## §6 Forcing with classes of conditions

The more interesting applications of forcing to admissible sets $M$ generally involve a class of conditions rather than an $M$-finite set. As an example:

<u>Theorem 1</u> If $M = \langle |M|; \in, A \rangle$ is a countable admissible structure and $A \subset On$, then there is an $F \subset On^3$ s.t. $N = \langle M[F]; A, F \rangle$ is admissible and every ordinal $\nu \in On_M$ is countable by a map $f \in N$.

proof.

Let $\rho_\nu$ ($\nu \in On$) enumerate monotonically the p.r. closed ordinals. The $F$ which we shall adjoin will have the form:

$$F = \{ \langle f_\nu(m), \rho_\nu, m \rangle \mid \nu \in On_M, m < \omega \},$$

where $f_\nu$ maps $\omega$ onto $\rho_\nu$.

Our conditions are pieces of information which fix the function F at finitely many places:

$\mathbb{P} =_{Df}$ the set of finite maps $p$ s.t. $\text{dom}(p) \subset \{ f_\nu \mid \nu \in On_M \} \times \omega$

and $p(f_\nu, m) < \rho_\nu$ whenever defined.

$p \leq q \longleftrightarrow_{Df} p \supset q$.

We also define:

$$\mathbb{P}_\nu =_{Df} \{ p \in \mathbb{P} \mid \text{dom}(p) \subset f_\nu \times \omega \}.$$

**Lemma 1**  If $C \subset \mathbb{P}_\nu$ is compatible in $\mathbb{P}_\nu$, then $C$ is compatible in $\mathbb{P}$.

proof.

Let $p \in \mathbb{P}$. Then $p' = p \restriction (f_\nu \times \omega)$ is compatible with some $q \in C$. It follows that $p$ is compatible with $q$.    QED

For $G \subset \mathbb{P}$ set: $G_\nu =_{Df} G \cap \mathbb{P}_\nu$.

<u>Lemma 2</u> If $G$ is $\mathbb{P}$-generic over $M$, then $G_\nu$ is $\mathbb{P}_\nu$-generic over $M$. More generally, if $M'$, $M''$ are transitive structures s.t. $\mathbb{P}_\nu$ is $M''$-definable and $\mathbb{P}$, $M''$ are $M'$-definable, then if $G$ is $\mathbb{P}$-generic over $M'$, $\mathbb{P}_\nu$ is $G_\nu$-generic over $M''$.

proof.

Let $C \subset \mathbb{P}_\nu$ be compatible in $\mathbb{P}_\nu$ and $M''$-definable. Then $C$ is compatible in $\mathbb{P}$ and $M'$-definable. Hence $G_\nu \cap C = G \cap C \neq \emptyset$

QED

Now set: $\mathfrak{M} =_{Df} \langle M^{\mathbb{P}}; I, E, \check{A}, \dot{G} \rangle$,

where $\check{A}(x) = \bigvee_{\nu \in A} [\![ x \equiv \check{\nu} ]\!]$

$\dot{G}(x) = \bigvee_{p \in \mathbb{P}} [\![ x \equiv \check{p} ]\!] \wedge [p]$ .

Clearly, if $G$ is $\mathbb{P}$-generic, then $N = \mathcal{M}/G$ has the form:

$$N = \langle |N|; A, G \rangle .$$

Define models $M_\nu$ ($\nu \in On_M$) by:

$$M_\nu = \langle |M_\nu|, A \cap \rho_\nu \rangle$$

where $|M_\nu| = \{ x \in M \mid \operatorname{rn}(x) < \rho_\nu \}$, $|M_\nu|$ is p.r. closed.

Define $BA(\mathbb{P}_\nu)$-valued structures

$$\mathcal{M}_\nu = \langle M_\nu^{\mathbb{P}_\nu}; I_\nu, E_\nu, \check{A}_\nu, \dot{G}_\nu \rangle$$

in analogy with $\mathcal{M}$, using $\mathbb{P}_\nu$, $A \cap \rho_\nu$ in place of $\mathbb{P}, A$.

Then $N_\nu = \mathcal{M}_\nu / G_\nu$ has the form:

$$N_\nu = \langle |N_\nu|; A \cap \rho_\nu, G_\nu \rangle .$$

Clearly: $G_\nu^*(x) = G^*(x)$ for $x \in M_\nu^{\mathbb{P}_\nu}$.

Hence $N_\nu \subseteq N_\mu \subseteq N$ for $\nu \le \mu < On_M$.

For $\nu = \tau + 1$, we have: $\pi_\nu \in M_\nu$,

hence: $|N_\nu| = M_\nu[G_\nu]$.

Thus:

$\underline{\text{Lemma 3}}$   $|N| = M[G]$ ; $N_\nu = M_\nu[G_\nu]$

for $\nu < On_M$.

proof.

For $\nu = \tau + 1$, the theorem is

proven.

$|N| \subset M[G]$, since $|N| = G^* ``M''$

and $f(x) = G^*(x)$ is p.r. in $G$.

But $M[G] = \bigcup_{\tau \in On_M} M_{\tau+1}[G_{\tau+1}] \subset |N|$.

Similarly we prove: $|N_\lambda| = M_\lambda[G_\lambda]$

for limit $\lambda$.                    QED

Let $\Vdash_{\mathfrak{M}}$ be the forcing relation of $\mathfrak{M}$ + let $\Vdash_{\mathfrak{M}}^{\Sigma_0}$ be its restriction to $\Sigma_0$ fmlae. Similarly for $\Vdash_{\mathfrak{M}_\nu}$, $\Vdash_{\mathfrak{M}_\nu}^{\Sigma_0}$. We do not yet know that $\Vdash_{\mathfrak{M}}^{\Sigma_0}$ is $M$-definable, but we can show:

Lemma 4 $\{\langle \nu, p, \varphi \rangle \mid p \Vdash_{\mathfrak{M}_\nu}^{\Sigma_0} \varphi \}$ is $\Delta_1$

proof. Since $\Vdash_{\mathfrak{M}_\nu}^{\Sigma_0}$ is uniformly p.r. in $\check{A}$, $\mathring{C}$, and the parameter $\mathbb{P}_\nu$, it suffices to note that the functions

$$f_0(\nu) = \mathbb{P}_\nu, \quad f_1(\nu, x) = \check{A}_\nu(x),$$
$$f_3(\nu, x) = \mathring{G}_\nu(x) \text{ are } \Delta_1. \quad \text{QED}$$

The fact that $\Vdash_{\mathfrak{M}}^{\Sigma_0}$ is $\Delta_1$ now follows from :

Lemma 5   If $\varphi$ is a $\Sigma_0$ fmla of $\mathfrak{M}_\nu$ and if $p \in \mathbb{P}_\nu$, then

$$p \Vdash_{\mathfrak{M}_\nu} \varphi \longleftrightarrow p \Vdash_{\mathfrak{M}} \varphi .$$

proof.

By Lemma 2 and the completeness theorem, it suffices to show that for every $G$ which is $\mathbb{P}$-generic over $M$ we have :

$$G \Vdash_{\mathfrak{M}} \varphi \longleftrightarrow G_\nu \Vdash_{\mathfrak{M}_\nu} \varphi ,$$

or equivalently :

$$\models_{\mathfrak{M}_\mathfrak{g}/G_\mathfrak{g}} \varphi^G \longleftrightarrow \models_{\mathfrak{M}_\nu/G_\nu} \varphi^{G_\nu} .$$

Since $G^*(x) = G_\nu^*(x)$ for $x \in M_\nu^{\mathbb{P}_\nu}$, we may conclude : $\varphi^G = \varphi^{G_\nu}$.

By Lemma 3 we have:

$$\mathcal{N}/G = \langle M[G] ; A, G \rangle$$

$$\mathcal{N}_\gamma / G_\gamma = \langle M_\gamma [G_\gamma] ; A \cap \rho_\gamma , G_\gamma \rangle .$$

But $A \cap M_\gamma [G_\gamma] = A \cap \rho_\gamma$ ; $G \cap M_\gamma [G_\gamma] = G_\gamma$ ,

Since $\varphi^G$ in $\Sigma_o$ , the conclusion

follows immediately.          QED

An immediate corollary is:

<u>Lemma 6</u>   $\Vdash_{\mathcal{N}}^{\Sigma_o}$ in $\Delta_1$ .

  proof.

$$p \Vdash_{\mathcal{N}} \varphi \longleftrightarrow p \Vdash_{\mathcal{N}_{\eta(p,\varphi)}} \varphi ,$$

where $\eta(p,\varphi) = $ the least $\eta$

s.t. $p \in \mathbb{P}_\eta$ , $\varphi \in Fml_{\mathcal{N}_\eta}$ .

                    QED

We must now show that $N = \mathfrak{N}/G$ is admissible - i.e. that $N$ satisfies the replacement axiom for $\Sigma_0$ - fmlae. In proving this, we shall make use of:

**Lemma 7** Let $D \subset \mathbb{P}$ be $\Sigma_1$ and compatible in $\mathbb{P}$. Let $\nu \in On_M$. Then for some $\beta \geq \nu$, $D_\beta = D \cap \mathbb{P}_\beta$ is compatible in $\mathbb{P}_\beta$ (hence in $\mathbb{P}$).

(Note that $D_\beta$ is not necessarily an element of $M$, since $D$ is $\Sigma_1$ rather than $\Delta_1$).

proof.

Set: $\sigma(p) = \mu\tau \; \forall q \in \mathbb{P}_\tau$ ($p$ is compatible with $q$)

$$\eta(\tau) = \sup_{p \in \mathbb{P}_\tau} \sigma(p)$$

Then $\beta = \eta^\omega(\nu)$ satisfies the conclusion of the lemma.    QED

**Lemma 8** If $\varphi$ is a $\Sigma_0$ $M$-fmla, then:

$$p \Vdash \bigvee \sigma\, \varphi \;\longrightarrow\; \bigvee x \in M^{\mathbb{P}}\; p \Vdash \bigvee \sigma \in x\; \varphi.$$

proof.

~~The direction~~ Let $p \Vdash \bigvee \sigma\, \varphi$. Then

$$D = \neg[p] \cup \{q \leq p \mid \bigvee y\; q \Vdash \varphi(\underline{y})\} \text{ is}$$

dense in $\mathbb{P}$. Let $p \in \mathbb{P}_\nu$. Then there is a $\beta \geq \nu$ s.t. $D \cap \mathbb{P}_\beta$ is dense in $\mathbb{P}_\beta$. Thus:

$$\bigwedge_{\substack{q \leq p \\ \in \mathbb{P}_\beta}} \bigvee y \bigvee_{\substack{q' \leq q \\ \in \mathbb{P}_\beta}} q' \Vdash \varphi(y).$$

By admissibility there is a $u \in M$ s.t.

$$\bigwedge_{\substack{q \leq p \\ \in \mathbb{P}_\beta}} \bigvee y \in u \bigvee_{\substack{q' \leq q \\ \in \mathbb{P}_\beta}} q' \Vdash \varphi(y).$$

~~But since every $p' \leq p$ it con-~~

~~We may complete:~~

$(*) = \bigwedge q$

Thus $C = \{ q \in \mathbb{P}_\beta \mid \forall y \in u \ \ q \Vdash \varphi(\underline{y}) \}$

is dense in $[p] \cap \mathbb{P}_\beta$, hence in $\mathbb{P}_\beta$.

Thus:

$(*)$ $\bigwedge q \leq p \ \bigvee q' \leq q \ \forall y \in u \ \ q' \Vdash \varphi(\underline{y})$.

Set: $x = \{ \langle 1, y \rangle \mid y \in u \wedge M^{\mathbb{P}} \}$. Then:

$$p \Vdash \forall v \in \underline{x} \ \varphi.$$

QED

The fact that $N$ is admissible follows by Lemma 8 and the fact that $\Vdash_{\mathcal{H}}^{\Sigma_0}$ is $\Delta_1$. We state this fact as a general lemma, for later reference:

<u>Lemma 9</u>. If $M$ is an admissible structure, $\mathbb{P}$ is a $\Delta_1$ system of conditions and $\mathcal{H} = \langle M^{\mathbb{P}}; I, E, A_1, \ldots, A_m \rangle$ is a $BA(\mathbb{P})$-valued model s.t. $\Vdash_{\mathcal{H}}^{\Sigma_0}$ is $\Delta_1$ ~~this~~ and

if $p \Vdash \forall x\, \varphi \rightarrow \forall u\, p \Vdash \forall x \varepsilon \underline{u}\, \varphi$ for every $\varphi \in \mathrm{Fml}_{\mathfrak{M}}^{\Sigma_0}$, then $\mathfrak{M}/G$ is admissible for $\mathbb{P}$-generic $G$.

proof.

It suffices to show that the $\Sigma_0$ replacement axiom is forced; i.e. if $\varphi$ is a $\Sigma_0$ fmla of $\mathfrak{M}$, then

$$\Vdash \left( \bigwedge_x \bigvee_y \varphi \rightarrow \bigwedge_u \bigvee_\sigma \bigwedge_{x \varepsilon u} \bigvee_{y \varepsilon \sigma} \varphi \right),$$

or equivalently:

$$p \Vdash \bigwedge_x \bigvee_y \varphi \longrightarrow p \Vdash \bigwedge_u \bigvee_\sigma \bigwedge_{x \varepsilon u} \bigvee_{y \varepsilon \sigma} \varphi$$

for $p \in \mathbb{P}$. Let $p \Vdash \bigwedge_x \bigvee_y \varphi$. Then:

$$\bigwedge_{x \in M^\mathbb{P}} \bigvee_{\sigma \in M^\mathbb{P}} p \Vdash \bigvee_{y \varepsilon \underline{\sigma}} \varphi(\underline{x}, y).$$

Let $u \in M^\mathbb{P}$. Then there is a $w \in M^\mathbb{P}$ s.t.

$$\bigwedge_{x \in \mathrm{dom}(u)} \bigvee_{\sigma \in w} p \Vdash \bigvee_{y \varepsilon \underline{\sigma}} \varphi(\underline{x}, y).$$

Set: $\sigma' = \bigcup w$. Then:

$$p \Vdash \bigwedge_{x \varepsilon \underline{u}} \bigvee_{y \varepsilon \underline{\sigma}'} \varphi. \qquad QED$$

Thus $N = \langle M[G], A, G \rangle$ is admissible.
The only remaining step ~~is~~ in the
proof of Thm 1 is to show that
$N$ is equivalent to a structure
$N' = \langle M[F]; A, F \rangle$, where $F \subset On_M^3$
is as above.

(Equivalence ~~here~~, of course, means:
$|N| = |N'|$ and the $\Delta_1$ relations of
$N$ are the $\Delta_1$ relations of $N'$).

$F = \cup G$ obviously has the re-
quired properties. F (as a
relation) is p.r. in G since:

$$\langle \varkappa, \nu, m \rangle \in F \longleftrightarrow \{\langle \varkappa, \nu, m \rangle\} \in G.$$

Conversely, G is p.r. in F
since:

$$p \in G \longleftrightarrow p \subset F.$$

This completes the proof of
Theorem 1.

**Def** Call $\alpha$ **admissible** in $A_1, \ldots, A_m \subset On$ iff

$$\langle L_\alpha[A_1, \ldots, A_m] ; A_1 \cap \alpha, \ldots, A_m \cap \alpha \rangle$$

is an admissible structure.

As a corollary to Thm 1, we obtain:

**Theorem 2** Let $\lambda$ be a $\overbrace{\text{countable}}$ limit of admissible ordinals. Let $A \subset \lambda$. Then there is a $B \subset \lambda$ s.t.

(a) If $\alpha < \lambda$ is admissible in $C \subset \alpha$ and $C$ is $\langle L_\lambda[A], A \rangle$ - definable then $\alpha$ is admissible in $B, C$.

(Thus, in particular, if $\alpha < \lambda$ is admissible in $A$, $\alpha$ is admissible in $A, B$).

(b) $\rho_\nu$ is countable by a map $f \in L_{\rho_{\nu+1}}[B]$ for $\nu < \lambda$.

proof of Thm 2.

As before, force with conditions:

$\mathbb{P}$ = the set of finite maps $p$
s.t. $\text{dom}(p) \subset \{p_\nu \mid \nu < \lambda\} \times \omega$
and $p(p_\nu, m) < p_\nu$ if defined.

Let $G$ be $\mathbb{P}$-generic over $\langle L_\lambda[A], A\rangle$.
Let $F = \bigcup G$. Then

$$F = \{\langle f_\nu(m), p_\nu, m\rangle \mid \nu < \lambda, m < \omega\},$$

where $f_\nu$ maps $\omega$ onto $p_\nu$.

Set: $B = \{\langle \alpha, \beta, \gamma\rangle \mid \langle \alpha, \beta, \gamma\rangle \in F\}$.

(b) follows immediately.

To prove (a), we note that
$G_\alpha$ is $\mathbb{P}_\alpha$-generic over $\langle L_\alpha[c], c\rangle$.
Hence, by the proof of Thm 1,
$\langle L_\alpha[c, B], c, B \cap \alpha\rangle$ is admissible.

.QED

We now give a second application
of forcing ~~of~~ with a class of con-
ditions to add, as before, a
class of ordinals to an ad-
missible structure M. This
application will differ from the
previous one, however, in that
no new <u>sets</u> will be added
to the structure.

<u>Def</u> $\alpha$ is <u>Mahlo</u> (or <u>recursively Mahlo</u>)
in $A \subset \alpha$ iff $\alpha$ is admissible in $A$ and
every ~~⬛~~ normal function which is
$\Delta_1$ in $\langle L_\alpha[A], A \bullet \rangle$ takes a value
which is admissible in $A$.

**Theorem 3**   Let $\alpha$ be Mahlo in $A$.
Assume that for all $\beta < \alpha$,
$\beta$ is countable in $L_{\beta'}[A]$,
where $\beta' =$ the least p.r. closed
$\beta' > \beta$. Then there is a $B \subset \alpha$
s.t.

(i) $\langle L_\alpha[A], A, B \rangle$ is admissible

(ii) If $\gamma < \alpha$ is admissible in $A$,
   then $\gamma$ is admissible in $A, B$

(iii) $\alpha$ is non Mahlo in $A, B$.

proof.
It suffices that $B$ satisfy
(i) and

(iv) $B$ is unbounded in $\alpha$, but if
   $\gamma < \alpha$ is admissible in $A$, then
   $B \cap \gamma \in L_\gamma[A]$.

(ii) obviously follows from (iv).
To see that (iii) follows,

note that the monotone enumeration
of the limit pts. of B in a
$\Delta_1$ normal function (in $\langle L_\alpha[A], A, B \rangle$)
~~containing no~~ none of whose
values are admissible in A.

Our conditions will describe
initial segments of the set
B which we wish to add:

$\mathbb{P} =_{Df}$ the collection of bounded
subsets $p$ of $\alpha$ s.t.

  (i) For all $\beta < \alpha$, $p \cap \beta \in L_{\beta'}[A]$,
    where $\beta' = $ the least p.s.
    closed $\beta' > \beta$.

  (ii) If $\beta \leq \alpha$ is admissible in A,
    then $p \cap \beta \in L_\beta[A]$.

  $p \leq q \longleftrightarrow_{Df} q = p \cap \sup\{\nu + 1 \mid \nu \in q\}$.

Set: $\mathcal{M} = \langle L_\alpha[A]^{\mathbb{P}}; \; I, E, \check{A}, \mathring{B} \rangle$,

where: $\check{A}(x) = \bigcup_{\nu \in A} [\![ x \equiv \check{\nu} ]\!]$

$$\mathring{B}(x) = \bigcup_{\nu < \alpha} \left( [\![ x \equiv \check{\nu} ]\!] \wedge \{ p \in \mathbb{P} \mid \nu \in p \} \right).$$

Then $N = \mathcal{M}/G = \langle |N|; A, B \rangle$, for $\mathbb{P}$-generic $G$, where $B$ is an unbounded subset of $\alpha$ satisfying (iv).

We must show that $N$ is admissible and that $|N| = L_\alpha[A]$.

**Lemma 1.** If $G$ is $\mathbb{P}$-generic, then

$$|\mathcal{M}/G| = L_\alpha[A].$$

proof.

Let $\beta < \alpha$ be p.r. closed.

Let $p = \bigcup G \cap \beta$. Then $p \in G$.

For $x \in L_\beta[A]^{\mathbb{P}}$ set:

$$p^*(x) = \{ p^*(y) \mid \forall q \geq p \; \langle q, y \rangle \in x \}.$$

Since $G \cap L_\beta[A] = \{ q \in L_\beta[A] \mid q \geq p \}$,

we have:

$p^*(x) = G^*(x)$ for $x \in L_\beta[A]^{\mathbb{P}}$.

But $p^* \restriction L_\beta[A]^{\mathbb{P}} \in L_\alpha[A]$.     QED

**Lemma 2** $\Vdash^{\Sigma_0}_{\mathfrak{M}}$ is $\Delta_1$ in $\langle L_\alpha[A], A \rangle$.

proof.

Let $\beta < \alpha$ be ~~a limit of~~ p.r. closed.
Let $\beta'$ be the next largest
p.r. closed ordinal. Let
$p \in L_{\beta'}[A]$, $p \in \mathbb{P}$ be s.t.
$\sup(p) \geq \beta$. Then for any $\mathbb{P}$-gen-
eric $G$ s.t $p \in G$ we have:

(1) $p^*(x) = G^*(x)$ for $x \in L_\beta[A]^{\mathbb{P}}$

($p^*$ being defined as in the
proof of lemma 1)

(2) $\mathbf{B} \cap \beta = p \cap \beta$ (where $\mathbf{B} = G^*(\mathring{B})$).

In particular, if $\varphi$ is a $\Sigma_0$
fmla of $\mathfrak{M}$ containing only

constants $\underline{x}$ s.t. $x \in L_\beta[A]^{\mathbb{P}}$, then

$$G \Vdash \varphi \iff \models_{\langle L_\beta[A, p \cap \beta], A \cap \beta, p \cap \beta \rangle} \varphi^{(p)}$$

(where $\varphi^{(p)}$ is the result of replacing $\underline{x}$ by $p^{*(x)}$).

Since this holds for _every_ $\mathbb{P}$-generic $G \ni p$, we conclude:

$$p \Vdash \varphi \iff \models_{\langle L_\beta[A, p], A \cap \beta, p \cap \beta \rangle} \varphi^{(p)}.$$

Now let $p' \in \mathbb{P} \cap L_\beta[A]$. Every $\mathbb{P}$-generic $G \ni p'$ contains a $p \in L_{\beta'}[A]$ s.t. $p' \geq p$ and $\sup(p) \geq \beta$. Hence:

$$p' \Vdash \varphi \iff \bigwedge p \in L_{\beta'}[A] \; (p \leq p' \wedge$$
$$\wedge \sup(p) \geq \beta \to \models_{\langle L_\beta[A, p], A \cap \beta, p \cap \beta \rangle} \varphi^{(p)}$$
$$\iff S(\varphi, p', \beta)$$

where $S$ is $\Delta_1$.

Hence:

$$p \Vdash \varphi \iff S(\varphi, p, \eta(\varphi, p)),$$

where $\eta(p, \varphi) =$ the least p.r. closed

$\beta$ s.t. $p, \varphi \in L_\beta[A]$.                    QED

Our main tool in showing that

$N = \mathfrak{M}/G$ is admissible will be:

**Lemma 3** Let $D_i$ $(i < \omega)$ be a

sequence of sets which are dense

in $\mathbb{P}$ and closed under extensions.

Let $\{ \langle p, i \rangle \mid p \in D_i \}$ be $\Sigma_1$ in

$\langle L_\alpha[A], A \rangle$. Then $\bigcap_i D_i$ is dense

in $\mathbb{P}$.

proof.

Let $p \in D_i \iff \forall x \, Rxip$,

where $R$ is $\Sigma_0$.

For $\delta < \alpha$ set:

$$D_i^\delta =_{df} \{ p \in \mathbb{P} \cap L_\delta[A] \mid \forall x \in L_\delta[A] \, Rxip \}$$

**Claim** There are arbitrarily large

p.r. closed $\delta < \alpha$ s.t.

(i) $D_i^\delta$ is dense in $\mathbb{P} \cap L_\delta [A]$ for $i < \omega$

(ii) $\delta$ is not admissible in $A$.

proof.

Let $\gamma < \alpha$ be s.t. $\vec{x} \in L_\gamma [A]$,
where $\vec{x}$ are the constants occur-
ring in the $\Sigma_o$ definition of $R$.
Define $\eta : \mathbb{P} \to \alpha$ by:

$\eta(p) = $ the least $\beta$ s.t.

$\bigwedge i < \omega \, \forall q \leq p \, \forall x \, (q, x \in L_\beta [A] \wedge R x i q)$.

Then $\eta$ is $\Delta_1$. Set:

$\bar{\eta}(\beta) = $ the least p.r. closed
$\beta' > \beta$ s.t. $\bigwedge p \in \mathbb{P} \cap L_\beta [A] \, \eta(p) < \beta'$

Then $\bar{\eta}$ is $\Delta_1$.

Let $\delta = \bar{\eta}^\omega (\gamma)$.

Then $\delta > \gamma$ is p.r. closed and
satisfies (i).

To show that $\delta$ satisfies (ii),
we note that $\eta \restriction (\mathbb{P} \cap L_\delta[A])$ is
$\Delta_1$ in $\langle L_\delta[A], A \cap \delta \rangle$. Thus,
if $\delta$ were admissible in $A$,
$\langle \bar\eta^i(\delta) \mid i < \omega \rangle$ would be $\Delta_1$ in
$\langle L_\delta[A], A \cap \delta \rangle$. Contradiction!

$$\text{QED (Claim)}$$

Now let $p \in \mathbb{P}$. We must show
that there is $q \leq p$ with $q \in \bigcap_i D_i$.
Pick $\delta$ s.t $p \in L_\delta[A]$, $D_i^\delta$
is dense in $\mathbb{P} \cap L_\delta[A]$ and
$\delta$ is not admissible in $A$.

~~Define $q_i \ (i < \omega)$ by:~~

~~$q_0 = p$~~

~~$q_{i+1} = \text{the least } q \leq q_i \ (\text{in the}$
$\text{canonical well ordering}$
$\text{of } L[A]) \text{ s.t } \sup(q)$~~

Let $f: \omega \leftrightarrow \delta$, $f \in L_{\delta'}[A]$, where $\delta'$ in the next largest p.r. closed ordinal. Define $q_i \in \mathbb{P} \cap L_{\delta}[A]$ $(i < \omega)$ by:

$$q_0 = P$$

$q_{i+1} = $ the least $q \leq q_i$ (in the canonical well ordering of $L[A]$) s.t. $q \in D_i^{\delta}$ and $\sup(q) \geq f(i)$.

Set $q = \bigvee_i q_i$. Then $q \in \mathbb{P}$, since $\sup(q) = \delta$ and $q \in L_{\delta'}[A]$. Hence $q \leq P \wedge q \in \bigcap_i D_i$.

$$\text{QED}$$

We are now ready to show:

$\underline{\text{Lemma 4}}$  If $G$ is $\mathbb{P}$-generic, then $\mathfrak{N}/G$ is admissible.

proof.

Since all sets are countable in $\mathfrak{N}/G$, it suffices to show that $\mathfrak{N}/G$ satisfies:

$$\bigwedge_{i<\omega} \bigvee_y R_{iy} \to \bigvee_u \bigwedge_{i<\omega} \bigvee_{y\in u} R_{iy}$$

for $\Sigma_0$ $R$. That is, we must prove:

$$\Vdash \bigwedge_{x\in\check\omega} \bigvee_y \varphi \to \bigvee_u \bigwedge_{x\in\check\omega} \bigvee_{y\in u} \varphi$$

for $\Sigma_0$ $\mathfrak{N}$-fmlae $\varphi$.

Let $p \Vdash \bigwedge_{x\in\check\omega} \bigvee_y \varphi$.

Set: $D_i = \{q \le p \mid \bigvee_y \ q \Vdash \varphi(\underline{i}, \underline{y})\} \cup$
$$\cup \neg[p]$$

for $i < \omega$. Then $D_i$ is dense in $\mathbb{P}$ and $\{\langle i, x\rangle \mid x \in D_i\}$ is $\Sigma_1$.

Hence $\bigcap_i D_i$ is dense in $\mathbb{P}$.

This means that for every $p' \leq p$
there is a $p'' \leq p'$ s.t,

$$\bigwedge_{i < \omega} \bigvee_y \quad p'' \Vdash \varphi(i, y).$$

By the replacement axiom there
is a ~~——~~ $v$ s.t.

$$\bigwedge_{i < \omega} \bigvee_{y \in v} \quad p'' \Vdash \varphi(i, y).$$

Set: $\tilde{v} = \{ \langle 1, y \rangle \mid y \in v \cap \mathbb{P} \}$.

Then:

$$p'' \Vdash \bigwedge_{x \in \check{\omega}} \bigvee_{y \in \tilde{v}} \varphi,$$

Hence $\quad p \Vdash \bigvee_v \bigwedge_{x \in \check{\omega}} \bigvee_{y \in v} \varphi,$

$$\text{QED}$$

We now turn to the major theorem of this section:

**Theorem 4**   Let $B \subset On$. Let $d_\nu \; (\nu < \delta)$ be a countable sequence of countable ordinals s.t. $d_\nu > \omega$ and $d_\nu$ is admissible in $B \cap d_\nu$, $\{d_i \mid i < \nu\}$ for $\nu < \delta$. Then there is a $b \subset \omega$ s.t.

(i)   $d_\nu$ is the $\nu$-th $d > \omega$ s.t. $d$ is admissible in $b$

(ii)   $B \cap d_\nu$ is $\Delta_1$ in $L_{d_\nu}[b]$

for $\nu < \delta$.

Thm 4 follows immediately from the conjunction of the following two theorems:

**Theorem 4.1** Let $\lambda$ be a countable limit of admissible ordinals. Let $B \subset \lambda$. Let $a \subset \lambda$ be s.t. $\omega \notin a$ and each $\alpha \in a$ is admissible in $B \cap \alpha$, and. Then there is a $C \subset \lambda$ s.t. for all $\alpha < \lambda$:

(i) $\alpha$ is admissible in $C$ iff $\alpha \in a$

(ii) If $\alpha$ is admissible in $C$, then $B \cap \alpha$, $a \cap \alpha$ are $\Delta_1$ in $\langle L_\alpha[C], C \cap \alpha \rangle$.

(iii) If $\alpha$ is p.r. closed and $\alpha'$ is the next largest p.r. closed ordinal, then $\alpha$ is countable in $L_{\alpha'}[C \cap \alpha]$.

**Theorem 4.2** Let $\lambda$ be a countable limit of admissible ordinals and let $C \subset \lambda$ satisfy (iii) of Thm 4.1. Then there is a $b \subset \omega$ s.t. for all $\alpha < \lambda$ s.t. $\alpha > \omega$:

(i) If $\alpha$ is admissible in $C$, then $\alpha$ is admissible in $b$

(ii) If $\alpha$ is admissible in $b$, then $C \cap \alpha$ is $\Delta_1$ in $L_\alpha[b]$

(Hence $\alpha$ is admissible in $b$ iff $\alpha$ is admissible in $C$).

$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$

We begin with the proof of Thm 4.1.

<u>Lemma 1</u> There is an $A \subset \lambda$ s.t. for $\alpha < \lambda$

(a) $\alpha$ is admissible in $B \cap \alpha$, and iff $\alpha$ is admissible in $A$

(b) If $\alpha$ is admissible in $A$, then $B \cap \alpha$, and are $\Delta_1$ in $\langle L_\alpha[A], A \rangle$

(c) If $\alpha$ is p.r. closed, then $\alpha$ is countable in $L_{\alpha'}[A]$, where $\alpha'$ is the next largest p.r. closed ordinal.

proof of Lemma 1.

By Thm 2 there is an $A' < \lambda$ satisfying (iii) s.t., whenever $\alpha < \lambda$ is admissible in $A \cap \alpha$, $B \cap \alpha$, then $\alpha$ is admissible in $A \cap \alpha$, $B \cap \alpha$, $A' \cap \alpha$. Set:

$$A = \{3 \cdot \nu \mid \nu \in a\} \cup \{3 \cdot \nu + 1 \mid \nu \in B\} \cup \{3 \cdot \nu + 2 \mid \nu \in A'\}$$

Then $A$ satisfies (i) - (iiii).            QED

Now ~~$\pm$~~ let $a^*$ be the set of $\alpha < \lambda$ s.t. $\alpha$ is admissible in $A$. (Hence $a \subset a^*$).

<u>Lemma 2</u> There is an $f : a^* \to \lambda$ s.t.

(i) $f(\alpha) < \alpha$    for $\alpha \in a^*$

(ii) $f$ is 1-1

(iii) $\{\alpha \in a^* \mid \kappa \mid f(\alpha) < \kappa\}$ is finite for $\kappa < \lambda$

(iv) If $\alpha \in a^*$, then $\alpha$ is admissible in $A, f \upharpoonright \alpha, \text{rng}(f \upharpoonright \alpha)$. (i.e. $\langle L_\alpha [A, f \upharpoonright \alpha, \text{rng}(f \upharpoonright \alpha)]; A \cap \alpha, f \upharpoonright \alpha, \text{rng}(f \upharpoonright \alpha)\rangle$ is admissible)

proof of Lemma 2.

Let $f_\nu$ $(\nu \leq \lambda)$ be the monotone enumeration of the p.r. closed ordinals $\leq \lambda$. Let $\Theta_\nu$ $(\nu \leq \lambda)$ be the set of maps $f$ $f$ defined on $a^* \cap f_\nu$ s.t. $f$ satisfies (i)-(iv) and $f \in L_{f_{\nu+1}}[A]$. Set:

$$f \leq f' \longleftrightarrow_{Df} \forall \nu \tau (\nu \leq \tau \wedge f \in \Theta_\nu \wedge f' \in \Theta_\tau \wedge$$
$$\wedge f \subset f' \wedge rng(f' \setminus f) \subset f_\tau \setminus f_\nu ),$$

By induction on $\kappa < \lambda$ we prove:

<u>Claim</u> $\Theta_\kappa \neq \emptyset$ ; moreover, if $\nu < \kappa$, $\nu \notin a^*$ and $f \in \Theta_\nu$ , then there is an $f' \in \Theta_\kappa$ s.t. $f \leq f'$.

proof.

<u>Case 1</u> $\kappa = 0$ : $\emptyset \in \Theta_0$.

<u>Case 2</u> $\kappa = \tau + 1$ ; $\tau \notin a^*$. Then $\Theta_\kappa = \Theta_\tau$

<u>Case 3</u> $\kappa = d + 1$ ; $d \in a^*$.
If $\nu < d$ , $\notin a^*$ , $f \in \Theta_\nu$ , then

pick $\bar{f} \in \Theta_\alpha$ s.t. $\bar{f} \geq f$. ~~let i ∈ α∖f~~

Let $i \in \alpha \setminus f$, s.t. $i \notin rng(\bar{f})$.

Set: $f' = f \cup \{\langle i, \alpha \rangle\}$.

## Case 4 $Lim(\kappa)$; $\kappa \notin a^*$.

Since $f_\kappa$ is countable in $L_{f_{\kappa+1}}[A]$, there

is a $g \in L_{f_{\kappa+1}}[A]$ s.t. $g : \omega \to \kappa$ is

monotone + cofinal in $\kappa$. We may

assume: $g(0) = \nu$ and $g(i) \notin a^*$ for

$i < \omega$. Define $f_i$ $(i < \omega)$ by:

$f_0 = f$

$f_{i+1} =$ the least $f' \in \Theta_{g(i+1)}$ ~~t.~~

(in the canonical well

ordering of $L[A]$) s.t.

$f' \geq f_i$.

Set: $f' = \bigcup_i f_i$. Then $f' \in \Theta_\kappa$ and

$f' \geq f$.

Case 5 $\kappa = \alpha \in a^*$.

By Thm 3 there is a $B \subset \alpha$ s.t.
$\alpha$ is admissible in $A, B \cdot$, $B$ is un-
bounded in $\alpha$, $B \cap \alpha'$ is bounded in
$\alpha'$ for every $\alpha' < \alpha$ s.t. $\alpha' \in a^*$ and
$B \cap \rho_\gamma \in L_{\gamma+1}^\rho [A]$ for $\gamma < \alpha$. From the
proof of Thm 3 it is clear that
we may assume: $B \in L_{\alpha+1}^\rho [A]$, for
$L_\alpha [A]$ is countable in $L_{\alpha+1}^\rho [A]$;
hence we can find $G \in L_{\alpha+1}^\rho [A]$ s.t.
$G$ is $\mathbb{P}$-generic over $\langle L_\alpha [A], A \rangle$,
$\mathbb{P}$ being the system of conditions
used in the proof of Thm 3.

Assume: $B \in L_{\alpha+1}^\rho [A]$ and let
$g : \alpha \to \alpha$ be the monotone enum-
eration of the limit points of $B$.
~~Then $\alpha$ is admissible~~
Then $g$ is a normal function

taking no ~~admissible~~ values in $a^*$,
$g \in L_{f_{\alpha+1}}[A]$ and $\alpha$ is admissible
in $A, g$. Moreover, $g \upharpoonright f_\gamma \in L_{f_{\gamma+1}}[A]$ for
$\nu < \alpha$. ~~$L_{\nu}[A]$ may~~

Let $\nu < \alpha$; ~~$\nu$~~ $f \in \Theta_\nu$. We may
assume without loss of generality
that $g(0) = \nu$. Define a sequence
$f_\nu \ (\nu < \alpha)$ by:

$\quad f_0 = f$

$\quad f_{\gamma+1} =$ the least $f' \in \Theta_{\gamma+1}$ (in the can-
$\qquad$ onical well ordering of $L[A]$
$\qquad$ s.t. $f' \geq f_\gamma$

$\quad f_\delta = \bigcup_{\nu < \delta} f_\nu \quad$ for limit $\delta$

$\qquad$ (hence $f_\delta \in \Theta_\delta$; $f_\delta \geq f_\nu$ for $\nu < \delta$).

$\quad$ ~~The~~ The sequence $f_\nu \ (\nu < \alpha)$ is
$\Delta_1$ in $\langle L_\alpha[A, g], A, g \rangle$.
Set: $f' = \bigcup_\nu f_\nu$.

Then $f'$ is $\Delta_1$ in $\langle L_\alpha[A,q], A, q \rangle$

since :

$$\nu = f'(\tau) \longleftrightarrow \nu = f_{\tau+1}(\tau) .$$

$rng(f')$ is $\Delta_1$ in $\langle L_\alpha[A,q], A, q \rangle$

since :

$$\iota \in rng(f') \longleftrightarrow \iota \in rng(f_{\iota+1}) .$$

Hence ~~But~~ $f' \geq f$ and $f' \in \Theta_\alpha$ .

QED (Claim).

Lemma 2 follows almost immediately from the claim : Let $\lambda_i$ $(i < \omega)$ be monotone + cofinal in $\lambda$. Assume : $\lambda_i \notin a^*$. Select $f_i$ $(i < \omega)$ s.t. $f_i \in \Theta_{\lambda_i}$ and $f_{i+1} \geq f_i$ . Set : $f = \bigcup_i f_i$ . Then $f$ satisfies (i) - (iv) of Lemma 2. QED

Now let $f$ be as in Lemma 2 and
define a map $\tilde{f}: (a^* \setminus a) \times \omega \to \lambda$
by: $\tilde{f}(d, i) = g_d(i)$, where:

$g_d = $ the least $g$ (in the can-
onical well ordering of $L[A]$)
s.t. $g$ is a monotone map
from $\omega$ to $d$ and $\sup_{i < \omega} g(i) = d$.

Set:

$$D = \{ \langle f(\beta), \tilde{f}(\beta, i) \rangle \mid \beta \in a^* \setminus a \land i < \omega \},$$

and

$$D_d = \{ \langle f(\beta), \tilde{f}(\beta, i) \rangle \mid \beta \in (a^* \setminus a) \cap d \land i < \omega \}$$

for $d < \lambda$. Then if $d \in a^*$, $D_d$ is

$\Delta_1$ in $\langle L_d[A], A \cap d, f \upharpoonright d, rng(f \upharpoonright d) \rangle$.

It follows that, if $d \in a$, $D \cap d$ is

also $\Delta_1$, since $D \cap d$ differs from $D_d$

by at most finitely many points.

On the other hand, if $d \in a^* \setminus a$,

$d$ is not admissible in $D \cap d$,

since, if it were, we should have:

$$rng(g_\alpha) = \{\nu \mid \langle f(\alpha), \nu \rangle \in D \cap d\} \text{ is}$$

$\Delta_1$ in $\langle L_\alpha[D], D \cap d \rangle$; hence $g_\alpha$, being

the monotone enumeration of $rng(g_\alpha)$,

would be $\Delta_1$. But this violates the

replacement axiom.

We now define the set $C$, whose

existence is asserted in Theorem 4.4.

$$C = \{4 \cdot \alpha \mid \alpha \in A\} \cup \{4 \cdot \langle \alpha, \beta \rangle + 1 \mid \alpha = f(\beta)\} \cup$$
$$\cup \{4 \cdot \alpha + 2 \mid \alpha \in rng(f)\} \cup \{4\alpha + 3 \mid \alpha \in D\}.$$

For $\alpha < \lambda$ we define $C_\alpha$ similarly,

using $A \cap \alpha$, $f \upharpoonright \alpha$, $rng(f \upharpoonright \alpha)$, $D_\alpha$ in

place of $A, f, rng(f), D$. Clearly,

if $\alpha \in a^*$, then $C_\alpha$ is $\Delta_1$ in

$\langle L_\alpha[A]; A \cap \alpha, f \upharpoonright \alpha, rng(f \upharpoonright \alpha) \rangle$. Hence

$C \cap \alpha$ is $\Delta_1$ for $\alpha \in a$, since $C_\alpha$ differs

from $C_\alpha$ by at most finitely many

elements. Thus $\alpha$ is admissible

in $C$ if $\alpha \in a$. On the other hand,
if $\alpha \notin a$, then $\alpha$ is not admissible
in $C$, since, ~~if it were, we should~~
~~have: $D \cap \alpha$ is $\Delta_1$ in $L_\alpha$~~, if it were,
we should have: $\alpha \in a^* \setminus a$, since
$A \cap \alpha$ is $\Delta_1$ in $\langle L_\alpha[C], C \cap \alpha \rangle$. But then
~~$D \cap \alpha$ is~~ $\alpha$ is not admissible in $C$,
since $D \cap \alpha$ is $\Delta_1$ in $\langle L_\alpha[C], C \cap \alpha \rangle$ and
$\alpha$ is not admissible in $D \cap \alpha$. This
establishes (i) of Theorem 4.1. (ii)
follows from Lemma 1 and the
fact that $A \cap \alpha$ is $\Delta_1$ in $\langle L_\alpha[C], C \cap \alpha \rangle$.
(iii) is established by the following
lemma:

**Lemma 3** $\rho_\alpha$ is countable in $L_{\rho_{\alpha+1}}[C \cap \alpha]$

for $\alpha < \lambda$.

proof of Lemma 3.

We assume the following facts:

(1) $\rho_0$ is countable in $L_{\rho_1}$

(2) $\rho_\nu$ is mappable onto $\rho_{\nu+1}$ by a function $g \in L_{\rho_{\nu+2}}$.

By (2) we get:

(3) If $\lim(\alpha)$ and $\rho_\alpha > \alpha$, then $\alpha$ is mappable onto $\rho_\alpha$ by a function $g \in L_{\rho_\alpha}$.

proof of (3).

Suppose not. Let $\kappa$ be the least $\kappa \leq \alpha$ s.t. $\alpha$ is not mappable onto $\rho_\kappa$ by a map $g \in L_{\rho_{\kappa+1}}$. For $\nu < \kappa$ let $g_\nu : \alpha \to \rho_\nu$ be the least $g \in L_{\rho_{\nu+1}}$ (in the canonical well ordering of $L$) s.t. $g$ maps $\alpha$ onto $\rho_\nu$, $\kappa$ is a limit ordinal by (2); hence the function

$$g(\langle \nu, \iota \rangle) = g_\nu(\iota)$$

maps $\kappa \times \alpha$ onto $\rho_\kappa$. It is easily seen that $g \in L_{\rho_{\kappa+1}}$. But since $\kappa \leq \alpha$ there is an $h \in L_{\rho_{\kappa+1}}$ which maps ~~$\kappa \times \alpha$ onto $\alpha$.~~ $\alpha$ onto $\kappa \times \alpha$. Hence: $g \circ h \in L_{\rho_{\kappa+1}}$ and $g \circ h$ maps $\alpha$ onto $\rho_\kappa$. Contradiction!         QED (3).

Now suppose Lemma 3 to be false. Let $\alpha$ be the least ordinal for which Lemma 3 fails. Then $\alpha = \rho_\alpha$ by (1)-(3). (hence $L_{\rho_\alpha}[C \cap \alpha] = L_\alpha[C]$).

(4) $\alpha$ is admissible in $C$. Moreover, if $R \subset L_\alpha[C]$ and $R \in L_{\rho_{\alpha+1}}[C \cap \alpha]$, then:

(*) $\bigwedge x \bigvee y \, Rxy \rightarrow \bigwedge u \bigvee v \bigwedge x \in u \bigvee y \in v \, Rxy$

holds in ~~$\rho$~~ $L_\alpha[C]$.

proof of (4).

Since each $x \in L_\alpha[C]$ is countable in $L_\alpha[C]$, it suffices to show:

$(**)$ $\bigwedge_{m < \omega} \bigvee_y R xy \rightarrow \bigvee_\sigma \bigwedge_{m < \omega} \bigvee_{y \in \sigma} Rmy$.

Suppose not. Set:

$$h(m) = \mu \gamma \bigvee_{y \in L_\gamma[C]} Rmy \qquad (m < \omega).$$

Then $\sup_{m < \omega} h(m) = \alpha$. Set:

$$g_m =_{Df} \text{the least } y \in L_\alpha[C] \text{ (in the canonical well ordering of } L[C]) \text{ s.t. } g \text{ maps } \omega \text{ onto } h(m).$$

Then $g(\langle m, m \rangle) = g_m(m)$ maps $\omega$ onto $\alpha$ and $g \in L_{\beta_{\alpha+1}}[C \cap \alpha]$. Contradiction!

$$\text{QED}$$

Using (4), we get:

(5) $\alpha$ is Mahlo in ~~in~~ $C$.

proof of (5)

Let $D$ be a closed, unbounded sub-
set of $d$ which is $\Delta_1$ in $\langle L_d[c], C \cap d \rangle$.
We must show: $\beta \in D$ for some
$\beta < d$ which is admissible in $C$.
Let $\vec{x}$ be all constants occurring in
the $\Delta_1$ definition of $D$. Let $u$ be the
set of $x \in L_d[c]$ which are
$\langle L_d[c], C \cap d \rangle$ - definable in the par-
ameters $\vec{x}$. Since there is an $\langle L_d[c], C \cap d \rangle$-
- definable well ordering of $L_d[c]$,
we have: ~~⟨u, C∩u⟩ is an~~
~~elementary submodel of~~

(*) $\langle u, C \cap u \rangle$ is an elementary sub-
        model of $\langle L_d[c], C \cap d \rangle$.

It follows that $u$ is transitive, for
by (*), if $x \in u$ ~~⟶~~ there is ~~onf~~
a $g \in u$ s.t. $g$ maps $\omega$ onto $x$.
Hence: $x = g''\omega \subseteq u$.

Let $\beta = d \cap u$. By the condensation lemma we have: $u = L_\beta[C]$.

By (*) $\langle L_\beta[C], C \cap \beta \rangle$ satisfies the admissibility axioms, since $\langle L_d[C], C \cap d \rangle$ does. Hence $\beta$ is admissible in $C$. We now claim: $\beta < d$. To see this, we let $\tilde{u}$ be the set of fmlae (with $\varepsilon, \equiv, \mathring{C}$) containing at most the constants $\vec{x}$ and the free vbl $v_0$. Set:

$$h(\varphi) =_{Df} \text{ the least } y \in L_d[C] \text{ (in the canonical well ordering of } L[C]) \text{ s.t. } \underset{\langle L_d[C], C \cap d \rangle}{\models} \varphi(v_0/y)$$

if such $y$ exists

$$=_{Df} \emptyset \text{ if not.}$$

Then $\tilde{u} \in L_d[C]$ and $h \in L_{d+1}[C \cap d]$.

By (4) there is a $v \in L_d[C]$ s.t. $u = h``\tilde{u} \subset v$. Hence $\beta = d \cap u < d$.

By (*), $D \cap \beta$ is closed and unbounded in $\beta$. Hence $\beta$ is a limit point of $D$. Hence $\beta \in D$ QED (5).

(5) leads to a contradiction, however, since :

~~(6) if $\alpha < \lambda$ is ad.~~

(6) $\alpha < \lambda$ is not Mahlo in C.

proof.

We may assume: $\alpha \in a$. Since $f \upharpoonright \alpha$, $rng(f \upharpoonright \alpha)$ are $\Delta_1$ in $\langle L_\alpha[c], C \cap \alpha \rangle$, we can define a $\Delta_1$ normal function $g$ by :

$g(0) = 0$

$g(\nu+1) = $ the least $\beta > g(\nu)$ s.t.

$\qquad f^{-1}(\nu) < \beta$ for all $\nu \in rng(f \upharpoonright \alpha) \cap g(\nu)$

$g(\delta) = \sup\limits_{\nu < \delta} g(\nu)$ for limit $\delta$.

Clearly, $g$ takes no values in $a^*$.

QED

This completes the proof of Thm 4.1.

We turn now to the proof of Thm 4.2.

Set: $M_\alpha = \langle L_{\rho_\alpha}[C \cap \alpha], C \cap \alpha \rangle$ $(\alpha < \lambda)$.

Define a sequence $a_\nu \subset \omega$ $(\nu < \lambda)$ by:

$a_\alpha =$ the least $a \in L_{\rho_{\alpha+1}}[C \cap \alpha]$ (in the

canonical well ordering of

$L[C \cap \alpha]$) s.t. $a$ is Cohen-

- generic over $M_\alpha$.

$(a_\alpha$ exists, since $M_\alpha$ is countable

in $L_{\rho_{\alpha+1}}[C \cap \alpha])$.

Let $\langle s_m \mid m < \omega \rangle$ be a recursive

enumeration (without repetition) of

all finite $0,1$ - sequences. For $a \subset \omega$

set:

$$S(a) =_{Df} \{ m \mid s_m \subset \chi_a \}$$

$(\chi_a$ being the characteristic

function of $a$).

The family $\langle S(a) \mid a \subset \omega \rangle$ is almost disjoint — that is:

$$a \neq a' \rightarrow S(a) \cap S(a') \text{ is finite.}$$

Our intention is to construct a $b \subset \omega$ s.t.

(1) $L_d[b]$ is admissible for $d \in a$

(2) $b \cap S(a_\nu)$ is finite iff $\nu \in C$.

Such $b$ will prove to satisfy (ii), (iii) of Thm 4.2.


$b$ will, of course, be constructed by forcing. We shall employ a general type of forcing devised by Solovay. We first give a general description of Solovay forcing, and then ~~decide to~~ consider its application to our particular case.

Let $Y \subset \mathcal{P}(\omega)$. The system of conditions $\mathbb{P}_Y$ is defined as follows:

$$\mathbb{P}_Y =_{Df} \text{the set of pairs } \langle u, U \rangle \text{ s.t.}$$
$$u, U \text{ are finite}; u \subset \omega; U \subset Y.$$

Each condition $\langle u, U \rangle$ contains a bit of information about a potential $b \subset \omega$; $\langle u, U \rangle$ says that $u \subset b$ and that $b \cap S(a) \subset u$ for $a \in U$. ~~From this~~ Thus, the extension relation is defined by:

$$\langle u', U' \rangle \leq \langle u, U \rangle \iff_{Df} u' \supset u \wedge$$
$$\wedge U' \supset U \wedge \bigwedge_{a \in U} u' \cap S(a) \subset u.$$

Suppose, now that $M = \langle |M|; B \rangle$ is a transitive p.r. closed structure and that $Y \subset M$ is $M$-definable. Then $\mathbb{P}_Y$ is $M$-definable. Let $G \subset \mathbb{P}_Y$ be $\mathbb{P}_Y$-generic over $M$. Set:

$$b = \bigcup \{ u \mid \langle u, \emptyset \rangle \in G \}.$$

Then $b \subseteq \omega$ is p.r. in $G$. $G$ is also p.r. in ~~$\mathbb{P}_Y$~~ $b, \mathbb{P}_Y$, since:

$$G = G_b = \{ \langle u, U \rangle \in \mathbb{P}_Y \mid u \subseteq b \wedge \bigwedge a \in U \; b \cap S(a) \subseteq u \}.$$

We call such $b \subseteq \omega$ a $\mathbb{P}_Y$-generic real over $M$.

Lemma 1  Let $M$, ~~■~~ $Y$ be as above and let $b$ be a $\mathbb{P}_Y$-generic real over $M$. Then for all $a \in M$ we have:

$$b \cap S(a) \text{ infinite} \iff a \in Y.$$

proof.

($\Longleftarrow$) By genericity, there is a $\langle u, U \rangle \in G_b$ s.t. $a \in U$. But then $u' \cap S(a) \subseteq u$ for all $\langle u', U' \rangle \in G_b$. Hence $b \cap S(a) \subseteq u$.

$(\to)$ Let $a \notin Y$. Let $m < \omega$. We claim that $m \in b \cap S(a)$ for some $n > m$. It suffices to note that, by almost disjointness, $\{\langle u, U \rangle \in \mathbb{P}_Y \mid \forall n > m \ n \in u\}$ is dense in $\mathbb{P}_Y$. QED

~~Now set:~~

~~$Y = \{a_\nu \mid \nu \in \delta\} \cup \{x \subseteq \omega \mid x \text{ is finite}\}$.~~

~~We force over $M = \langle L_\lambda[c], \in \rangle$,~~

~~using the conditions $\mathbb{P}_Y$. It is~~
~~obvious that, if $b$ is $\mathbb{P}_Y$-generic~~
~~over $M$, then:~~

~~$b \cap S(a_\nu)$ infinite $\iff \nu \in c$~~

Now set:

$$Y = \{a_\nu \mid \nu \in C\} \cup \{x < \omega \mid x \text{ is finite}\}.$$

We force over $M = \langle L_\chi[c], c \rangle$, using the conditions $\mathbb{P}_Y$. It is obvious that, if $b$ is $\mathbb{P}_Y$-generic, then:

$$b \cap S(a_\nu) \text{ is finite} \iff \nu \in C.$$

We must show that $L_d[b]$ is admissible for $d \in a$. The proof will be largely a recapitulation of the proof of ~~≡~~ Theorem 1.

Set: $Y_d = Y \cap M_d$ for $d < \lambda$. Then:

$$Y_d = \{a_\nu \mid \nu \in C \cap d\} \cup \{x < \omega \mid x \text{ is finite}\}.$$

The following Lemma corresponds to Lemma 1 in the proof of Thm 1.

Lemma 2 If $D \subset \mathbb{P}_{y_\alpha}$ is $M_\alpha$-definable
and compatible in $\mathbb{P}_{y_\alpha}$, then $D$
is compatible in $\mathbb{P}_{y_\beta}$ for $\beta \geq \alpha$
(hence in $\mathbb{P}_y = \bigcup_{\beta \geq \alpha} \mathbb{P}_{y_\beta}$).

proof. (By induction on $\beta \geq \alpha$).

Case 1. $\beta = \alpha$ trivial

Case 2. $Lim(\beta)$ trivial

Case 3. $\beta = \gamma + 1$.

$D$, $\mathbb{P}_{y_\beta}$ are definable in:

$$M'_\gamma = \langle M_\gamma [a_\gamma] ; C \cap \gamma \rangle.$$

Suppose that $D$ is not compatible in
$\mathbb{P}_{y_\beta}$. Then there is a $p \in \mathbb{P}_{y_\beta}$ s.t $p$
is incompatible with every $q \in D$. $p$
must have the form: $p = \langle u, U \cup \{a_\gamma\} \rangle$,
where $\langle u, U \rangle \in \mathbb{P}_{y_\gamma}$. Since $a_\gamma$ is
Cohen-generic over $M_\gamma$, this state
of affairs must be forced by some

condition on $a_\gamma$. Thus there is a Cohen-condition $r$ s.t.

(*)  $r \Vdash (\langle \check{u}, \check{u} \cup \{ \mathring{a} \} \rangle$ is incompatible with every $q \in \check{D})$.

We shall derive a contradiction from (*). Set:

$$x = \{ n < \omega \mid n \in dom(r) \wedge r(n) = 1 \},$$

Then $x \in Y_\gamma$, since $x$ is finite. Since $D$ is compatible in $\mathbb{P}_{Y_\gamma}$, ~~th~~ we can pick $q \in D$, $\langle u', u' \rangle \leq \langle u, u \cup \{x\} \rangle$ s.t. $\langle u', u' \rangle \in \mathbb{P}_{Y_\gamma}$ and $\langle u', u' \rangle \leq q$. Let $n > max(u' \cup x)$. Let $r' : n \to 2$ s.t. $r'(i) = 1 \Longleftrightarrow i \in x$. Then $r'$ is a Cohen condition, $r' \leq r$ and:

$$r' \Vdash (\langle \check{u'}, \check{u'} \cup \{ \mathring{a} \} \rangle \leq \langle \check{u}, \check{u} \cup \{ \mathring{a} \} \rangle \wedge$$
$$\wedge \langle \check{u'}, \check{u'} \cup \{ \mathring{a} \} \rangle \leq \check{q} \wedge \check{q} \in \check{D}).$$

Contradiction!                    QED

An immediate corollary to Lemma 2 is:

**Lemma 3** If $G \subset \mathbb{P}_y$ is $\mathbb{P}_y$-generic over $M$, then $G_\alpha = G \cap \mathbb{P}_{y_\alpha}$ is $\mathbb{P}_{y_\alpha}$-generic over $M_\alpha$ for $\alpha < \lambda$.

Now set:

$$\mathcal{M} = \langle M^{\mathbb{P}_y} ; I, E \rangle$$

$$\mathcal{M}_\alpha = \langle M_\alpha^{\mathbb{P}_{y_\alpha}} ; I_\alpha, E_\alpha \rangle \qquad (\alpha < \lambda).$$

Let $G \subset \mathbb{P}_y$ be $\mathbb{P}_y$-generic over $M$. For $\alpha = \beta + 1$, we have: $\mathbb{P}_{y_\alpha} \in M_\alpha$;

hence: $\mathcal{M}_\alpha / G_\alpha = M_\alpha[G_\alpha]$. Hence $\mathcal{M}_\alpha / G_\alpha = M_\alpha[G_\alpha]$ for limit $\alpha$ as well, since:

$$\mathcal{M}_\alpha / G_\alpha = G_\alpha^* {}^{\text{``}} M_\alpha^{\mathbb{P}_{y_\alpha}} = \bigcup_{\nu < \alpha} G_{\nu+1}^* {}^{\text{``}} M_{\nu+1}^{\mathbb{P}_{y_{\nu+1}}} =$$

$$= \bigcup_{\nu < \alpha} M_{\nu+1}[G_{\nu+1}] = M_\alpha[G_\alpha].$$

Similarly, $\mathcal{M} / G = M[G]$. Set:

$$b = \bigcup \{ u \mid \langle u, \emptyset \rangle \in G \}.$$

Then $b$ is p.r. in $G$ and $G$ is p.r.

in $b$, $\mathbb{P}_\gamma$. Hence $\mathcal{M}/G = M[G] = M[b]$.

Similarly, $b$ is p.r. in $G_\alpha$ and $G_\alpha$ is p.r. in $b$, $\mathbb{P}_{\gamma_\alpha}$ for $\alpha < \lambda$; hence $\mathcal{M}_\alpha/G_\alpha = M_\alpha[G_\alpha] = M_\alpha[b]$. ~~We must~~

~~now show that $M_\alpha[b]$ is ad~~

We must now show that $M_\alpha[b]$ is admissible for $\alpha \in a$.

<u>Lemma 4</u> $\{ \langle \nu, p, \varphi \rangle \mid \nu < \alpha, p \Vdash_{\mathcal{M}_\gamma}^{\Sigma_0} \varphi \}$

is $\Delta_1$ in $M_\alpha$ for $\alpha \in a$.

proof. Exactly like ~~the proof~~ ~~of~~ Lemma 4 in the proof of Thm 1.

<u>Lemma 5</u> If $\varphi$ is a $\Sigma_0$ fmla of $\mathcal{M}_\gamma$ ~~$\Longleftrightarrow$~~ , $p \in \mathbb{P}_\gamma$, and $\alpha \geq \nu$,

then: $p \Vdash_{\mathcal{M}_\gamma} \varphi \Longleftrightarrow p \Vdash_{\mathcal{M}_\alpha} \varphi$.

proof. Exactly like Lemma 5 in the proof of Thm 1.

Lemma 6 $\Vdash^{\Sigma_0}_{\mathcal{M}_\alpha}$ in $\Delta_1$ — in $M_\alpha$

for $\alpha \in a$. ~~$\overline{\phantom{xxxx}}$~~

proof. From Lemmas 4, 5 as in the proof of Thm 1.

~~By Lemma 6 above~~

Lemma 7 $M_\alpha[b]$ is admissible for $\alpha \in a$.

proof.
By Lemma 6 above and by Lemma 9 in the proof of Thm 1, it suffices to show:

$$p \Vdash_{\mathcal{M}_\alpha} \forall x \, \varphi \rightarrow \forall u \in M_\alpha^{\mathbb{P}\gamma_\alpha} \; p \Vdash \forall x \varepsilon \underline{u} \, \varphi$$

for $\Sigma_0$ fmlae $\varphi$.

Let $p \Vdash \forall x \, \varphi$. Set:

$$D = \{q \leq p \mid \forall x \; q \Vdash \varphi(\underline{x})\}.$$

Then $D$ is dense in $[p]$. Since $D$ is $\Sigma_1$, we have:

$$q \in D \longleftrightarrow \forall z\; R_z q,$$

where $R$ is $\Sigma_0$. For $\beta < \alpha$ set:

$$D_\beta = \{q \in \mathbb{P}_{Y_\beta} \mid \forall z \in M_\beta\; R_z q\}.$$

**Claim** There is a $\beta < \alpha$ s.t. $p \in \mathbb{P}_{Y_\beta}$ and $D_\beta$ is dense in $[p] \cap \mathbb{P}_{Y_\beta}$.

proof. For $q \leq p$ set:

$$\eta(q) =_{Df} \mu\gamma\; \forall q' \in \mathbb{P}_{Y_\gamma}\; \forall x \in M_\gamma\; (q' \leq q \;\wedge$$
$$\wedge\; q' \Vdash \varphi(\underline{x})),$$

$\eta$ is $\Delta_1$ and defined on $[p]$.

Set: $\bar\eta(\gamma) =_{Df} \sup\limits_{q \in [p] \cap \mathbb{P}_{Y_\gamma}} \eta(\gamma)$.

Let $p \in \mathbb{P}_{Y_\gamma}$ and set: $\beta = \bar\eta^\omega(\gamma)$.

$\beta$ clearly has the desired property.

Thus $D_\beta \cup (\mathbb{P}_{Y_\beta} \cap \neg [p])$ is $M_\beta$ – definable

and dense in $\mathbb{P}_{Y_\beta}$. It follows that

$D_\beta \cup (\mathbb{P}_{Y_\beta} \cap \neg [p])$ is compatible in

$\mathbb{P}_{Y_\alpha}$. Hence $D_\beta$ is compatible in $[p]$.

Since: $\bigwedge q \in D_\beta \bigvee x \; q \Vdash \varphi(x)$, there

is a $u \in M_\alpha$ s.t.

$$\bigwedge q \in D_\beta \bigvee x \in u \; q \Vdash \varphi(x).$$

Since $D_\beta$ is compatible in $[p]$,

we conclude:

$$\bigwedge q \leq p \bigvee q' \leq q \bigvee x \in u \; q \Vdash \varphi(x).$$

Set: $\tilde{u} = \{ \langle 1, x \rangle \mid x \in u \cap M_\alpha^{\mathbb{P}_{Y_\alpha}} \}$. Then:

$$p \Vdash \bigvee x \in \tilde{u} \; \varphi.$$
QED

An immediate corollary of Lemma 7 is :

Corollary 8    $L_\alpha[b]$ is admissible for $\alpha \in a$.

This proves (i) of Thm 4.2.

It remains only to prove :

Lemma 9    If $\alpha < \lambda$ is admissible in $b$, then $C \cap \alpha$ is $\Delta_1$ in $L_\alpha[b]$.

proof.

By Lemma 1, we have :

$$\nu \in C \iff b \cap S(a_\nu) \text{ is finite}.$$

Set : $h(\nu) = \langle a_\nu, C \cap \nu \rangle$ $(\nu < \alpha)$.

Then $h(\nu)$ is defined by the recursion :

$a_\nu$ = the least $a \in L_{\rho_\nu}[C \cap \nu]$ s.t.

$\quad$ $a$ is Cohen-generic over

$\quad\quad$ $\langle L_{\rho_\nu}[C \cap \nu], C \cap \nu ]$

$C \cap \nu = \{ \iota < \nu \mid b \cap S(a_\iota) \text{ is finite} \}$.

Hence $h$ is $\Delta_1$ in $L_\alpha[b]$. $It$
follows that $C \cap \alpha$ is $\Delta_1$, since:

$$\nu \in C \cap \alpha \longleftrightarrow b \cap S(a_\nu) \text{ infinite.}$$

QED