Correction to §1 of Robust Extenders

We gave a seriously flawed proof of Thm 4, which stated that the 1-small robust K^c model is universal if it is iterable. We also prove the same fact for the full robust K^c, assuming it to exist. In fact we shall show:

Thm A Let $K = K^c$ be the full robust K^c-model. Let P be a premouse. There is no coiteration of P, K of length ∞.

Thm 4 will then be a corollary of the proof of Thm A. Thm A follows from:

Thm B Let K, P be as above. Let $\overline{\overline{P}} < \Theta = (2^\beta)^+$, where $\beta \geq 2^\omega$. There is no coiteration of P, K of length $\Theta + 1$.

We prove this in ZFC.

Suppose not. Set $Q = K \| \Theta$. Then P, Q have a coiteration $\langle \mathcal{J}^P, \mathcal{J}^Q \rangle$ of length $\Theta + 1$, where

$$\mathcal{J}^P = \langle \langle P_i \rangle, \langle \nu_i \rangle, \langle \pi^P_{ij} \rangle, T^P \rangle$$

$$\mathcal{J}^Q = \langle \langle Q_i \rangle, \langle \nu_i \rangle, \langle \pi^Q_{ij} \rangle, T^Q \rangle$$

Let $\Omega > \overline{\overline{H}}_\theta$ be regular. Pick $X \prec H_\Omega$ s.t. $y^P, y^Q \in X$, $\overline{\overline{X}} < \theta$, $\bar\theta = X \cap \theta$ is transitive, $[X]^\omega \subset X$, and $V_{\omega+2} \subset X$. (This is obviously possible by our assumptions on θ.) Let $\sigma : \bar H \overset{\sim}{\longleftrightarrow} X$, where $\bar H$ is transitive. Then $\sigma : \bar H \prec H$, $\bar\theta = \mathrm{crit}(\sigma)$, $\sigma(\bar\theta) = \theta$. Let

$\sigma(\bar y^P) = y^P$, $\sigma(\bar y^Q) = y^Q$, where

$\bar y^P = \langle \langle \bar P_i \rangle, \langle \bar\nu_i \rangle, \langle \bar\pi^P_{ij} \rangle, \bar T^P \rangle$

$\bar y^Q = \langle \langle \bar Q_i \rangle, \langle \bar\nu_i \rangle, \langle \bar\pi^Q_{ij} \rangle, \bar T^Q \rangle$.

Set $H_{\bar\theta} =_{\mathrm{Df}} (H_{\bar\theta})^{\bar H}$. Then $\sigma \restriction H_{\bar\theta} = \mathrm{id}$. On both sides of the coiteration we have:

(1) (a) $i \underset{\bar T}{\leq} \bar\theta \longleftrightarrow i \underset{T}{\leq} \theta$ for $i < \hat\theta$

(b) $i \underset{\bar T}{\leq} j \longleftrightarrow i \underset{T}{\leq} j$ for $i, j < \bar\theta$,

since $\sigma \restriction H_{\bar\theta} = \mathrm{id}$, $\sigma(\bar\theta) = \theta$

(c) $\bar\theta \underset{T}{\leq} \theta$, since $\bar\theta$ is a limit point of the branch $\{i \mid i \underset{T}{\leq} \theta\}$.

There are at most finitely many truncations on the branch $\{i \mid i \underset{T}{\leq} \theta\}$, hence the last truncation lies below $\bar\theta$. Hence:

(d) There is $i_0 < \bar\theta$ s.t. π_{ij} is a total fcn on P_i (Q_i) for $i_0 < i \underset{T}{\leq} j \underset{T}{\leq} \theta$,

(e) $\bar\pi_{ij} = \sigma^{-1}(\pi_{ij})$ is a total function on $\bar P_i$ ($\bar Q_i$) for $i_0 < i \underset{\bar T}{\leq} j \leq \bar\theta$.

Clearly $P_i \in H_\theta$ for $i < \theta$ and $\varkappa_i \leq ht(P_i)$. Hence $\theta = \sup_{i<\theta} \nu_i$ and $\bar{\theta} = \sup_{i<\bar{\theta}} \bar{\nu}_i$, where $\bar{\nu}_i = \sigma^{-1}(\nu_i) = \nu_i$ for $i < \bar{\theta}$. We then get:

(f) $\bar{\theta} \leq \mathrm{crit}(\pi_{\bar{\theta}\theta})$

proof.

Note that $E^P_{\nu_i} \neq \emptyset \vee E^Q_{\nu_i} \neq \emptyset$ for all $i < \theta$. Hence ~~on~~ at least one side of the coiteration we have:

(*) $E_{\nu_i} \neq \emptyset$ for arbitrarily large $i \leq_T \theta$.

But then $E_{\nu_i} \neq \emptyset$ for arb. large $i < \bar{\theta}$.

Let $j \geq \bar{\theta}$ be least s.t. $\bar{\theta} \leq_T j+1 \leq_T \theta$, and $E_j \neq \emptyset$. Then $\kappa_j = \mathrm{crit}(\pi_{\bar{\theta}\theta})$ and $\kappa_j \geq \sup_{i<\bar{\theta}} \lambda_i = \bar{\theta}$. If (*) does not hold, then $\pi_{i\theta} = \mathrm{id}$ for suffi. large $i \leq_T \theta$. But then the last place where $\pi_{i\theta} \neq \mathrm{id}$ must lie below $\bar{\theta}$ and $\pi_{\bar{\theta}\theta} = \mathrm{id}$.

QED (f)

Now set: $\tau = \bar{\theta}^+ \cup J^E_{\lambda_{\bar{\theta}}}$. (Note that $(J^E_{\lambda_h})^{P_l} = (J^E_{\lambda_h})^{Q_l}$ for all $l \geq h$.)

Then:

(g) $\tau = \bar\theta + P_{\bar\theta} = \bar\theta + Q_{\bar\theta}$.

prf. Suppose not.
We first note that (*) holds on the P-side, since otherwise $P_i = P_{i_0}$ for $i_0 < i \leq_T \theta$, where $i_0 < \bar\theta$. Hence $\lambda_i \in P_{i_0}$ for $i < \bar\theta$. Contr! Let j again be least s.t. $\bar\theta \leq_T j+1 \leq_T \theta$ and $E_{\nu_j} \neq \emptyset$ on the P-side. Let $\zeta = T(j+1)$. Then ζ is the least $\zeta \geq \bar\theta$ s.t. $E_{\nu_\zeta} \neq \emptyset$, $P_\zeta = P_{\bar\theta}$ and $\kappa_j < \lambda_\zeta$. Since $J^E_{\lambda_\zeta} = P_{\bar\theta} \| \lambda_\zeta$ and $\bar\theta + P_{\bar\theta} > \tau$, there is a subset of $\bar\theta \leq \kappa_j$ in $P_{\bar\theta} \setminus J^E_{\lambda_\zeta}$. Hence a truncation occurs at $j+1$, violating the fact that all truncations on the main branch occured below $\bar\theta$. Contr! If (*) holds on the Q-side the same argument goes through. Now let (*) fail on the Q-side. Then $Q_\theta = Q_{\bar\theta}$, $ht(Q_\theta) = \theta$, and J is a segment of Q_θ for $i < \theta$. Hence $Q_{\bar\theta} = Q_{\bar\theta} = \bigcup_{i<\theta} J^E_{\lambda_i}$. Hence $\tau = \bar\theta + Q_{\bar\theta}$. QED (g)

We now look more closely at the P-side of the coiteration.

Since $P_i \in H_\theta$ for $i < \theta$, we have $\bar{P}_i \in \bar{H}_{\bar\theta}$ for $i < \bar\theta$. But $\sigma \restriction H_{\bar\theta} = \mathrm{id}$. Hence:

(2) (a) $\bar{P}_i = \sigma(\bar{P}_i) = P_i$ for $i < \bar\theta$. Similarly:

(b) $\bar\pi^P_{ij} = \pi^P_{ij}$ for $i \leq_T j < \bar\theta$.

But then:

(c) $\bar{P}_{\bar\theta} = P_{\bar\theta}$, $\bar\pi^P_{i\bar\theta} = \pi^P_{i\bar\theta}$ for $i \leq_T$,

since $\langle \bar{P}_{\bar\theta}, \langle \bar\pi^P_{i\bar\theta} \mid i \leq_T \bar\theta \rangle\rangle =$

$= \lim(\langle P_i \mid i <_T \bar\theta \rangle, \langle \pi^P_{ij} \mid i \leq_T j <_T \bar\theta \rangle\rangle$

$= \langle P_{\bar\theta}, \langle \pi^P_{i\bar\theta} \mid i < \bar\theta \rangle\rangle$ by (a), (b) and (1) (a), (b). QED(c)

Hence:

(d) $\sigma \restriction P_{\bar\theta} = \pi^P_{\bar\theta\theta}$,

since, letting $x \in P_{\bar\theta}$, $x = \pi_{i\bar\theta}(x')$, $i <_T \bar\theta$,

we have: $\sigma(x) = \sigma(\pi_{i\bar\theta}(x')) = \pi_{i\theta}(x') = \pi_{\bar\theta\theta}(x)$.

QED (2)(c).

We then conclude:

(3) $\bar\theta \in P_{\bar\theta}$

prf. Suppose not.

Clearly $\bar\theta = \sup_{i<\bar\theta} \lambda_i \leq ht(P_{\bar\theta})$. Hence $ht(P_{\bar\theta}) = \bar\theta$, where $J^{E^{P_{\bar\theta}}}_{\bar\theta} = J^{E^{Q_{\bar\theta}}}_{\bar\theta}$. Since $\bar\theta$ is a limit of cardinals in $P_{\bar\theta}$, $Q_{\bar\theta}$, it cannot index an extender in these models. Hence $P_{\bar\theta} = Q_{\bar\theta} \| \bar\theta$ and the coiteration terminates at $\bar\theta$. Contr!

QED (3)

We now attempt to apply this analysis to the Q-side, bearing in mind that $ht(Q) = \theta$. Since $Q_i \subset H_\theta$ for $i < \theta$ we certainly have: $\bar Q_i \subset H_{\bar\theta}$ for $i < \theta$. Since $\sigma \restriction H_{\bar\theta} = id$ we have: $\bar Q_i \| \xi = \sigma(\bar Q_i \| \xi) = Q_i \| \xi$ for $i, \xi < \bar\theta$. Hence $J_{\bar\theta}^{EQ_i} = J_{\bar\theta}^{E\bar Q_i}$ for $i < \theta$. Since $\bar\theta$ is a limit of cardinals in both Q_i and $\bar Q_i$, we conclude:

(4) (a) $\bar Q_i = Q_i \| \bar\theta$ for $i < \bar\theta$. Similarly

(b) $\bar\pi^Q_{ij} = \pi^Q_{ij} \restriction \bar Q_i$ for $i, j < \bar\theta$

Set: $\tilde\theta = ht(\bar Q_{\bar\theta})$. Then $\tilde\theta = \bigcup_{i \leq_T \bar\theta} \pi_{i\bar\theta}{}''\bar\theta$.

(c) $\bar Q_{\bar\theta} = Q_{\bar\theta} \| \tilde\theta$ and $\pi^{\bar Q}_{i\bar\theta} = \pi^Q_{i\bar\theta} \restriction \bar Q_i$

for $i <_T \bar\theta$, since

$\langle \bar Q_{\bar\theta}, \langle \bar\pi_{i\bar\theta} \mid i \leq_T \bar\theta \rangle\rangle =$

$= \lim(\langle Q_i \| \bar\theta \mid i <_T \bar\theta \rangle, \langle \pi_{ij} \restriction (Q_i \| \bar\theta) \mid i \leq_T j \leq_T \bar\theta \rangle\rangle$

$= \langle Q_{\bar\theta} \| \tilde\theta, \langle \pi_{i\bar\theta} \restriction (Q_i \| \bar\theta) \mid i \leq_T \bar\theta \rangle\rangle =$

$= \langle Q_{\bar\theta} \| \tilde\theta, \langle \pi_{i\bar\theta} \restriction \bar Q_i \mid i \leq_T \bar\theta \rangle\rangle$

Exactly as in (2)(d) we conclude:

(d) $\sigma \upharpoonright \overline{Q}_{\bar\theta} = \pi^{Q}_{\bar\theta\theta} \upharpoonright \overline{Q}_{\bar\theta}$.

Using this we prove:

(5) There is no truncation on the main branch $b^Q = \{i \mid i \leq_{T^Q} \theta\}$ of $\mathcal{J}^Q$.

prf. Suppose not.

The last truncation occurs below $\bar\theta$. Hence by (4)(a)-(c) we know that for sufficiently large $i \leq_T \bar\theta$ we have $\overline{Q}_i = Q_i$ and $\bar\pi^Q_{ij} = \pi^Q_{ij}$ for $i \leq_T j \leq_T \bar\theta$. It then follows exactly as in (3) that $\bar\theta \in Q_{\bar\theta}$. By (1)(g) $\mathcal{P}(\bar\theta) \cap P_{\bar\theta} = \mathcal{P}(\bar\theta) \cap Q_{\bar\theta} = \mathcal{P}(\bar\theta) \cap \overline{Q}_{\bar\theta}$.

Using (2)(d) and (4)(d) we then imitate the classical proof that the coiteration of two mice terminates: Let $j = j_P$ be least s.t. $\bar\theta \leq_{T^P} j+1 <_{T^P} \theta$ and $E^P_{\nu_j} \neq \emptyset$. Then $E^P_{\nu_j}(X) = \sigma(X) \cap \lambda_{j_0}$ by (2)(d). Similarly if j_Q has the same definition on the Q-side, then $E^Q_{\nu_{j_Q}}(X) = \sigma(X) \cap \lambda_{j_0}$ by (4)(d). The usual proof then shows $j_P = j_Q = j$; hence $E^Q_{\nu_j} = E^P_{\nu_j}$, which is impossible in a coiteration. QED(5)

Carrying the argumentation of (5) a bit further, we get:

(6) $\bar\theta = \tilde\theta$.

prf. Suppose not.

Then $\bar\theta \in \bar Q_{\bar\theta}$ and $\bar\theta = \mathrm{crit}(\pi^{Q}_{\bar\theta\theta})$ by (4)(d).

Clearly $\tilde\theta = \bigcup_{i <_T \bar\theta} \pi_{i\bar\theta}{}''\bar\theta = \pi_{j\bar\theta}(\bar\theta)$ for $j <_T \bar\theta$.

Recall that $\tau = \bar\theta^{+P_{\bar\theta}} = \bar\theta^{+Q_{\bar\theta}}$ and that $J_\tau^{E^{P_{\bar\theta}}} = J_\tau^{E^{Q_{\bar\theta}}}$. Hence it suffices to show:

<u>Claim</u> $\tau \leq \tilde\theta$,

since then $\mathbb{P}(\bar\theta) \cap P_{\bar\theta} = \mathbb{P}(\bar\theta) \cap Q_{\bar\theta} = \mathbb{P}(\bar\theta) \cap \bar Q_\theta$ and we can use (2)(d), (4)(d) to get a contradiction just as before.

Clearly the claim holds if $\tilde\theta$ is a cardinal in $Q_{\bar\theta}$, so suppose not. Then $\bar\theta = \pi^{-1}_{0\bar\theta}(\tilde\theta)$ is not a cardinal in $K\|\theta = Q_0$. Let α be the largest $\alpha < \bar\theta$ which is a cardinal in $K\|\theta$. Then $\pi_{0\bar\theta}(\alpha)$ is the largest cardinal in $Q_{\bar\theta}$. Hence $\tau \leq \pi_{0\bar\theta}(\alpha) < \tilde\theta$, since τ is a cardinal in $Q_{\bar\theta}$.

QED (6)

The following key observation is due to Mitchell and Schindler ([MSch]):

(7) $\mathbb{P}(\bar\theta) \cap K \subset \bar H$

prf.
Let $X \in \mathbb{P}(\bar\theta) \cap K$. Then $X \in K\|\theta = Q_0$. Let $Y = \pi_{0\bar\theta}(X)$. Then $Y \in J_{\tau}^{E^{Q_{\bar\theta}}} = J_{\tau}^{E^{P_{\bar\theta}}} \subset \bar H$ and $X = \pi_{0\bar\theta}^{-1}$ "$Y = \bar\pi_{0\bar\theta}^{-1}$ "Y, since $\pi_{0\bar\theta} \restriction \bar\theta = \bar\pi_{0\bar\theta} \restriction \bar\theta$. QED (7)

It follows of course that $K\|\alpha \in \bar H$ for $\alpha < \bar\tau$, since $K\|\alpha$ is coded by an $X \in \mathbb{P}(\bar\theta) \cap K$. Pursuing this line of reasoning we in fact get:

(8) Let $\pi_{0\bar\theta}(\bar\tau) = \tau$. Then $K\|\bar\tau \in \bar H$ and is $\bar H$-definable in the parameter $K\|\bar\theta$.

proof.
Set Z = the set of $\alpha > \bar\theta$ s.t. $\rho^{\omega}_{K\|\alpha} = \bar\theta$.
Then Z is unbounded in $\bar\tau$. Set:
$U = \langle K\|\alpha \mid \alpha \in Z \rangle$. Since $K\|\bar\tau = \bigcup \mathrm{rng}(U)$ and U is a function on a subset of $\bar\tau \leq \tau \in \bar H$, it suffices to show:

<u>Claim</u> U = the set of $\langle N, \alpha \rangle \in \bar H$ s.t. $\alpha > \bar\theta$, N is a core premouse, $\alpha = \mathrm{ht}(N)$, $N\|\bar\theta = K\|\bar\theta$ and N obeys the following condensation principle wrt $K\|\bar\theta$:

(*) Let $\pi : \bar{N} \prec N$, $\text{crit}(\pi) \leq \bar{\theta}$ and $\bar{\alpha} = ht(\bar{N})$. Then $\bar{N} = K\|\bar{\alpha}$.

(Note such π, if it exists, lies in $\bar{H}$, since it is obtained by collapsing $H^N(\gamma)$ = the set of $x \in N$ which are N-definable in parameters $< \gamma$.)

($\subset$) If $\alpha \in Z$, then $\langle K\|\alpha, \alpha \rangle \in U$, since $K\|\alpha$ is a mouse & hence satisfies (*) by the condensation lemma for mice (noting that $\bar{\theta}$ is a limit cardinal in $K\|\alpha$), and, as remarked, $K\|\alpha \in \bar{H}$.

($\supset$) Let $\langle N, \alpha \rangle \in U$. Set $\langle N', \alpha' \rangle = \sigma(\langle N, \alpha \rangle)$. Then $N'\|\theta = K\|\theta$ and N' has the condensation property (*) wrt. $K\|\theta$. But $\sigma \upharpoonright N : N \prec N'$ and $\bar{\theta} = \text{crit}(\sigma \upharpoonright N)$. Hence $N = K\|\alpha$ and $\langle N, \alpha \rangle \in Z$. QED (8)

(9) $\pi^Q_{0\bar{\theta}} \upharpoonright (K\|\bar{\tau}) \in \bar{H}$ is $\bar{H}$-definable in the parameters $K\|\bar{\theta}$, $\bar{\pi}^Q_{0\bar{\theta}}$.

proof.

For $\alpha \in Z$ set: $\pi_\alpha = \pi^Q_{0\bar{\theta}} \upharpoonright (K\|\alpha)$;

$N_\alpha = \pi^Q_{0\bar{\theta}}(K\|\alpha) = Q_{\bar{\theta}}\|\alpha' = P_{\bar{\theta}}\|\alpha'$, where $\pi_{0\bar{\theta}}(\alpha) = \alpha'$. Then $N_\alpha \in \bar{H}$ and $\pi_\alpha \in \bar{H}$, since π_α is obtained by collapsing the N_α-definable hull of $\mathrm{rng}(\overline{\pi}^{Q}_{0\overline{\theta}})$.

It suffices to show:

$\langle\langle N_\alpha, \pi_\alpha\rangle \mid \alpha \in Z\rangle$ = the set of triples $\langle N, \pi, \alpha\rangle$ and $\pi : K\|\alpha \prec N$ and $\pi \upharpoonright (K\|\overline{\theta}) = \overline{\pi}^{Q}_{0\overline{\theta}}$ and $\pi(\overline{\theta}) = \overline{\theta}$.

proof.

($\subset$) follows by the above remarks.

($\supset$) Let $\langle N_\alpha, \pi_\alpha\rangle$ be as above.

<u>Claim</u> $N_\alpha = \pi_{0\overline{\theta}}(K\|\alpha)$, $\pi = \pi_{0\overline{\theta}} \upharpoonright (K\|\alpha)$.

Let f be a $K\|\alpha$-definable map of onto $K\|\alpha$. Let f' have the same def. over N_α and let $\tilde{f}$ have the same def. over $\tilde{N} = \pi_{0\overline{\theta}}(K\|\alpha)$. Pick $\delta < \overline{\theta}$ s.t. ~~Let~~ $\xi_1, \ldots, \xi_n < \overline{\theta}$, $\xi_1, \ldots, \xi_n < \delta$. ~~Let~~ φ be a 1-st order formula in the language of $K\|\alpha$. Set:

$a = \{\langle \eta_1, \ldots, \eta_n\rangle \in \delta^n \mid K\|\alpha \models \varphi[\vec{\eta}]\}$,

Then $a \in K\|\overline{\theta}$ and:

$N_\alpha \models \varphi[f'(\xi_1), \ldots, f'(\xi_n)] \longleftrightarrow \tilde{N} \models \varphi[\tilde{f}(\vec{\xi})]$,

$\longleftrightarrow \langle \vec{\xi}\rangle \in \overline{\pi}_{0\overline{\theta}}(a)$

Hence there is an isomorphism $k : N_\alpha \overset{\sim}{\longleftrightarrow} \tilde{N}$ defined by $k(f'(\xi)) = \tilde{f}(\xi)$. Hence $N_\alpha = \tilde{N}$. But $\pi(f(\xi)) = \tilde{f}(\bar{\pi}_{0\bar{\theta}}(\xi)) =$
$\dot{=} \pi_{0\theta}(f(\xi))$. Hence $\pi = \check{\pi}_{0\bar{\theta}} \upharpoonright (K \| \alpha)$.

QED (9)

(10) $\bar{H} \models cf(\bar{\tau}) < \bar{\theta}$.

proof.
$\pi^{Q}_{0\bar{\theta}}$ takes $\bar{\tau}$ cofinally to τ since the iteration is by Σ_0 ultraproducts on the main branch and $\bar{\tau}$ is a successor cardinal in K. Since $\pi^{Q}_{0\bar{\theta}} \upharpoonright \bar{\tau} \in \bar{H}$ it suffices to show that $cf(\tau) < \bar{\theta}$ in $\bar{H}$. Since $\sigma : \bar{H} \prec H_\Omega$, it suffices to show that $cf(\tau^*) < \theta$, where $\tau^* = \sigma(\tau)$.
For this it suffices to show:
<u>Claim 1</u> σ takes τ cofinally to τ^*,
since then $cf(\tau^*) \le \tau < \theta$.
We use the fact that $\pi^{P}_{\bar{\theta}\theta} = \sigma \upharpoonright P_{\bar{\theta}}$.
There is a $i_0 <_T \bar{\theta}$ s.t. no truncation occurs on the main branch above i_0.
Hence π_{ij} is total on P_i for $i_0 \le i \le_T j \le_T \theta$.
Let $\eta =$ the maximal $\eta \le \omega$ s.t.
$\kappa_i < \rho^{\eta}_{P_i}$ for all $i \in [i_0, \theta)_T$.

<u>Subclaim 2</u> If $n > \eta$, then $\varkappa_i > \rho^n_{P_i}$ for all $i \in [\bar{\theta}, \theta)_T$.

proof.

Let i_1 be the least $i \leq_T \theta$ s.t. $\varkappa_i \geq \eta + 1$. Then $i_1 < \bar{\theta}$, since $\sigma : \bar{H} \prec H_\Omega$.

It follows easily that $\rho^{\eta+1}_{P_i} = \rho^{\eta+1}_{P_{i_1}}$ and hence $\varkappa_i > \rho^{\eta+1}_{P_i}$ for $i_1 \leq_T i \leq_T \theta$, since the $\varkappa_i$'s are increasing. QED

But then, if $\bar{\theta} \leq_T i = T(j+1)$, $j+1 \leq_T \theta$, $J^{E_{j+1}}_{\rho^\eta_{P_{j+1}}}$ is a Σ_0 ultraproduct of $J^{E_i}_{\rho^\eta_{P_i}}$ by E_{ν_j}. This means, in particular, that successor cardinals in $J^{E_i}_{\rho^\eta_{P_i}}$ are taken cofinally to their images. (This holds for $\rho^\eta_{P_i}$ as well if it is a successor, since $\rho^\eta_{P_{j+1}}$ is a cardinal in P_{j+1}.)

Clearly $\tau \leq \rho^\eta_{P_{\bar{\theta}}}$, since $\rho^\eta_{P_{\bar{\theta}}}$ is a cardinal in $P_{\bar{\theta}}$. By induction on i it follows easily that $\pi^P_{\bar{\theta} i}$ takes τ cofinally to its image for $\bar{\theta} \leq_T i \leq_T \theta$. QED(10)

Now let $X' \prec H_\Omega$ be defined like X, but with $X \cup \{\theta\} \subset X'$. Let $\sigma' : H' \xrightarrow{\sim} X'$, where H' is transitive. Let $\theta' = \theta \cap X'$. Then, as before, $\theta' = \mathrm{crit}(\sigma')$, $\sigma'(\theta') = \theta$, where $\sigma' : H' \prec H_\Omega$. Set $\tilde{\sigma} = \sigma'^{-1}\sigma$. Then $\tilde{\sigma} : H \prec H'$. Let $\tilde{\sigma}^{-1}(y^P, y^Q) =$
$= y'^P, y'^Q$, $y'^P = \langle\langle P'_i\rangle, \langle \nu'_i\rangle, \langle \pi'^P_{ij}\rangle, T'^P\rangle$,
$y'^Q = \langle\langle Q'_i\rangle, \ldots, T'^Q\rangle$. As before:
$T' = T \cap (\theta' + 1)^2$, $\pi'^P_{ij} = \pi^P_{ij}$ for $i, j \leq \theta'$, $\theta' \leq_T \theta$, $\pi'^Q_{ij} = \pi^Q_{ij} \upharpoonright (Q_i \| \theta')$ for $i, j \leq \theta'$. Since $\bar{\theta}, \theta' \in \{i \mid i \leq_T \theta\}$ and $\bar{\theta} < \theta'$, we have $\bar{\theta} \leq_T \theta'$.
Clearly $\tilde{\sigma}(\bar{\theta}) = \theta'$ and $\bar{\theta} = \mathrm{crit}(\tilde{\sigma})$. Moreover $\tilde{\sigma}(\pi^Q_{i\bar{\theta}}) = \pi'^Q_{i\theta'}$. Set:
$\tau' = \theta' + \kappa$. As before, $K \| \tau' \in H'$ and is H'-definable from $\pi'^Q_{i\theta'}$ by the same def. as $K \| \bar{\tau}$ from $\bar{\pi}^Q_{i\bar{\theta}}$ in $\bar{H}$. Hence $\tilde{\sigma}(K \| \bar{\tau}) = K \| \tau'$. By (10) there is $f \in \bar{H}$ mapping a $\mu < \bar{\theta}$ cofinally to $\bar{\tau}$.

But then $\tilde{\sigma}(f)$ maps μ cofinally to τ'. It follows that $\tilde{\sigma}\upharpoonright\bar{\tau}$ maps $\bar{\tau}$ cofinally to τ'. Set $\pi = \tilde{\sigma}\upharpoonright(K\|\bar{\tau})$, $F = \tilde{\sigma}\upharpoonright\mathbb{P}(\kappa)$.

Then $\pi : K\|\bar{\tau} \xrightarrow[F]{} K\|\tau'$. The structure $N = \langle K\|\tau', F\rangle$ has all the properties of a premouse except, perhaps, the initial segment condition. Hence it makes sense to ask whether F is robust in N. We prove:

(11) F is robust in $N = \langle K\|\tau', F\rangle$.

proof.

Let $U \subset \theta'$, $W \subset \mathbb{P}(\bar{\theta}) \cap K$ be countable.

Let $g : \omega \leftrightarrow U$, $X : \omega \xrightarrow{\text{onto}} W$, where $X = \langle X_i \mid i < \omega\rangle$. Set:

$D = \{\langle i_1, \ldots, i_n, j\rangle \mid \langle g(i_1), \ldots, g(i_n)\rangle \in F(X_j)\}$

$A = \{\langle a_1, \ldots, a_n, \varphi\rangle \mid \varphi$ is a Σ_1 fmla $\wedge$ $\wedge\ a_1, \ldots, a_n \subset \omega \wedge C^{E^K}_{c,\infty} \models \varphi(g''a_1, \ldots, g''a_n)\}$

where $c = \sup U$

The claim reduces to:

(*) There is $\bar{g} : \omega \to \bar{\theta}$ s.t.

(a) For all $i_1, \ldots, i_n < \omega$, $j < \omega$:

$\langle \bar{g}(\vec{i}) \rangle \in X_j \leftrightarrow \langle \vec{i}, j \rangle \in D$

(b) For all $a_1, \ldots, a_n \subset \omega$ and all Σ_1 fmlae φ we have:

$C^{E^K}_{\bar{c}, \theta} \models \varphi(\bar{g}``a_1, \ldots, \bar{g}``a_n) \leftrightarrow \langle \vec{a}, \varphi \rangle \in A$,

where $\bar{c} = \sup \bar{g}``\omega$.

Note that $D, A \in \bar{H}$, since $\mathcal{P}(\omega) \in \bar{H}$, and $\tilde{\sigma}(D) = D$, $\tilde{\sigma}(A) = A$. Moreover, each $\bar{g} : \omega \to \bar{\theta}$ belongs to $\bar{H}$, since $[X]^\omega \subset X$ + hence $[\bar{H}]^\omega \subset \bar{H}$.

Thus we need only show that (*) holds in $\bar{H}$. This is equivalent to saying that (*) holds of D, A, $\theta' = \tilde{\sigma}(\bar{\theta})$, $\tilde{\sigma}(X) = \langle \tilde{\sigma}(X_i) \mid i < \omega \rangle$ in in H'. That statement is, in fact, verified by g in place of $\bar{g}$, since $g : \omega \to \theta'$, $\tilde{\sigma}(X_i) = F(X_i)$

(a) $\langle g(\vec{i}) \rangle \in \tilde{\sigma}(X_j) \leftrightarrow \langle \vec{i}, j \rangle \in D$

(b) Let $a_1, \ldots, a_n \subset \omega$, $c = \sup g``\omega$ (hence $c = \sup U$). Then:

$C^{E^K}_{c,\theta'} \models \varphi(g''a_1, \ldots, g''a_n) \longleftrightarrow$

$C^{E^K}_{c,\theta} \models \varphi(g''a_1, \ldots, g''a_n) \longleftrightarrow$

(since $\sigma''(C^{E^K}_{c,\theta'}) = C^{E^K}_{c,\theta}$)

$\longleftrightarrow C^{E^K}_{c,\infty} \models \varphi(g''a_1, \ldots, g''a_n)$

(since $C^{E^K}_{c,\theta} \prec_{\Sigma_1} C^{E^K}_{c,\infty}$)

$\longleftrightarrow \langle \vec{a}, \varphi \rangle \in A$. QED (11)

[Note "F is robust in $\langle J^{E^K}_{\tau'}, F \rangle$" involves only $J^{E^K}_{\theta'}$. If $\bar{\tau} \leq \xi \leq \theta'$ we define $F|\xi : \mathbb{P}(\bar{\tau}) \cap K \to \mathbb{P}(\xi)$ by: $\mathrm{dom}(F|\xi) = \mathrm{dom}(F)$, $(F|\xi)(X) = F(X) \cap \xi$. It then follows that $F|\xi$ is robust w.r.t. $J^{E^K}_{\xi}$ in the same sense.]

We finish the proof exactly as before. Clearly $F \notin K$, since otherwise $\pi \in K$, where $\pi = \tilde{\sigma} \restriction J^E_{\bar{\tau}} : J^E_{\bar{\tau}} \xrightarrow{F} J^E_{\tau'}$; hence $\mathrm{cf}(\tau') \leq \bar{\tau} < \tau'$ in K, where $\tau' = \theta' + \kappa$. Contr! Hence there must be a least $\lambda \leq \theta'$ s.t. $\bar{F} = F|\lambda \notin K$

and there is $\bar{\pi}: J_{\bar{\tau}}^{E^K} \xrightarrow[\bar{F}]{} J_{\bar{\nu}}^{\bar{E}}$ with $\bar{\pi}(\bar{\theta}) = \lambda$. The structure $\bar{M} = \langle J_{\bar{\nu}}^{\bar{E}}, \bar{F} \rangle$ will then satisfy the initial segment condition. But there is $\delta : \bar{M} \xrightarrow[\Sigma_0]{} M = \langle J_{\bar{\tau}}^{E^K}, F \rangle$ defined by $\delta(\bar{\pi}(f)(\alpha)) = \pi(f)(\alpha)$. Then $\delta(\lambda) = \theta'$, $\delta \upharpoonright \lambda = \mathrm{id}$. If $\lambda < \xi < \nu$ s.t. $\rho^1_{J_\xi^{\bar{E}}} = \lambda$, then $\rho^1_{J^{E^K}_{\delta(\xi)}} = \theta'$. By the condensation lemma for mice we conclude: $J_\xi^{\bar{E}} = J_\xi^{E^K}$. Since this holds for arbitrarily large $\xi < \nu$, we conclude: $M = \langle J_\nu^{E^K}, \bar{F} \rangle$. Since λ is a limit cardinal in K, it follows by standard methods (cf [MI] §1, Fact 5 - Fact 9) that there is $\eta < \infty$ with $N_\eta = \langle J_\nu^{E^K}, \emptyset \rangle$ and $\omega\rho^\omega_{N_i} \geq \lambda$ for all $i \geq \eta$. By the robustness of $\bar{F}$ we have: $N_{\eta+1} = \bar{M} = M_{\eta+1} = N_i \| \nu$ for all $i > \eta$. Hence $M = K \| \nu$ and $\bar{F} \in K$. Contr!

QED (Lemma A)

Theorem 4 now follows from:

Thm C Let $K = K^c$ be the robust K^c of 1-small mice. Let P be a 1-small premouse. There is no coiteration of P, K of length ∞.

proof.
Let θ, β be as in Thm B and repeat the proof of Thm B. Everything goes through as before except for the final conclusion, where we must prove that $\bar{M} = \langle J_\nu^{\bar{E}}, \bar{F} \rangle$ is 1-small. Suppose not. Then there is $\bar{\kappa} < \bar{\theta}$ s.t. $J_{\bar{\theta}}^{\bar{E}} \models \bar{\kappa}$ is Woodin. Hence $K \models \bar{\kappa}$ is Woodin. Let $\kappa = \pi^Q_{0\bar{\theta}}(\bar{\kappa})$. Then $Q_{\bar{\theta}} \models \kappa$ is Woodin, where $\kappa < \bar{\theta}$. Since $Q_{\bar{\theta}}$ is 1-small, there is no $\nu > \kappa$ s.t. $E_\nu^{Q_{\bar{\theta}}} \neq \emptyset$. But $P_{\bar{\theta}} \| \bar{\theta} = Q_{\bar{\theta}} \| \bar{\theta}$, where $\bar{\theta}$ is a cardinal in $P_{\bar{\theta}}$. Hence $P_{\bar{\theta}} \models \kappa$ is Woodin. Hence, by 1-smallness, there is no $\nu > \kappa$ s.t. $E_\nu^{P_{\bar{\theta}}} \neq \emptyset$. Since $ht(P_{\bar{\theta}}) < \theta = ht(Q_{\bar{\theta}})$, $P_{\bar{\theta}}$ is an initial segment of $Q_{\bar{\theta}}$ and the coiteration terminates at $\bar{\theta}$. Contr!

QED