§1 Iterability and MS-mice

In this section we discuss more fully the difference between our notion of iterability and that used by Mitchell and Steel in [MS]. There are in fact three points of difference in the overall approach:
(a) The fine structural concepts.
(b) The indexing of extenders.
(c) The iterability requirements.
In (a) the differences are essentially verbal, since their concepts are translatable into ours and conversely. In point (b) there is a difference, but, in fact, we could have used the Mitchell-Steel indexing with our version of (a) + (c), just as they could have used our indexing with their version. The most significant

difference is in (c), where we appear to require more of a mouse than they do. In order to make this difference clear, I shall now define "MS-iterable mice", retaining our version of (a), (b) and reformulating their version of (c) in the new setting. Let M be acceptable and let F be an extender on M at κ, λ. For each $k \leq \omega$ Mitchell and Steel define a "k-ultrapower" as follows: $Ult^k(M, F)$ = the (transitivized) ultrapower of M by F using the class of functions $\Gamma^k(M, \kappa)$ = the set of $f : \kappa \to M$ s.t. either $f \in M$ or f is a good $\Sigma_1^{(n)}$ map, where $\omega\rho_M^{n+1} > \kappa$ and $n \leq k$, (Then $\Gamma^0(M,\kappa) = \Gamma(M,\kappa)$ and $\Gamma^\omega(M,\kappa) = \Gamma^*(M,\kappa)$.) We also write $\pi : M \xrightarrow[F]{k} M'$ in the obvious sense.

We define a *generalized k-iteration* $\mathcal{J} = \langle\langle M_i\rangle, \langle \nu_i \mid i \in D\rangle, \langle \eta_i\rangle, \langle \pi_{ij}\rangle, T\rangle$ exactly like our earlier notion of generalized $*$-iteration (in §4) except that we require $\pi_{\xi, i+1} : M_\xi \xrightarrow[F_{\nu_i}]{k} M_{i+1}$ whenever $i \in D$, $\xi = T(i+1)$ and $\pi_{0,i+1}$ is total. ~~If~~ If $\pi_{0,i+1}$ is not total (i.e. $\eta_j < ht(M_{T(j+1)})$ for a $j+1 \le_T i+1$), we require:

$\pi_{\xi, i+1} : M_\xi \xrightarrow[F_{\nu_i}]{*} M_{i+1}$.

The notions: 'standard', 'direct', 'normal' carry over mutatis mutandis. We say that N is *normally MS-iterable* iff N ~~is~~ is normally k-iterable for every $k \le \omega$. By the methods of §4 it follows easily that if $\mathcal{I}$ is a normal k-iteration, then π_{ij} is $\Sigma_0^{(k)}$ - preserving whenever $i \leq_T j$ and π_{0j} is total. If π_{0j} is not total, then π_{ij} is Σ^* - preserving. By a slight modification of §3 Lemma 1 we get:

<u>Fact 1</u> Let $\langle \pi, g \rangle : \langle \bar{M}, \bar{F} \rangle \longrightarrow \langle M, F \rangle$ and let π be $\Sigma_0^{(n)}$ - preserving for all n s.t. $n \leq k$ and $\omega\rho_{\bar{M}}^{n} > \bar{\kappa} = \mathrm{crit}(\bar{F})$. Let $\sigma : M \longrightarrow M'$ be $\Sigma_0^{(n)}$ preserving for the same n and let $F = (\sigma \restriction \mathbb{P}(\kappa)) | \lambda$, where $\kappa = \mathrm{crit}(F)$, $\lambda = lh(F)$. Then there exists $\bar{\sigma} : \bar{M} \longrightarrow_{\bar{F}}^{k} \bar{M}'$. Moreover, there is a unique π s.t.

(i) $\pi' : \bar{M}' \longrightarrow_{\Sigma_0^{(n)}} M'$ for all $n \leq k$ s.t. $\omega\rho_{\bar{M}}^{n} > \bar{\kappa}$,

(ii) $\pi'\bar{\sigma} = \sigma\pi$ and $\pi \restriction \bar{\lambda} = g$.

(π' is defined by: $\pi'(\bar{\sigma}(f)(\alpha)) = \sigma\pi(f)(g(\alpha))$ for $\alpha < \bar{\lambda}$, $f \in \Gamma^{k}(\bar{\kappa}, \bar{M})$.)

This is, indeed, the lemma on which the copying arguments in [MS] are based. Using this and the methods of §5 it follows straightforwardly that if M is normally k-iterable and $\omega \geq k \geq h$, then M is normally h-iterable. Since ω-iterations are the same as $*$-iterations, it follows that M is normally MS-iterable iff it is normally iterable in our sense.

Although Mitchell and Steel don't develop a general theory of good iterations, they do occasionally "stack" normal iterations and need to prove iterability in that sense. The first approximation to a good MS-iteration would be one that resolves into a good sequence $\langle\langle M_i \mid i < \Gamma\rangle, \langle \mathcal{J}_i \mid i < \Gamma\rangle, \langle \pi_{ij} \mid i \leq j < \Gamma\rangle\rangle$ in the sense of §4 – with the difference that each $\mathcal{J}_i$ is a normal k_i-iteration for some $k_i \leq \omega$. (As in §4 the good sequence is represented by a single iteration with markers.)

We need, however, to place some further restriction on the k_i's.

Def A good k-iteration is an iteration $\mathcal{I}$ with markers which resolve it into a good sequence

$\langle\langle M_i \mid i < \Gamma\rangle, \langle \mathcal{I}_i \mid i < \Gamma\rangle, \langle \pi_{ij} \mid i \leq j < \Gamma\rangle\rangle$

where each $\mathcal{I}_i$ is a normal k_i-iteration and the following hold:

(a) If $M_0^{\mathcal{I}_i}$ is a proper segment of M_i, then $k_i = \omega$.

(b) Let $M_0^{\mathcal{I}_i} = M_i$. Then:

(i) $k_i = k$ if $i = 0$

(ii) If $\lim(i)$, then

$k_i = \min\{k_j \mid j < i \text{ and } \pi_{ji} \text{ is total}\}$

(iii) If $i = j+1$, $\delta = lh(\mathcal{I}_j)$, then $k_i =$ the maximal k s.t. $k \leq k_j$ if $\pi_{0\delta}^{\mathcal{I}_j}$ is total and $\omega\rho_{M_i}^k > \lambda_h^{\mathcal{I}_j}$ for $h < \delta$.

These conditions appear arcane at first glance. They are, however, implicit in [MS] for the following reason: The version of the k-ultrapower $Ult^k(M,F)$ used by Mitchell and Steel is defined only in the case that M is sound above $\omega\rho^n_M$ whenever $n \leq k$ and $\omega\rho^n_M > crit(F)$. If this soundness requirement is imposed and the k_i's are chosen maximally for $i > 0$, then (a) and (b) (ii), (iii) must follow. (We shall, of course not impose the soundness requirement, since our k-ultrapowers are always defined.) We then set:

<u>Def</u> M is k-<u>iterable</u> iff M has a strategy for k-iteration.

M is MS-<u>iterable</u> (an MS-<u>mouse</u>) iff M is k-iterable for all $k \leq \omega$.

(Using the methods of §5 it is not too hard to show that every mouse in our sense is an MS-mouse.

(<u>Note</u> This notion of MS-iterability is less restrictive than that actually used by Mitchell and Steel. Because of the above soundness requirement, their version must read: Every k-core of M is k-iterable.)

(Note It does not follow from the definition that an MS-iterate of an MS-mouse is MS-iterable.)

An advantage of this notion of iterability is that it is easier to verify. This can be seen fr. ins. in the proof that smooth MS-iterability implies MS-iterability. (A smooth k-iteration is defined as above with the additional condition: $M_i = M_0^{y_i}$ for all i.) We can carry out the proof in §9 using only the above copying principle Fact 1 and with no use of pseudo projecta.

The same simplifying effect occurs in the iterability proof of §10, where we verified that N_ξ is weakly iterable below θ for $\xi < \theta$

We first recall the definition:
N is *weakly iterable* (a *weak mouse*) iff whenever $\sigma : Q \xrightarrow{\Sigma^*} N$ and Q is countable, then Q is countably iterable. (When we use this notion, we generally assume:

(A0) Either $\#(\omega_1)$ is closed under # or ω_1 is not Woodin in an inner model.

Either of these assumptions implies that Q is $\omega_1 + 1$ iterable (if Q is 1-small). We can then prove comparison for countable iterable countable premice and show that the results of §9, §10 (solidity + condensation) hold for all 1-small weak mice)

The obvious corresponding notion is:

Def N is *weakly MS-iterable* (a *weak MS-mouse*) iff whenever $\sigma : Q \xrightarrow{\Sigma^{(k)}} N$ + Q is countable, then Q is k-iterable ($k \leq \omega$).

Defining N_ξ, M_ξ as in §10 we can then prove the weak MS-iterability of N_ξ using only Fact 1 as copying lemma and without use of pseudo projecta. Our new version of §10 Lemma 1 now reads:

<u>Lemma 1</u> Let $\sigma : Q \to_{\Sigma_0^{(k)}} N_\xi$, where Q is a countable premouse. Then Q has a countable normal k-iteration strategy S. Moreover if $\mathcal{J} = \langle\langle Q_i\rangle, \langle \nu_i\rangle, \langle \eta_i\rangle, \langle \pi_{ij}\rangle, T\rangle$ is a countable normal k-iteration of length $\theta+1$, then:

(i) There is $\sigma' : Q_\theta \to N_\gamma$ for a $\gamma \leq \xi$

s.t.

(ii) If $\pi_{0\theta}$ is not total, then $\gamma < \xi$ and σ' is $\Sigma_1^{(n)}$ preserving whenever $\omega\rho^n_{M_\theta} \geq \sup\limits_{i<\theta} \lambda_i$

(iii) If $\pi_{0\theta}$ is total, then $\gamma = \xi$ and σ' is $\Sigma_1^{(n)}$ preserving whenever $\omega\rho^n_{M_\theta} \geq \sup\limits_{i<\theta} \lambda_i$ and $n \leq k$.

This suffices for the purpose. (Note Since Mitchell and Steel were constrained to form k-ultrapowers only when their soundness condition held, they gave a direct proof of: The k-th core of N_ξ is k-iterable. This introduced further complications.)