ell}_{g,1}(A, J) < 0$, this contradicts Corollary 5.32.

The space

$$\mathcal{J}^{\mathrm{reg}} := \bigcap_{K \in \mathbb{N}} \mathcal{J}^{\mathrm{reg},K} \subset \mathcal{J}(M, \omega\,; \mathcal{U}, J_{\mathrm{fix}})$$

is now a Baire subset. If $J \in \mathcal{J}^{\mathrm{reg}}$ and there exists a simple J-holomorphic curve $u : (\Sigma, j) \to (M, J)$ of genus g with an injective point $\zeta \in u^{-1}(\mathcal{U}) \subset \Sigma$ at which Petri's condition is not satisfied to infinite order, then we can define u as an element of $\mathcal{M}^*_{g,1}(A, J)$ by calling ζ the marked point. Since nontrivial elements $t \in \ker \mathbf{D}^N_u \otimes \ker(\mathbf{D}^N_u)^*$ have finite rank and cannot vanish to infinite order at any point, we can then normalize t and thus find an element $(u, \zeta, t) \in \mathcal{N}^K(J)$ for K sufficiently large, which is a contradiction. This proves that for $J \in \mathcal{J}^{\mathrm{reg}}$, all simple curves $v : (\Sigma, j) \to (M, J)$ satisfy Petri's condition to infinite order at every injective point in $v^{-1}(\mathcal{U})$, which is an open and dense subset of $v^{-1}(\mathcal{U})$. It follows that the condition is also satisfied for all multiple covers $u = v \circ \varphi$ at points in $u^{-1}(\mathcal{U}) = \varphi^{-1}(v^{-1}(\mathcal{U}))$ that are not branch points and are preimages of injective points; that is likewise an open and dense subset of $u^{-1}(\mathcal{U})$. ☐

Remark 5.34. The proof above would work equally well to find generic *families* of almost complex structures depending on finitely many parameters such that Petri's condition is always satisfied. The key point is that for the parametric moduli spaces analogous to $\mathcal{M}^{k,r,\ell}_{g,1}(A, J)$ and $\widehat{\mathcal{M}}^{k,r,\ell}_{g,1}(A, J)$, the codimension of the former in the latter grows quadratically with k, while the dimension of the larger space grows only linearly, so that the space analogous to $\mathcal{M}^{k,r,\ell}_{g,1}(A, J)$ will always turn out to be empty for generic choices if k is made sufficiently large, no matter how many extra dimensions are added to the original moduli space by introducing parameters. The extension to families is important for the bifurcation theory discussed in §2.4.

5.5. **A global application.** We now give an application of Petri's condition which will be crucial for the proof of Theorem D. The setting is as follows: assume E and F are smooth real vector bundles over a smooth (not necessarily compact) manifold M, with chosen bundle metrics $\langle\ ,\ \rangle_E$, $\langle\ ,\ \rangle_F$ and a chosen volume from μ on M which are used to define L^2-pairings

$$\langle \eta, \eta' \rangle_{L^2} := \int_M \langle \eta, \eta' \rangle_E \, \mu, \qquad \langle \xi, \xi' \rangle_{L^2} := \int_M \langle \xi, \xi' \rangle_F \, \mu$$

for $\eta, \eta' \in \Gamma(E)$ and $\xi, \xi' \in \Gamma(F)$. The product $\langle \eta, \eta' \rangle_{L^2}$ is well defined for two (not necessarily smooth or compactly supported) sections η, η' of E whenever the function $\langle \eta, \eta' \rangle_E$ belongs to $L^1(M, \mu)$, and in this case we will say they are **L^2-orthogonal** if $\langle \eta, \eta' \rangle_{L^2} = 0$; an analogous definition applies for sections of F. Consider a linear partial differential operator $\mathbf{D} : \Gamma(E) \to \Gamma(F)$ and its formal adjoint $\mathbf{D}^* : \Gamma(F) \to \Gamma(E)$ defined via $\langle \xi, \mathbf{D}\eta \rangle_{L^2} = \langle \mathbf{D}^*\xi, \eta \rangle_{L^2}$ for all smooth sections η, ξ with compact support. We will consider the extensions of both of these operators to certain Banach space completions,

$$\mathbf{D} : \mathbf{X}(E) \to \mathbf{Y}(F), \qquad \mathbf{D}^* : \mathbf{X}^*(F) \to \mathbf{Y}^*(E),$$

where $\mathbf{X}(E)$ and $\mathbf{Y}^*(E)$ are Banach spaces of sections of E in some regularity class defined almost everywhere, while $\mathbf{Y}(F)$ and $\mathbf{X}^*(F)$ are likewise Banach spaces of sections of F. In this functional-analytic setting, we impose the following assumptions:

(1) $\mathbf{D}$ and $\mathbf{D}^*$ are Fredholm operators whose kernels consist only of smooth sections;
(2) $\ker \mathbf{D}^* \subset \mathbf{Y}(F)$, and the L^2-product $\langle \xi, \xi' \rangle_{L^2}$ is well defined whenever $\xi \in \mathbf{Y}(F)$ and $\xi' \in \ker \mathbf{D}^*$, so in particular it is well defined whenever both are in $\ker \mathbf{D}^*$;
(3) $\mathbf{Y}(F) = \operatorname{im} \mathbf{D} \oplus \ker \mathbf{D}^*$, where the two factors in this splitting are closed L^2-orthogonal subspaces.

We shall denote the natural projection resulting from the third assumption by

$$\pi : \mathbf{Y}(F) \to \ker \mathbf{D}^*.$$

Remark 5.35. In the setting of §3.2, the assumptions above are satisfied for a Cauchy-Riemann type operator $\dot{\mathbf{D}} : \Gamma(\dot{E}) \to \Gamma(\dot{F})$ over a punctured Riemann surface $\dot{\Sigma}$, using the weighted Sobolev spaces $\mathbf{X}(\dot{E}) := W^{k,p,-\boldsymbol{\delta}}(\dot{E})$ and $\mathbf{Y}(\dot{F}) := W^{k-1,p,-\boldsymbol{\delta}}(\dot{F})$ for $k \in \mathbb{N}$, $p \in (1,\infty)$ and exponential weights $\boldsymbol{\delta} = \{\delta_w > 0\}_{w \in \Theta}$; recall that $\dot{\mathbf{D}}$ is Fredholm if all δ_w are chosen to be sufficiently small. For the formal adjoint $\dot{\mathbf{D}}^*$, we then define $\mathbf{X}^*(\dot{F}) := W^{k,p,\boldsymbol{\delta}}(\dot{F})$ and $\mathbf{Y}^*(\dot{E}) := W^{k-1,p,\boldsymbol{\delta}}(\dot{E})$, so that Proposition 3.13 provides the necessary splitting of $\mathbf{Y}(\dot{F})$.

Lemma 5.36. *Given the assumptions above, suppose $\mathcal{U} \subset M$ is an open subset such that $\mathbf{D}$ satisfies Petri's condition over $\mathcal{U}$. Assume moreover that $V \subset \Gamma(\operatorname{Hom}(E,F))$ is a linear subspace satisfying the following conditions:*
(1) *$\Phi\eta \in \mathbf{Y}(F)$ for all $\Phi \in V$ and $\eta \in \ker \mathbf{D}$.*
(2) *There exists a dense subset $\Delta \subset \mathcal{U}$ with the following property: for every $z \in \Delta$ and $\Phi_0 \in \operatorname{Hom}(E_z, F_z)$, there exists a $\Phi \in \Gamma(\operatorname{Hom}(E,F))$ satisfying $\Phi(z) = \Phi_0$ such that for every neighborhood $\mathcal{U}' \subset \mathcal{U}$ of z, $\beta\Phi \in V$ for some smooth function $\beta : M \to [0,1]$ with compact support in $\mathcal{U}'$ satisfying $\beta(z) = 1$.*

Then the linear map $\mathbf{L} : V \to \operatorname{Hom}(\ker \mathbf{D}, \ker \mathbf{D}^*)$ defined by $\mathbf{L}(\Phi)\eta = \pi(\Phi\eta)$ is surjective.

Proof. Fix bases $\eta_1, \dots, \eta_m \in \ker \mathbf{D}$ and $\xi_1, \dots, \xi_n \in \ker \mathbf{D}^*$. Since $\operatorname{im} \mathbf{D} = \ker \pi$ is L^2-orthogonal to $\ker \mathbf{D}^*$, we then have

$$\langle \mathbf{L}(\Phi)\eta_i, \xi_j \rangle_{L^2} = \langle \Phi\eta_i, \xi_j \rangle_{L^2} \quad \text{for all} \quad i = 1, \dots, m, \ j = 1, \dots, n,$$

and these matrix elements determine $\mathbf{L}(\Phi) : \ker \mathbf{D} \to \ker \mathbf{D}^*$. Now if $\mathbf{L}$ is not surjective, there exists a nontrivial linear map $\Psi : \ker \mathbf{D} \to \ker \mathbf{D}^*$ which is "orthogonal" to every $\mathbf{L}(\Phi)$ in the sense that its matrix elements $\Psi^{ij} := \langle \Psi\eta_i, \xi_j \rangle_{L^2} \in \mathbb{R}$ satisfy

$$\sum_{i,j} \Psi^{ij} \langle \Phi\eta_i, \xi_j \rangle_{L^2} = 0$$

for every $\Phi \in V$. We can rewrite this as

$$0 = \sum_{i,j} \Psi^{ij} \int_{\mathcal{U}} \langle \Phi\eta_i, \xi_j \rangle_F \, \mu = \int_{\mathcal{U}} \langle \ , \ \rangle_F \circ (\Phi \otimes \mathbb{1}) \circ \left(\sum_{i,j} \Psi^{ij} \, \eta_i \otimes \xi_j \right) \mu,$$

where $\sum_{i,j} \Psi^{ij} \eta_i \otimes \xi_j$ is regarded as a section of $E \otimes F$. Since the Ψ^{ij} are not all zero, this section is the image of a nontrivial element of $\ker \mathbf{D} \otimes \ker \mathbf{D}^*$ under the Petri map, so by assumption, it does not vanish identically on $\mathcal{U}$. Now choose a point $z \in \Delta$ at which this section is nonzero. Lemma 5.37 below provides a linear map $\Phi_0 : E_z \to F_z$ such that the integrand is positive near z for any $\Phi \in V$ satisfying $\Phi(z) = \Phi_0$, and we can then make the entire integral positive after multiplying Φ by smooth bump functions with sufficiently small support. □

We used:

Lemma 5.37. *Suppose V and W are real finite-dimensional vector spaces, $\langle \ , \ \rangle : W \otimes W \to \mathbb{R}$ is an inner product on W, and $T \in V \otimes W$ is nonzero. Then there exists a linear map $\Phi : V \to W$ such that $\langle \ , \ \rangle \circ (\Phi \otimes \mathbb{1})(T) > 0$.*

Proof. Choosing a basis $v_1, \dots, v_n$ of V, we have $T = \sum_{j=1}^n v_j \otimes w_j$ for unique vectors $w_1, \dots, w_n \in W$, which do not all vanish since $T \neq 0$. Choosing $\Phi : V \to W$ such that $\Phi(v_j) = w_j$ for all j then gives $\langle \ , \ \rangle \circ (\Phi \otimes \mathbb{1})(T) = \sum_j \langle w_j, w_j \rangle > 0$. □

6. Proof of the stratification theorem

We are now in a position to prove Theorem D. The main idea behind the proof is standard, though some details are less so: we will write down a universal moduli space with a projection to a suitable Banach manifold of perturbed data whose regular values have the property stated in the theorem. The hard part is of course to prove that the universal moduli space is a smooth Banach manifold—this follows from the implicit function theorem after proving that some version of the operator defined in (3.24) is surjective, and that is where the results of the previous section on Petri's condition are needed.

Fix $J_{\mathrm{ref}} \in \mathcal{J}(M, \omega\,; \mathcal{U}, J_{\mathrm{fix}})$ and consider again the space $\mathcal{J}_\varepsilon$ of Floer C_ε-small perturbations of J_{ref} as constructed in §5.4 via a choice of decaying positive sequence $\varepsilon = \{\varepsilon_\nu\}_{\nu=0}^\infty \in \boldsymbol{\mathcal{E}}$. For each of the choices of data in the statement of Theorem D, we define a universal moduli space

$$\mathscr{U}^d(\mathcal{J}_\varepsilon\,; \ell_1, \dots, \ell_m)$$

consisting of pairs (u, J) with $J \in \mathcal{J}_\varepsilon$ and u belonging to the isosymmetric stratum

$$\mathcal{M}^d(J\,; \ell_1, \dots, \ell_m) := \mathcal{M}^d_{\mathbf{b}}(\mathcal{M}_{g,m}(A, J\,; \ell_1, \dots, \ell_m)).$$

We shall denote elements of $\mathcal{M}^d(J\,; \ell_1, \dots, \ell_m)$ by $u = v \circ \varphi$, where we have chosen parametrizations of the underlying simple curve $v : (\Sigma, j) \to (M, J)$ and the d-fold branched cover $\varphi : (\Sigma', j') \to (\Sigma, j)$. Recall from §3 that for every such element $u = v \circ \varphi$, there is a unique isomorphism class of minimal regular presentations for φ, giving rise to a regular cover

$$\widehat{\varphi} : (\widehat{\Sigma}, \widehat{\jmath}) \to (\dot{\Sigma}, j)$$

with automorphism group $G := \mathrm{Aut}(\widehat{\varphi})$, where $\dot{\Sigma}$ is the punctured surface obtained from Σ by removing the critical values of φ. We can then consider the J-holomorphic curve $\widehat{u} := v \circ \widehat{\varphi} : (\widehat{\Sigma}, \widehat{\jmath}) \to (M, J)$ and its normal Cauchy-Riemann operator $\dot{\mathbf{D}}^N_{\widehat{u}}$, defined as in §3.2 on a Sobolev space of sections of $E := N_{\widehat{u}}$ over the punctured domain $\widehat{\Sigma}$ with negative exponential weights close to zero. Recall that its formal adjoint $(\dot{\mathbf{D}}^N_{\widehat{u}})^*$ is defined on a similar Sobolev space of sections of $F := \overline{\mathrm{Hom}}_{\mathbb{C}}(T\widehat{\Sigma}, N_{\widehat{u}})$, but with corresponding positive exponential weights. The notation associating to each $(u = v \circ \varphi, J) \in \mathscr{U}^d(\mathcal{J}_\varepsilon\,; \ell_1, \dots, \ell_m)$ a regular covering map $\widehat{\varphi}$ of potentially larger degree and corresponding J-holomorphic curve $\widehat{u} = v \circ \widehat{\varphi}$ will be used consistently in the following.

Definition 6.1. Given integers $k, c \geqslant 0$ and an almost complex structure J, we define the subset

$$\mathcal{M}^d(J\,; \ell_1, \dots, \ell_m\,; k, c) := \left\{ u \in \mathcal{M}^d(J\,; \ell_1, \dots, \ell_m) \;\middle|\; \dim\ker \dot{\mathbf{D}}^N_{\widehat{u}} = k \text{ and } \dim\operatorname{coker} \dot{\mathbf{D}}^N_{\widehat{u}} = c \right\}.$$

This gives rise to a universal moduli space

$$\mathscr{U}^d(\mathcal{J}_\varepsilon\,; \ell_1, \dots, \ell_m\,; k, c) \subset \mathscr{U}^d(\mathcal{J}_\varepsilon\,; \ell_1, \dots, \ell_m)$$

consisting of all pairs (u, J) such that $J \in \mathcal{J}_\varepsilon$ and $u \in \mathcal{M}^d(J\,; \ell_1, \dots, \ell_m\,; k, c)$.

By the results of §3.5, in particular Lemma 3.24, the connected components of the subsets $\mathcal{M}^d(J\,; \ell_1, \dots, \ell_m\,; k, c)$ for individual values of k and c are precisely the walls described in Theorem D (see also Remark 2.14). We would thus be able to apply the standard Sard-Smale argument toward a proof of Theorem D if we could show that $\mathscr{U}^d(\mathcal{J}_\varepsilon\,; \ell_1, \dots, \ell_m\,; k, c) \subset \mathscr{U}^d(\mathcal{J}_\varepsilon\,; \ell_1, \dots, \ell_m)$ is a smooth Banach submanifold of the correct finite codimension on each component. What we will actually show is that this is true for a certain open subset of $\mathscr{U}^d(\mathcal{J}_\varepsilon\,; \ell_1, \dots, \ell_m\,; k, c)$, which suffices due to the genericity of Petri's condition.

Definition 6.2. An element $u = v \circ \varphi \in \mathcal{M}^d(J\,; \ell_1, \dots, \ell_m\,; k, c)$ will be called **Petri regular** if for the regular covering map $\widehat{\varphi}$ and corresponding J-holomorphic curve $\widehat{u} = v \circ \widehat{\varphi}$ described above, the operator $\dot{\mathbf{D}}^N_{\widehat{u}}$ satisfies Petri's condition over $\widehat{u}^{-1}(\mathcal{U})$. We will denote the set of Petri regular curves by

$$\mathcal{M}^d_\Pi(J\,; \ell_1, \dots, \ell_m\,; k, c) \subset \mathcal{M}^d(J\,; \ell_1, \dots, \ell_m\,; k, c),$$

and define the corresponding universal moduli space

$$\mathscr{U}^d_\Pi(\mathcal{J}_\varepsilon\,;\,\ell_1,\dots,\ell_m\,;\,k,c) \subset \mathscr{U}^d(\mathcal{J}_\varepsilon\,;\,\ell_1,\dots,\ell_m\,;\,k,c)$$

to be the set of pairs $(u,J) \in \mathscr{U}^d(\mathcal{J}_\varepsilon\,;\,\ell_1,\dots,\ell_m\,;\,k,c)$ such that u belongs to the moduli space $\mathcal{M}^d_\Pi(J\,;\,\ell_1,\dots,\ell_m\,;\,k,c)$.

Remark 6.3. The condition defining $\mathcal{M}^d_\Pi(J\,;\,\ell_1,\dots,\ell_m\,;\,k,c)$ is clearly satisfied by any curve $u = v\circ\varphi$ for which $\mathbf{D}^N_v$ satisfies the local Petri condition on $v^{-1}(\mathcal{U})$, thus by Theorem 5.26, there is a Baire subset in $\mathcal{J}(M,\omega\,;\,\mathcal{U},J_{\text{fix}})$ for which $\mathcal{M}^d_\Pi(J\,;\,\ell_1,\dots,\ell_m\,;\,k,c) = \mathcal{M}^d(J\,;\,\ell_1,\dots,\ell_m\,;\,k,c)$.

The next several results are aimed at proving that for suitable choices of the sequence ε, $\mathscr{U}^d_\Pi(\mathcal{J}_\varepsilon\,;\,\ell_1,\dots,\ell_m\,;\,k,c)$ is a finite-codimensional Banach submanifold of $\mathscr{U}^d(\mathcal{J}_\varepsilon\,;\,\ell_1,\dots,\ell_m)$.

Lemma 6.4. *For $\varepsilon \in \boldsymbol{\mathcal{E}}$ with sufficiently rapid decay, $\mathscr{U}^d(\mathcal{J}_\varepsilon\,;\,\ell_1,\dots,\ell_m)$ carries a smooth Banach manifold structure such that every $(u_0 = v_0\circ\varphi_0, J_0) \in \mathscr{U}^d(\mathcal{J}_\varepsilon\,;\,\ell_1,\dots,\ell_m)$ admits a neighborhood $\mathcal{V} \subset \mathscr{U}^d(\mathcal{J}_\varepsilon\,;\,\ell_1,\dots,\ell_m)$ with a smooth family of vector bundle isomorphisms*

$$v_0^*TM \xrightarrow{\cong} v^*TM, \qquad \textit{for} \qquad (u = v\circ\varphi, J) \in \mathcal{V}$$

mapping N_{v_0} isomorphically to N_v.

Proof. For each $(u_0 = v_0\circ\varphi_0, J_0) \in \mathscr{U}^d(\mathcal{J}_\varepsilon\,;\,\ell_1,\dots,\ell_m)$, the underlying simple curve $v_0 : (\Sigma, j_0) \to (M,J)$ lives in the universal moduli space $\mathscr{U}^*(\mathcal{J}_\varepsilon)$ defined in Appendix A, more specifically in the subset

$$\mathscr{U}^*(\mathcal{J}_\varepsilon\,;\,\ell_1,\dots,\ell_m) \subset \mathscr{U}^*(\mathcal{J}_\varepsilon)$$

of this space defined by the condition that the ith marked point should have critical order ℓ_i and curves are immersed everywhere else. If ε has sufficiently rapid decay, then $\mathscr{U}^*(\mathcal{J}_\varepsilon)$ is a smooth Banach manifold, and $\mathscr{U}^*(\mathcal{J}_\varepsilon\,;\,\ell_1,\dots,\ell_m)$ is an open subset of the space $\widehat{\mathscr{U}}^*(\mathcal{J}_\varepsilon\,;\,\ell_1,\dots,\ell_m) \subset \mathscr{U}^*(\mathcal{J}_\varepsilon)$, which is shown in Lemma A.3 to be a smooth finite-codimensional submanifold of $\mathscr{U}^*(\mathcal{J}_\varepsilon)$. In particular, we can identify an open neighborhood of the element (v_0, J_0) in $\mathscr{U}^*(\mathcal{J}_\varepsilon\,;\,\ell_1,\dots,\ell_m)$ with a smooth finite-codimensional submanifold

$$X_\varepsilon \subset \bar\partial^{-1}(0) \subset \mathcal{T}\times\mathcal{B}\times\mathcal{J}_\varepsilon$$

of the zero-set of the nonlinear Cauchy-Riemann operator $\bar\partial$, where $\mathcal{T}$ denotes a Teichmüller slice through j_0 in the space of complex structures on Σ, and $\mathcal{B}$ is a suitable Banach manifold of maps $v : \Sigma \to M$.

We claim that there exists a neighborhood $\mathcal{V}_0 \subset X_\varepsilon$ of (j_0, v_0, J_0) that parametrizes a smooth family of bundle isomorphisms $v_0^*TM \to v^*TM$ sending N_{v_0} to N_v. Note that this would be clearly false if we did not impose the critical point constraints on v, as e.g. v_0 might then have critical points while v is immersed, in which case N_{v_0} and N_v would have different topological types. Assuming $N_v \subset v^*TM$ is always defined as the symplectic orthogonal complement of $T_v \subset v^*TM$ with $T_v := \operatorname{im} dv$ away from critical points, let us recall from [Wen10] how the latter is defined at critical points. We have a smooth family of bundles v^*TM carrying linearized Cauchy-Riemann operators $\mathbf{D}_v$, whose complex-linear parts $\mathbf{D}^{\mathbb{C}}_v$ define a smooth family of holomorphic structures on v^*TM. The crucial observation is then that $dv \in \Gamma(\operatorname{Hom}_{\mathbb{C}}(T\Sigma, v^*TM))$ is always a holomorphic section with respect to the holomorphic bundle structures on v^*TM and $T\Sigma$, so choosing a smooth family of holomorphic trivializations and holomorphic coordinates near the ith marked point, each dv is represented by some holomorphic function of the form

$$f^{(i)}_v : \mathbb{D} \to \mathbb{C}^m, \qquad f^{(i)}(z) = z^{\ell_i} g^{(i)}_v(z),$$

where $g^{(i)}_v : \mathbb{D} \to \mathbb{C}^m$ is another family of holomorphic functions which depend smoothly on $(j,v,J) \in X_\varepsilon$ but also are nonzero at 0. The main point here is that the critical orders ℓ_i do not vary with v. The span of $g^{(i)}_v(0)$ thus defines the fibers of T_v near each critical point, so we deduce smooth dependence of T_v on $(j,v,J) \in X_\varepsilon$, and therefore also of N_v.

We can parametrize a neighborhood of φ_0 in $\mathcal{M}^d_{\mathbf{b}}(j_0)$ as explained in Examples 3.6 and 3.8, meaning that if $\Theta = \{w_1,\dots,w_r\} \subset \Sigma$ is the set of critical values of φ_0, we choose a smooth family of diffeomorphisms $\psi_\tau : \Sigma \to \Sigma$ parametrized by $\tau \in B^{2r}$ which are holomorphic near Θ and supported on a slightly larger neighborhood of Θ such that $\psi_0 = \mathrm{Id}$ and

$$B^{2r} \to \Sigma^{\times r} : \tau \mapsto (\psi_\tau(w_1), \dots, \psi_\tau(w_r))$$

is a diffeomorphism onto an open set. The neighborhood of (u_0, J_0) in the space $\mathscr{U}^d(\mathcal{J}_\varepsilon\,;\,\ell_1, \dots, \ell_m)$ can now be identified with $B^{2r} \times X_\varepsilon$ by associating to each $(\tau, (j, v, J)) \in B^{2r} \times X_\varepsilon$ the curve $v \circ (\psi_\tau \circ \varphi_0)$, making $\mathscr{U}^d(\mathcal{J}_\varepsilon\,;\,\ell_1, \dots, \ell_m)$ a smooth fiber bundle over $\mathscr{U}^*(\mathcal{J}_\varepsilon\,;\,\ell_1, \dots, \ell_m)$. □

Lemma 6.5. *The subset* $\mathscr{U}^d_\Pi(\mathcal{J}_\varepsilon\,;\,\ell_1, \dots, \ell_m\,;\,k, c) \subset \mathscr{U}^d(\mathcal{J}_\varepsilon\,;\,\ell_1, \dots, \ell_m\,;\,k, c)$ *is open.*

Proof. Lemma 6.4 implies that the operators $\dot{\mathbf{D}}^N_{\hat{u}}$ and $(\dot{\mathbf{D}}^N_{\hat{u}})^*$ can both be understood as varying continuously with $(u, J) \in \mathscr{U}^d(\mathcal{J}_\varepsilon\,;\,\ell_1, \dots, \ell_m)$, and the dimensions of their kernels are locally constant as long as (u, J) moves only in the subset $\mathscr{U}^d(\mathcal{J}_\varepsilon\,;\,\ell_1, \dots, \ell_m\,;\,k, c)$. It follows that the family of Petri maps defined on $\ker \mathbf{D}^N_{\hat{u}} \otimes \ker(\mathbf{D}^N_{\hat{u}})^*$ and then restricted to $\hat{u}^{-1}(\mathcal{U})$ depends continuously on $(u, J) \in \mathscr{U}^d(\mathcal{J}_\varepsilon\,;\,\ell_1, \dots, \ell_m\,;\,k, c)$, and since their domains are finite dimensional, the injectivity of these maps is an open condition. □

Following Example 3.8, the smooth family of operators $\mathbf{D}^N_v$ parametrized by $\mathscr{U}^d(\mathcal{J}_\varepsilon\,;\,\ell_1, \dots, \ell_m)$ can now be fit into the general picture from §3 of a parametrized family of bundles with Cauchy-Riemann operators. In particular, we choose the parameter space P to be the local model of $\mathscr{U}^d(\mathcal{J}_\varepsilon\,;\,\ell_1, \dots, \ell_m)$ near (u_0, J_0) described in the proof of Lemma 6.4 above,

$$P := B^{2r} \times X_\varepsilon \subset B^{2r} \times \bar{\partial}^{-1}(0) \subset B^{2r} \times (\mathcal{T} \times \mathcal{B} \times \mathcal{J}_\varepsilon),$$

and in the notation of §3, associate to each $\tau = (\sigma, (j, v, J)) \in P$ the data

$$\psi_\tau := \psi_\sigma, \qquad j_\tau := j, \qquad (E_\tau, J_\tau) := (N_v, J), \qquad \mathbf{D}_\tau := \mathbf{D}^N_v.$$

If $(u_0, J_0) \in \mathscr{U}^d(\mathcal{J}_\varepsilon\,;\,\ell_1, \dots, \ell_m\,;\,k, c)$, then using the setup in §3.5, we now find a smooth map

$$\mathbf{F}_\varepsilon : B^{2r} \times X_\varepsilon \to \mathrm{Hom}_G\left(\ker \dot{\mathbf{D}}^N_{\hat{u}_0}, \ker(\dot{\mathbf{D}}^N_{\hat{u}_0})^*\right) \tag{6.1}$$

whose zero-set is a neighborhood of (u_0, J_0) in $\mathscr{U}^d(\mathcal{J}_\varepsilon\,;\,\ell_1, \dots, \ell_m\,;\,k, c)$.

Definition 6.6. We will say that $(u_0, J_0) \in \mathscr{U}^d(\mathcal{J}_\varepsilon\,;\,\ell_1, \dots, \ell_m\,;\,k, c)$ is **ε-regular** if $\varepsilon \in \boldsymbol{\mathcal{E}}$ has sufficiently rapid decay to satisfy the conclusions of Lemma 6.4 and, additionally, the linearization of the map (6.1) at $(0, (j_0, v_0, J_0))$ is surjective. Given $J \in \mathcal{J}(M, \omega\,;\,\mathcal{U}, J_{\mathrm{fix}})$ and $\varepsilon \in \boldsymbol{\mathcal{E}}$, an element u in the space $\mathcal{M}^d(J\,;\,\ell_1, \dots, \ell_m\,;\,k, c)$ will similarly be called **ε-regular** if $J \in \mathcal{J}_\varepsilon$ and (u, J) is ε-regular.

In analogy with Definition 5.30, ε-regularity for an element $(u_0, J_0) \in \mathscr{U}^d(\mathcal{J}_\varepsilon\,;\,\ell_1, \dots, \ell_m\,;\,k, c)$ just means that a neighborhood of (u_0, J_0) in this space is a smooth Banach submanifold with the "correct" finite codimension in $\mathscr{U}^d(\mathcal{J}_\varepsilon\,;\,\ell_1, \dots, \ell_m)$. It could be phrased alternatively as the condition that (u_0, J_0) is a transverse intersection of the map $(u, J) \mapsto \dot{\mathbf{D}}^N_{\hat{u}}$ from $\mathscr{U}^d(\mathcal{J}_\varepsilon\,;\,\ell_1, \dots, \ell_m)$ to the relevant space of G-equivariant Fredholm operators with the finite-codimensional submanifold $\{\mathbf{T} \mid \dim\ker\mathbf{T} = \dim\ker\dot{\mathbf{D}}^N_{\hat{u}_0}\}$; expressed in this way, ε-regularity is clearly an open condition and is independent of the choices involved (except of course for the choice of $\varepsilon \in \boldsymbol{\mathcal{E}}$).

Let us define the analogous condition for moduli spaces with fixed J. Note that if the simple curve v_0 is regular for the *constrained* moduli space $\mathcal{M}_{g,m}(A, J_0\,;\,\ell_1, \dots, \ell_m)$ as defined in Appendix A, then the set

$$X(J_0) := \left\{(j, v, J_0) \in X_\varepsilon \;\middle|\; j \in \mathcal{T},\ v \in \mathcal{B}\right\} \subset \bar{\partial}^{-1}_{J_0}(0)$$

is independent of $\varepsilon \in \boldsymbol{\mathcal{E}}$ and is a smooth finite-dimensional submanifold parametrizing a neighborhood of v_0 in $\mathcal{M}_{g,m}(A, J_0\,;\,\ell_1, \dots, \ell_m)$. A neighborhood of u_0 in $\mathcal{M}^d(J\,;\,\ell_1, \dots, \ell_m)$ is then parametrized by the submanifold $B^{2r} \times X(J_0) \subset B^{2r} \times X_\varepsilon$. We will say that $u_0 \in \mathcal{M}_{g,m}(A, J_0\,;\,\ell_1, \dots, \ell_m\,;\,k, c)$ is **regular in its stratum** if regularity of v_0 in the sense above holds and, additionally, the restricted linearization

$$T_{(0,(j_0,v_0,\varphi_0))}(B^{2r} \times X(J_0)) \xrightarrow{d\mathbf{F}_\varepsilon(0,(j_0,v_0,\varphi_0))} \mathrm{Hom}_G\left(\ker \dot{\mathbf{D}}^N_{\hat{u}_0}, \ker(\dot{\mathbf{D}}^N_{\hat{u}_0})^*\right)$$

is surjective. This can also be rephrased as a transverse intersection condition in the space of Fredholm operators, and is thus open and independent of choices (including ε). Our goal is to show that all curves satisfy this condition for generic J.

Lemma 6.7. *If $u = v \circ \varphi \in \mathcal{M}^d(J_{\mathrm{ref}}, ; \ell_1, \dots, \ell_m ; k, c)$ is Petri regular, then it is ε-regular for all $\varepsilon \in \boldsymbol{\mathcal{E}}$ with sufficiently rapid decay.*

Proof. Clearly $(u, J_{\mathrm{ref}}) \in \mathscr{U}^d_{\Pi}(\mathcal{J}_\varepsilon ; \ell_1, \dots, \ell_m ; k, c)$ for every $\varepsilon \in \boldsymbol{\mathcal{E}}$, and we shall assume ε has sufficiently rapid decay so that $\mathscr{U}^d(\mathcal{J}_\varepsilon ; \ell_1, \dots, \ell_m)$ is a smooth Banach manifold. By Lemma 5.27, there is a large space of smooth perturbations $Y \in T_{J_{\mathrm{ref}}}\mathcal{J}(M, \omega ; \mathcal{U}, J_{\mathrm{fix}})$ that give rise via (5.21) to smooth 1-parameter families $J_\tau := J_{\tau Y} \in \mathcal{J}(M, \omega ; \mathcal{U}, J_{\mathrm{fix}})$ for which v remains J_τ-holomorphic, and the normal Cauchy-Riemann operator $\mathbf{D}_v^N$ is perturbed in the direction of an arbitrary smooth zeroth-order term A_Y with support in $v^{-1}(\mathcal{U})$ away from the discrete set of critical and double points of v. Such a perturbation defines a tangent vector $(0, Y) \in T_{(u,J_{\mathrm{ref}})}\mathscr{U}^d(\mathcal{J}_\varepsilon ; \ell_1, \dots, \ell_m)$ whenever ε has sufficiently rapid decay for Y to be of class C_ε. Assuming this for the moment, the resulting perturbation to $\dot{\mathbf{D}}^N_{\widehat{u}}$ is

$$\dot{\mathbf{D}}^N_{\widehat{u}} \rightsquigarrow \dot{\mathbf{D}}^N_{\widehat{u}} + \widehat{\varphi}^* A_Y,$$

hence differentiating $\mathbf{F}_\varepsilon$ in the direction $(0, Y)$ produces a G-equivariant linear map $\mathbf{L}(Y) : \ker \dot{\mathbf{D}}^N_{\widehat{u}} \to \ker(\dot{\mathbf{D}}^N_{\widehat{u}})^*$ given by (3.24), namely

$$\mathbf{L}(Y)\eta = \pi\big((\widehat{\varphi}^* A_Y)\eta\big),$$

in terms of the projection

$$\pi : W^{k-1,p,-\widehat{\varphi}^*\boldsymbol{\delta}}(N_{\widehat{u}}) = \operatorname{im}(\dot{\mathbf{D}}^N_{\widehat{u}}) \oplus \ker(\dot{\mathbf{D}}^N_{\widehat{u}})^* \to \ker(\dot{\mathbf{D}}^N_{\widehat{u}})^*.$$

We claim that Y can be chosen to make $\mathbf{L}(Y)$ equal to any given element

$$\Psi \in \operatorname{Hom}_G\big(\ker \dot{\mathbf{D}}^N_{\widehat{u}}, \ker(\dot{\mathbf{D}}^N_{\widehat{u}})^*\big).$$

Indeed, let us abbreviate $E = N_v$ and $F = \overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, N_v)$, and let $\Delta \subset v^{-1}(\mathcal{U}) \subset \Sigma$ denote the set of injective points of v that are not critical values of $\widehat{\varphi}$ and have image in $\mathcal{U}$; these form an open and dense subset of $v^{-1}(\mathcal{U})$. Since $\dot{\mathbf{D}}^N_{\widehat{u}}$ satisfies Petri's condition over $\widehat{u}^{-1}(\mathcal{U})$, Lemma 5.36 then provides for any given Ψ a section $\widehat{A} \in \Gamma(\operatorname{Hom}_{\mathbb{R}}(\widehat{\varphi}^*\dot{E}, \widehat{\varphi}^*\dot{F}))$ with compact support in the open and dense subset $\widehat{\varphi}^{-1}(\Delta) \subset \widehat{u}^{-1}(\mathcal{U})$ such that

$$\langle \xi, \widehat{A}\eta \rangle_{L^2} = \langle \xi, \Psi\eta \rangle_{L^2}$$

for all $\xi \in \ker(\dot{\mathbf{D}}^N_{\widehat{u}})^*$ and $\eta \in \ker \dot{\mathbf{D}}^N_{\widehat{u}}$. Note that we are free to assume the L^2-product is invariant under the action of G via deck transformations. Then since Ψ is G-equivariant, we also have for every $g \in G$,

$$\langle \xi, (g\widehat{A})\eta \rangle_{L^2} = \langle g^{-1}\xi, \widehat{A}(g^{-1}\eta) \rangle_{L^2} = \langle g^{-1}\xi, \Psi(g^{-1}\eta) \rangle_{L^2} = \langle g^{-1}\xi, g^{-1}(\Psi\eta) \rangle_{L^2} = \langle \xi, \Psi\eta \rangle_{L^2},$$

implying that the symmetrization $\widehat{A}_G := \frac{1}{|G|}\sum_{g \in G} g\widehat{A}$ also satisfies

$$\langle \xi, \widehat{A}_G\eta \rangle_{L^2} = \langle \xi, \Psi\eta \rangle_{L^2}$$

for all ξ, η. But the G-invariance of $\widehat{A}_G$ implies $\widehat{A}_G = \widehat{\varphi}^* A$ for some $A \in \Gamma(\operatorname{Hom}_{\mathbb{R}}(\dot{E}, \dot{F}))$ with compact support in Δ, hence $A = A_Y$ for some $Y \in T_{J_{\mathrm{ref}}}\mathcal{J}(M, \omega ; \mathcal{U}, J_{\mathrm{fix}})$, and this proves the claim. We can now choose any finite collection of perturbations $Y_1, \dots, Y_N \in T_{J_{\mathrm{ref}}}\mathcal{J}(M, \omega ; \mathcal{U}, J_{\mathrm{fix}})$ such that the $\mathbf{L}(Y_i)$ span $\operatorname{Hom}_G\big(\ker \dot{\mathbf{D}}^N_{\widehat{u}}, \ker(\dot{\mathbf{D}}^N_{\widehat{u}})^*\big)$, and choose $\varepsilon \in \boldsymbol{\mathcal{E}}$ so that all of them are of class C_ε. $\square$

By the implicit function theorem, the open set of ε-regular elements in

$$\mathscr{U}^d(\mathcal{J}_\varepsilon ; \ell_1, \dots, \ell_m ; k, c) \subset \mathscr{U}^d(\mathcal{J}_\varepsilon ; \ell_1, \dots, \ell_m)$$

is a smooth Banach submanifold whose codimension near any given element (u, J) is given by the formula in (3.23), and thus matches $\operatorname{codim}(u)$ as specified by Definition 2.11. We can then apply the Sard-Smale theorem to the projection

$$\mathscr{U}^d(\mathcal{J}_\varepsilon\,;\,\ell_1, \dots, \ell_m\,;\,k, c) \to \mathcal{J}_\varepsilon : (u, J) \mapsto J$$

and thus find a Baire subset $\mathcal{J}_\varepsilon^{\mathrm{reg}} \subset \mathcal{J}_\varepsilon$ such that for all $J \in \mathcal{J}_\varepsilon^{\mathrm{reg}}$, all ε-regular elements of $\mathcal{M}^d(J\,;\,\ell_1, \dots, \ell_m\,;\,k, c)$ are regular in their stratum.

To turn this into a Baire subset of $\mathcal{J}(M, \omega\,;\,\mathcal{U}, J_{\mathrm{fix}})$ and drop the ε-regularity condition, we now apply another variation on the Taubes trick that was used in the proof of Theorem 5.26, i.e. we exhaust the moduli space of Petri regular curves by a countable collection of compact subsets

$$\mathcal{N}^K(J) \subset \mathcal{M}^d_\Pi(J\,;\,\ell_1, \dots, \ell_m\,;\,k, c), \qquad K \in \mathbb{N},$$

in order to define open and dense subsets of $\mathcal{J}(M, \omega\,;\,\mathcal{U}, J_{\mathrm{fix}})$ whose intersection has the desired properties. As in §2.2, let $h \geqslant 0$ denote the genus of d-fold branched covers of a genus g surface as determined by the branching data $\mathbf{b}$ and the Riemann-Hurwitz formula. We shall again write $\mathbf{b} = (\mathbf{b}_1, \dots, \mathbf{b}_r)$ for some $r \geqslant 0$, where each individual $\mathbf{b}_i$ is a tuple $(b_i^1, \dots, b_i^{q_i})$ of natural numbers satisfying $\sum_{j=1}^{q_i} b_i^j = d$. Now fix a closed model surface Σ_g of genus g along with an ordered set of distinct points $\Theta = (x_1, \dots, x_m)$ in Σ_g and a continuous function $F_g : \Sigma_g \to [0, \infty)$ that is positive on $\Sigma \backslash \Theta$ and, using local complex coordinates z to identify a neighborhood of each x_j with $\mathbb{D} \subset \mathbb{C}$ so that x_j becomes $0 \in \mathbb{D}$, satisfies

$$F_g(z) = |z|^{\ell_j} \text{ near } x_j, \qquad j = 1, \dots, m.$$

Similarly, fix a closed model surface Σ_h of genus h, an ordered set of distinct points

$$\Theta' = (\zeta_1^1, \dots, \zeta_1^{q_1}, \dots, \zeta_r^1, \dots, \zeta_r^{q_r})$$

in Σ_h, and a continuous function $F_h : \Sigma_h \to [0, \infty)$ that is positive on $\Sigma_h \backslash \Theta'$ and takes the form

$$F_h(z) = |z|^{b_i^j - 1} \text{ near } \zeta_i^j, \qquad j = 1, \dots, q_i,\ i = 1, \dots, r$$

in suitable local coordinates. We also make arbitrary choices of Riemannian metrics on Σ_g, Σ_h and M so as to define the various distance functions $\operatorname{dist}(\ ,\ )$ and norms referred to below. We then define $\mathcal{N}^K(J)$ to consist of every element in $\mathcal{M}^d(J\,;\,\ell_1, \dots, \ell_m\,;\,k, c)$ that admits a representative of the form $u = v \circ \varphi : (\Sigma_h, j') \to (M, J)$, with $v : (\Sigma_g, j) \to (M, J)$ simple and $\varphi : (\Sigma_h, j') \to (\Sigma, j)$ a d-fold holomorphic branched cover, such that v is critical of order ℓ_i at x_i for $i = 1, \dots, m$ and φ has branching order b_i^j at ζ_i^j for $j = 1, \dots, q_i$ and $i = 1, \dots, r$, and the following quantitative conditions are also satisfied:

(1) *Domains do not degenerate:* Using the compact sets of complex structures provided by Lemma 5.33, $j \in \mathcal{J}^K(\Sigma_g, \Theta)$ and $j' \in \mathcal{J}^K(\Sigma_h, \Theta')$.

(2) *Bubbles do not form:* $\sup_{z \in \Sigma_g} |dv(z)| \leqslant K$ and $\sup_{z \in \Sigma_h} |d\varphi(z)| \leqslant K$.

(3) *Injective points do not disappear:* There exists a point $\zeta \in \Sigma_g$ such that

$$|dv(\zeta)| \geqslant \frac{1}{K}, \qquad \inf_{z \in \Sigma_g \backslash \{\zeta\}} \frac{\operatorname{dist}(v(\zeta), v(z))}{\operatorname{dist}(\zeta, z)} \geqslant \frac{1}{K}, \qquad \text{and} \qquad \operatorname{dist}(v(\zeta), M \backslash \mathcal{U}) \geqslant \frac{1}{K}.$$

(4) *Critical orders do not increase:*

$$\inf_{z \in \Sigma_g \backslash \Theta} \frac{|dv(z)|}{F_g(z)} \geqslant \frac{1}{K} \qquad \text{and} \qquad \inf_{z \in \Sigma_h \backslash \Theta'} \frac{|d\varphi(z)|}{F_h(z)} \geqslant \frac{1}{K}.$$

(5) *Images of branch points do not collide:* There exist distinct points $w_i = \varphi(\zeta_i^1) = \dots = \varphi(\zeta_i^{q_i}) \in \Sigma_g$ for $i = 1, \dots, r$ such that

$$\operatorname{dist}(w_i, w_j) \geqslant \frac{1}{K} \qquad \text{for all} \qquad i, j = 1, \dots, r \text{ with } i \neq j.$$

(6) *Kernels do not get larger:* Writing $\dot{E} := N_{\widehat{u}}$ and $\dot{F} := \overline{\mathrm{Hom}}_{\mathbb{C}}(T\widehat{\Sigma}, N_{\widehat{u}})$ for the canonically defined regular cover $\widehat{u} : \widehat{\Sigma} \to M$ of v, the operator $\dot{\mathbf{D}}^N_{\widehat{u}} : W^{k,p,-\boldsymbol{\delta}}(\dot{E}) \to W^{k-1,p,-\boldsymbol{\delta}}(\dot{F})$ satisfies

$$\left\| \dot{\mathbf{D}}^N_{\widehat{u}} \eta \right\|_{W^{k-1,p,-\boldsymbol{\delta}}} \geqslant \frac{1}{K} \inf_{\xi \in \ker \dot{\mathbf{D}}^N_{\widehat{u}}} \|\eta - \xi\|_{W^{k,p,-\boldsymbol{\delta}}} \quad \text{for all} \quad \eta \in W^{k,p,-\boldsymbol{\delta}}(N_{\widehat{u}}).$$

(7) *Curves remain Petri regular:* For the regular cover $\widehat{u}$, the Petri map $\Pi : \ker \dot{\mathbf{D}}^N_{\widehat{u}} \otimes \ker(\dot{\mathbf{D}}^N_{\widehat{u}})^* \to \Gamma(\dot{E} \otimes \dot{F})$ satisfies the estimate

$$\|\Pi(t)\|_{C^0(\widehat{\Sigma}^K)} \geqslant \frac{1}{K} \|t\|,$$

where $\widehat{\Sigma}^K := \left\{ z \in \widehat{\Sigma} \;\middle|\; \mathrm{dist}(\widehat{u}(z), M \backslash \mathcal{U}) \geqslant 1/K \right\}$ and the norm on the tensor product $\ker \dot{\mathbf{D}}^N_{\widehat{u}} \otimes \ker(\dot{\mathbf{D}}^N_{\widehat{u}})^*$ is defined via any norms on $\ker \dot{\mathbf{D}}^N_{\widehat{u}}$ and $\ker(\dot{\mathbf{D}}^N_{\widehat{u}})^*$ that vary continuously with $u \in \mathcal{M}^d(J\,;\ell_1,\dots,\ell_m\,;k,c)$.

Clearly every element of $\mathcal{M}^d_{\Pi}(J\,;\ell_1,\dots,\ell_m\,;k,c)$ belongs to some $\mathcal{N}^K(J)$ for $K \in \mathbb{N}$ sufficiently large. Now define

$$\mathcal{J}^{\mathrm{reg},K} \subset \mathcal{J}(M,\omega\,;\mathcal{U},J_{\mathrm{fix}})$$

via the property that $J \in \mathcal{J}^{\mathrm{reg},K}$ if and only if every element of $\mathcal{N}^K(J)$ is regular in its stratum.

We claim that $\mathcal{J}^{\mathrm{reg},K}$ is open in $\mathcal{J}(M,\omega\,;\mathcal{U},J_{\mathrm{fix}})$. Indeed, suppose $J_\nu \in \mathcal{J}(M,\omega\,;\mathcal{U},J_{\mathrm{fix}})$ is a sequence converging to $J \in \mathcal{J}^{\mathrm{reg},K}$ as $\nu \to \infty$ such that for every ν, there exists a curve $u_\nu \in \mathcal{N}^K(J_\nu)$ that is not regular in its stratum. Given parametrizations $u_\nu = v_\nu \circ \varphi_\nu$ with $v_\nu : (\Sigma_g, j_\nu) \to (M, J_\nu)$ and $\varphi_\nu : (\Sigma_h, j'_\nu) \to (\Sigma_g, j_\nu)$ satisfying the conditions above, conditions 1 and 2 imply via standard elliptic regularity arguments that there are C^∞-convergent subsequences $v_\nu \to v$, $j_\nu \to j$, $\varphi_\nu \to \varphi$ and $j'_\nu \to j'$, so that u_ν itself converges to the composition of a J-holomorphic curve $v : (\Sigma_g, j) \to (M, J)$ and another d-fold holomorphic branched cover $\varphi : (\Sigma_h, j') \to (\Sigma, j)$. Since all conditions in the definition of $\mathcal{N}^K(J)$ are closed, they are also satisfied for the limit u. Condition 3 then guarantees that v has an injective point mapped into $\mathcal{U}$, conditions 4 and 5 ensure that both v and φ satisfy the given constraints on critical orders and branching data, and condition 6 implies via Lemma 6.8 below that $\dim\ker \dot{\mathbf{D}}^N_{\widehat{u}} = \dim\ker \dot{\mathbf{D}}^N_{\widehat{u}_\nu}$. It follows that $u \in \mathcal{M}^d(J\,;\ell_1,\dots,\ell_m\,;k,c)$, thus u also belongs to $\mathcal{N}^K(J)$ and must therefore be regular in its stratum. Regularity must then also hold for u_ν with ν sufficiently large, since it is an open condition, and this is a contradiction.

The use of condition 6 in the above argument depends on interpreting it in terms of the injective map induced by $\dot{\mathbf{D}}^N_{\widehat{u}}$ on the quotient of its domain by its kernel, and then feeding this into the following functional-analytic lemma:

Lemma 6.8. *Suppose X and Y are Banach spaces, $\mathbf{T}_n : X \to Y$ is a sequence of Fredholm operators converging to a Fredholm operator $\mathbf{T} : X \to Y$, and there exists a constant $c > 0$ such that*

$$\|\mathbf{T}_n x\|_Y \geqslant c \|\pi_n x\|_{X/\ker \mathbf{T}_n},$$

where $\pi_n : X \to X/\ker \mathbf{T}_n$ is the quotient projection. Then $\dim\ker \mathbf{T}_n = \dim\ker \mathbf{T}$ for all n sufficiently large.

Proof. One can use the same trick as in the proof of Lemma 3.25 to find a sequence of Banach space isomorphisms $\boldsymbol{\Phi}_n : X \to X$ converging to $\mathbb{1}$ such that $\ker \mathbf{T}_n \subset \boldsymbol{\Phi}_n(\ker \mathbf{T})$ for every n sufficiently large. Then if $\dim\ker \mathbf{T}_n < \dim\ker \mathbf{T}$ for all n, we can find a bounded sequence $x_n \in \boldsymbol{\Phi}_n(\ker \mathbf{T})$ such that the norm of $\pi_n(x_n)$ in $X/\ker \mathbf{T}_n$ is bounded away from zero. Equivalently, $x_n = \boldsymbol{\Phi}_n(v_n)$ for a bounded sequence $v_n \in \ker \mathbf{T}$, which then has a subsequence convergent to some $v_\infty \in \ker \mathbf{T}$ since $\dim\ker \mathbf{T} < \infty$, implying a corresponding subsequence $x_n \to x_\infty$ and thus $\mathbf{T}_n x_n \to 0$. The latter contradicts the estimate in the hypothesis. $\square$

We claim that $\mathcal{J}^{\mathrm{reg},K}$ is also dense in $\mathcal{J}(M,\omega\,;\mathcal{U},J_{\mathrm{fix}})$. Since the reference structure $J_{\mathrm{ref}} \in \mathcal{J}(M,\omega\,;\mathcal{U},J_{\mathrm{fix}})$ can be chosen arbitrarily, it suffices to find some $\varepsilon \in \boldsymbol{\mathcal{E}}$ and a sequence $J_\nu \in \mathcal{J}^{\mathrm{reg},K}$ such that $J_\nu \to J_{\mathrm{ref}}$ in the C_ε-topology. The argument used above for openness shows that $\mathcal{N}^K(J_{\mathrm{ref}})$ is compact, and condition 7 implies that every curve in $\mathcal{N}^K(J_{\mathrm{ref}})$ is Petri regular, so by Lemma 6.7, one can choose a lower bound for a finite set of choices $\varepsilon \in \boldsymbol{\mathcal{E}}$ and thus assume that every curve in $\mathcal{N}^K(J_{\mathrm{ref}})$ is ε-regular. Now pick a sequence $J_\nu \in \mathcal{J}^{\mathrm{reg}}_\varepsilon$ with $J_\nu \to J_{\mathrm{ref}}$, and arguing by contradiction, suppose $J_\nu \notin \mathcal{J}^{\mathrm{reg},K}$, meaning there exists a sequence $u_\nu \in \mathcal{N}^K(J_\nu)$ such that each u_ν is not regular in its stratum. After passing to a subsequence, the previous compactness argument shows that u_ν converges to some $u \in \mathcal{N}^K(J_{\mathrm{ref}})$, implying that u_ν is ε-regular for all ν sufficiently large. That contradicts the definition of $\mathcal{J}^{\mathrm{reg}}_\varepsilon$ and thus proves the claim.

To conclude, $\bigcap_{K\in\mathbb{N}} \mathcal{J}^{\mathrm{reg},K}$ is now a Baire subset of $\mathcal{J}(M,\omega\,;\mathcal{U},J_{\mathrm{fix}})$ containing almost complex structures J such that every Petri regular curve in $\mathcal{M}^d(J\,;\ell_1,\dots,\ell_m\,;k,c)$ is regular in its stratum. By Theorem 5.26, we can intersect this with another Baire subset in order to assume that every curve in $\mathcal{M}^d(J\,;\ell_1,\dots,\ell_m\,;k,c)$ is Petri regular. The resulting Baire subset depends on the choices of data d, $\mathbf{b}$, G, g, m, A, $\ell_1,\dots,\ell_m$, k, but since there are only countably many such choices, a further countable intersection of Baire subsets now produces a Baire subset of almost complex structures for which the result of Theorem D holds. The proof of Theorem D is thus complete.

7. Super-rigidity in dimension four

We now prove the 4-dimensional case of Theorem A, using intersection-theoretic arguments that are essentially unrelated to the rest of the paper. Throughout this section, assume (M,J) is an almost complex manifold with

$$\dim M = 4.$$

The genus zero case is an "automatic" phenomenon, i.e. it does not require any genericity condition except for ensuring that the index 0 simple curve is immersed:

Proposition 7.1. *Every simple immersed J-holomorphic sphere $v : (S^2,i) \to (M,J)$ of index 0 in an almost complex 4-manifold is super-rigid.*

Proof. Assume $\varphi : (\Sigma',j') \to (S^2,i)$ is a d-fold branched cover and $u = v\circ\varphi$. Since v is immersed, the Riemann-Roch formula implies

$$0 = \operatorname{ind}(v) = \operatorname{ind}\mathbf{D}^N_v = \chi(S^2) + 2c_1(N_v),$$

hence $c_1(N_v) = -1$. Then $c_1(N_u) = c_1(\varphi^* N_v) = -d$, so if $\eta \in \ker \mathbf{D}^N_u$ is nontrivial, its algebraic count of zeroes is negative, violating the similarity principle. □

For the genus one case, we use a variant of the "magic trick" proposed by Hutchings [Hut] in the context of Embedded Contact Homology.

Proposition 7.2. *A simple immersed J-holomorphic torus $v : (\mathbb{T}^2,j) \to (M,J)$ of index 0 in an almost complex 4-manifold is super-rigid if and only if all its unbranched covers are Fredholm regular.*

Proof. We will assume for most of the proof that $v : (\Sigma,j) \to (M,J)$ has unspecified genus $g \geqslant 1$. Since v is immersed with index 0, it is regular if and only if its normal Cauchy-Riemann operator $\mathbf{D}^N_v$ is injective, so given this and the assumption that the same holds for all unbranched covers $u = v\circ\varphi$, we need to show that $\mathbf{D}^N_u$ is also injective for $u = v\circ\varphi$ where $\varphi : (\Sigma',j') \to (\Sigma,j)$ is *any* holomorphic branched cover. We will prove this by induction on the degree $d := \deg(\varphi)$, thus assume it is true for all covers up to degree $d-1$. Note that since $\operatorname{ind}(v) = 0$, we have

$$\operatorname{ind}\mathbf{D}^N_v = \chi(\Sigma) + 2c_1(N_v) = 0, \tag{7.1}$$

and if φ has branch points, then Σ' has genus $g' > 1$ by the Riemann-Hurwitz formula.

By the construction in the proof of Proposition B.1, one can endow the total space of the normal bundle $\pi : N_v \to \Sigma$ with an almost complex structure J_N such that J_N-holomorphic curves $u_\eta : (S, i) \to (N_v, J_N)$ correspond to sections $\eta \in \ker \mathbf{D}^N_{v \circ \psi}$ along holomorphic branched covers $\psi = \pi \circ u_\eta : (S, i) \to (\Sigma, j)$. If $\ker \mathbf{D}^N_u$ contains a nontrivial element η, the inductive hypothesis implies that the corresponding J_N-holomorphic curve u_η is somewhere injective. We can view v itself as a J_N-holomorphic embedding into N_v, and u_η is homologous to its d-fold cover, so applying the adjunction formula to both u_η and v as J_N-holomorphic curves in N_v,

$$\begin{aligned} u_\eta \bullet u_\eta &= 2\delta(u_\eta) + c_1(u_\eta^* TN_v) - \chi(\Sigma') = 2\delta(u_\eta) + d \cdot c_1(v^* TN_v) - \chi(\Sigma') \\ &= d^2(v \bullet v) = d^2 \cdot c_1(N_v) = d^2 \cdot c_1(v^* TN_v) - d^2 \cdot \chi(\Sigma), \end{aligned}$$

where $\delta(u_\eta) \geqslant 0$ denotes the algebraic count of double points and critical points of u_η. Solving for $\delta(u_\eta)$ and plugging in (7.1) to compute $c_1(v^* TN_v) = \chi(\Sigma) + c_1(N_v) = \frac{1}{2}\chi(\Sigma) = 1 - g$, we have

$$\begin{aligned} 2\delta(u_\eta) &= d(d-1) \cdot c_1(v^* TN_v) - d^2 \cdot \chi(\Sigma) + \chi(\Sigma') \\ &= d(d-1)(1-g) - 2d^2(1-g) + 2 - 2g' = d(d+1)(g-1) - 2(g'-1) \end{aligned}$$

Plugging in $g = 1$ and the fact that $g' > 1$, this gives a contradiction since $\delta(u_\eta)$ cannot be negative. □

Remark 7.3. In the spirit of §2.4, the two results above show that the story of super-rigidity and bifurcations is simpler in dimension four. In the genus zero case bifurcations can be avoided altogether: since having a critical point is a codimension 2 condition (see Appendix A), index 0 simple curves for generic 1-parameter families of almost complex structures can be assumed immersed, and therefore super-rigid by Prop. 7.1. This is no longer true in the genus one case since regularity of some unbranched cover might fail under a generic homotopy, producing the birth-death or degree-doubling bifurcations in [Tau96a], but Prop. 7.2 implies that this is the only danger—the only bifurcations that can happen involve unbranched covers with $g' = 1$ and $d \in \{1, 2\}$, and they are already described in [Tau96a].

Appendix A. Moduli spaces with prescribed orders of critical points

The proposition below is well known to experts, but a proof of it is difficult to find in the literature, so we will sketch one here.

Fix a symplectic manifold (M, ω) of dimension $2n$, $n \in \mathbb{N}$, and suppose $J \in \mathcal{J}(M, \omega)$. Recall that if (Σ, j) is a connected Riemann surface and $u : (\Sigma, j) \to (M, J)$ is a nonconstant J-holomorphic curve with a critical point $du(z) = 0$, then the critical point is isolated and has a well-defined positive **order**,

$$\mathrm{ord}(du; z) \in \mathbb{N},$$

characterized by the property that $\mathrm{ord}(du; z) = \ell$ if z is a zero of order ℓ for the section $du \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(T\Sigma, u^*TM))$, where the latter is viewed as a holomorphic section with respect to a natural holomorphic bundle structure on u^*TM determined by the linearized Cauchy-Riemann operator, see e.g. [Wen10, §3.3]. When (Σ, j) is closed, we denote the resulting algebraic count of critical points by

$$Z(du) := \sum_{\{z \in \Sigma \mid du(z) = 0\}} \mathrm{ord}(du; z) \geqslant 0,$$

and note that it vanishes if and only if u is immersed. Given integers $g, m \geqslant 0$, a homology class $A \in H_2(M)$ and a tuple of positive integers $(\ell_1, \ldots, \ell_m)$, let

$$\mathcal{M}_{g,m}(A, J\,; \ell_1, \ldots, \ell_m) \subset \mathcal{M}_{g,m}(A, J)$$

denote the following subset of the moduli space of unparametrized J-holomorphic curves homologous to A with genus g and m marked points: a map $u : (\Sigma, j) \to (M, J)$ with marked points

$\zeta_1, \ldots, \zeta_m \in \Sigma$ representing an element of $\mathcal{M}_{g,m}(A, J)$ belongs to $\mathcal{M}_{g,m}(A, J\,;\,\ell_1, \ldots, \ell_m)$ if and only if it is critical at all marked points,

$$\operatorname{ord}(du; \zeta_j) = \ell_j \quad \text{for} \quad j = 1, \ldots, m,$$

and it is immersed everywhere else.

Proposition A.1. *Fix an open subset $\mathcal{U} \subset M$ with compact closure and a compatible almost complex structure $J_{\text{fix}} \in \mathcal{J}(M, \omega)$. There exists a Baire subset*

$$\mathcal{J}^{\text{reg}} \subset \mathcal{J}(M, \omega\,;\,\mathcal{U}, J_{\text{fix}})$$

such that for all $J \in \mathcal{J}^{\text{reg}}$ and all $g, m \geqslant 0$, $A \in H_2(M)$ and $(\ell_1, \ldots, \ell_m) \in \mathbb{N}^m$, the open subset of $\mathcal{M}_{g,m}(A, J\,;\,\ell_1, \ldots, \ell_m)$ consisting of somewhere injective curves that pass through $\mathcal{U}$ is a smooth manifold with dimension equal to its virtual dimension, where

$$\operatorname{vir-dim} \mathcal{M}_{g,m}(A, J\,;\,\ell_1, \ldots, \ell_m) = \operatorname{vir-dim} \mathcal{M}_g(A, J) - \sum_{i=1}^{m} (2n\ell_i - 2).$$

Corollary A.2. *For generic compatible J in any closed symplectic $2n$-manifold, all closed, connected and somewhere injective J-holomorphic curves u with $m \geqslant 0$ critical points satisfy $\operatorname{ind}(u) \geqslant 2nZ(du) - 2m$.*

One well-known consequence of this result is that for generic J, somewhere injective index 0 curves in almost complex manifolds of dimension at least four are always immersed. Another proof of this is given in [OZ09], though it is analytically somewhat more complicated than the one given below.

It will suffice to prove that the same statement as in Prop. A.1 holds for the slightly larger moduli space

$$\widehat{\mathcal{M}}_{g,m}(A, J\,;\,\ell_1, \ldots, \ell_m)$$

characterized by the condition $\operatorname{ord}(du; \zeta_j) \geqslant \ell_j$ for all $j = 1, \ldots, m$ without requiring u to be immersed outside the marked points. Indeed, $\mathcal{M}_{g,m}(A, J\,;\,\ell_1, \ldots, \ell_m) \subset \widehat{\mathcal{M}}_{g,m}(A, J\,;\,\ell_1, \ldots, \ell_m)$ is an open subset. We shall borrow from Zehmisch [Zeh15] the notion of *holomorphic jets*: given a point p in an almost complex manifold (M, J) and an integer $r > 0$, a **holomorphic r-jet** at p is an equivalence class of J-holomorphic curves

$$u : (\mathbb{D}_\epsilon, i) \to (M, J)$$

with $u(0) = p$, where $(\mathbb{D}_\epsilon, i)$ denotes the ϵ-disk in $\mathbb{C}$, and two curves are considered equivalent if their partial derivatives at 0 match up to order r. The nonlinear Cauchy-Riemann equation implies that the holomorphic r-jet represented by u is determined by the holomorphic part of its Taylor polynomial of degree r (see [Wena, Prop. 2.99]), and moreover, every holomorphic Taylor polynomial of degree r is realizable as the r-jet of a local J-holomorphic curve ([Wena, Theorem 2.100]). Thus the space of all holomorphic r-jets at p is a real $2rn$-dimensional vector space, and the union of these spaces for all $p \in M$ forms a smooth manifold

$$\operatorname{Jet}^r_J(M)$$

of real dimension $2n(r + 1)$.

We shall analyze the local structure of $\widehat{\mathcal{M}}_{g,m}(A, J\,;\,\ell_1, \ldots, \ell_m)$ following a minor modification of the scheme outlined in [Wena, Chapter 4]. For simplicity, we shall assume in this exposition that $2g + m \geqslant 3$, so that we only need to deal with *stable* marked Riemann surfaces. (For the finitely many non-stable cases, see Remark A.5.) Given $(\Sigma, j_0, \Theta, u_0)$ representing an element of $\widehat{\mathcal{M}}_{g,m}(A, J\,;\,\ell_1, \ldots, \ell_m)$, with marked points $\Theta := (\zeta_1, \ldots, \zeta_m)$, choose a **Teichmüller slice** through j_0: this means a smooth $(6g - 6 + 2m)$-dimensional family $\mathcal{T}$ of complex structures on Σ that includes j_0 and parametrizes a neighborhood of $[j_0]$ in the Teichmüller space of complex structures modulo diffeomorphisms that are homotopic to the identity and fix Θ. The tangent space $T_{j_0}\mathcal{T}$ is also required to define a closed complement of the image of the canonical Cauchy-Riemann operator on $T\Sigma$ restricted to the space of vector fields vanishing at Θ, cf. [Wena, Definition 4.29]. Moreover, we can arrange for $\mathcal{T}$ to have the following two properties (cf. [Wen10, Lemmas 3.3 and 3.4]):

- $\mathcal{T}$ is invariant under the action of the group $\mathrm{Aut}(\Sigma, j_0, \Theta)$ of biholomorphic maps on (Σ, j_0) fixing Θ;
- There exists a neighborhood of Θ on which every $j \in \mathcal{T}$ matches j_0.

Now let $r := \max\{\ell_1, \dots, \ell_m\}$, and choose any $k \in \mathbb{N}$ and $p \in (1, \infty)$ such that

$$(k - r)p > 2, \tag{A.1}$$

so the Sobolev embedding theorem implies that functions of class $W^{k,p}$ on Σ are also in C^r. We define the Banach manifold

$$\mathcal{B} := W^{k,p}(\Sigma, M)$$

and smooth Banach space bundle $\mathcal{E} \to \mathcal{T} \times \mathcal{B}$ with fibers

$$\mathcal{E}_{(j,u)} := W^{k-1,p}\big(\overline{\mathrm{Hom}}_{\mathbb{C}}((T\Sigma, j), (u^*TM, J))\big),$$

so that

$$\bar{\partial}_J : \mathcal{T} \times \mathcal{B} \to \mathcal{E} : (j, u) \mapsto Tu + J \circ Tu \circ j$$

defines a smooth section. We say that $(\Sigma, j_0, \Theta, u_0)$ is **Fredholm regular** if the linearization

$$D\bar{\partial}_J(j_0, u_0) : T_{j_0}\mathcal{T} \oplus W^{k,p}(u_0^*TM) \to W^{k-1,p}\big(\overline{\mathrm{Hom}}_{\mathbb{C}}((T\Sigma, j_0), (u_0^*TM, J))\big)$$

of this section at (j_0, u_0) is surjective, in which case a neighborhood of (j_0, u_0) in $\bar{\partial}_J^{-1}(0)$ is a smooth finite-dimensional manifold, and its quotient by the natural action of $\mathrm{Aut}(\Sigma, j_0, \Theta)$ can be identified naturally with a neighborhood of $[(\Sigma, j_0, \Theta, u_0)]$ in $\mathcal{M}_{g,m}(A, J)$. To incorporate the critical point condition, fix holomorphic coordinates identifying a neighborhood of each marked point ζ_j with the standard unit disk $(\mathbb{D}, i)$; note that this can be done for all $j \in \mathcal{T}$ at once since they are assumed to match j_0 near Θ. Then since $\mathcal{B}$ has a continuous inclusion into $C^r(\Sigma, M)$, there is a well-defined and smooth[11] *jet evaluation map*

$$\mathrm{ev} : \bar{\partial}_J^{-1}(0) \to \mathrm{Jet}_J^{\ell_1}(M) \times \dots \times \mathrm{Jet}_J^{\ell_m}(M),$$

whose ith factor for $i = 1, \dots, m$ is the holomorphic ℓ_i-jet represented by u in its parametrization by $(\mathbb{D}, i)$ at ζ_i. We will say that $(\Sigma, j_0, \Theta, u_0)$ is **regular for the constrained moduli space** $\widehat{\mathcal{M}}_{g,m}(A, J\,;\, \ell_1, \dots, \ell_m)$ if it is Fredholm regular and the jet evaluation map is transverse to the submanifold

$$Z \subset \mathrm{Jet}_J^{\ell_1}(M) \times \dots \times \mathrm{Jet}_J^{\ell_m}(M)$$

consisting of m-tuples of jets of constant maps. Note that this condition does not depend on the chosen holomorphic coordinates near the marked points, as it is equivalent to the condition that u should have vanishing derivatives up to order ℓ_i at ζ_i for each $i = 1, \dots, m$. Whenever the regularity condition is satisfied, $\mathrm{ev}^{-1}(Z) \subset \bar{\partial}_J^{-1}(0)$ inherits the structure of a smooth submanifold with real codimension $2n \sum_i \ell_i$, so $\widehat{\mathcal{M}}_{g,m}(A, J\,;\, \ell_1, \dots, \ell_m)$ in general becomes an orbifold near $[(\Sigma, j_0, \Theta, u_0)]$, with

$$\begin{aligned}
\dim \widehat{\mathcal{M}}_{g,m}(A, J\,;\, \ell_1, \dots, \ell_m) &= \dim \mathcal{M}_{g,m}(A, J) - 2n \sum_i \ell_i \\
&= \dim \mathcal{M}_g(A, J) + 2m - 2n \sum_i \ell_i \\
&= \dim \mathcal{M}_g(A, J) - \sum_{i=1}^m (2n\ell_i - 2).
\end{aligned}$$

[11]The smoothness of ev is clear because it is the restriction to $\bar{\partial}_J^{-1}(0)$ of a map $\mathcal{B} \to \mathrm{Jet}_J^{\ell_1}(M) \times \dots \times \mathrm{Jet}_J^{\ell_m}(M)$ which in the natural Banach manifold charts provided by [Elĭ67] looks like a linear map evaluating derivatives of functions at the fixed points $\Theta \subset \Sigma$. This works because we are choosing to represent elements of $\mathcal{M}_{g,m}(A, J)$ by maps with marked points at fixed positions; of course there is no actual constraint on the movement of the marked points, but this freedom is seen in our setup by varying j in $\mathcal{T}$ instead of varying the points $\zeta_1, \dots, \zeta_m$. This is a notable difference from the setup in [OZ09].

To prove that the constrained regularity condition can be achieved generically, fix $J_{\mathrm{ref}} \in \mathcal{J}(M,\omega\,;\mathcal{U},J_{\mathrm{fix}})$ and a suitable sequence of positive numbers $\varepsilon_\nu \to 0$, and consider a Banach manifold $\mathcal{J}_\varepsilon$ of almost complex structures in $\mathcal{J}(M,\omega\,;\mathcal{U},J_{\mathrm{fix}})$ that are C_ε-close to J_{ref} (cf. §5.4). This gives rise to two universal moduli spaces,

$$\mathscr{U}^*(\mathcal{J}_\varepsilon) := \left\{(u,J) \;\middle|\; J \in \mathcal{J}_\varepsilon \text{ and } u \in \mathcal{M}^*_{g,m}(A,J)\right\}$$

and

$$\widehat{\mathscr{U}}^*(\mathcal{J}_\varepsilon\,;\ell_1,\dots,\ell_m) := \left\{(u,J) \;\middle|\; J \in \mathcal{J}_\varepsilon \text{ and } u \in \widehat{\mathcal{M}}^*_{g,m}(A,J\,;\ell_1,\dots,\ell_m)\right\},$$

where we abbreviate by

$$\mathcal{M}^*_{g,m}(A,J) \subset \mathcal{M}_{g,m}(A,J),$$
$$\widehat{\mathcal{M}}^*_{g,m}(A,J\,;\ell_1,\dots,\ell_m) \subset \widehat{\mathcal{M}}_{g,m}(A,J\,;\ell_1,\dots,\ell_m)$$

the subspaces defined via the condition that u be somewhere injective and pass through $\mathcal{U}$. As is well known, $\mathscr{U}^*(\mathcal{J}_\varepsilon)$ is a separable and metrizable smooth Banach manifold if ε_ν converges to 0 fast enough, and for $[(\Sigma,j_0,\Theta,u_0)] \in \mathcal{M}^*_{g,m}(A,J_0)$, a neighborhood of (u_0,J_0) in $\mathscr{U}^*(\mathcal{J}_\varepsilon)$ can be identified with the zero-set of a smooth section

$$\bar\partial : \mathcal{T} \times \mathcal{B} \times \mathcal{J}_\varepsilon \to \mathcal{E} : (j,u,J) \mapsto \bar\partial_J(u),$$

where $\mathcal{E}$ now denotes the Banach space bundle with fibers

$$\mathcal{E}_{(j,u,J)} = W^{k-1,p}\big(\overline{\mathrm{Hom}}_{\mathbb{C}}((T\Sigma,j),(u^*TM,J))\big).$$

The tangent space $T_{(u_0,J_0)}\mathscr{U}^*(\mathcal{J}_\varepsilon)$ is the kernel of the surjective operator

$$\begin{aligned}\mathbf{L} := D\bar\partial(j_0,u_0,J_0) : T_{j_0}\mathcal{T} \oplus W^{k,p}(u_0^*TM) \oplus T_{J_0}\mathcal{J}_\varepsilon &\to W^{k-1,p}(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma,u_0^*TM))\\ (y,\eta,Y) &\mapsto J_0 \circ Tu_0 \circ y + \mathbf{D}_{u_0}\eta + Y \circ Tu_0 \circ j_0,\end{aligned}$$

where $\mathbf{D}_{u_0}$ is the linearized Cauchy-Riemann operator associated to $u_0 : (\Sigma,j_0) \to (M,J_0)$. We can again define the smooth jet evaluation map

$$\mathrm{ev} : \bar\partial^{-1}(0) \to \mathrm{Jet}^{\ell_1}_J(M) \times \ldots \times \mathrm{Jet}^{\ell_m}_J(M) \tag{A.2}$$

and identify a neighborhood of (u_0,J_0) in $\widehat{\mathscr{U}}^*(\mathcal{J}_\varepsilon\,;\ell_1,\dots,\ell_m)$ with $\mathrm{ev}^{-1}(Z)$. The main technical ingredient behind Proposition A.1 is now the following.

Lemma A.3. *If $\varepsilon_\nu \to 0$ fast enough, then the jet evaluation map* (A.2) *is a submersion.*

Proof. We need to show that for any $X \in T_{\mathrm{ev}(u_0)}(\mathrm{Jet}^{\ell_1}_J(M) \times \ldots \times \mathrm{Jet}^{\ell_m}_J(M))$, there exists an element $(y,\eta,Y) \in \ker \mathbf{L}$ with

$$d\,\mathrm{ev}(u_0)\eta = X.$$

Let us first observe that this problem can be solved locally near the marked points: in fact, there exists a smooth section $\eta \in \Gamma(u_0^*TM)$ with

$$\mathbf{D}_{u_0}\eta = 0 \text{ near } \Theta \quad\text{ and }\quad d\,\mathrm{ev}(u_0)\eta = X.$$

This follows from the local existence theorem for J-holomorphic curves with prescribed holomorphic derivatives at a point, cf. [Wena, Theorem 2.100]. More precisely, choose a smooth path $\gamma = (\gamma_1,\dots,\gamma_m) : (-\delta,\delta) \to \mathrm{Jet}^{\ell_1}_J(M) \times \ldots \times \mathrm{Jet}^{\ell_m}_J(M)$ with $\gamma(0) = \mathrm{ev}(u_0)$ and $\dot\gamma(0) = X$. Then the local existence theorem provides for each $i = 1,\dots,m$ a smooth family of J-holomorphic curves $u^{(i)}_\tau : \mathbb{D}_\epsilon \to M$ defined on sufficiently small disks $\mathbb{D}_\epsilon \subset \mathbb{C}$ such that the holomorphic ℓ_i-jet represented by $u^{(i)}_\tau$ is $\gamma_i(\tau)$ for each τ. The desired section $\eta \in \Gamma(u_0^*TM)$ can now be constructed by writing it in our chosen holomorphic coordinates near each marked point ζ_i as $\partial_\tau u^{(i)}_\tau|_{\tau=0}$ and then extending it arbitrarily outside these neighborhoods.

Given η as above, we aim now to find a pair $(\xi,Y) \in W^{k,p}(u_0^*TM) \oplus T_{J_0}\mathcal{J}_\varepsilon$ such that

$$\mathbf{L}(0,\eta+\xi,Y) = \mathbf{L}(0,\xi,Y) + \mathbf{D}_{u_0}\eta = 0 \quad\text{ and }\quad d\,\mathrm{ev}(u_0)\xi = 0,$$

in which case $(0, \eta + \xi, Y) \in T_{(u_0,J_0)}\mathscr{U}^*(\mathcal{J}_\varepsilon)$ and $d\,\mathrm{ev}(u_0, J_0)(0, \eta + \xi, Y) = X$. We will use the weighted Sobolev spaces described in §3.2. Let $\dot{\Sigma} := \Sigma \backslash \Theta$, and assume without loss of generality that $u_0^{-1}(\mathcal{U}) \subset \Sigma$ is disjoint from Θ; this can be achieved at the cost of shrinking $\mathcal{U}$ and therefore the space of perturbations $\mathcal{J}_\varepsilon$. As a consequence, $Y \circ Tu_0 \circ j_0$ now has compact support in $\dot{\Sigma}$ for any $Y \in T_{J_0}\mathcal{J}_\varepsilon$. Using the fixed holomorphic coordinates on neighborhoods of marked points $\zeta_i \in \Theta$, we can identify them biholomorphically with half-cylinders $[0, \infty) \times S^1$ and fix trivializations of u_0^*TW on these neighborhoods to define weighted Sobolev norms and a bounded linear map

$$\dot{\mathbf{D}}_{u_0} : W^{k,p,\delta}(u_0^*TM|_{\dot{\Sigma}}) \to W^{k-1,p,\delta}\big(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, u_0^*TM)|_{\dot{\Sigma}}\big),$$

where sections η of class $W^{k,p,\delta}$ are required to satisfy $e^{\delta s}\eta \in W^{k,p}([0, \infty) \times S^1)$ when expressed in the chosen trivialization and holomorphic coordinates $(s, t) \in [0, \infty) \times S^1$ on each cylindrical end near Θ. As explained in §3.2, $\dot{\mathbf{D}}_{u_0}$ is asymptotic to the trivial asymptotic operator at each puncture and is thus Fredholm for any $\delta \in \mathbb{R} \backslash 2\pi\mathbb{Z}$. We claim that whenever this condition is satisfied, the linear map

$$\begin{aligned} \mathbf{L}_\delta : W^{k,p,\delta}(u_0^*TM|_{\dot{\Sigma}}) \oplus T_{J_0}\mathcal{J}_\varepsilon &\to W^{k-1,p,\delta}\big(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, u_0^*TM)|_{\dot{\Sigma}}\big) \\ (\xi, Y) &\mapsto \dot{\mathbf{D}}_{u_0}\xi + Y \circ Tu_0 \circ j_0 \end{aligned}$$

is surjective. The proof is more or less standard: we start with the case $k = 1$ and note that since $\dot{\mathbf{D}}_{u_0}$ is Fredholm, $\mathbf{L}_\delta$ has closed range, so it is not surjective if and only if there exists a nontrivial section $\lambda \in (L^{p,\delta})^* = L^{q,-\delta}$ for $1/p + 1/q = 1$ which is L^2-orthogonal to the images of both $\eta \mapsto \dot{\mathbf{D}}_{u_0}\eta$ and $Y \mapsto Y \circ Tu_0 \circ j_0$. Since u_0 has an injective point $z_0 \in \dot{\Sigma}$ with $u(z_0) \in \mathcal{U}$, the latter implies that λ vanishes near z_0; this depends on ε_ν converging to 0 fast enough for $T_{J_0}\mathcal{J}_\varepsilon$ to contain an abundance of bump functions with arbitrarily small support. The former implies in turn that λ is a weak solution to the formal adjoint equation $\dot{\mathbf{D}}_{u_0}^*\lambda = 0$ and is therefore smooth with isolated zeroes, giving a contradiction. The case of general $k \in \mathbb{N}$ follows from this via elliptic regularity, namely Lemma 3.11.

With this claim in place, we observe that $-\mathbf{D}_{u_0}\eta$ vanishes near Θ and thus restricts to $\dot{\Sigma}$ as a section of class $W^{k-1,p,\delta}$ for any $\delta > 0$, thus we can find $\xi \in W^{k,p,\delta}(u_0^*TM|_{\dot{\Sigma}})$ and $Y \in T_{J_0}\mathcal{J}_\varepsilon$ such that

$$\mathbf{L}(0, \xi, Y) = -\mathbf{D}_{u_0}\eta \quad \text{on} \quad \dot{\Sigma}.$$

Since Y has compact support in $\dot{\Sigma}$ and $\mathbf{D}_{u_0}\eta = 0$ near Θ, this equation implies $\mathbf{D}_{u_0}\xi = 0$ near Θ. The continuous inclusion $W^{k,p,\delta} \hookrightarrow C^0$ implies that ξ also has a continuous extension over Σ that vanishes on Θ; moreover, since (A.1) implies a continuous inclusion $W^{k,p} \hookrightarrow C^1$, ξ has a bounded first derivative on the cylindrical ends, implying via a short computation that for $1 < q < 2$, the L^q-norm of its derivative on punctured disk-like neighborhoods of Θ is finite. It follows that the extension of ξ over the punctures is in $W^{1,q}$ on Σ, and elliptic regularity then implies that it is smooth everywhere. Finally, the exponential weight condition implies that in each holomorphic coordinate system identifying the neighborhood of a marked point $\zeta_i \in \Theta$ with $\mathbb{D}$ such that ζ_i is at the origin, we have

$$|\xi(z)| \leqslant c|z|^{\delta/2\pi}$$

for some constant $c > 0$. But the choice of $\delta > 0$ in this discussion was arbitrary, so choosing it large enough, we can arrange for ξ to have vanishing derivatives of arbitrarily large finite order at Θ, proving $d\,\mathrm{ev}(u_0)\xi = 0$. $\square$

The lemma implies that $\widehat{\mathscr{U}}^*(\mathcal{J}_\varepsilon\,;\, \ell_1, \ldots, \ell_m)$ is a separable and metrizable smooth Banach manifold, so we can now apply the Sard-Smale theorem to the projection

$$\widehat{\mathscr{U}}^*(\mathcal{J}_\varepsilon\,;\, \ell_1, \ldots, \ell_m) \to \mathcal{J}_\varepsilon : (u, J) \mapsto J,$$

giving a Baire subset of $\mathcal{J}_\varepsilon$ for which $\widehat{\mathcal{M}}^*_{g,m}(A, J\,;\, \ell_1, \ldots, \ell_m)$ is a manifold of the correct dimension, and the countable intersection of these subsets for all g, m, A and $(\ell_1, \ldots, \ell_m)$ is again comeager in $\mathcal{J}_\varepsilon$, proving that there is a C^∞-dense subset of $\mathcal{J}(M,\omega\,;\mathcal{U},J_{\text{fix}})$ for which the statement of the theorem holds. To turn this into a Baire subset of $\mathcal{J}(M,\omega\,;\mathcal{U},J_{\text{fix}})$, one can use the standard Taubes trick (see e.g. [Wena, §4.4.2]): present $\widehat{\mathcal{M}}^*_{g,m}(A,J\,;\ell_1,\dots,\ell_m)$ as a countable union of compact subsets, and associate to each one a set of regular almost complex structures, which is open by construction and dense due to the argument above, so its intersection is comeager.

Remark A.4. Lemma A.3 implies that for generic J, the jet evaluation map can be made transverse to any given submanifold, hence this method can be used to understand any moduli space of holomorphic curves with marked points satisfying conditions on their derivatives, e.g. the incidence/tangency conditions studied by Cieliebak-Mohnke [CM07, CM18] or McDuff-Siegel [MS].

Remark A.5. The assumption $2g + m \geqslant 3$ misses only four special cases, and for these the discussion above is modified as follows:

(1) The automorphism group $\mathrm{Aut}(\Sigma, j_0, \Theta)$ is not finite, but is instead a nontrivial Lie group;
(2) The usual formula $\dim \mathcal{T} = 6g-6+2m$ for the dimension of Teichmüller space is wrong.

In fact, these two differences cancel each other out in the sense that

$$\dim \mathcal{T} - \dim \mathrm{Aut}(\Sigma, j_0, \Theta) = 6g - 6 + 2m,$$

which is why the stated formulas for the virtual dimensions of the moduli spaces $\mathcal{M}_{g,m}(A,J)$ and $\mathcal{M}_{g,m}(A,J\,;\ell_1,\dots,\ell_m)$ remain correct in these non-stable cases. In the cases with genus zero, Teichmüller space is trivial and there is thus no need to include a Teichmüller slice in the argument; the only difference is then the fact that dividing $\bar\partial_J^{-1}(0)$ by $\mathrm{Aut}(\Sigma, j_0, \Theta)$ changes its dimension. There is no need to discuss the non-stable genus one case here since that case also has $m = 0$, and thus does not involve critical point constraints.

Appendix B. Super-rigid curves are isolated

In this appendix we prove the following precise version of the statement that the multiple covers of a super-rigid curve form an open and closed subset of the ambient moduli space.

Proposition B.1. *Suppose (M, J_k) is a sequence of almost complex manifolds with $J_k \to J_\infty$ in C^∞ on some compact subset containing a super-rigid J_∞-holomorphic curve $u_\infty : (\Sigma, j_\infty) \to (M, J_\infty)$. Then for sufficiently large k, there exists a sequence of J_k-holomorphic curves $u_k : (\Sigma, j_k) \to (M, J_k)$ with $j_k \to j_\infty$ and $u_k \to u_\infty$ in C^∞, and if v_k is any sequence of smooth closed J_k-holomorphic curves Gromov-convergent to a stable nodal J_∞-holomorphic curve with image contained in $u_\infty(\Sigma)$, then for all k sufficiently large, every v_k is either a biholomorphic reparametrization or a multiple cover of u_k.*

Note that this statement belongs to the almost complex category and makes no reference to any symplectic structure. Other than that detail, a nearly identical statement has been proved before by Zinger, see [Zin11, Prop. 3.2]. The proof given below is essentially the same and is included mainly for the sake of completeness; it just requires the extra step of introducing an auxiliary symplectic structure in order to use Gromov's compactness theorem. Recall from §2.1 that if $u \in \mathcal{M}_g(A,J)$ and $d \geqslant 1$ and $h \geqslant 0$ are integers, we denote by

$$\overline{\mathcal{M}}_h(d;u) \subset \overline{\mathcal{M}}_h(dA,J)$$

the moduli space of all stable nodal d-fold covers of u with arithmetic genus h.

Suppose $J_k \to J_\infty$ is a C^∞-convergent sequence of almost complex structures on a manifold M, and $[(\Sigma, j_\infty, u_\infty)] \in \mathcal{M}_g(A, J_\infty)$ is a super-rigid curve. Then u_∞ is Fredholm regular with index 0, so the implicit function theorem implies the existence of curves $u_k : (\Sigma, j_k) \to (M, J_k)$ for sufficiently large k such that $j_k \to j_\infty$ and $u_k \to u_\infty$ in C^∞; these curves are unique up to biholomorphic reparametrization, and are also simple and immersed for sufficiently large k. Assume $v_k \in \mathcal{M}_h(dA, J_k)$ is a sequence of J_k-holomorphic curves converging to a nodal cover

$\tilde{u} \in \overline{\mathcal{M}}_h(d; u_\infty)$ for some $d > 0$. We will show that if the curves v_k are not covers of u_k for all sufficiently large k, then rescaling the normal fibers near u_k as $k \to \infty$ gives rise to a nontrivial section in the kernel of the normal Cauchy-Riemann operator on some cover of u_∞, contradicting super-rigidity.

Choose a convergent sequence of J_k-invariant Riemannian metrics and corresponding Levi-Civita connections ∇^k. Since the maps u_k are immersed, we can define J_k-invariant normal bundles $N_{u_k} \to \Sigma$ as the orthogonal complements of $\operatorname{im} du_k$. These are all isomorphic as real vector bundles, so we can identify them all with the real bundle $N := N_{u_\infty} \subset u_\infty^* TM$ carrying a sequence of complex structures

$$(N, J_k) \xrightarrow{\pi} (\Sigma, j_k),$$

and then use the sequence of exponential maps determined by ∇^k to define a C^∞-convergent sequence of immersions

$$\Psi_k : \mathcal{N}(\Sigma) \to M$$

of some fixed neighborhood $\mathcal{N}(\Sigma) \subset N$ of the zero section $\Sigma \subset N$ onto some neighborhood of $u_k(\Sigma)$, such that $\Psi_k|_\Sigma = u_k$. Let $\widehat{J}_k = \Psi_k^* J_k$ for $k = 1, 2, 3, \ldots, \infty$, so that for k sufficiently large, the curves v_k can be identified with $\widehat{J}_k$-holomorphic curves in the total space of N, and each u_k is identified with the zero section.

Let $\pi_N : u_\infty^* TM \to N$ denote the normal projection, so that $\widehat{\nabla} := \pi_N \circ \nabla^\infty$ induces a connection on $N \to \Sigma$ (as a *real* vector bundle), and thus defines a splitting into horizontal and vertical subbundles

$$TN = HN \oplus VN.$$

This splitting is invariant under the diffeomorphisms on N defined by real scalar multiplication. For $z \in \Sigma$ and $\eta \in N_z$, the fibers in the splitting admit canonical identifications

$$H_{(z,\eta)} N = T_z \Sigma, \qquad V_{(z,\eta)} N = N_z,$$

and we can write $\widehat{J}_k$ with respect to the splitting as

$$\widehat{J}_k(z,\eta) = \begin{pmatrix} \alpha_k(z,\eta) & \beta_k(z,\eta) \\ \gamma_k(z,\eta) & \delta_k(z,\eta) \end{pmatrix},$$

for some smoothly varying linear maps $\alpha_k(z,\eta) : T_z\Sigma \to T_z\Sigma$, $\beta_k(z,\eta) : N_z \to T_z\Sigma$ and so forth. Since $u_k : (\Sigma, j_k) \to (M, J_k)$ is J_k-holomorphic and the fibers of N_{u_k} are J_k-invariant along u_k, we have

$$\alpha_k(z,0) = j_k(z), \quad \delta_k(z,0) = J_k(u_k(z)), \quad \beta_k(z,0) = 0, \quad \gamma_k(z,0) = 0.$$

Now for any constant $r > 0$, the diffeomorphism

$$\Phi_r : N \to N : (z,\eta) \mapsto (z, r\eta)$$

transforms $\widehat{J}_k$ to

$$\widehat{J}_k^r(z,\eta) := \Phi_r^* \widehat{J}_k|_{(z,\eta)} = \begin{pmatrix} \alpha_k(z,r\eta) & r\beta_k(z,r\eta) \\ \frac{1}{r}\gamma_k(z,r\eta) & \delta_k(z,r\eta) \end{pmatrix},$$

so given any positive sequence $r_k \to 0$, the sequence $\widehat{J}_k^{r_k}$ converges in C^∞ on compact subsets of N to

$$\widehat{J}_\infty^0(z,\eta) := \begin{pmatrix} j_\infty(z) & 0 \\ d\gamma_\infty(z,0)\eta & J_\infty(u_\infty(z)) \end{pmatrix}. \tag{B.1}$$

Lemma B.2. *A neighborhood of Σ in N admits a symplectic form ω that tames $\widehat{J}_\infty^0$.*

Proof. We use a variation on Thurston's method for constructing symplectic forms on fibrations (cf. [MS17, Theorem 6.1.4]). For any open subset $\mathcal{U} \subset \Sigma$, let $\Lambda(\mathcal{U})$ denote the space of smooth 1-forms λ on $\pi^{-1}(\mathcal{U})$ satisfying the following conditions:

(i) At any point $(z,0) \in \mathcal{U} \subset N|_{\mathcal{U}}$ in the zero section,

$$\lambda|_{(z,0)} = 0 \quad \text{and} \quad d\lambda|_{T_z\Sigma \times N_z} = 0;$$

(ii) The restriction of $d\lambda$ to fibers in $\pi^{-1}(\mathcal{U})$ defines a symplectic vector bundle structure on $N|_{\mathcal{U}}$ taming J_∞.

We observe that $\Lambda(\mathcal{U})$ is nonempty whenever there exists a complex trivialization of (N, J_∞) over $\mathcal{U}$, and moreover, it is C^∞*-convex* in the sense that if $\lambda_0, \lambda_1 \in \Lambda(\mathcal{U})$, then

$$(\psi \circ \pi)\lambda_1 + (1 - \psi \circ \pi)\lambda_0 \in \Lambda(\mathcal{U})$$

for every smooth function $\psi : \mathcal{U} \to [0, 1]$. It follows that an element of $\Lambda(\Sigma)$ can be constructed by patching together local constructions via a partition of unity.

Now given $\lambda \in \Lambda(\Sigma)$, choose an area form σ on Σ taming j_∞. Then for a sufficiently large constant $K > 0$,

$$\omega := K\pi^*\sigma + d\lambda$$

is a closed 2-form that tames $\widehat{J}^0_\infty$ at Σ and hence also in a neighborhood of Σ. □

Remark B.3. The above proof did not use any special properties of $\widehat{J}^0_\infty$ except that the zero section is pseudoholomorphic and the normal fibers along the zero section are also complex. The same argument shows that for any embedded closed J-holomorphic curve in any almost complex manifold (M, J), a neighborhood of the curve admits a symplectic form that tames J.

Lemma B.4. *Suppose $\psi : \widetilde{\Sigma} \to \Sigma$ is a smooth map, $\tilde{j}$ is a complex structure on $\widetilde{\Sigma}$, and $\xi \in \Gamma(\psi^* N)$ is a smooth section along ψ. Then the map $z \mapsto \xi(z)$ from $\widetilde{\Sigma}$ into the total space of N is a pseudoholomorphic map $(\widetilde{\Sigma}, \tilde{j}) \to (N, \widehat{J}^0_\infty)$ if and only if $\psi : (\widetilde{\Sigma}, \tilde{j}) \to (\Sigma, j_\infty)$ is holomorphic and $\xi \in \ker \mathbf{D}^N_{u_\infty \circ \psi}$.*

Proof. Denote by $v : \widetilde{\Sigma} \to N$ the smooth map into the total space of N defined by $v(z) := \xi(z) \in N_{\psi(z)} \subset N$. Then using (B.1), the equation $Tv + \widehat{J}^0_\infty \circ Tv \circ \tilde{j} = 0$ translates into the two equations

$$d\psi(z) + j_\infty(\psi(z)) \circ d\psi(z) \circ \tilde{j}(z) = 0,$$

and

$$\widehat{\nabla}\eta(z) + J_\infty(u_\infty(\psi(z))) \circ \widehat{\nabla}\eta(z) \circ \tilde{j} + [d\gamma_\infty(\psi(z), 0)\eta(z)]\, d\psi(z) \circ \tilde{j} = 0$$

for $z \in \widetilde{\Sigma}$. The first equation says that $\psi : (\widetilde{\Sigma}, \tilde{j}) \to (\Sigma, j_\infty)$ is holomorphic, and under this assumption, the second matches $\mathbf{D}^N_{u_\infty \circ \psi}\eta = 0$ after observing

$$[d\gamma_\infty(\psi, 0)\eta] \circ d\psi \circ \tilde{j} = \pi_N \circ (\nabla_\eta J_\infty) \circ T(u_\infty \circ \psi) \circ \tilde{j}.$$

□

We now prove Proposition B.1 as follows. Arguing by contradiction, assume after taking a subsequence that the curves $v_k : (\widetilde{\Sigma}, \tilde{j}_k) \to (M, J_k)$ are not covers of u_k for any k as $k \to \infty$. Choose a symplectic form ω near the zero section in $N = N_{u_\infty}$ as given by Lemma B.2, and choose $\delta > 0$ such that ω tames $\widehat{J}^0_\infty$ on $\{\eta \in N \mid |\eta| < 2\delta\}$. Writing $v_k(z) = \xi_k(\psi_k(z))$ for sequences $\psi_k : \widetilde{\Sigma} \to \Sigma$ and $\xi_k \in \Gamma(\psi_k^* N)$, we have

$$r_k := \frac{1}{\delta} \max_{z \in \widetilde{\Sigma}} |\xi_k(z)| > 0$$

and $r_k \to 0$ by assumption. Then

$$w_k := \Phi_{r_k}^{-1} \circ v_k : (\widetilde{\Sigma}, \tilde{j}_k) \to (N, \widehat{J}^{r_k}_k)$$

is a sequence of smooth pseudoholomorphic curves in a compact subset of the neighborhood $\{\eta \in N \mid |\eta| < 2\delta\}$, which can be written as $w_k(z) = \eta_k(\psi_k(z))$ where $\eta_k = \frac{1}{r_k}\xi_k$ satisfies

$$\max_{z \in \widetilde{\Sigma}} |\eta_k(z)| = \delta. \tag{B.2}$$

Note that since v_k converges to a nodal curve in $\overline{\mathcal{M}}_h(d; u_\infty)$, we can also assume the maps $\psi_k : \widetilde{\Sigma} \to \Sigma$ have fixed degree d. Then since $\widehat{J}^{r_k}_k \to \widehat{J}^0_\infty$ and the latter is tamed by ω in the region under consideration, Gromov compactness applies to w_k and yields a subsequence convergent to a stable nodal curve $w_\infty \in \overline{\mathcal{M}}_h(d[\Sigma], \widehat{J}^0_\infty)$. By Lemma B.4, each smooth component w of w_∞ has the form $w(z) = \eta(\psi(z))$ where $\psi : (\widetilde{\Sigma}, \tilde{j}) \to (\Sigma, j_\infty)$ is holomorphic and $\mathbf{D}^N_{u_\infty \circ \psi}\eta = 0$. We claim there must be at least one such component for which $\deg(\psi) > 0$ and $\eta \neq 0$. Indeed, (B.2) implies that there is at least one component with $\eta \neq 0$. If every such component also satisfies $\deg(\psi) = 0$, then η is a nonzero constant on this component, as the normal operator $\mathbf{D}^N_{u_\infty \circ \psi}$ is simply the standard Cauchy-Riemann operator on a trivial bundle when ψ is constant. But since $\deg(\psi_k) = d > 0$, any component with $\deg(\psi) = 0$ is necessarily connected by a chain of nodes to another component with $\deg(\psi) > 0$, and on this component, η is nonzero at the nodal point. This implies the existence of a nontrivial element $\eta \in \ker \mathbf{D}^N_{u_\infty \circ \psi}$ for some positive degree holomorphic cover ψ, and thus violates super-rigidity. The proof of Proposition B.1 is complete.

Appendix C. The Sard-Smale theorem for C^∞-subvarieties

The proof of Petri's condition in §5 requires a version of the Sard-Smale theorem for objects that are not Banach manifolds but are almost as nice in some analytically quantifiable sense. The results in this appendix are easy consequences of standard results in the analysis of smooth Banach manifolds, but expressed in a slightly more general framework.

Suppose X is a smooth Banach manifold and $Y \subset X$ is a subset. Given $k \in \mathbb{N}$, we will say that Y is a **C^∞-subvariety of codimension at least** k if for every $x \in Y$, there exists a neighborhood $\mathcal{U} \subset X$ of x, a finite-dimensional vector space V and a smooth map $f : \mathcal{U} \to V$ such that:

(1) $Y \cap \mathcal{U} = f^{-1}(0)$;
(2) $\operatorname{rank} df(x) \geqslant k$.

Proposition C.1. *If $Y \subset X$ is a C^∞-subvariety of codimension at least k, then for every $x \in Y$, there exists a smooth Banach submanifold $\widetilde{Y} \subset X$ of codimension k such that a neighborhood of x in Y is contained in $\widetilde{Y}$.*

Proof. Given $x \in Y$, we have $Y \cap \mathcal{U} = f^{-1}(0)$ for some open neighborhood $x \in \mathcal{U} \subset X$ and smooth map $f : \mathcal{U} \to V$, with V a finite-dimensional vector space and $\dim \operatorname{im} df(x) \geqslant k$. Then we can choose a linear map $\Lambda : V \to \mathbb{R}^k$ whose restriction to $\operatorname{im} df(x) \subset V$ is surjective onto $\mathbb{R}^k$, hence $\Lambda \circ df(x) : T_x X \to \mathbb{R}^k$ is surjective. Define $\widetilde{Y} \subset X$ to be a neighborhood of x in $(\Lambda \circ f)^{-1}(0)$. The implicit function theorem implies that this is a Banach submanifold of codimension k if the neighborhood is taken sufficiently small. □

The discussion so far makes sense under a very unrestrictive definition of the term "Banach manifold," e.g. in [Lan99], such objects need not even be Hausdorff. In practice, of course, the Banach manifolds one encounters in applications are typically at least metrizable (hence Hausdorff and paracompact) and separable. The latter is the condition required for the Sard-Smale theorem [Sma65]. We will need the following standard bit of general topology:

Lemma C.2. *If X is a paracompact and separable topological space, then every open cover of X has a countable subcover.* □

The following is the main result of this appendix. The proof of Theorem 5.9 uses the special case in which all manifolds are finite dimensional, so the Fredholm assumption is automatic and only the finite-dimensional version of Sard's theorem is needed. The infinite-dimensional version with the Sard-Smale theorem is required for the proof of Theorem 5.26.

Proposition C.3. *Assume $\mathscr{U}$ and Z are separable and metrizable smooth Banach manifolds, $\pi : \mathscr{U} \to Z$ is a smooth Fredholm map, and $X \subset \mathscr{U}$ is a C^∞-subvariety of codimension at least $k \in \mathbb{N}$. For each $z \in Z$, denote*

$$\mathcal{M}(z) := \pi^{-1}(z) \subset \mathscr{U}, \qquad X(z) := X \cap \mathcal{M}(z) \subset \mathcal{M}(z),$$

and let $Z^{\mathrm{reg}}_\pi \subset Z$ denote the Baire subset consisting of regular values of π. Then there exists a further Baire subset $Z^{\mathrm{reg}}_X \subset Z$ such that for all $z \in Z^{\mathrm{reg}}_\pi \cap Z^{\mathrm{reg}}_X$, $X(z)$ is a C^∞-subvariety of codimension at least k in $\mathcal{M}(z)$.

Proof. Suppose $x \in X$, so by assumption, there exists a neighborhood

$$x \in \mathscr{U}_x \subset \mathscr{U},$$

a finite-dimensional vector space V_x and a smooth map $f_x : \mathscr{U}_x \to V_x$ such that $f_x^{-1}(0) = X \cap \mathscr{U}_x$ and $\operatorname{rank} df_x(x) \geqslant k$. After possibly shrinking $\mathscr{U}_x$ to a smaller neighborhood of x, we can use the argument in the proof of Proposition C.1 to find a linear map $\Lambda_x : V_x \to \mathbb{R}^k$ such that $0 \in \mathbb{R}^k$ is a regular value of $\Lambda_x \circ f_x : \mathscr{U}_x \to \mathbb{R}^k$ and

$$\widetilde{\mathscr{U}_x} := (\Lambda_x \circ f_x)^{-1}(0) \subset \mathscr{U}$$

is a smooth Banach submanifold of codimension k containing $X \cap \mathscr{U}_x$.

Since $\mathscr{U}$ is metrizable and separable, X also has both of these properties, thus Lemma C.2 implies that we can find a sequence $\{x_n\}_{n=1}^{\infty}$ of points in X such that every $x \in X$ lies in at least one of the neighborhoods $\mathscr{U}_{x_n}$. Let $Z_n^{\mathrm{reg}} \subset Z$ denote the set of regular values of the projection

$$\widetilde{\mathscr{U}_{x_n}} \xrightarrow{\pi} Z,$$

The latter is a smooth Fredholm map since $\widetilde{\mathscr{U}_{x_n}}$ is a smooth finite-codimensional submanifold of $\mathscr{U}$. The Sard-Smale theorem thus implies that $Z_n^{\mathrm{reg}} \subset Z$ is a Baire subset, and consequently,

$$Z_X^{\mathrm{reg}} := \bigcap_{n=1}^{\infty} Z_n^{\mathrm{reg}} \subset Z$$

is also a Baire subset.

Now for any $z \in Z_X^{\mathrm{reg}} \cap Z_\pi^{\mathrm{reg}}$ and $x \in X(z)$, pick $n \in \mathbb{N}$ such that $x \in \mathscr{U}_{x_n}$, and consider the restricted map

$$g_n : \mathcal{M}(z) \cap \mathscr{U}_{x_n} \to V_{x_n} : x \mapsto f_{x_n}(x),$$

whose zero-set is a neighborhood of x in $X(z)$. Regularity and the implicit function theorem imply that $\widetilde{\mathscr{U}_{x_n}} \subset \mathscr{U}$ and $\mathcal{M}(z) \subset \mathscr{U}$ are transverse submanifolds, so that 0 is also a regular value of $\Lambda_{x_n} \circ g_n : \mathcal{M}(z) \cap \mathscr{U}_{x_n} \to \mathbb{R}^k$. It follows that $\Lambda_{x_n} \circ dg_n(x) : T_x\mathcal{M}(z) \to \mathbb{R}^k$ is surjective, and thus $\operatorname{rank} dg_n(x) \geqslant k$. □

The results of this discussion combine to yield the following useful consequence:

Corollary C.4. *In the setting of Proposition C.3, if the smooth Fredholm map $\pi : \mathscr{U} \to Z$ satisfies* $\operatorname{ind} d\pi(x) < k$ *for all $x \in \mathscr{U}$, then $X(z)$ is empty for generic $z \in Z$.* □

Appendix D. History of errors

This appendix has been added (at the suggestion of an anonymous referee) in the interest of transparency: its purpose is to clarify more precisely what went wrong with previous attempts to prove Theorem A, and how those attempts are related to the proof in this paper. There were at least two claims of proofs of super-rigidity that were publicized and then withdrawn before I ever started thinking about the problem, but since it is not my place to comment on those, I will only discuss the attempts that I have been involved in.

D.1. **Analytic perturbation theory.** The original version of [GW17] was a preprint under a different title [GW], which claimed a proof of Theorem A (also in dimension four) for embedded index 0 curves that are fully contained in the perturbation domain $\mathcal{U} \subset M$. The ideas behind that argument were almost totally disjoint from those of the present paper, excepting the superficial feature that both derive originally from (separate) ideas developed in Taubes's work on the Gromov invariant. The literature on the Gromov invariant contains two quite different methods to prove transversality for the doubly covered tori that must be counted: one (from [Tau96a]) is based on a splitting of Cauchy-Riemann type operators with respect to irreducible representations, and gives rise to dimension-counting arguments that provided the original inspiration for this paper. The other, from [Tau96b, Proof of Prop. 7.1, Step 7], is in some respects more novel: it is based on a Weitzenböck formula for Cauchy-Riemann type operators and analytic perturbation theory. In the setting of [Tau96b], where one needs to prove that a $\mathbb{Z}_2$-equivariant index 0 Cauchy-Riemann type operator $\mathbf{D} : \Gamma(E) \to \Gamma(F)$ on a trivial line bundle $E \to \mathbb{T}^2$ can always be perturbed equivariantly to one that is invertible, these two ingredients combine in the following way:

(1) The Weitzenböck formula implies that for any complex-antilinear bundle isomorphism $A : E \to F$, the deformed operator $\mathbf{D}_\tau := \mathbf{D} + \tau A$ is invertible for all $\tau \gg 0$.
(2) Since the deformed operators $\mathbf{D}_\tau$ depend analytically on the parameter $\tau \in \mathbb{R}$, analytic perturbation theory as in [Kat95] implies that the set $\{\tau \in \mathbb{R} \mid \mathbf{D}_\tau \text{ is not invertible}\}$ is either $\mathbb{R}$ or is discrete. The first possibility has already been ruled out via the Weitzenböck formula, so it follows that $\mathbf{D}_\tau$ is invertible for all $\tau \neq 0$ in some neighborhood of 0.

This technique has the appealing feature that it does not care how symmetric the perturbation term $A \in \Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(E, F))$ is, thus it can work equally well for simple holomorphic curves and multiple covers. The preprint [GW] was motivated by the insight that both parts of the argument can be made to work somewhat more generally: the operator $\mathbf{D}$ can have negative index if we talk about *injectivity* of $\mathbf{D}_\tau$ instead of invertibility, and E can also be a higher-rank bundle if A is required to satisfy an extra condition which, for topological reasons, can be assumed without loss of generality. Applying the argument to normal Cauchy-Riemann operators of branched covers then produces the following result:

Lemma D.1 ([GW17]). *Suppose* $\dim M \geqslant 4$, $J \in \mathcal{J}(M, \omega\,; \mathcal{U}, J_{\mathrm{fix}})$, $v : (\Sigma, j) \to (M, J)$ *is an embedded closed* J*-holomorphic curve of index* 0 *with image contained in* $\mathcal{U}$, *and* $u = v \circ \varphi$ *where* $\varphi : (\widetilde{\Sigma}, \widetilde{j}) \to (\Sigma, j)$ *is a holomorphic branched cover of degree* $d \in \mathbb{N}$ *between closed connected Riemann surfaces. Then there exists a smooth* 1*-parameter family* $\{J_\tau \in \mathcal{J}(M, \omega\,; \mathcal{U}, J_{\mathrm{fix}})\}_{\tau \in (-\epsilon, \epsilon)}$ *such that* $J_0 = J$, v *and* u *are* J_τ*-holomorphic for every* τ, *and the resulting normal Cauchy-Riemann operators* $\mathbf{D}^N_{u,\tau}$ *for* u *with respect to* J_τ *are injective for all* $\tau \neq 0$. □

A proof of generic super-rigidity would follow via relatively straightforward topological arguments if one instead had the following stronger statement:[12]

Lemma(?) D.2. *In the setting of Lemma D.1, the family of almost complex structures* $\{J_\tau \in \mathcal{J}(M, \omega\,; \mathcal{U}, J_{\mathrm{fix}})\}_{\tau \in (-\epsilon, \epsilon)}$ *can be chosen so that for some neighborhood* $\mathcal{O}(\varphi)$ *of* φ *in the moduli space of* d*-fold holomorphic branched covers, the normal Cauchy-Riemann operators* $\mathbf{D}^N_{v \circ \varphi', \tau}$ *are injective for all* $\tau \neq 0$ *and* $\varphi' \in \mathcal{O}(\varphi)$.

Unfortunately, Lemma D.1 does not imply Lemma D.2, as analytic perturbation theory gives no obvious way to control the size of the range of parameter values $\tau \in (-\epsilon, \epsilon) \backslash \{0\}$ for which injectivity is guaranteed as φ varies in the moduli space of branched covers. This detail was overlooked in [GW]; the crucial gap in our argument was pointed out by Ionel and Parker. What can still be salvaged from Lemma D.1, and eventually appeared as the main result of the published paper [GW17], is a result similar to Theorem B about transversality for *unbranched* covers: in the unbranched case there is no distinction between Lemmas D.1 and D.2 because the moduli space that φ lives in is discrete.

I currently believe the proof of Theorem A originally attempted in [GW] to be unsalvageable. There are also strong philosophical arguments for preferring the approach of the present paper over analytic perturbation theory: notably, the use of the Weitzenböck formula requires a more *global* class of perturbations (u must be contained in the perturbation domain $\mathcal{U} \subset M$ rather than merely intersecting it), and the whole strategy seems completely unsuitable for studying the wall-crossing phenomena mentioned in §2.4. On the other hand, the Weitzenböck argument (minus analytic perturbation theory) has been usefully exploited by other authors in certain special settings where geometric information removes the need to assume $\tau \gg 0$; see [LP07, IP18].

[12] The question mark in the statement indicates that I do not know whether Lemma D.2 is true, and I do not have a strong enough opinion about it to call it a conjecture.

D.2. **Earlier versions of the present paper.** The main ideas behind the proofs of Theorems A–D have changed very little since the first version of this paper appeared on the arXiv, but one important technical detail has changed a lot: the proof that generic Cauchy-Riemann type operators satisfy Petri's condition.[13] The intuition from the beginning had been that Petri's condition was the main analytical lemma needed for the proof of Theorem D (on which Theorems A–C all depend), and that it should hold due to unique continuation except for some special class of non-generic Cauchy-Riemann type operators. Up to version 3 on the arXiv [Wenc], a much more naive approach to this lemma was taken, in which the word "generic" was given a precise characterization:

(False) Lemma D.3 ([Wenc, Corollary 5.2 and Lemma 3.11]). *Suppose $E, F \to \Sigma$ are complex vector bundles and $\mathbf{D} : \Gamma(E) \to \Gamma(F)$ is a Cauchy-Riemann type operator such that the bundle map $\mathbf{D}^{0,1} \in \Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(E, F))$ given by the complex-antilinear part of $\mathbf{D}$ defines an invertible map $E_z \to F_z$ at some point $z \in \Sigma$. Then $\mathbf{D}$ satisfies Petri's condition to infinite order at z.*

It is relatively easy to show (see [Wenc, Lemma 6.2]) that the hypothesis on invertibility of complex-antilinear parts is generic, i.e. all normal Cauchy-Riemann operators of J-holomorphic curves satisfy it for generic (and necessarily non-integrable) J. The benefit of this condition is that it forces $\ker \mathbf{D} \subset \Gamma(E)$ and $\ker \mathbf{D}^* \subset \Gamma(F)$ to be totally real subspaces, meaning that any real-linearly independent set of vectors in one of these spaces is also complex-linearly independent. The original reason to believe in Lemma D.3 was the elementary observation mentioned in Example 5.5 that for complex-linear Cauchy-Riemann type operators, which can always be expressed locally as the standard one, the complex version of Petri's condition (involving complex tensor products) does hold to infinite order at every point; a proof of this may be found on page 48 of [Wenc]. Lemma D.3 was thus an attempt to fit real-linear Cauchy-Riemann type operators into a complex-linear context with the aid of the totally real hypothesis. The proof was destroyed by a careless mistake in linear algebra: Equations (5.3) and (5.4) in [Wenc] define certain functions η^ν_α and ξ^μ_β that are meant to be in $\ker \mathbf{D}$ and $\ker \mathbf{D}^*$ respectively because they are linear combinations of functions in those spaces, but in fact, the coefficients in those linear combinations are complex rather than real, while $\mathbf{D}$ and $\mathbf{D}^*$ are only real-linear. Similarly, the claim in the final paragraph of that proof that certain linear combinations $\sum_i c^{ij} \xi_i$ and $\sum_j c^{ij} \eta_j$ satisfy linear Cauchy-Riemann or anti-Cauchy-Riemann equations does not hold, again because the coefficients c^{ij} are complex instead of real. These errors were noticed by Doan and Walpuski while working on their own alternative exposition of the super-rigidity proof [DWb]. Example 5.7 was found later, showing that Lemma D.3 is in fact false.

After Lemma D.3 fell apart, the intuition remained that the failure of the local Petri condition for a Cauchy-Riemann type operator should be overdetermined in some sense, and the jet space approach in the current §5 was then developed to make this intuition precise. Lemma D.3 has now been replaced by Corollary 5.10, whose proof is completely different from what was attempted in [Wenc], and has an additional advantage over the earlier approach in that the jet space formalism can potentially be applied to more general classes of operators beyond Cauchy-Riemann (§5.2 has been written with this in mind). A more detailed informal discussion of the fix may be found in the blog post [Wend].

For completeness, I should mention a somewhat serious but non-fatal error that was also pointed out by Doan and Walpuski but corrected between arXiv versions 2 and 3 of this paper. The definition of the walls appearing in Theorem D was slightly wrong in earlier versions, because it was overlooked that in the splitting of the normal Cauchy-Riemann operator $\mathbf{D}^N_u$ into summands $\dot{\mathbf{D}}^N_{u,\boldsymbol{\theta}_i}$ corresponding to irreducible representations $\boldsymbol{\theta}_i$, the kernels and cokernels of these summands are always modules over the equivariant endomorphism algebra ($\mathbb{R}$, $\mathbb{C}$ or

[13] The term "Petri's condition" did not appear in the first three versions of this paper on the arXiv, but the same notion was there under the label of "unique continuation for tensor products" and has sometimes also been advertised as "quadratic unique continuation". The current terminology was introduced by Doan and Walpuski [DWb] after the first version of this paper appeared.

$\mathbb{H}$) of $\boldsymbol{\theta}_i$, and this structure must be respected in talking about their dimensions. The result was a mistake in [Wenb, Theorem D] that was hard to spot, because the statement looked the same as in the current version, but its meaning was different. The source of the problem was an erroneous representation-theoretic dimension calculuation in [Wenb, Corollary 3.23], which was stated without proof. A corrected version of that result appears in this version as Corollary 3.23, with a proof given in the preceding paragraph.

References

[ACGH85] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris, *Geometry of algebraic curves. Vol. I*, Grundlehren der Mathematischen Wissenschaften, vol. 267, Springer-Verlag, New York, 1985.

[BS] S. Bai and M. Swaminathan, *Bifurcations of embedded curves and an extension of Taubes' Gromov invariant to Calabi-Yau 3-folds*. Preprint arXiv:2106.01206.

[BEH+03] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, and E. Zehnder, *Compactness results in symplectic field theory*, Geom. Topol. **7** (2003), 799–888.

[BKL01] J. Bryan, S. Katz, and N. C. Leung, *Multiple covers and the integrality conjecture for rational curves in Calabi-Yau threefolds*, J. Algebraic Geom. **10** (2001), no. 3, 549–568.

[BP01] J. Bryan and R. Pandharipande, *BPS states of curves in Calabi-Yau 3-folds*, Geom. Topol. **5** (2001), 287–318 (electronic).

[BP05] ———, *Curves in Calabi-Yau threefolds and topological quantum field theory*, Duke Math. J. **126** (2005), no. 2, 369–396.

[BP06] ———, *On the rigidity of stable maps to Calabi-Yau threefolds*, The interaction of finite-type and Gromov-Witten invariants (BIRS 2003), Geom. Topol. Monogr., vol. 8, Geom. Topol. Publ., Coventry, 2006, pp. 97–104.

[BP08] ———, *The local Gromov-Witten theory of curves*, J. Amer. Math. Soc. **21** (2008), no. 1, 101–136. With an appendix by Bryan, C. Faber, A. Okounkov and Pandharipande.

[CM07] K. Cieliebak and K. Mohnke, *Symplectic hypersurfaces and transversality in Gromov-Witten theory*, J. Symplectic Geom. **5** (2007), no. 3, 281–356.

[CM18] K. Cieliebak and K. Mohnke, *Punctured holomorphic curves and Lagrangian embeddings*, Invent. Math. **212** (2018), no. 1, 213–295.

[DIW] A. Doan, E.-N. Ionel, and T. Walpuski, *The Gopakumar-Vafa finiteness conjecture*. Preprint arXiv:2103.08221.

[DWa] A. Doan and T. Walpuski, *Castelnuovo's bound and rigidity in almost complex geometry*. Preprint arXiv:1809.04731.

[DWb] ———, *Equivariant Brill-Noether theory for elliptic operators and super-rigidity of J-holomorphic maps*. Preprint arXiv:2006.01352.

[Don11] S. Donaldson, *Riemann surfaces*, Oxford Graduate Texts in Mathematics, vol. 22, Oxford University Press, Oxford, 2011.

[Eft16] E. Eftekhary, *On finiteness and rigidity of J-holomorphic curves in symplectic three-folds*, Adv. Math. **289** (2016), 1082–1105.

[EGH00] Y. Eliashberg, A. Givental, and H. Hofer, *Introduction to symplectic field theory*, Geom. Funct. Anal., Special Volume (2000), 560–673.

[Eli67] H. I. Eliasson, *Geometry of manifolds of maps*, J. Differential Geometry **1** (1967), 169–194.

[Fab13] O. Fabert, *Local symplectic field theory*, Internat. J. Math. **24** (2013), no. 5, 1350041, 16.

[Flo88] A. Floer, *The unregularized gradient flow of the symplectic action*, Comm. Pure Appl. Math. **41** (1988), no. 6, 775–813.

[FHS95] A. Floer, H. Hofer, and D. Salamon, *Transversality in elliptic Morse theory for the symplectic action*, Duke Math. J. **80** (1995), no. 1, 251–292.

[FO99] K. Fukaya and K. Ono, *Arnold conjecture and Gromov-Witten invariant*, Topology **38** (1999), no. 5, 933–1048.

[GW] C. Gerig and C. Wendl, *Super-rigidity for closed pseudoholomorphic curves*. Preprint arXiv:1407.0678v1, abandoned and later revised into [GW17].

[GW17] ———, *Generic transversality for unbranched covers of closed pseudoholomorphic curves*, Comm. Pure Appl. Math. **70** (2017), no. 3, 409–443.

[GV] R. Gopakumar and C. Vafa, *M-theory and topological strings–II*. Preprint arXiv:hep-th/9812127.

[Gro85] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), no. 2, 307–347.

[HLS97] H. Hofer, V. Lizan, and J.-C. Sikorav, *On genericity for holomorphic curves in four-dimensional almost-complex manifolds*, J. Geom. Anal. **7** (1997), no. 1, 149–159.

[HWZ95] H. Hofer, K. Wysocki, and E. Zehnder, *Properties of pseudo-holomorphic curves in symplectisations. II. Embedding controls and algebraic invariants*, Geom. Funct. Anal. **5** (1995), no. 2, 270–328.

[HWZ96] ______, *Properties of pseudoholomorphic curves in symplectisations. I. Asymptotics*, Ann. Inst. H. Poincaré Anal. Non Linéaire **13** (1996), no. 3, 337–379.

[HWZ99] ______, *Properties of pseudoholomorphic curves in symplectizations. III. Fredholm theory*, Topics in nonlinear analysis, 1999, pp. 381–475.

[HWZ17] ______, *Applications of polyfold theory I: The polyfolds of Gromov-Witten theory*, Mem. Amer. Math. Soc. **248** (2017), no. 1179, v+218.

[Hut14] M. Hutchings, *Lecture Notes on Embedded Contact Homology*, Contact and Symplectic Topology, Bolyai Society Mathematical Studies, vol. 26, Springer, 2014, pp. 389–484.

[Hut] ______, *A magic trick for defining obstruction bundles.* Blog post, available at `http://floerhomology.wordpress.com/2013/09/04/`.

[IP18] E.-N. Ionel and T. H. Parker, *The Gopakumar-Vafa formula for symplectic manifolds*, Ann. of Math. (2) **187** (2018), no. 1, 1–64.

[Kat95] T. Kato, *Perturbation theory for linear operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[Kaw79] T. Kawasaki, *The Riemann-Roch theorem for complex V-manifolds*, Osaka Math. J. **16** (1979), no. 1, 151–159.

[Lan99] S. Lang, *Fundamentals of differential geometry*, Springer-Verlag, New York, 1999.

[LP07] J. Lee and T. H. Parker, *A structure theorem for the Gromov-Witten invariants of Kähler surfaces*, J. Differential Geom. **77** (2007), no. 3, 483–513.

[LP12] ______, *An obstruction bundle relating Gromov-Witten invariants of curves and Kähler surfaces*, Amer. J. Math. **134** (2012), no. 2, 453–506.

[LZ07] J. Li and A. Zinger, *Gromov-Witten invariants of a quintic threefold and a rigidity conjecture*, Pacific J. Math. **233** (2007), no. 2, 417–480.

[LT98a] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties*, J. Amer. Math. Soc. **11** (1998), no. 1, 119–174.

[LT98b] ______, *Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds*, Topics in symplectic 4-manifolds (Irvine, CA, 1996), First Int. Press Lect. Ser., I, Int. Press, Cambridge, MA, 1998, pp. 47–83.

[MS12] D. McDuff and D. Salamon, *J-holomorphic curves and symplectic topology*, 2nd ed., American Mathematical Society Colloquium Publications, vol. 52, American Mathematical Society, Providence, RI, 2012.

[MS17] ______, *Introduction to symplectic topology*, 3rd ed., Oxford University Press, 2017.

[MS] D. McDuff and K. Siegel, *Counting curves with local tangency constraints.* Preprint arXiv:1906.02394, to appear in J. Topol.

[OZ09] Y.-G. Oh and K. Zhu, *Embedding property of J-holomorphic curves in Calabi-Yau manifolds for generic J*, Asian J. Math. **13** (2009), no. 3, 323–340.

[Pan99] R. Pandharipande, *Hodge integrals and degenerate contributions*, Comm. Math. Phys. **208** (1999), no. 2, 489–506.

[PT14] R. Pandharipande and R. P. Thomas, *13/2 ways of counting curves*, Moduli spaces, London Math. Soc. Lecture Note Ser., vol. 411, Cambridge Univ. Press, Cambridge, 2014.

[Par16] J. Pardon, *An algebraic approach to virtual fundamental cycles on moduli spaces of pseudo-holomorphic curves*, Geom. Topol. **20** (2016), no. 2, 779–1034.

[PS98] G. Pólya and G. Szegő, *Problems and theorems in analysis. II*, Classics in Mathematics, Springer-Verlag, Berlin, 1998. Theory of functions, zeros, polynomials, determinants, number theory, geometry; Translated from the German by C. E. Billigheimer; Reprint of the 1976 English translation.

[Rua99] Y. Ruan, *Virtual neighborhoods and pseudo-holomorphic curves*, Proceedings of 6th Gökova Geometry-Topology Conference, 1999, pp. 161–231.

[Sch95] M. Schwarz, *Cohomology operations from S^1-cobordisms in Floer homology*, Ph.D. Thesis, ETH Zürich, 1995.

[Ser77] J.-P. Serre, *Linear representations of finite groups*, Springer-Verlag, New York-Heidelberg, 1977. Translated from the second French edition by Leonard L. Scott; Graduate Texts in Mathematics, Vol. 42.

[Sie] B. Siebert, *Gromov-Witten invariants of general symplectic manifolds.* Preprint arXiv:9608005.

[Sie08] R. Siefring, *Relative asymptotic behavior of pseudoholomorphic half-cylinders*, Comm. Pure Appl. Math. **61** (2008), no. 12, 1631–1684.

[Sma65] S. Smale, *An infinite dimensional version of Sard's theorem*, Amer. J. Math. **87** (1965), 861–866.

[Tau96a] C. H. Taubes, *Counting pseudo-holomorphic submanifolds in dimension* 4, J. Differential Geom. **44** (1996), no. 4, 818–893.

[Tau96b] C. H. Taubes, SW $\Rightarrow$ Gr*: from the Seiberg-Witten equations to pseudo-holomorphic curves*, J. Amer. Math. Soc. **9** (1996), no. 3, 845–918.

[Wen10] C. Wendl, *Automatic transversality and orbifolds of punctured holomorphic curves in dimension four*, Comment. Math. Helv. **85** (2010), no. 2, 347–407.

[Wena] ———, *Lectures on holomorphic curves in symplectic and contact geometry.* Preprint arXiv:1011.1690v2.

[Wenb] ———, *Transversality and super-rigidity for multiply covered holomorphic curves.* Preprint arXiv:1609.09867v2, withdrawn earlier version of this paper.

[Wenc] ———, *Transversality and super-rigidity for multiply covered holomorphic curves.* Preprint arXiv:1609.09867v3, withdrawn earlier version of this paper.

[Wend] ———, *Why Petri's condition is generic.* Blog post, available at `https://symplecticfieldtheorist.wordpress.com/2019/05/27/`.

[Wene] ———, *Lectures on Symplectic Field Theory.* Preprint arXiv:1612.01009v2, to appear in EMS Series of Lectures in Mathematics.

[Zeh15] K. Zehmisch, *Holomorphic jets in symplectic manifolds*, J. Fixed Point Theory Appl. **17** (2015), no. 2, 379–402.

[Zin11] A. Zinger, *A comparison theorem for Gromov-Witten invariants in the symplectic category*, Adv. Math. **228** (2011), no. 1, 535–574.

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