§4 Barwise Theory

Jon Barwise worked out the syntax and model theory of certain infinitary (but M-finitary) languages on countable admissible structures M. In so doing he created a powerful and flexible tool for set theorists, which enables us to construct transitive structures using elementary model theory. In this chapter we give an introduction to this theory, whose potential for set theory has, we feel, been sadly neglected.

Let M be admissible. Barwise uses a first order theory in which arbitrary M-finite conjunctions and disjunctions are allowed. The predicates, however, have only a finite number of argument places and there are no infinite strings of quantifiers. The arithmetization of this 1st order theory is much like that of the M-language in §1. However, we need M-infinitely many variables, since otherwise a single formula might exhaust all variables, so we set:

$$v_x = \langle 0, x \rangle \quad (x \in M).$$

We want not only to formalize the language of M, but also to have a full 1st order logic, so in addition to the special predicates ε, $\equiv$, $\dot{A}_j$ ($j = 1, \ldots, n$), we adopt for each $x \in M$ and each $n \in \omega$ an n-place predicate $P^n_x =_{Df} \langle 20, \langle n, x \rangle \rangle$. In addition to the special constants $\underline{x}$ ($x \in M$) we adopt constants $c_x =_{Df} \langle 21, x \rangle$ ($x \in M$). ($\underline{x}$ will, as before, be employed only as a name for x.) Finally, if $\langle \varphi_i \mid i \in u \rangle \in M$ is a sequence of formulae, we define the infinite (but M-finite) conjunction and disjunction by:

$$\bigwedge_{i \in u} \varphi_i =_{Df} \langle 22, \langle \varphi_i \mid i \in u \rangle \rangle$$

$$\bigvee_{i \in u} \varphi_i =_{Df} \langle 23, \langle \varphi_i \mid i \in u \rangle \rangle.$$

The set PFml of primitive formulae is defined as before and the set Fml of all formulae is defined as before as the closure of PFml under quantification and sentential operations (including those just defined).

M-*finite predicate logic* has as axioms all instances of the usual predicate logical axiom schemata together with:

$$\bigwedge_{i\in u} \varphi_i \to \varphi_j \quad , \quad \varphi_j \to \bigvee_{i\in u} \varphi_i \quad \text{for } j\in u \in M.$$

The *rules of inference* are:

$$\frac{\varphi \, , \, \varphi \to \psi}{\psi} \quad \text{(modus ponens)}$$

$$\frac{\varphi \to \psi}{\varphi \to \bigwedge x \psi} \quad , \quad \frac{\psi \to \varphi}{\bigvee x \psi \to \varphi} \quad \text{for } x \notin Fr(\varphi)$$

$$\frac{\varphi \to \psi_i \ (i \in u)}{\varphi \to \bigwedge_{i\in u} \varphi_i} \quad , \quad \frac{\psi_i \to \varphi \ (i\in u)}{\bigvee_{i\in u} \psi_i \to \varphi}$$

We say that φ is *provable* from a set of statements A if φ is in the smallest set which contains A and the axioms and is closed under the rules of inference. We write $A \vdash \varphi$ to mean that φ is provable from A.
(Note: By the last rule, $\bigvee\emptyset \to \varphi$ for every φ, hence $\vdash \neg \bigvee \emptyset$. Similarly $\vdash \bigwedge \emptyset$.)

a formula is provable if and only if it has a proof. Because we have not assumed choice to hold in our admissible structure M, we must use a somewhat unorthodox concept of proof, however.

<u>Def</u> By a <u>proof</u> <u>from</u> A we mean a sequence $\langle p_i \mid i < \alpha \rangle$ s.t. $\alpha \in On$ and for each $i < \alpha$, if $\psi \in p_i$, then either $\psi \in A$ or ψ is an axiom or ψ follows from $\bigcup_{h<i} p_h$ by a single application of one of the rules.

$p = \langle p_i \mid i < \alpha \rangle$ is a <u>proof</u> <u>of</u> φ iff $\varphi \in \bigcup_{i<\alpha} p_i$.

If A is $\Sigma_1(M)$ in a parameter q it follows easily that $\{p \in M \mid p \text{ is a proof from } A\}$ is $\Sigma_1(M)$ in the same parameter.

It is also easily seen that $A \vdash \varphi$ iff there exists a proof of φ from A.

A more interesting conclusion is:

Lemma 1 Let A be $\Sigma_1(M)$. Then $A \vdash \varphi$ iff there is an M-finite proof of φ from A.

proof.

$(\leftarrow)$ is trivial. We prove $(\rightarrow)$.

Let X = the set of φ s.t. there exists a $p \in M$ which proves φ from A.

Claim $\{\varphi \mid A \vdash \varphi\} \subset X$.

proof.

We know that $A \subset X$ and all axioms lie in X. Hence it suffices to show that X is closed under the rules of proof. This must be demonstrated rule by rule. As an example we show:

Claim Let $\varphi \to \psi_i \in X$ for $i \in u$, where $u \in M$. Then $\varphi \to \bigwedge_{i \in u} \psi_i \in X$.

prf.

Let $P(p, \psi)$ mean: p is a proof of ψ from A. Then P is $\Sigma_1(M)$. By our assumption:

(1) $\bigwedge i \in u \bigvee p\ P(p, \varphi \to \psi_i)$.

Now let $P(p, \psi) \leftrightarrow \bigvee z\, P'(z, p, \psi)$, where P' is Σ_0. We then have:

(2) $\bigwedge i \in u\ \bigvee z\ \bigvee p\ P'(z, p, \varphi \to \psi_i)$

whence follows easily that there is $v \in M$ with:

(3) $\bigwedge i \in u\ \bigvee z \in v\ \bigvee p \in v\ P'(z, p, \varphi \to \psi_i)$

Set $w = \{p \in v \mid \bigvee i \in u\ \bigvee z \in v\ P'(z, p, \psi)\}$.

Then

(4) $\bigwedge i \in u\ \bigvee p \in w.\ P(p; \varphi \to \psi_i)$ and w consists of proofs from A.

Let $\alpha \in M$, $\alpha \geq \mathrm{dom}(p)$ for all $p \in w$. Define a proof p^* of length $\alpha + 1$ by:

$$p^*(i) = \begin{cases} \bigcup\{p_i \mid p \in w \wedge i \in \mathrm{dom}(p_i)\} & \text{for } i < \alpha \\ \{\varphi \to \bigwedge_{i \in u} \psi_i\} & \text{for } i = \alpha \end{cases}$$

Then $p^* \in M$ proves $\varphi \to \bigwedge_{i \in u} \psi_i$ from A. QED (Lemma 1)

From this we get the M-<u>finiteness lemma</u>:

<u>Lemma 2</u> Let A be $\Sigma_1(M)$. Then $A \models \varphi$ iff there is $u \in M$ s.t. $u \subset A$ and $u \models \varphi$.

prf.

$(\leftarrow)$ is trivial. We prove $(\rightarrow)$.

Let $p \in M$ be a proof of φ from A. Let u = the set of ψ s.t. for some $i \in \mathrm{dom}(p)$, $\psi \in p_i$, but ψ is not an axiom and does not follow from $\bigcup_{h<i} p_h$ by a single application of a rule. Then $u \in M$, $u \subset A$, and p is a proof from u. Hence $u \vdash \varphi$.

QED (Lemma 2)

Another consequence of Lemma 1 is

<u>Lemma 3</u> Let A be $\Sigma_1(M)$ in q. Then $\{\varphi \mid A \vdash \varphi\}$ is $\Sigma_1(M)$ in the same parameter q (uniformly in the Σ_1 definition of A from q).

proof of Lemma 3

$\{\varphi \mid A \vdash \varphi\} = \{\varphi \mid \bigvee p \in M \ p \text{ proves } \varphi \text{ from } A\}$.

QED

<u>Corollary 4</u> Let A be $\Sigma_1(M)$ in q. Then "A is consistent" is $\Pi_1(M)$ in the same parameter q (uniformly in the Σ_1 definition of A from q).

Note that, since $u \in M$ is uniformly $\Sigma_1(M)$ in itself, we have:

<u>Corollary 5</u> $\{\langle u, \varphi\rangle \mid u \in M \wedge u \vdash \varphi\}$ is $\Sigma_1(M)$

Similarly:

<u>Corollary 6</u> $\{u \in M \mid u \text{ is consistent}\}$ is $\Pi_1(M)$

<u>Note</u> Call a proof p <u>strict</u> iff $\bar{\bar{p}}_i = 1$ for $i \in \mathrm{dom}(p)$. This corresponds to the more usual notion of proof. If M satisfies the axiom of choice in the form: Every set is enumerable by an ordinal, then Lemma 1 holds with "strict proof" in place of "proof". We leave this to the reader.

<u>Languages</u>

We will normally not employ all of the predicates and constants in our M-finitary first order logic, but cut down to a smaller set of symbols which we intend to interpret in a model. Thus we define a <u>language</u> to be a set $\mathcal{L}$ of predicates and constants. By a <u>model</u> of $\mathcal{L}$ we mean a structure

$$\mathfrak{A} = \langle |\mathfrak{A}|, \langle t^{\mathfrak{A}} \mid t \in \mathcal{L} \rangle \rangle$$

s.t. $|\mathfrak{A}| \neq \emptyset$, $P^{\mathfrak{A}} \subset |\mathfrak{A}|^n$ whenever P is an n-place predicate, and $c^{\mathfrak{A}} \in |\mathfrak{A}|$ whenever $|\mathfrak{A}|$ is a constant. By a <u>variable</u> <u>assignment</u> we mean a map $f: Vbl \to \mathfrak{A}$ (Vbl being the set of all variables). The <u>satisfaction</u> <u>relation</u> $\mathfrak{A} \models \varphi[f]$ is defined in the usual way, where $\mathfrak{A} \models \varphi[f]$ means that the formula φ becomes true in $\mathfrak{A}$ if the free variables in φ are interpreted by f. We leave the definition to the reader, remarking only that:

$$\mathfrak{A} \models \bigwedge_{i \in u} \varphi_i [f] \quad \text{iff} \quad \bigwedge i \in u \ \mathfrak{A} \models \varphi_i [f]$$

$$\mathfrak{A} \models \bigvee_{i \in u} \varphi_i [f] \quad \text{iff} \quad \bigvee i \in u \ \mathfrak{A} \models \varphi_i [f].$$

We adopt the usual conventions of model theory, writing $\mathfrak{A} = \langle |\mathfrak{A}|, t_1^{\mathfrak{A}}, \ldots \rangle$ if we think of the predicates and constants of $\mathcal{L}$ as being arranged in a fixed sequence $t_1, t_2, \ldots$. Similarly, if $\varphi = \varphi(v_1, \ldots, v_n)$ is a formula in which at most the variables $v_1, \ldots, v_n$ occur free, we write: $\mathfrak{A} \models \varphi[x_1, \ldots, x_n]$ for: $\mathfrak{A} \models \varphi[f]$ where $f(v_i) = x_i$ $(i = 1, \ldots, n)$. If φ is a statement, we write: $\mathfrak{A} \models \varphi$. If A is a set of statements we write: $\mathfrak{A} \models A$ to mean: $\mathfrak{A} \models \varphi$ for all $\varphi \in A$.

The <u>correctness theorem</u> says that if A is a set of $\mathcal{L}$-statements and $\mathfrak{A} \models A$, then A is consistent. (We leave this to the reader.)

<u>Barwise' Completeness Theorem</u> says that the converse holds if our admissible structure M is countable:

Thm 7 Let M be a countable admissible structure. Let A be a set of statements in the M-language $\mathcal{L}$. If A is consistent in M-finite predicate logic, then A has a model $\mathfrak{A}$.

proof (sketch)

We make use of the following theorem of Rasiowa and Sikorski:

Let $\mathbb{B}$ be a Boolean algebra. Let $X_i \subset \mathbb{B}$ $(i<\omega)$ s.t. the Boolean union $\bigvee X_i = b_i$ exists in the sense of $\mathbb{B}$. Then $\mathbb{B}$ has an ultrafilter U s.t.

$$b_i \in U \longleftrightarrow X_i \cap U \neq \emptyset$$

for $i<\omega$.

(proof. Successively choose c_i $(i<\omega)$ by $c_0 = 1$, $c_{i+1} = c_i \cap b \neq 0$, where $b \in X_i \cup \{\neg b_i\}$. Let $\bar{U} = \{a \in \mathbb{B} \mid \bigvee_i\, c_i \subset a\}$. Then $\bar{U}$ is a filter and extends to an ultrafilter on $\mathbb{B}$.)

Extend the language $\mathcal{L}$ by adding an M-infinite set C of new constants. Call the extended language $\mathcal{L}^*$ and set:

$[\varphi] = \{\psi \mid A \vdash \psi \leftrightarrow \varphi\}$

for $\mathcal{L}^*$-statements φ. Then

$\mathbb{B} = \{[\varphi] \mid \varphi \in St_{\mathcal{L}^*}\}$

is the Lindenbaum algebra of $\mathcal{L}^*$ with the operations:

$[\varphi] \cup [\psi] = [\varphi \vee \psi]$

$[\varphi] \cap [\psi] = [\varphi \wedge \psi]$

$\neg[\varphi] = [\neg\varphi]$

$\bigcup_{i \in u} [\varphi_i] = [\bigvee_{i \in u} \varphi_i] \quad (u \in M)$

$\bigcap_{i \in u} [\varphi_i] = [\bigwedge_{i \in u} \varphi_i] \quad (u \in M)$

$\bigcup_{c \in C} [\varphi(c)] = [\bigvee v\, \varphi(v)]$

$\bigcap_{c \in C} [\varphi(c)] = [\bigwedge v\, \varphi(v)]$

The last two equations hold because the constants in C, which do not occur in the axioms A, behave like free variables. By Rasiowa and Sikorski there is then an ultrafilter U on $\mathbb{B}$ which respects the above operations. We define a model $\mathfrak{A} = \langle |\mathfrak{A}|, \langle t^{\mathfrak{A}} \mid t \in \mathcal{L} \rangle \rangle$ as follows:

For $c \in C$ set $[c] = \{c' \in C \mid [c = c'] \in U\}$.
If $P \in \mathcal{L}$ is an n-place predicate, set:

$P^{\mathfrak{A}}([c_1], \ldots, [c_n]) \longleftrightarrow [P c_1 \ldots c_n] \in U$,

If $t \in \mathcal{L}$ is a constant set:

$t^{\mathfrak{A}} = [c]$, where $c \in C$, $[t = c] \in U$.

A straightforward induction then shows:

$\mathfrak{A} \models \varphi[[c_1], \ldots, [c_n]] \longleftrightarrow [\varphi(c_1, \ldots, c_n)] \in U$

for formulae $\varphi = \varphi(v_1, \ldots, v_n)$ with at most the free variables $v_1, \ldots, v_n$. In particular

$\mathfrak{A} \models \varphi \longleftrightarrow [\varphi] \in U$ for $\mathcal{L}^*$-statements φ. Hence $\mathfrak{A} \models A$, since $[\varphi] = 1$ for all $\varphi \in A$. QED (Thm 7)

Combining the completeness thm with the M-finiteness lemma, we get the well known <u>Barwise compactness theorem</u>:

<u>Cor 8</u> Let M be countable. Let $\mathcal{L}$ be Δ_1 and A be Σ_1. If every M-finite subset of A has a model, then so does A.

By a <u>theory</u> or <u>axiomatized language</u> we mean a pair $\mathcal{L} = \langle \mathcal{L}_0, A \rangle$ s.t. $\mathcal{L}_0$ is a language and A a set of $\mathcal{L}_0$-statements. We say that $\mathfrak{A}$ <u>models</u> $\mathcal{L}$ iff $\mathfrak{A}$ is a model of $\mathcal{L}_0$ and $\mathfrak{A} \models A$. We also write:
$\mathcal{L} \vdash \varphi$ for $(A \vdash \varphi \wedge \varphi \in \mathrm{Fml}_{\mathcal{L}_0})$
We say that $\mathcal{L} = \langle \mathcal{L}_0, A \rangle$ is $\Sigma_1(M)$ (in the parameter p) iff $\mathcal{L}_0$ is $\Delta_1(M)$ (in p) and A is $\Sigma_1(M)$ (in p).
Similarly for: $\mathcal{L}$ is $\Delta_1(M)$ (in p).

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We now consider the class of axiomatized languages containing a fixed predicate $\dot{\in}$, the special constants $\underline{x}$ $(x \in M)$ and the <u>basic axioms</u>

- Extensionality
- $\bigwedge v (v \dot{\in} \underline{x} \longleftrightarrow \bigvee_{z \in x} v = \underline{z})$ $(x \in M)$

(Further predicates, constants, and axioms are allowed, of course.)
We call any such theory an "$\in$-theory".

Then:

<u>Lemma 9</u> Let $\mathfrak{A}$ be a solid model of the $\in$-theory $\mathcal{L}$. Then $\underline{x}^{\mathfrak{A}} = x \in \mathrm{wfc}(\mathfrak{A})$ for $x \in M$.

proof. $\in$-induction on x.

<u>Def</u> Let $\mathcal{L}$ be an $\in$-theory.
$ZF^-_{\mathcal{L}}$ is the set of (really) finite $\mathcal{L}$-statements which are axioms of $\mathcal{L}$.
(Similarly for $ZFC^-_{\mathcal{L}}$).
We write $\mathcal{L} \vdash ZF^-$ for $\mathcal{L} \vdash ZF^-_{\mathcal{L}}$. (Similarly for $\mathcal{L} \vdash ZFC^-$.)

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$\in$-theories are a useful tool in set theory. We now bring some typical applications.
We recall that an ordinal α is called <u>admissible</u> if L_α is admissible and <u>admissible in</u> $a \subset \alpha$ if $L^a_\alpha = \langle L_\alpha[a], a\rangle$ is admissible.

<u>Lemma 10</u> Let $\alpha > \omega$ be a countable admissible ordinal. There is $a \subset \omega$ s.t. α is the least ordinal admissible in a.

This follows straightforwardly from:

<u>Lemma 11</u> Let M be a countable admissible structure. Let $\mathcal{L}$ be a consistent $\Sigma_1(M)$ $\mathcal{G}$-theory s.t. $\mathcal{L} \vdash ZF^-$. Then $\mathcal{L}$ has a solid model $\mathfrak{A}$ s.t. $On \cap wfc(\mathfrak{A}) = On \cap M$.

We first show that Lemma 11 implies Lemma 10, and then prove Lemma 11.

Take $M = L_\alpha$, where α is as in Lemma 10.

Let $\mathcal{L}$ be the M-theory with:

<u>Predicate</u> $\dot{\in}$, <u>Constants</u> $\underline{x}$ $(x \in M)$, $\mathring{a}$

<u>Axioms</u> Basic axioms $+$ ZF, and
$\underline{\beta}$ is not admissible in $\mathring{a}$ $(\beta < \alpha)$.

Then $\mathcal{L}$ is consistent, since $\langle H_{\omega_1}, \in, a\rangle$ is a model, where a is any $a \subset \omega$ which codes a well ordering of type $\geq \alpha$ (and $\underline{x}$ is interpreted by x for $x \in M$). Now let $\mathfrak{A}$ be a solid model of $\mathcal{L}$ s.t. $On \cap wfc(\mathfrak{A}) = \alpha$. Then $wfc(\mathfrak{A})$ is admissible by §1 Lemma 22. Hence so is L_α^a, where $a = \mathring{a}^{\mathfrak{A}}$. But β is not admissible in a for $\omega < \beta < \alpha$, since "β is admissible in a" is $\Sigma_1(L^a_\alpha)$; hence the same Σ_1 statement would hold of β in $\mathfrak{A}$. Contr! QED (Lemma 10)

<u>Note</u> Pursuing this method a bit further we can prove: Let $\omega < \alpha_0 < \cdots < \alpha_{n-1}$ be a sequence of countable admissible ordinals. There is $a \subset \omega$ s.t. α_i = the i-th $\alpha > \omega$ which is admissible in a $(i < n)$. A similar Theorem holds for countable infinite sequences, but the proof requires forcing and is much more complex. It is given in §5, §6.

We now prove Lemma 11 by modifying the proof of the completeness theorem. Let $\Gamma(\sigma)$ be the set of formulae $\sigma \in On$, $\sigma > \underline{\beta}$ $(\beta \in M)$. Add an M-infinite (but $\Delta_1(M)$) set E of new constants to $\mathcal{L}$. Let $\mathcal{L}'$ be $\mathcal{L}$ with the new constants and the new axioms $\Gamma(e)$ $(e \in E)$. Then $\mathcal{L}'$ is consistent, since any M-finite subset of the axioms can be modeled by interpreting the new constants as ordinals in $wfc(\mathfrak{A})$,

$\mathfrak{M}$ being any solid model of $\mathcal{L}$. As in the proof of completeness we then add a new class C of constants which is not M-finite. We assume, however, that C is $\Delta_1(M)$. We add no further axioms, so the elements of C behave like free variables. The so extended language $\mathcal{L}''$ is clearly $\Sigma_1(M)$. Now set:

$$\Delta(v) = \{v \notin On\} \cup \bigcup_{\beta \in M} \{v \leq \beta\} \cup \bigcup_{e \in E} \{e < v\}.$$

<u>Claim</u> Let $c \in C$. Then $\bigcup \{[\varphi] \mid \varphi \in \Delta(c)\} = 1$ in the Lindenbaum algebra of $\mathcal{L}''$

proof. Suppose not. Set $\Delta' = \{\neg\varphi \mid \varphi \in \Delta(\ )\}$. Then there is an $\mathcal{L}''$ statement ψ s.t. $[\psi] \neq 1$ (i.e. $A \cup \{\psi\}$ is consistent, where $\mathcal{L}'' = \langle \mathcal{L}_0'', A \rangle$) and $A \cup \{\psi\} \vdash \Delta'$.
Pick an $e \in E$ which does not occur in ψ. Let A^* be the result of omitting the axioms $\Gamma(e)$ from A. Then $A^* \cup \{\psi\} \cup \Gamma(e) \vdash c \leq e$. By the M-finiteness lemma there is $\beta \in M$ s.t. $A^* \cup \{\psi\} \cup \{\beta \leq e\} \vdash c \leq e$.

But e behaves here like a free variable, so $A^* \cup \{\psi\} \vdash c \leq \underline{\beta}$. But $A \supset A^*$ and $A \cup \{\psi\} \vdash \underline{\beta} < c$. Thus $A \cup \{\psi\}$ is inconsistent. Contr! QED (Claim)

Now let $\mathcal{U}$ be an ultrafilter on the Lindenbaum algebra of $\mathcal{L}''$ which respects both the operations listed in the proof of the completeness theorem and the unions $\bigcup\{[\varphi] \mid \varphi \in \Delta(c)\}$ for $c \in C$. Let $X = \{\varphi \mid [\varphi] \in \mathcal{U}\}$. Then, as before, $\mathcal{L}''$ has a model $\mathfrak{A}$, all of whose elements have the form $c^{\mathfrak{A}}$ for a $c \in C$ and such that $\mathfrak{A} \models \varphi \Longleftrightarrow \varphi \in X$ for $\mathcal{L}''$-statements φ. We assume w.l.o.g. that $\mathfrak{A}$ is solid. It suffices to show that $Y = \{x \in \mathfrak{A} \mid x > \underline{\nu}$ in $\mathfrak{A}$ for all $\nu \in M\}$ has no minimal element in $\mathfrak{A}$. Let $x \in Y$, $x = c^{\mathfrak{A}}$. Then $\mathfrak{A} \models e < c$ for some $e \in E$. But $e^{\mathfrak{A}} \in Y$.

QED (Lemma 11)

Another - very typical - application is:

<u>Lemma 12</u> Let W be an inner model of ZFC. Suppose that, in W, U is a normal measure on κ. Let $\tau > \kappa$ be regular in W and set $M = \langle H_\tau^W, U \rangle$. Assume that M is countable in V. Then for any $\alpha \leq \kappa$ there is $\bar{M} = \langle \bar{H}, \bar{U} \rangle$ s.t. $\bar{U}$ is a normal measure in $\bar{M}$ and $\bar{M}$ iterates to M in exactly α many steps. (Hence $\bar{M}$ is iterable, since M is.)

proof.

The case $\alpha = 0$ is trivial, so assume $\alpha > 0$. Let δ be least s.t. $L_\delta(M)$ is admissible. Then $N = L_\delta(M)$ is countable. Let $\mathcal{L}$ be the $\in$-theory on N with:

<u>Predicate</u>: $\dot{\in}$ <u>Constants</u> $\underline{x}$ $(x \in N)$, $\dot{M}$

<u>Axioms</u> The basic axioms; ZFC$^-$; $\dot{M} = \langle \dot{H}, \dot{U} \rangle$ is a transitive ZFC$^-$ model; $\dot{M}$ iterates to $\underline{M}$ in $\underline{\alpha}$ many steps.

It suffices to prove:

<u>Claim</u> $\mathcal{L}$ is consistent.

We first show that the Claim implies the theorem. Let $\mathfrak{A}$ be a solid model of $\mathcal{L}$. Then $N \subset \mathrm{wfc}(\mathfrak{A})$

Hence $M, \bar{M} \in wfc(\mathfrak{A})$, where $\bar{M} = \dot{M}^{\mathfrak{A}}$. There is $\langle \bar{M}_i \mid i < \alpha \rangle$ which, in $\mathfrak{A}$, is an iteration from $\bar{M}$ to M. But then $\langle \bar{M}_i \mid i < \alpha \rangle \in$ $\in wfc(\mathfrak{A})$ really is an iteration by absoluteness. QED

We now prove the Claim.

<u>Case 1</u> $\alpha < \kappa$.

Iterate $\langle W, U \rangle$ α many times, getting $\langle W_i, U_i \rangle$ $(i \leq \alpha)$ with iteration maps $\pi_{ij} : \langle W_i, U_i \rangle \prec \langle W_j, U_j \rangle$.

Set $M_i = \pi_{0i}(M)$. Then $\langle M_i, U_i \rangle$ $(i \leq \alpha)$ is the iteration of $\langle M, U \rangle$ with maps $\pi'_{ij} = \pi_{ij} \upharpoonright M_i$. It suffices to show that $\mathcal{L}_\alpha = \pi_{0\alpha}(\mathcal{L})$ is consistent. This is clear, however, since $\langle H_{\tau^+}, M \rangle$ models $\mathcal{L}_\alpha$ (with M interpreting the constant $\dot{M}_\alpha = \pi_{0\alpha}(\dot{M})$).

QED (Case 1)

<u>Case 2</u> $\alpha = \kappa$.

This time we iterate $\langle W, U \rangle$ β many times where $\pi_{0\beta}(\kappa) = \beta$ and $\beta \leq \kappa^+$. $\langle H_{\tau^+}, M \rangle$ again models $\mathcal{L}_\beta$.

QED (Lemma 12)

Barwise theory is useful in situations where one is given a transitive structure Q and wishes to find a transitive structure $\bar{Q}$ with similar properties inside an inner model. Another tool used in such situations is Schoenfield's lemma, which, however requires coding $\bar{Q}$ by a real. Unsurprisingly, Schoenfield's lemma can itself be derived from Barwise theory. We first note the well known fact that every Σ^1_2 condition on a real is equivalent to a $\Sigma_1(H_{\omega_1})$ condition, and conversely. Thus it suffices to show:

<u>Lemma 13</u> Let $H_{\omega_1} \models \varphi[a]$, $a \subset \omega$, where φ is Σ_1. Then

$$H_{\omega_1}^{L[a]} \models \varphi[a]$$

proof.

Let $\varphi = \bigvee z\, \psi$, where ψ is Σ_0. Let $H_{\omega_1} \models \psi[z, a]$, where $rn(z) < \alpha$ and α is admissible in a. Let $\mathcal{L}$ be the language on $L_\alpha(a)$ with:

<u>Predicate</u> $\dot{\in}$, <u>Constants</u> $\dot{z}$, $\underline{x}$ $(x \in L_\alpha(a))$

<u>Axioms</u> Basic axioms, ZFC^-, $\psi(\dot{z}, \underline{a})$.

Then $\mathcal{L}$ is consistent in $\langle H_{\omega_1}, \dot{z} \rangle$ in a model. Applying Löwenheim - Skolem in $L(a)$, we find a countable $\bar{\alpha}$ and a map $\pi : L_{\bar{\alpha}}(a) \prec L_\alpha(\alpha)$. Let w.l.o.g. $\pi(\dot{z}) = \dot{z}$ and let $\bar{\mathcal{L}}$ be defined over $L_{\bar{\alpha}}(a)$ like $\mathcal{L}$ over $L_\alpha(a)$. Then $\bar{\mathcal{L}}$ is consistent and has a solid model $\mathfrak{A}$ in $L(a)$. Let $z = \dot{z}^{\mathfrak{A}}$. Then $z \in L(a)$ and $H_{\omega_1} \models \psi[z, a]$ in $L(a)$. QED (Lemma 13)