§1.2 <u>Def</u> $\mathbb{B}$ is <u>almost subcomplete</u> (ASC) <u>as witnessed by</u> $\langle\theta, \mathbb{S}\rangle$ iff

(I) $\mathbb{S} = \{\langle\bar{N}, \bar{\theta}, \bar{\mathbb{B}}, \bar{G}\rangle \mid H_{\omega_1} \models \varphi[\bar{N}, \bar{\theta}, \bar{\mathbb{B}}, \bar{G}]\}$

where $\varphi$ is a $\Sigma_1$ formula.

(II) $\mathbb{B} \in H_\theta$. Let $N = L_\tau^A$ be a ZFC$^-$ model s.t. $H_\theta \subset N$, $\theta < \tau$. Let $\pi : \bar{N} \prec N$, where $\bar{N}$ is countable, transitive, and almost full.

Let $\pi(\bar{\theta}, \bar{\mathbb{B}}) = \theta, \mathbb{B}$.

Then $S = \mathbb{S}(\bar{N}, \bar{\theta}, \bar{\mathbb{B}}) =: \{\bar{G} \mid \langle\bar{N}, \bar{\theta}, \bar{\mathbb{B}}, \bar{G}\rangle \in \mathbb{S}\}$ is a set of weakly generic $\bar{G} \subset \bar{\mathbb{B}}$ over $\bar{N}$ s.t.

(a) If $a \in \bar{\mathbb{B}} \setminus \{0\}$, there is $\bar{G} \in S$ with $a \in \bar{G}$

(b) Let $\bar{G} \in S$, $\bar{s} \in N$, $\pi(\bar{s}) = s$. Then there is $b \in \mathbb{B} \setminus \{0\}$ s.t. whenever $G \ni b$ is $\mathbb{B}$-generic over $V$, then there is $\sigma \in V[G]$ with:

- $\sigma : \bar{N} \prec N$
- $\sigma(\bar{\theta}, \bar{\mathbb{B}}, \bar{s}) = \theta, \mathbb{B}, s$
- $C_\delta^N(\operatorname{rng}\sigma) = C_\delta^N(\operatorname{rng}\pi)$ where $\delta = \delta(\mathbb{B})$
- $\bar{G} = \sigma^{-1}{}''G$.

(Hence $\bar{G}$ is well founded as well as weakly generic.)

<u>Def</u> $\mathbb{B}$ is <u>almost subcomplete</u> iff it is ASC as witnessed by some $\langle\theta, \mathbb{S}\rangle$.

<u>Note</u> By a Löwenheim–Skolem argument we can restrict ourselves to ZFC$^-$ models $N$ s.t. $\bar{\bar{N}} = \bar{\bar{H}}_\theta$ in the above definition. This means that if

Note: This section has a page with number 28A.

we take any $\Omega > \theta$ s.t. $2^\theta < \Omega$, then the definition of "$\mathbb{S}$ is subcomplete as witnessed by $\Omega$" relativizes to $H_\Omega$. Hence it relativizes to $M$ for any transitive ZFC$^-$ model $M$ s.t. $M \supset H_\Omega$.

<u>Def</u> $\mathbb{B}$ is ASC as <u>verified</u> by $\langle \theta, \mathbb{S} \rangle$ iff $\mathbb{B}$ is ASC as witnessed by $\langle \theta', \mathbb{S} \rangle$ for all $\theta' \geq \theta$.

<u>Lemma 1</u> Let $\mathbb{B}$ be ASC as witnessed by $\langle \theta, \mathbb{S} \rangle$. There is $\mathbb{S}'$ s.t. $\mathbb{B}$ is verified by $\langle (2^\theta)^+, \mathbb{S}' \rangle$.

proof.
Assume w.l.o.g. that $\theta$ is least with the property that $\langle \theta, \mathbb{S} \rangle$ witnesses that $\mathbb{B}$ is ASC. Let $M \supset H_\Omega$ where $\Omega \geq (2^\theta)^+$ and $M = L_\mu^B$ is a ZFC$^-$ model. $\mathbb{S}$ is uniformly $M$-definable, since it is $\Sigma_1(H_{\omega_1})$. But then $\theta$ is uniformly $M$-definable in $\mathbb{B}$ – i.e. there is $\varphi$ s.t.

$$\theta = \text{the unique } \theta \text{ s.t. } M \models \varphi[\theta, \mathbb{B}]$$

for all such $M$.

We then define:

Def $\mathbb{S}' =$ the set of $\langle \bar{M}, \bar{\Omega}, \bar{\mathbb{B}}, \bar{G} \rangle$ s.t. $\bar{\Omega} \in \bar{M}$, $\bar{\mathbb{B}} \in H_{\bar{\Omega}}^{\bar{M}}$ is a complete BA in $M$, there is a unique $\bar{\theta}$ s.t. $\bar{M} \models \varphi[\bar{\theta}, \bar{\mathbb{B}}]$, and $\langle \bar{M}, \bar{\theta}, \bar{\mathbb{B}}, \bar{G} \rangle \in \mathbb{S}$.

Note that, since the uniform definition $\varphi$ does not depend on the particular $\Omega$ chosen, neither does the definition of $\mathbb{S}'$.

Claim Let $\Omega > \theta$, $2^{\theta} < \Omega$. Then $\mathbb{B}$ is ASC as witnessed by $\langle \Omega, \mathbb{S}' \rangle$.

proof.

We clearly have:

(I) $\mathbb{S}'$ is $\Sigma_1(H_{\omega_2})$.

We now verify II. Clearly $\mathbb{B} \in H_{\Omega}$.

Now let $M = L_{\mu}^{B}$ be a ZFC$^-$ model s.t. $H_{\Omega} \subset M$, $\Omega < \mu$. Let $\pi : \bar{M} \prec M$ where $\bar{M}$ is countable and almost full. Let $\pi(\bar{\Omega}, \bar{\mathbb{B}}) = \Omega, \mathbb{B}$. Then $\pi(\bar{\theta}) = \theta$, where $\bar{\theta}$ is the unique $\bar{\theta}$ s.t. $\bar{M} \models \varphi[\bar{\theta}, \bar{\mathbb{B}}]$. Hence $S' = \mathbb{S}'(\bar{M}, \bar{\Omega}, \bar{\mathbb{B}}) = \mathbb{S}(\bar{M}, \bar{\theta}, \bar{\mathbb{B}}) = S$. Hence $S'$ is a set of weakly generic $\bar{G} \subset \bar{\mathbb{B}}$ over $M$.

 

(a) is then immediate. We prove (b).
Let $\bar{G} \in S' = S$, $\bar{\imath} \in \bar{M}$, $\pi(\bar{\imath}) = \imath$. Then
$\pi(\langle \bar{\Omega}, \bar{\imath} \rangle) = \langle \Omega, \imath \rangle$ and there is
$b \in \mathbb{B} \setminus \{0\}$ s.t. whenever $G \ni b$ is $\mathbb{B}$ -
- generic over $V$, then there is $\sigma \in V[G]$ with:

- $\sigma : \bar{M} \prec M$
- $\sigma(\bar{\Theta}, \bar{\mathbb{B}}, \bar{\imath}, \bar{\Omega}) = \Theta, \mathbb{B}, \imath, \Omega$
- $\sigma_{\delta}^{M}(\operatorname{rng} \sigma) = C_{\delta}^{M}(\operatorname{rng} \pi)$ where $\delta = \delta(\mathbb{B})$
- $\bar{G} = \sigma^{-1}{}'' G$.

QED (Lemma 1)

We can weaken the definition of ASC by introducing a parameter:

<u>Def</u> $\langle \Theta, p, \mathbb{S} \rangle$ witnesses the ASC-ness of $\mathbb{B}$ iff

(I) $\mathbb{S} = \{\langle \bar{N}, \bar{\Theta}, \bar{\mathbb{B}}, \bar{p}, \bar{G} \rangle \mid H_{\omega_1} \models \varphi[\bar{N}, \bar{\Theta}, \bar{\mathbb{B}}, \bar{p}, \bar{G}]\}$
where $\varphi$ is $\Sigma_1(H_{\omega_1})$

(II) as before, requiring $\pi(\bar{p}) = p$ and with
$S = \{\bar{G} \mid \langle \bar{N}, \bar{\Theta}, \bar{\mathbb{B}}, \bar{p}, \bar{G} \rangle \in \mathbb{S}\}$, and with
$\sigma(\bar{\Theta}, \bar{\mathbb{B}}, \bar{p}, \bar{\imath}) = \sigma(\Theta, \mathbb{B}, p, \imath)$.

This notion appears easier to satisfy, but in fact:

**Lemma 2** Let $\mathbb{B}$ be ASC as witnessed by $\langle \theta, p, \$ \rangle$. Then there is $\$'$ s.t. the ASC-ness of $\mathbb{B}$ is verified by $\langle (2^{\theta})^{+}, \$' \rangle$.

(The proof turns on the fact that there is a pair of formulae $\psi, \chi$ s.t.,
letting $\theta$ be minimal s.t.
$V_p(\langle \theta, p, \$ \rangle$ witnesses the ASC-ness of $\mathbb{B}$,
then for any $M \supset H_{\Omega}$ where $\Omega \geq (2^{\theta})^{+}$, and $M = L^{B}_{\mu}$ is a ZFC$^{-}$ model, then:

$$\theta = \text{the unique } \theta \text{ s.t. } M \models \psi[\theta, \mathbb{B}]$$

$$p_M = \text{the unique } p \text{ s.t. } M \models \chi[p, \mathbb{B}]$$

where $p_M =:$ the $M$-least $p$ s.t. $\langle \theta, p, \$ \rangle$ the ASC-ness of $\mathbb{B}$.

We then set:

$\$' =$ the set of $\langle \bar{M}, \bar{\Omega}, \bar{\mathbb{B}}, \bar{G} \rangle$ s.t. $\bar{\Omega} \in \bar{M}$, $\bar{\mathbb{B}} \in H^{\bar{M}}_{\bar{\Omega}}$ is a complete BA in $M$, there are unique $\bar{\theta}, \bar{p}$ s.t. $\bar{M} \models (\psi(\bar{\theta}, \bar{\mathbb{B}}) \wedge \chi(\bar{p}, \bar{\mathbb{B}}))$, and $\langle \bar{M}, \bar{\theta}, \bar{p}, \bar{\mathbb{B}}, G \rangle \in \$$.

The proof is then exactly as before.)

We shall often tacitly use Lemma 2 in verifying ASC-ness. We define:

**Def** $\mathbb{B}$ is ASC as verified by $\$$ iff there is $\theta$ s.t. $\mathbb{B}$ is ASC as verified by $\langle \theta, \$ \rangle$.

The iteration thm for ASC-ness reads:

Thm 3 Let $\mathbb{B} = \langle \mathbb{B}_i \mid i < \alpha \rangle$ be an RCS-iteration s.t. $\mathbb{B}_0 = \mathbb{1}$ and:

(a) $\mathbb{B}_i \neq \mathbb{B}_j$ for $i < j$

(b) $\Vdash_i (\check{\mathbb{B}}_{i+1}/\dot{G}$ is ASC) for $i < \alpha$

(c) $\Vdash_{i+1} \mathrm{card}(\delta(\check{\mathbb{B}}_i)) \leq \omega_1$ for $i < \alpha$.

Then each $\mathbb{B}_i$ is ASC.

proof.

Let $\mathbb{B} \in H_\Omega \prec_{\Sigma_n} V$ for a sufficiently large $n < \omega$. (The requirement on $n$ will become clear in the course of the proof.) Then $H_\Omega = V_\Omega$ and $\Omega$ is a strong limit cardinal. Moreover, if

$$\mathbb{S} = \mathbb{S}_\varphi = \{\langle \bar{N}, \bar{\theta}, \bar{\mathbb{A}}, \bar{G} \rangle \mid H_{\omega_1} \models \varphi[\bar{N}, \bar{\theta}, \bar{\mathbb{A}}, \bar{G}]\}$$

and $\mathbb{A} \in H$, then the statement "$\mathbb{S}$ verifies the ASC-ness of $\mathbb{A}$" is absolute in $H_\Omega$. Hence $\mathbb{S}$ verifies the ASC-ness of $\mathbb{A}$ iff $\langle \Omega, \mathbb{S} \rangle$ does.

Claim There is a single

$$\mathbb{S} = \mathbb{S}_\varphi = \{\langle \bar{N}, \bar{\Omega}, \bar{\mathbb{A}}, \bar{\mathbb{B}}, \bar{G} \rangle \mid H_{\omega_1} \models \varphi[\bar{N}, \bar{\Omega}, \bar{\mathbb{A}}, \bar{\mathbb{B}}, \bar{G}]\}$$

s.t. $\langle \Omega, \mathbb{B}, \mathbb{S} \rangle$ witnesses the ASC-ness of $\mathbb{B}_i$ for all $i < \alpha$.

In fact we shall construct;

$$\mathbb{S}' = \{\langle \bar{N}, \bar{\Omega}, \bar{\mathbb{B}}, \nu, \bar{G}\rangle \mid H_{\omega_1} \models \varphi'[\bar{N}, \bar{\Omega}, \bar{\mathbb{B}}, \nu, \bar{G}]\}$$

s.t.

$$\mathbb{S} = \{\langle \bar{N}, \Omega, \bar{A}, \bar{\mathbb{B}}, \bar{G}\rangle \mid \bigvee_\nu (\bar{A} = \bar{\mathbb{B}}_\nu \wedge \langle \bar{N}, \bar{\Omega}, \bar{\mathbb{B}}, \nu, \bar{G}\rangle \in \mathbb{S}')\}$$

does the job.

$\mathbb{S}'$ is the set of $\langle \bar{N}, \bar{\Omega}, \bar{\mathbb{B}}, \nu, \bar{G}\rangle$ satisfying the following conditions (A), (B), (C), (D):

(A) $\bar{N} = L_{\bar{\tau}}^{\bar{A}}$ is a ZF$^-$C$^-$ model satisfying:

- $H_{\bar{\Omega}} = V_{\bar{\Omega}}$ where $\bar{\Omega} > \omega$ is a strong limit cardinal
- $\bar{\mathbb{B}} = \langle \bar{\mathbb{B}}_i \mid i < \bar{\alpha}\rangle \in H_{\bar{\Omega}}$
- each $\bar{\mathbb{B}}_i$ is a complete Boolean Algebra; $\bar{\mathbb{B}}_0 = 1$
- $\bar{\mathbb{B}}_i \subseteq \bar{\mathbb{B}}_j$ ($\bar{\mathbb{B}}_i$ is completely contained in $\bar{\mathbb{B}}_j$) for $i \leq j < \bar{\alpha}$
- $\Vdash_{\bar{\mathbb{B}}_i} \check{\bar{\mathbb{B}}}_{i+1}/\dot{G}$ is ASC in $H_{\bar{\Omega}}$ for $i+1 < \bar{\alpha}$.

(B) $\nu < \bar{\alpha}$ and $\bar{G}$ is well founded and weakly $\bar{\mathbb{B}}_\nu$ - generic over $\bar{N}$

The conditions (A), (B) are easily seen to be $\Sigma_1(H_{\omega_1})$. Our next condition (C) will have the form: $\bigwedge_{j<\nu} R(j, \bar{N}, \bar{\mathbb{B}}, \nu, \bar{G})$ where $R$ is $\Sigma_1(H_{\omega_1})$. Hence (C) will also be $\Sigma_1(H_{\omega_1})$. To formulate this condition, we let $j < \nu$ and define:

Set: $\bar{G}_i =: \bar{G} \cap \bar{\mathbb{B}}_j$. (Hence $\bar{G}_i$ is well founded and weakly $\bar{\mathbb{B}}_i$-generic over $\bar{N}$.) Let $\langle e_i, N', G' \rangle$ be the completion of $\bar{N}, \bar{G}_i$ in the sense of §1.1. Set: $\mathbb{B}'_h =: e_i(\bar{\mathbb{B}}_h)$ for $h < \bar{\alpha}$. In $\bar{N}$ we have: $\Vdash_{\bar{\mathbb{B}}_i} (\check{\bar{\mathbb{B}}}_{i+1}/\dot{G}$ is ASC in $H_{\check{\bar{\Omega}}})$. Hence in $N'$:

$$\Vdash_{\mathbb{B}'_i} (\check{\mathbb{B}}'_{i+1}/\dot{G} \text{ is ASC in } H_{\check{\Omega}'}) \text{ where } \Omega' = e_i(\bar{\Omega}).$$

Set: $\tilde{\mathbb{B}}_j =: \mathbb{B}'_{j+1}/G'$; $\tilde{N}_i = N'^{G'} =: \langle N'[G'], G' \rangle$. Then $\tilde{\mathbb{B}}_j$ is ASC in $H_{\Omega'}^{\tilde{N}}$. But then there is a least $\Sigma_1$ formula $\varphi = \varphi_i$ s.t. $\mathbb{S}_\varphi$ verifies the ASC-ness of $\tilde{\mathbb{B}}_j$ in $H_{\Omega'}^{\tilde{N}_i}$. Our next condition reads:

(C) For all $j < \nu$ there is $\tilde{G} \subset \tilde{\mathbb{B}}$ s.t.

(a) $\tilde{G}_i$ is $\tilde{\mathbb{B}}_j$-generic over $\tilde{N}_j$

(b) $\bar{G}_{i+1} = \{ b \in \bar{\mathbb{B}}_{i+1} \mid (e_j(b)/G') \in \tilde{G} \}$

(c) $H_{\omega_1} \models \varphi_i [\tilde{N}_i, \Omega', \tilde{\mathbb{B}}_i, \tilde{G}]$

(Note: The meaning of (c) is clearer if we consider a map $\pi: \bar{N} \prec N$ s.t. $\pi(j) = j^*$, $\pi(\bar{\Omega}, \bar{\mathbb{B}}) = \Omega, \mathbb{B}$. Let $\sigma = \mathrm{int}(\pi, \bar{G}_i)$ in the sense of §1.1. Suppose that $G$ is $\mathbb{B}_{i^*}$-generic over $V$ with $\pi''\bar{G}_i \subset G$. Then $\sigma''G' \subset G$ and $\sigma$ extends uniquely to a $\sigma: N'[G'] \prec N[G]$ with $\sigma(G') = G$, (By an abuse of notation we do not distinguish between $\sigma$ and its extension.) Then $\sigma(\tilde{\mathbb{B}}_j) = \mathbb{B}^* =: \mathbb{B}^*_{j+1}/G$. Thus $\mathbb{S}_{\varphi_i}$ verifies the ASC-ness of $\mathbb{B}^*$ in $H_\Omega[\cdot]$. Hence $\langle \Omega, \mathbb{S}_{\varphi_i} \rangle$ verifies the ASC-ness of $\mathbb{B}^*$.

Condition (C) (c) says that $\langle \tilde{N}_i, \tilde{\Omega}, \tilde{\mathbb{B}}_i, \tilde{G} \rangle$ lies in $\mathbb{S}_{\varphi_i}$.)

Our final condition reads:

(D) Let $\lambda \leq \nu$ be a limit ordinal. For $i < \lambda$ let $\langle e_i, N_i', G_i' \rangle$ be the completion of $\bar{N}, \bar{G}_i$ in the sense of § 1.1 (where $\bar{G}_i = \bar{G} \cap \mathbb{B}_i$). For $i \leq j \leq \lambda$ let $e_{i,j} = \mathrm{int}(\bar{G}_i, \bar{G}_j)$ in the sense of §1.1. Then $N_\lambda', \langle e_{i,\lambda} \mid i < \lambda \rangle$ is the direct limit of:

$$\langle N_i' \mid i < \lambda \rangle, \langle e_{i,j} \mid i \leq j < \lambda \rangle.$$

(In other words, $\bigwedge x \in N_\lambda' \; \bigvee i < \lambda \; x \in \mathrm{rng}(e_{i,\lambda})$)

Set: $S_\nu = S(\bar{N}, \bar{\Omega}, \nu) =: \{G \mid \langle \bar{N}, \bar{\Omega}, \nu, G \rangle \in \mathbb{S}'\}$.

It is evident that:

<u>Fact</u> Let $\bar{G} \in S_\nu$. Then $\bar{G}_j \in S_j$ for $j \leq \nu$.

Now let $N = L^A_\tau$ be a ZFC$^-$ model s.t. $H_\Omega \subset N$, $\Omega < \tau$. Let $\pi : \bar{N} \prec N$, where $\bar{N}$ is countable and almost full. Let $\pi(\bar{\Omega}, \bar{\mathbb{B}}) = \Omega, \mathbb{B}$. Set:

$$S_\nu = \{G \mid \langle \bar{N}, \bar{\Omega}, \bar{\mathbb{B}}, \nu, G\rangle \in \mathbb{S}'\},$$

where $\mathbb{S}'$ is defined as above.

Let: $\bar{\mathbb{B}} = \langle \bar{\mathbb{B}}_i \mid i < \bar{\alpha}\rangle$.

Defining $\mathbb{S}$ from $\mathbb{S}'$ as above, we see that Theorem 3 follows from:

<u>Main Claim</u> Let $\nu < \bar{\alpha}$. Then

(i) $\bigwedge a \in \bar{\mathbb{B}}_\nu \setminus \{0\}\ \bigvee G \in S_\nu\ a \in G$

(ii) If $\bar{G} \in S_\nu$, $\pi(\nu) = \nu^*$, and $u \in \bar{N}$ is finite, there is $b \in \mathbb{B}_{\nu^*}$ s.t. whenever $G \ni b$ is $\mathbb{B}_{\nu^*}$-generic, then there is $\sigma \in V[G]$ s.t.

- $\sigma : \bar{N} \prec N$
- $\sigma(\bar{\Omega}, \bar{\mathbb{B}}) = \Omega, \mathbb{B}$ and $\sigma \restriction u = \pi \restriction u$
- $C^N_\delta(\operatorname{rng} \sigma) = C^N_\delta(\operatorname{rng} \pi)$ where $\delta = \delta(\mathbb{B}_{\nu^*})$
- $\bar{G} = \sigma^{-1}{}''G$.

We shall prove this by induction on $\nu$, but will need a stronger induction hypothesis. To facilitate its formulation we define:

**Def** Let $j < \bar{\alpha}$, $j^* < \alpha$, $\bar{G} \in S_j$. Let $G$ be $\mathbb{B}_{j^*}$-generic over $V$ and $\sigma \in V[G]$.

$\langle \sigma, G \rangle$ <u>witnesses</u> $\bar{G}$ iff

- $\sigma : \bar{N} \prec N$
- $\sigma(\bar{\Omega}, \bar{\mathbb{B}}, j) = \Omega, \mathbb{B}, j^*$
- $\bar{G} = \sigma^{-1}{}''G$

(<u>Note</u> $j$ is uniquely determined by $\bar{G}$, since $\mathbb{B}_i \neq \mathbb{B}_j$ if $i \neq j$. Similarly $j$ is uniquely determined by $G$.)

(<u>Note</u> By our definition of $S'$ it is easily seen that if $G \in S_j$ and $i \leq j$, then $G \cap \mathbb{B}_i \in S_i$.)

**Def** Let $j \leq i < \bar{\alpha}$, $\bar{G}' \in S_i$, $\bar{G} = \bar{G}' \cap \mathbb{B}_j$. Let $\langle \sigma, G \rangle$ witness $\bar{G}$ and $\langle \sigma', G' \rangle$ witness $G'$.

$\langle \sigma', G' \rangle$ is <u>strong over</u> $\langle \sigma, G \rangle$ iff the following hold:

- $\sigma'(j) = \sigma(j)$ and $G = G' \cap \mathbb{B}_{\sigma(j)}$
- $C^N_\delta(\mathrm{rng}\,\sigma') = C^N_\delta(\mathrm{rng}\,\sigma)$, where $\delta = \delta(\mathbb{B}_{\sigma'(i)})$.

We can now formulate our induction hypothesis:

(I) Let $j < \nu$, $G \in S_j$. Let $b \in \overline{\mathbb{B}}_\nu$ s.t. $h_j(b) \in G$. (We write $h_i(b) =: h_{\mathbb{B}_i}(b)$ for $i \leq \nu$). There is $G' \in S_\nu$ s.t. $G \cap \{b\} \subset G'$.

(II) Let $j < \nu$ and $\bar{G} \in S_\nu$. Let $j^* < \nu^* < \alpha$. Let $a \in \mathbb{B}_{j^*} \setminus \{0\}$, $\dot{\sigma} \in V^{\mathbb{B}_{j^*}}$, and $u \subset \bar{N}$ be finite s.t. whenever $G \ni a$ is $\mathbb{B}_{j^*}$-generic ~~and~~ $\sigma = \dot{\sigma}^G$, then: $\sigma(j,\nu) = j^*, \nu^*$ and:

$\langle \sigma, G \rangle$ witnesses $\bar{G}_j = \bar{G} \cap \mathbb{B}_j$.

Then there is $b \in \mathbb{B}_{\nu^*}$ s.t. $h_{j^*}(b) = a$ and whenever $G' \ni b$ is $\mathbb{B}_{\nu^*}$-generic, then, letting $G = G' \cap \mathbb{B}_{j^*}$, $\sigma = \dot{\sigma}^G$, we have:

- $\langle \sigma', G' \rangle$ witnesses $\bar{G}$
- $\langle \sigma', G' \rangle$ is strong over $\langle \sigma, G \rangle$
- $\sigma \restriction u = \sigma' \restriction u$.

<u>Note</u> By §1.1 Fact 16, I implies:

Let $j < \nu$, $\bar{G} \in S_j$. Let $\langle e, \hat{N}, \hat{G} \rangle$ be the completion of $\bar{N}, \bar{G}$. Let $b \in e(\mathbb{B}_\nu)$ s.t. $h_{e(\mathbb{B}_j)}(b) \in \hat{G}$. There is $\bar{G}' \in S_\nu$ s.t. $\bar{G} = \bar{G}' \cap \mathbb{B}_j$ and if $\langle e', N', G' \rangle$ is the completion of $\bar{N}, \bar{G}'$, then $i(b) \in G'$, where $i = \mathrm{int}(\bar{G}, \bar{G}')$.

Note Since $|B_0| = |\bar{B}_0| = 2$, we have: $\{1\} \in S_0$ and $\{1\}$ is 2-generic. (i), (ii) in Claim 1 are trivial for $\nu = 0$. If $\nu > 0$ we have:

Applying (I) to $j = 0$ with $\bar{G} = \{1\}$ gives (i)

Applying (II) to $j = 0$ with $\bar{G} = G = \{1\}$, $\langle \sigma, G \rangle = \langle \pi, \{1\} \rangle$, $\dot{\sigma} = \check{\pi}$, $\dot{u} = \check{u}$ gives us (ii).

---

Def We say that I, II hold at $\langle j, \nu \rangle$ iff $j < \nu < \bar{\alpha}$ and I, II hold as stated for the specific $j$. We say that I, II hold at $\nu$ iff they hold at $\langle j, \nu \rangle$ for all $j < \nu$.

We shall show by induction on $\nu$ that I, II hold at $\nu$ for all $\nu < \bar{\alpha}$. We first note:

Lemma 3.2 Let I, II hold at $\eta$, where $\eta < \nu$. Let them hold at $\langle \eta, \nu \rangle$. Then they hold at $\langle j, \nu \rangle$ for all $j < \eta$.

proof.

(I) Let $i < \eta$, $G \in S_i$, $b \in \bar{B}_\nu$, $h_j(b) \in G$. Pick $G' \supset G$ s.t. $G' \in S_\eta$ and $h_\eta(b) \in G'$. Then there is $G'' \supset G'$ s.t. $G'' \in S_\nu$, $b \in G''$.

QED(I)

Before proving this for II, we note:

<u>Sublemma 3.3</u> II is equivalent to the statement that the following is forced by $\mathrm{coll}(\theta, \omega)$ for sufficiently large $\theta$:

(II)′ Let $j < \nu$, $\bar{G} \in S_\nu$, let $j^* < \nu^* < \alpha$, let $\langle \sigma, G \rangle$ witness $\bar{G}_j = \bar{G} \cap \mathbb{B}_j$ s.t. $\sigma(j, \nu) = j^*, \nu^*$, let $u \subset \vec{N}$ be finite.

Then there are $G' \supset G$, $\sigma' \in V[G']$ s.t. $\langle \sigma', G' \rangle$ witness $\bar{G}$, $\sigma'(\nu) = \nu^*$, $\langle \sigma', G' \rangle$ is strong over $\langle \sigma, G \rangle$, and $\sigma' \restriction u = \sigma \restriction u$.

proof.
$(\rightarrow)$ is trivial. We prove $(\leftarrow)$.
Let $a$ be as in the assumption of II.
Then for every $a' \subset a$ s.t. $a' \neq 0$ in $\mathbb{B}_{j^*}$ there is a $\mathbb{B}_{j^*}$-generic $G \ni a'$. Hence, letting $\sigma = \dot{\sigma}^G$, we have: $G, \sigma, u$ satisfy the assumption of II′. Hence there is $G' \supset G$, $\sigma' \in V[G']$ satisfying the conclusion of II′. But then there is a $b \in G'$ s.t. $b \subset a'$ and $b$ forces that whenever $G_1' \ni b$ is $\mathbb{B}_{\nu^*}$-generic, then

the conclusion of II holds. In other words, the set $\Delta$ of $b \in \mathbb{B}_{\nu^*} \setminus \{0\}$ with this property is dense below $a$ in $\mathbb{B}_{\nu^*}$. Let $A$ be a maximal antichain in $\{a' \subset a \mid \bigvee b \in \Delta\ a' \subset h_{j^*}(b)\}$. Then $\bigcup A = a$. For each $a' \in A$ choose $b_{a'} \in \Delta$ s.t. $a' \subset h_{j^*}(b_{a'})$. Set

$$b = \bigcup_{a' \in A} a' \wedge b_{a'}.$$

Then $h_{j^*}(b) = a$ and $b$ has the desired properties.

QED (Sublemma 3.3)

Using this we complete the proof of Lemma 3.2 by showing that II' holds at $\langle j, \nu \rangle$ if $j < \eta$. Let $G \in \mathbb{B}_{j^*}$, $\sigma \in V[G]$ s.t. $\sigma(j, \nu) = j^*, \nu^*$ and $\langle \sigma, G \rangle$ witnesses $\bar{G}_j$. Let $\sigma(\eta) = \eta^*_N$. Then $j^* < \eta^* < \nu^*$.

Let $u \subset \bar{N}$ be finite. Let $G' \supset G$ be $\mathbb{B}_{\eta *}$-generic and $\sigma' \in V[G']$ s.t. $\langle \sigma', G' \rangle$ witnesses $\bar{G}_\eta$, $\langle \sigma', G' \rangle$ is strong over $\langle \sigma, G \rangle$ and $\sigma' \upharpoonright u = \sigma \upharpoonright u$. Assume w.l.o.g. that $\sigma'(\nu) = \nu^*$ (we could take $\nu \in u$). Let $G'' \supset G'$ be $\mathbb{B}_{\nu *}$-generic and $\sigma'' \in V[G']$ s.t. $\langle \sigma'', G'' \rangle$ witnesses $\bar{G}_\nu$, $\langle \sigma'', G'' \rangle$ is strong over $\langle \sigma', G' \rangle$, $\sigma'' \upharpoonright u = \sigma' \upharpoonright u$, and $\sigma''(j) = \sigma'(j) = j^*$. $\langle \sigma'', G'' \rangle$ is easily seen to have the desired properties. QED (Lemma 3.2)

We now prove I, II by induction on $\nu$. Case $\nu = 0$ is trivial. There remain two cases:

<u>Case 1</u> $\nu = \eta + 1$.
By Lemma 3.2 it suffices to prove the assertion for $j = \eta$.
(I) Let $G \in S_\eta$, $b \in \mathbb{B}_\nu$, $h_\eta(b) \in G$. Let $\langle \hat{\sigma}, \hat{N}, \hat{G} \rangle$ be the completion of $N, G$. Then $h_{\hat{\eta}}(\hat{b}) \in \hat{G}$, where $\hat{\eta} = \hat{\sigma}(\eta)$.
Set $\hat{b} = \hat{\sigma}(b)$. Set $\hat{\mathbb{B}} = \sigma(\mathbb{B}_\nu)$, $\tilde{\mathbb{B}} = \hat{\mathbb{B}}/\hat{G}$. Set $N' = \hat{N}^{\hat{G}} = L^{\hat{A}, \hat{G}}_{\hat{\tau}}$ where $\hat{N} = L^{\hat{A}}_{\hat{\tau}}$.
Since $\pi : \bar{N} \prec N$, $\pi(\bar{\mathbb{B}}) = \mathbb{B} = \langle \mathbb{B}_i \mid i < \alpha \rangle$, we know that $\Vdash_\eta (\check{\mathbb{B}}_\nu / \dot{G}$ is ASC) in $H^{\bar{N}}_{\bar{\Omega}}$.

Hence the same holds of $\hat{\mathbb{B}} = \sigma(\bar{\mathbb{B}}_\nu)$ in $H_{\hat{\Omega}}^{\hat{N}}$ where $\sigma(\bar{\Omega}) = \hat{\Omega}$. But then $\tilde{\mathbb{B}}$ is ASC in $H_{\hat{\Omega}}^{\hat{N}}[\hat{G}] = H_{\hat{\Omega}}^{\hat{N}[\hat{G}]}$, since $\hat{G}$ is $\sigma(\mathbb{B}_\eta)$-generic over $\hat{N}$. Let $\varphi = \varphi_\eta =:$ the minimal $\varphi$ s.t. $\$_\varphi$ verifies the ASC-ness of $\tilde{\mathbb{B}}$ in $H_{\hat{\Omega}}^{\hat{N}}[\hat{G}]$. Applying the induction hypothesis II′ at $\eta$ gives us a $\langle \sigma^*, G^* \rangle$ which witnesses $\bar{G}_\eta$ + s.t. $\sigma^*(\eta) = \bar{\pi}(\eta)$. ($G^*$ is of course found in the generic collapse of a sufficiently large cardinal.) Let $\eta^* = \sigma^*(\eta)$. Then $\sigma^*(\nu) = \nu^* = \eta^* + 1$. Now let $i = \mathrm{int}(\sigma^*, G^*)$ (where $\bar{G}_\eta = \sigma^{*-1}{}''G^*$). Then $i : \hat{N} \prec N$ s.t. $i \circ \sigma = \sigma^*$ and $i''\hat{G} \subset G^*$. Hence $i$ extends to a unique $i^* : \hat{N}[\hat{G}] \prec N[G^*]$ s.t. $i^*(\hat{G}) = G^*$. Hence $i^*(\tilde{\mathbb{B}}) = \mathbb{B}_{\nu^*}/G^*$ and $\mathbb{B}_{\nu^*}/G^*$ is ASC in $H_\Omega[G^*] = H_\Omega^{V[G^*]}$ as verified by $\$_\varphi$. Hence $\mathbb{B}_{\nu^*}/G^*$ is ASC in $V[G^*]$ as verified by by $\$_\varphi$, since

since $H_\Omega[G^*] \prec_{\Sigma_n} V[G^*]$. Applying this to the map $i^* : \hat{N}[\hat{G}] \prec N[G^*]$, we see that for any $b' \in (\mathbb{B}_{\gamma^*}/G^*) \setminus \{0\}$

$\tilde{G}$ s.t. $\langle \hat{N}[\hat{G}], \hat{\Omega}, \hat{\mathbb{B}}, \tilde{G} \rangle \in \mathbb{S}_\varphi$ and $b' \in \tilde{G}$. In particular, we can take $b' = \hat{\sigma}(b)/\hat{G}$, where $b \in \bar{\mathbb{B}}_\nu$ s.t. $h_\eta(b) \in \bar{G}_\eta$ (hence $\hat{\sigma}(b)/\hat{G} \neq 0$).

Set: $G' = \hat{G} * \tilde{G} =: \{b \in \hat{\mathbb{B}} \mid b/\hat{G} \in \tilde{\mathbb{B}}\}$,

$\bar{G}' =: \sigma^{-1}{}''G'$

Then $\langle \bar{N}, \bar{\Omega}, \bar{\mathbb{B}}_\nu, \bar{G}' \rangle \in \mathbb{S}_\nu$ and $b \in \bar{G}'$, since (A)–(C) are satisfied.

QED (I)

Note In order to literally satisfy (C) we should have written $\hat{N}^{\hat{G}}$ and $N^{G^*}$ instead of $\hat{N}[\hat{G}]$ and $N[G^*]$, where:

$$\left(L_\tau^{A_1,\ldots,A_n}\right)^B =: L_\tau^{A_1,\ldots,A_n,B}.$$

However, this is an abuse of notation which we shall often commit.

We now turn to:

(II) We verify II$'$ of Sublemma 3.3. Let $\langle \sigma, G \rangle$ witness $\bar{G}_\eta = \bar{G} \cap \mathbb{B}_\eta$, where $\sigma(\eta) = \eta^*$ and $G$ is $\mathbb{B}_{\eta^*}$-generic. Then $\sigma(\nu) = \nu^* = \eta^* + 1$. Let $u \subset \bar{N}$ be finite. Let $\langle \hat{\sigma}, \hat{N}, \hat{G} \rangle$ be the completion of $\bar{N}, \bar{G}_\eta$. Set $u$
$\hat{\eta}, \hat{u}, \hat{\Omega}, \hat{\mathbb{B}}, \hat{\nu} =: \hat{\sigma}(\eta, u, \bar{\Omega}, \mathbb{B}, \nu)$.
Let $\pi = \mathrm{int}(\sigma, G)$. Then $\sigma = \pi \circ \hat{\sigma}$ and $\pi : \hat{N}[\hat{G}] \prec N[G]$, $\pi(\hat{G}) = G$.
Set: $\mathbb{B}^* = \mathbb{B}_{\nu^*}/G$, $\hat{\mathbb{B}}^* = \hat{\mathbb{B}}_{\hat{\nu}}/\hat{G}$. Then $\pi(\hat{\mathbb{B}}^*) = \mathbb{B}^*$. Set $\hat{N}_\eta = L^{\hat{A}, \hat{G}}_{\hat{\tau}}$ where $\hat{N} = L^{\hat{A}}_{\hat{\tau}}$, and $N_\eta = L^{A, G}_\tau$ where $N = L^A_\tau$. Then $\pi : \hat{N}_\eta \prec N_\eta$. By the definition of $S_\nu$ there is $\hat{G}^*$ s.t. $\hat{G}^* \subset \hat{\mathbb{B}}^*$ is $\hat{\mathbb{B}}^*$-generic over $\hat{N}_\eta$ and:

- $\bar{G} = \hat{\sigma}^{-1}{}''\hat{G}'$, where $\hat{G}' = \hat{G} * \hat{G}^* = \{ b \mid b/\hat{G} \in \hat{G}^* \}$,
- $H_{\omega_1} \models \varphi_\eta [\hat{N}_\eta, \hat{\Omega}, \hat{\mathbb{B}}^*, \hat{G}^*]$
(i.e. $\langle \hat{N}_\eta, \hat{\Omega}, \hat{\mathbb{B}}^*, \hat{G}^* \rangle \in S_{\varphi_\eta}$).

But, in $V[G]$, $\mathbb{B}^*$ is ASC as verified by $\langle \Omega, S_\varphi \rangle$. Hence there are

$G^* \subset \mathbb{B}^*$, $\pi^* \in V[G^*]$ s.t.

(a) $\pi^* : \hat{N}_\eta \prec N_\eta$

(b) $\pi^* \restriction \hat{u} = \pi \restriction \hat{u}$

(c) $C^{N_\eta}_{\delta^*}(\mathrm{rng}\, \pi^*) = C^{N_\eta}_{\delta^*}(\mathrm{rng}\, \pi)$,
where $\delta^* = \delta(\mathbb{B}^*)$ in $V[G]$

(d) $\pi^* {}'' \hat{G}^* \subset G^*$.

Set: $G' = G * G^* = \{b \mid b/G \in G^*\}$. Then $G'$ is $\mathbb{B}_{\nu^*}$ - generic and $G' \supset G$. Set $\sigma' = \pi^* \circ \hat{\sigma}$.

It suffices to show:

Claim
(A) $\langle \sigma', G' \rangle$ witnesses $\bar{G}$
(B) $\langle \sigma', G' \rangle$ is strong over $\langle \sigma, \bar{G}_\eta \rangle$
(C) $\sigma' \restriction u = \sigma \restriction u$

prf:
(A), (C) are immediate. We prove (B).
All conditions are trivial except:
(1) $C^N_\delta(\mathrm{rng}\, \sigma') = C^N_\delta(\mathrm{rng}\, \sigma)$ where $\delta = \delta(\mathbb{B}_{\nu^*})$.
Note first that $\delta \geq \delta^*$. Hence
(2) $C^{N_\eta}_\delta(\mathrm{rng}\, \pi^*) = C^{N_\eta}_\delta(\mathrm{rng}\, \pi)$
Set: $\tilde{\pi} = \pi \restriction \hat{N}$, $\tilde{\pi}^* = \pi^* \restriction \hat{N}$. Then
$\sigma = \tilde{\pi} \cdot \hat{\sigma}$, $\sigma' = \tilde{\pi}^* \cdot \hat{\sigma}$. We prove:
(3) $C^N_\delta(\mathrm{rng}\, \tilde{\pi}^*) = C^N_\delta(\mathrm{rng}\, \tilde{\pi})$.

(1) will then follow from (3), since by §1.1 Fact 11 we know that:

(4) $\hat{N} = C^{\hat{N}}_{\hat{\delta}}(\mathrm{rng}\,\hat{\sigma})$ where $\hat{\delta} = \delta(\mathbb{B}_{\gamma})$.

But then $C^{N}_{\delta}(\mathrm{rng}\,\sigma) \supset \mathrm{rng}(\tilde{\pi})$ and $C^{N}_{\delta}(\mathrm{rng}\,\sigma') \supset \mathrm{rng}(\tilde{\pi}^*)$. The conclusion is immediate. It remains only to prove (3).

We first show:

(5) $C^{N}_{\delta}(\mathrm{rng}\,\tilde{\pi}) = N \cap C^{N_\eta}(\mathrm{rng}\,\pi)$

($\subset$) is trivial. We prove ($\supset$).

Let $x \in N \cap C^{N_\eta}(\mathrm{rng}(\pi))$. Then $x = \pi(f)(\xi)$ where $f \in \hat{N}[\hat{G}]$, $\xi < \delta$. Let $f = \dot{f}^{\hat{G}}$. Then $\pi(f) = \tilde{\pi}(\dot{f})^{G}$. Clearly $\delta = \delta(\mathbb{B}_{\gamma*}) \in \mathrm{rng}\,\tilde{\pi}$. Since $\mathbb{B}_{\eta*}$ has a dense set of size $\leq \delta$, there is $k \in \mathrm{rng}(\tilde{\pi})$ mapping $\delta$ onto such a dense set $\Delta$. But then for some $\zeta < \delta$ we have:

$x =$ the unique $x$ s.t. $k(\zeta) \Vdash^{N}_{\mathbb{B}} \tilde{\pi}(\dot{f})(\check{\xi}) = \check{x}$.

Hence $x \in C^{N}_{\delta}(\mathrm{rng}\,\tilde{\pi})$. QED (5)

An entirely similar proof yields:

(6) $C^{N}_{\delta}(\mathrm{rng}\,\tilde{\pi}^*) = N \cap C^{N_\eta}(\mathrm{rng}\,\tilde{\pi}^*)$.

(2) follows immediately

QED (Case 1)

Case 2 $\nu = \lambda$ is a limit ordinal.

By the induction hypothesis and Lemma 3.2 it suffices to prove I, II at $j, \lambda$ for sufficiently large $j$. We shall in fact prove it for $j$ with the property:

(1) If there is $i < \lambda$ s.t. $cf(\lambda) < \delta(\mathbb{B}_i)$ in $\bar{N}$, then $j > i$ for some such $i$.

As a preliminary to proving I we show:

**Lemma 3.4** Let $\bar{G}_i \in S_i$ for $i < \lambda$ s.t. $\bar{G}_h = \mathbb{B}_h \cap \bar{G}_i$ for $h \leq i < \lambda$. Let $\langle e_i, \hat{N}_i, \hat{G}_i \rangle$ be the completion of $\bar{N}, \bar{G}_i$. Let $e_{ij} = \text{int}(\bar{G}_i, \bar{G}_j)$ for $i \leq j < \lambda$. Then the commutative system $e_{ij} : \hat{N}_i \prec \hat{N}_j$ has a well founded direct limit.

proof

Let $N^*, \langle e_i^* \mid i < \lambda \rangle$ be a direct limit of $\langle \hat{N}_i \mid i < \lambda \rangle, \langle e_{ij} \mid i \leq j < \lambda \rangle$

Let $\langle \xi_i \mid n < \omega \rangle$ be a monotone cofinal sequence in $\lambda$ with $\xi_0 = 0$.

Let $\langle x_i \mid i < \omega \rangle$ enumerate $\bar{N}$.

Working in the generic collapse of a sufficient cardinal, we make successive application of II′ to get $\langle \sigma_n, G_n \rangle$ s.t.

- $\langle \sigma_n, G_n \rangle$ witnesses $G_{\xi_n}$
- $\langle \sigma_m, G_m \rangle$ is strong over $\langle \sigma_h, G_h \rangle$ for $h < m$
- $\sigma_m(x_i) = \sigma_h(x_i)$ for $i \leq h \leq m$.
- $\sigma_m(\xi_i) = \sigma_h(\xi_i)$ for $i \leq h \leq m$.

Then for each $x \in \bar{N}$ we can set:

$$\tilde{\sigma}(x) = \sigma_n(x) \text{ if } \sigma_m(x) = \sigma_n(x) \text{ for } m \geq n.$$

It follows easily that $\tilde{\sigma} : \bar{N} \prec N$.

Since $\sigma_n : \bar{N} \prec N$ and $\sigma_n{}''\bar{G}_{\xi_n} \subset G_n$, we can set:

$$\sigma_n^* = \operatorname{int}(\sigma_n, G_n) \text{ for } n < \omega.$$

Then $\sigma_n^* : \hat{N}_{\xi_n} \prec N$, $\sigma_n^*{}''\hat{G}_{\xi_n} < G_n$ and $\sigma_n^* \circ e_{\xi_n} = \sigma_n$.

<u>Claim</u> Let $x \in N^*$. Let $x_n = e_{\xi_n}^{*-1}(x)$ if defined. For sufficiently large $n$ we have:

$$\sigma_n^*(x_n) = \sigma_m^*(x_m) \text{ for } n \leq m.$$

proof.

Let $x_h$ be defined for $h < \omega$. Note that $\sigma_h(\xi_h) = \sigma_m(\xi_h)$ for $h \leq m$. Hence $\sigma_h(\mathbb{B}_h) = \sigma_m(\mathbb{B}_h) = \mathbb{B}_{\tilde{\sigma}(\xi_h)}$ for $h \leq m$. Set: $\hat{G}_{hm} =: \hat{G}_{\xi_m} \cap e_{\xi_m}(\mathbb{B}_{\xi_h})$ for $h \leq m$. Then:

$$\sigma_m^* {}'' \hat{G}_{hm} \subset G_m \cap \mathbb{B}_{\tilde{\sigma}(\xi_h)} = G_h .$$

But $\hat{G}_{hm}$ is $e_{\xi_m}(\mathbb{B}_{\xi_h})$ - generic over $\hat{N}$ and $G_h$ is $\mathbb{B}_{\tilde{\sigma}(\xi_h)}$ - generic over $N$,

where $\sigma^*(e_{\xi_m}(\mathbb{B}_{\xi_h})) = \sigma_m(\mathbb{B}_{\xi_h}) = \mathbb{B}_{\tilde{\sigma}(\xi_h)}$.

Hence there is $\tilde{\sigma}_m : \hat{N}[\hat{G}_{hm}] \prec N[G_h]$ s.t. $\tilde{\sigma}_m(\hat{G}_{hm}) = G_h$. Now let $x_h = e_{\xi_h}(t)^{\hat{G}_{\xi_h}}$. Pick $n > h$ big enough that $\sigma_m(t) = \tilde{\sigma}(t)$ for $m \geq n$. For $m \geq n$ we have:

$$\sigma_m^*(x_m) = \tilde{\sigma}_m(e_{\xi_m}(t)^{\hat{G}_{hm}}) = \tilde{\sigma}(t)^{G_h}$$

QED (Claim)

If we then set:

$\sigma^*(x) =: \sigma^*_n(x_n)$ whenever $\sigma^*_n(x_n) = \sigma^n_m(x_m)$ for $n \leq m$,

then: $\sigma^*: N^* \prec N$. Hence $N^*$ is well founded.

QED (Lemma 3.4)

We now verify I at $j', \lambda$. Let $\bar{G} \in S_{j'}$.
Let $b \in \mathbb{B}_\lambda$ s.t. $a = h_{j'}(b) \in \bar{G}$. We must construct $\bar{G}' \in S_\lambda$ s.t. $\bar{G} \subset \bar{G}'$ and $b \in \bar{G}'$. Let $\langle \xi_n \mid n < \omega \rangle$ be monotone and cofinal in $\lambda$. We shall construct a sequence $\langle \bar{G}_n, b_n \rangle$ s.t.

(a) $\bar{G}_n \in S_{\xi_n}$, $\bar{G}_0 = \bar{G}$, $\bar{G}_n \subset \bar{G}_{n+1}$

Let $\langle e_n, \hat{N}_n, \hat{G}_n \rangle$ be the completion of $\bar{N}, \bar{G}_n$ and set $e_{nm} = \mathrm{int}(\bar{G}_n, \bar{G}_m)$ for $n \leq m < \omega$. We shall have:

(b) $b_n \in e_n(\bar{\mathbb{B}}_\lambda)$ s.t. $b_0 = e_0(b)$

and $b_m \subset b_n$ for $h \leq m$;

For $i, n < \omega$ and $d \in e_n(\bar{\mathbb{B}}_\lambda)$ set:

$h^n_i(d) =: h_{e_n(\bar{\mathbb{B}}_i)}(d)$ in $\hat{N}_n$,

Set $a_n = h^n_n(b_n)$. Then

(c) $e_{in}(a_i) = h^n_i(b_n)$ for $i \leq n$

(d) $a_n \in \hat{G}_n$

$\bar{G}_0$, $b_0$ are given. If we have $G_n$, $b_n$ and $b'_n \subset b_n$ in $e_n(\bar{\mathbb{B}}_\lambda)$ s.t. $h^n_n(b^*_n) = a_n = h^n_1(b_n)$, then the induction hypothesis I at $\xi_{n+1}$ and Fact 16 of §1.1 give us a $\bar{G}_{n+1} \in S_{\xi_{n+1}}$ s.t.

(e) $e_{n,n+1}(h^n_{n+1}(b'_n)) \in \hat{G}_{n+1}$.

If we set: $b_{n+1} =: e_{n,n+1}(b'_n)$,

Then (a) – (d) will be satisfied at $n+1$.

It remains only to define $b'_n$.

Let $\langle \Delta^i_j \mid j < \omega \rangle$ enumerate the $\Delta \in \hat{N}_i$ which are strongly dense subsets of $e_i(\mathbb{B}_\lambda)$. Let $\langle \langle i_n, j_n \rangle \mid n < \omega \rangle$ enumerate $\omega^2$ with $i_n, j_n \leq n$.

Set: $\Delta_n =: e_{i_n n}(\Delta^{i_n}_{j_n})$. Then $\Delta_n$ is a strongly dense subset of $e_n(\mathbb{B}_\lambda)$. Set:

$$B_n =: \{ h^n_n(b') \mid b' \subset b_n \wedge b' \in \Delta_n \}.$$

Let $A_n \in \hat{N}_n$ be a maximal antichain in $B_n$.

Then $\bigcup A_n = a_n$. (If not, let $a = a_n \setminus \bigcup A_n$.
Then $a \cap b_n \neq 0$. Pick $b' \subset a \cap b_n$ s.t.
$b' \in \Delta_n$. Set $a' = h_n^m(b')$. Then $a' \in B_m$
and $a' \cap \bigcup A_n = 0$. Contradiction!.)
For each $a \in A_n$ pick $b^a \subset b_m$ s.t.
$b^a \in \Delta_n$. Set: $b_m' = \bigcup_{a \in A_n} b^a$.
$b_m'$ has the required properties.

This completes the construction of
$\bar{G}_m, b_m$ $(n < \omega)$. Now let
$\dot{N}^*, \langle e_m^* \mid n < \omega \rangle$ be the transitivized
direct limit of:
$\langle \hat{N}_m \mid n < \omega \rangle, \langle e_{nm} \mid n \leq m < \omega \rangle$.

Since $e_{nm} \dot{e}_n = e_m$, we can define:
$e^* : N \prec N^*$ by: $e^* = e_n^* e_m$ $(n < \omega)$.
Let $G^*$ be the filter on $e^*(\mathbb{B}_\lambda)$
generated by the set of
$e_m^*(a \cap b_m')$ s.t. $a \in \hat{G}_m$. <u>We claim</u>:
(2) $G^*$ is a generic ultrafilter on $e^*(\mathbb{B}_\lambda)$.
proof
Let $\Delta \in N^*$ be strongly dense
in $e^*(\mathbb{B}_\lambda)$. Let $\Delta = e_i^*(\Delta_i^i)$.

Then $\Delta = e_n^*(\Delta_n)$ where $\langle \delta, \gamma' \rangle = \langle \delta_n, \gamma_n' \rangle$. Since $\hat{G}_n$ is $e_n(\mathbb{B}_{\xi_n})$ - generic over $\hat{N}_n$, there is $a \in \hat{G}_n$ s.t. $a \in A_n$. Hence $a \wedge b_n' \in \Delta_n$. Hence $e_n^*(a \wedge b_n) \in \Delta$. QED (2).

Set: $\bar{G} = e^{*-1}{}''G^*$. Then $\bar{G}$ is well founded and weakly $\mathbb{B}_\lambda$ - generic over $\bar{N}$. But $\bar{G} \cap \mathbb{B}_{\bar{\xi}_n} = \bar{G}_n$ for $n < \omega$. From this it easily follows that $\bar{G}$ satisfies (A), (B), (C), (D) and is, therefore, an element of $S_\lambda$. Finally we note that

$$e^*(b) = e_0^* e_0(b) = e_0^*(b_0) \in G^*,$$

hence $b \in \bar{G}$.

This proves I in Case 2

We now prove $\underline{\text{II}}$ at $\lambda$. By Sublemma 3.3.7 it suffices to prove the equivalent version $\underline{\text{II}}'$. Let $j < \lambda$ satisfy (1). Let $\bar{G} \in S_\lambda$. Let $\langle \sigma^0, G^0 \rangle$ witness $\bar{G}_j = \bar{G} \wedge B_j$ where $\sigma^0(j) = j^*$ and $\sigma(\lambda) = \lambda^*$. Let $u \subset \bar{N}$ be finite. ($u$ can be taken as large as we want. We shall in fact later assume that $t^0 \in u$ for a specific $t^0 \in \bar{N}$.)

$\underline{\text{Claim}}$ There is (in the generic collapse of a sufficiently large cardinal) a pair $\langle \sigma', G' \rangle$ s.t. $G' \supset G^0$ and:

- $\langle \sigma', G' \rangle$ witnesses $\bar{G}$
- $\langle \sigma', G' \rangle$ is strong over $\langle \sigma^0, G^0 \rangle$ and $\sigma'(\lambda) = \lambda^*$
- $\sigma' \restriction u = \sigma^0 \restriction u$

Let $\eta =: \sup \sigma^{0\,\prime\prime} \lambda$. Let $\langle \bar{\xi}_i \mid i < \omega \rangle$ be monotone and cofinal in $\lambda$ s.t. $\bar{\xi}_0 = j$. Let $\sigma^0 = \mathring{\sigma}^{G^0}$, where $\mathring{\sigma} \in V^{\mathbb{B}_{j^*}}$. Set:

$$\eta_i = \sigma^0(\bar{\xi}_i) \quad \text{for } i < \omega.$$

Then $\langle \eta_i \mid i < \omega \rangle$ is monotone and cofinal in $\eta$ and $\eta_0 = j^*$.

($\underline{\text{Note}}$ The sequence $\langle \eta_i \mid i < \omega \rangle$ is fixed for the rest of the proof. We may, however, consider pairs $\langle G, \sigma \rangle$ different from $\langle G^0, \sigma^0 \rangle$ s.t. $\langle G, \sigma \rangle$ witnesses $\bar{G}_j$ and $\sigma = \mathring{\sigma}^G$. We then do not necessarily have $\sigma(\bar{\xi}_i) = \eta_i$.)

Let $f = \sigma^\circ \upharpoonright u$. We know that $\langle \sigma^\circ, G^\circ \rangle$ witnesses $\bar{G}_j$, $\sigma^\circ(j) = j^*$, $\sigma^\circ(\lambda) = \lambda^*$, $\sigma^\circ \upharpoonright u = f$, and $\sigma^\circ(\bar{\xi}_1) = \eta_1$ and $\sup \sigma^{\circ\prime\prime} \lambda = \eta$. Hence there is $a \in G^\circ$ which forces all of this – i.e.

$$a \Vdash_{\mathbb{B}_{j^*}} (\langle \dot{\sigma}, \dot{G} \rangle \text{ witnesses } \check{\bar{G}}_j \wedge \dot{\sigma}(\check{j}) = \check{j}^* \wedge \dot{\sigma}(\check{\lambda}) = \check{\lambda}^* \wedge \dot{\sigma} \upharpoonright \check{u} = \check{f} \wedge \dot{\sigma}(\check{\bar{\xi}}_1) = \check{\eta}_1 \wedge \sup \dot{\sigma}'' \check{\lambda} = \check{\eta})$$

where $\dot{G}$ is the canonical name for the generic set (i.e. $[\![ b \in \dot{G} ]\!] = b$ for $b \in \mathbb{B}_{j^*}$).

<u>Lemma 3.5</u> Let $j^* < i^* < \eta$, $j < i < \lambda$. Let $\langle G', \sigma' \rangle$ witness $\bar{G}_i$, where $G'$ is $\mathbb{B}_{i^*}$ – generic and $\sigma'(j, i, \lambda) = j^*, i^*, \lambda^*$. Let $G = G' \cap \mathbb{B}_{j^*}$. Assume: $a \in G$. Set $\sigma = \dot{\sigma}^G$. (Hence $\langle \sigma, G \rangle$ witnesses $\bar{G}_j$.) If $\langle \sigma', G' \rangle$ is strong over $\langle \sigma, G \rangle$, then $\eta = \sup \sigma'' \lambda$.

proof

<u>Case 1</u> $\mathrm{cf}(\lambda) > \delta(\bar{\mathbb{B}}_h)$ for all $h < \lambda$ in $\bar{N}$.

Then $\lambda^*$ is inaccessible and $\delta(\mathbb{B}_h) < \lambda^*$ for $h < \lambda^*$.

<u>Claim</u> Let $\mu < \eta$. Then

$\sup \sigma'' \lambda = \sup \lambda^* \cap C_\mu^N(\mathrm{rng}\, \sigma)$

proof

$\leq$ is trivial. We prove $\geq$.

Let $\zeta \in \lambda^* \cap C^N_\mu(\operatorname{rng}\sigma)$. Then $\zeta = \sigma(g)(\bar\zeta)$ where $g \in \bar N$, $\operatorname{rng}(g) \subset \lambda$, and $\bar\zeta \in \operatorname{dom}(g)$. Let $\sigma(\tau) \geq \mu$, where $\tau < \lambda$. Then
$\zeta \leq \sup \sigma(g)''\sigma(\tau) = \sigma(\sup g''\tau) \in \operatorname{rng}\sigma$.

QED (Claim)

But the same proof shows that

$$\sup \sigma'{}''\lambda = \sup \lambda^* \cap C^N_\mu(\operatorname{rng}\sigma').$$

Hence for $\delta = \delta(\mathbb{B}_{i^*})$ we have:

$$\eta = \sup \sigma''\lambda = \sup \lambda^* \cap C^N_\delta(\operatorname{rng}\sigma) =$$

$$= \sup \lambda^* \cap C^N_\delta(\operatorname{rng}\sigma') = \sup \sigma'{}''\lambda.$$

QED (Case 1)

<u>Case 2</u> Case 1 fails.

Then there is $h < \lambda$ s.t. $\mathrm{cf}(\lambda) < \delta(\mathbb{B}_h)$ in $\bar N$. Hence by (1) $j > h$ for some such $h$. Hence, letting $\tau = \mathrm{cf}(\lambda)^{\bar N}$, we have: $\Vdash^{\bar N}_{\mathbb{B}_j} \operatorname{card}(\tau) \leq \omega_1$.

Hence there is $\mathring{f} \in \bar N^{\mathbb{B}_j}$ s.t.

$$\Vdash^{\bar N}_{\mathbb{B}_j} (\operatorname{dom}(\mathring{f}) \leq \omega_1 \wedge \mathring{f} \text{ is a normal function whose range lies cofinally in } \check\lambda).$$

Let $\mathring{f}$ be the $\bar N$-least such. Then $\sigma(\mathring{f}) = \sigma'(\mathring{f})$, since $\sigma(j, \lambda) = \sigma'(j, \lambda) = j^*, \lambda^*$

By weak genericity there is a $\bar{\gamma}$ s.t. $\bar{G}_1 \Vdash \check{\gamma} = \mathrm{dom}(\mathring{f})$ and for each $\xi < \lambda$ there is a unique $\nu = \nu(\xi)$ s.t. $\bar{G}_1 \Vdash \mathring{f}(\check{\nu}) \leq \check{\xi} < \mathring{f}(\check{\nu}+1)$. Set:

$$f = \sigma(\mathring{f})^G = \sigma'(\mathring{f})^G, \; \gamma = \sigma(\bar{\gamma}) = \sigma'(\bar{\gamma}).$$

$f : \gamma \to \lambda^*$ is a cofinal normal function and $f(\nu) \leq \sigma(\xi) = \sigma'(\xi) < f(\nu+1)$ for $\xi < \lambda$, $\nu = \nu(\xi)$. Hence:

$$\sup \sigma''\lambda = \sup \sigma'{}''\lambda = \sup f''\bar{\gamma}.$$

QED (Lemma 3.5)

As a corollary of the proof:

<u>Lemma 3.6</u> $\sup e_\nu''\lambda = e_\nu(\lambda)$ for $\nu \leq \lambda$.

proof.

Apply exactly the same argument to $\nu, \hat{N}_\nu, e_\nu, e_\nu(\nu)$ in place of $i, N, \sigma, i^*$. In Case 1 we have:

$$\sup(C_\delta^{\hat{N}_\nu}(\mathrm{rng}\, e_\nu)) \cap e_\nu(\lambda) = \sup e_\nu''\lambda$$

where $\hat{N}_\nu = C_\delta^{\hat{N}_\nu}(\mathrm{rng}\, e_\nu)$, $\delta = e_\nu(\delta(\mathbb{B}_\nu))$.

In Case 2: $e_\nu''\lambda = \sup e_\nu(f)''\bar{\gamma}$, where $\bar{\gamma} \leq \omega_1^{\bar{N}}$; hence $e_\nu(\bar{\gamma}) = \bar{\gamma}$ and $e_\nu \upharpoonright \bar{\gamma} = \mathrm{id}$, since $\bar{\gamma} \leq \omega_1^{\bar{N}}$ and $\omega_1^{\hat{N}_\nu} = \omega_1^{\bar{N}}$. QED (Lemma 3.6)

We shall make use of the induction hypothesis in constructing sequences $\langle a_i \mid i<\omega\rangle$, $\langle \dot{\tau}_i \mid i<\omega\rangle$, $\langle \dot{\Gamma}_i \mid i<\omega\rangle$ s.t.

(I) (a) $a_i \in \mathbb{B}_{\eta_i}$, $h_{\eta_l}(a_i)=a_l$ for $l\le i$ (where $h_\nu =: h_{\mathbb{B}_\nu}$)

(b) $\dot{\tau}_i \in V^{\mathbb{B}_{\eta_i}}$, $\dot{\tau}_0=\dot{\sigma}$

(II) Let $G\ni a_i$ be $\mathbb{B}_{\eta_i}$-generic. Set:

$G_\xi = G\cap \mathbb{B}_\xi$ $(\xi\le\eta_i)$, $\tau_l=\dot{\tau}_l^{G_{\eta_l}}$ $(l\le i)$. Then:

(a) $\tau_i : \bar{N} \prec N$

(b) $\tau_i(\bar{\theta},\bar{\mathbb{B}},\lambda)=\theta,\mathbb{B},\lambda^*$

(c) $\tau_i \upharpoonright u = \tau_l(u)$ $(l\le i)$

(d) $\sup \tau_i{}''\lambda = \eta$

(e) Set: $n_l =:$ the maximal $n$ s.t. $\tau_l(\bar{\xi}_n)\le\eta_{l+1}$

$n_l^+ =:$ the least $n$ s.t. $\tau_l(\bar{\xi}_n)\ge\eta_{l+1}$

Then $\tau_i(\bar{\xi}_l)=\tau_h(\bar{\xi}_l)$ for $h\le i$, $l\le n_h^+$

(III) $\dot{\Gamma}_i \in V^{\mathbb{B}_{\eta_i}}$. Let $G\ni a_i$ be as in II, S.

Set: $\Gamma_l =: \dot{\Gamma}_l^{G_{\eta_l}}$ for $l\le i$. Then

$\Gamma_i = \langle\langle \xi_j, \dot{\sigma}_j, b_j, \dot{u}_j\rangle \mid j\le n_i\rangle$ where:

(a) $\Gamma_l \subset \Gamma_i$ for $l\le i$

(b) $\xi_j = \tau_i(\bar{\xi}_j)$ for $j\le n_i$

(c) $b_j \in \mathbb{B}_{\xi_j}$, $b_0=a$, $b_l=h_{\eta_l}(b_j)$ for $l\le j\le i$

(d) $\dot{\sigma}_j, \dot{u}_j \in V^{\mathbb{B}_{\xi_j}}$, $\dot{\sigma}_0=\dot{\sigma}$

( Note $a_i, \dot{\tau}_i, \dot{\Gamma}_i \in V$ and are in fact $V$-definable in the parameters $\bar{\theta}, \bar{\mathbb{B}}, \lambda, \theta, \mathbb{B}, \lambda^*, \bar{N}, N$, and $\eta_0, \dots, \eta_i$. The sequences $\langle a_i | i<\omega\rangle$, $\langle \dot{\tau}_i | i<\omega\rangle$, $\langle \dot{\Gamma}_i | i<\omega\rangle$ then lie in $V[\langle \eta_i | i<\omega\rangle]$ and hence in $V[G^0]$. For the moment, however, we make no further use of $G^0$ and regard the sequence $\langle \eta_i | i<\omega\rangle$ as given. )

We shall also have:

IV Let $G, \Gamma_i$ be as in III. Let $G' \supset G$ be $\mathbb{B}_{\xi_{n_i}}$-generic s.t. $b_{n_i} \in G'$. Set:

$G'_\nu =: G' \cap \mathbb{B}_\nu$ for $\nu \le \xi_{n_i}$, $\sigma_h =: \dot{\sigma}_h^{G'_{\xi_h}}$, $u_h =: \dot{u}_h^{G'_{\xi_h}}$ for $h \le n_i$. Then for all $j \le n_i$ we have:

(a) $\langle \sigma_j, G'_{\xi_j}\rangle$ witnesses $\bar{G}_{\bar{\xi}_j}$ and $\sigma_j(\lambda) = \lambda^*$

(b) $\langle \sigma_j, G'_{\xi_j}\rangle$ is strong over $\langle \sigma_h, G'_{\xi_h}\rangle$ for $h<j$

(c) $\sigma_h \upharpoonright u_h = \sigma_j \upharpoonright u_h$ for $h<j$

(d) $u_j$ is finite and $u \cup u_h \cup \{x_h, \omega_h\} \subset u_j$ for $h<j$, where $\langle x_h | h<\omega\rangle$ is a fixed enumeration of $\bar{N}$ and $\omega_h$ is defined as follows:

$\omega_h =:$ the $\bar{N}$-least $\omega$ s.t. $\bar{\omega} \le \delta(\mathbb{B}_{\bar{\xi}_h})$ in $\bar{N}$ and $\sigma_0(x_h) \subset \sigma_h(\omega)$.

(e) $\{\bar{\xi}_0, \dots, \bar{\xi}_{n_i^+}\} \subset u_j$ if $\eta_i \le j$

Finally we shall need:

V Let $G$ be as in II. Then

(a) $h_{\eta_i}(b_j^i) \in G$ for $j \leq m_i$

(b) $\tau_i = (\sigma^{*}_{m_k})^G$ if $i = k+1$.

(Note The existence of $\omega_h$ in IV (d) follows from $C_\delta^N(\operatorname{rng}\sigma^0) = C_\delta^N(\operatorname{rng}\sigma_h)$ where $\delta = \delta(\mathbb{B}_{\xi_h})$.)

(Note It is entirely possible that $m_i = m_j$ for a $j > i$. This will occur if

$$\tau_i(\bar{\xi}_{m_i}) \leq \eta_{i+1} < \eta_{j+1} < \tau_i(\eta_j),$$

In this case we will have:

$$\dot{\Gamma}_i^{G \cap \mathbb{B}_{\eta_i}} = \dot{\Gamma}_j^G$$

where $G$ is $\mathbb{B}_{\eta_j}$ - generic.)

We now prove the Claim on the assumption that the sequences $\langle a_i \rangle$, $\langle \dot{\tau}_i \rangle$, $\langle \dot{\eta}_i \rangle$ have been defined and satisfy I–V. We must show (in the generic collapse of a sufficient cardinal) that there is a pair $\langle \sigma', G' \rangle$ s.t. $G' \supset G^0$, $\langle \sigma', G' \rangle$ witnesses $\bar{G}$ and is strong over $\langle \sigma^0, G^0 \rangle$, where $\sigma'(\lambda) = \lambda^*$ and $\sigma' \upharpoonright u = \sigma^0 \upharpoonright u$.

$\langle \eta_i \mid i < \omega \rangle$ lies in $V[G^0]$ and hence so do the sequences $\langle a_i \rangle$, $\langle \dot{\tau}_i \rangle$, $\langle \dot{\eta}_i \rangle$. The sequence $\langle a_i / G^0 \mid i < \omega \rangle$ then lies in $V[G^0]$. It is easily seen that $h_{\eta_i}(a_j / G^0) = a_i / G^0$ for $i \leq j < \omega$ (where $h_{\eta_i} =: h_{(\mathbb{B}_{\eta_i} / G^0)}$). Since $\mathbb{B}_\eta$ is the RSC limit of $\langle \mathbb{B}_\nu \mid \nu < \eta \rangle$, it follows that in $V[G^0]$ the BA $\mathbb{B}_\eta / G^0$ is the countable support limit of $\langle \mathbb{B}_{\eta_i} / G^0 \mid i < \omega \rangle$. Hence $a^* \in (\mathbb{B}_{\eta_i} / G^0) \setminus \{0\}$, where $a^* = \bigcap_{i<\omega} a_i / G^0$. But then $a^* = a' / G^0$

for some $a' \in \mathbb{B}_\eta \setminus \{0\} \subset \mathbb{B}_{\lambda^*} \setminus \{0\}$.

Assume w.l.o.g. that $a' \subset a = a_0$.

Let $G^*$ be $\mathbb{B}_{\lambda^*}$ - generic over $V[G_0]$ s.t. $a^* \in G^*$. Set:

$G' = G^0 * \dot{G}^* = \{ b \mid b/G^0 \in G^* \}$. Then $a' \in G'$ and $G'$ is $\mathbb{B}_{\lambda^*}$-generic over $V$. We shall construct a $\sigma' \in V^{\mathbb{B}_{\lambda^*}}$ s.t. $\langle \sigma', G' \rangle$ satisfies our Claim.

Set $G'_\nu = G' \cap \mathbb{B}_\nu$ $(\nu \le \lambda^*)$. Set:

$\Gamma_i = \dot{\Gamma}_i^{G'_{\eta_i}}$, $\tau_i = \dot{\tau}_i^{G'_{\eta_i}}$ for $i < \omega$.

Then II (a)-(e) hold for $i < \omega$, as do II (a)-(d).

Set:

$\Gamma =: \bigcup_i \Gamma_i = \langle \langle \xi_j, \dot{\sigma}_j, b_j, u_j \rangle \mid j < \omega \rangle$.

Define $m_i, m_i^+$ as before from $\Gamma_i, \eta_{i+1}$.

Then $\tau_i(\overline{\xi}_\ell) = \tau_h(\overline{\xi}_\ell) = \overline{\xi}_\ell$ for $h \le i, \ell \le m_h^+$.

We note that:

(2) $\sup_i \xi_{m_i} = \eta$ (hence $\sup_i \xi_i = \eta$ and $\sup_i m_i = \omega$).

proof.

$\sup_i \xi_{m_i^+} = \eta$ since $\xi_{m_i^+} = \tau_i(\overline{\xi}_i) < \lambda$ and $\eta_{i+1} \le \xi_{m_i^+}$. But if $\xi_{m_i^+} < \eta_{j+1}$, then $\xi_{m_i^+} \le \xi_{m_j}$. Hence $\sup_i \xi_{m_i} = \eta$.

QED (2)

(3) $b_j \in G$ for all $b_j < \omega$.

proof.

We show $b_{n_i} \in G$ for $i < \omega$. Let $n_k > n_i$. Then

$$b_{n_i} = b^i_{\xi_{n_i}}(b^i_{n_k}) \subset b_{\eta_k}(b_{n_k}) \in G_{\eta_k} \quad \text{since } \xi_{n_i} \leq \eta_h$$

and by V (a). QED (3)

Set: $\sigma_i =: \mathring{\sigma}_i^{G_{\xi_i}}$, $u_i =: \mathring{u}_i^{G_{\xi_i}}$ for $i < \omega$.

Then IV (a)-(e) hold for all $j < \omega$, as do V (a), (b).

Since $\sigma_i(x_i) = \sigma_j(x_i)$ for $i \leq j < \omega$, we can define in $V[G']$ a new map $\sigma : \bar{N} \prec N$ by:

$$\sigma'(x) =: \sigma_i(x) \quad \text{if } \sigma_i(x) = \sigma_j(x) \text{ for all } j \geq i.$$

But then, since $\sigma_i(\bar{\Omega}, \bar{\mathbb{B}}, j, \lambda) = \Omega, \mathbb{B}, j^*, \lambda^*$, and $\sigma_i \restriction u = \sigma^0 \restriction u$ for $i < \omega$, we conclude:

(4) $\sigma'(\bar{\Omega}, \bar{\mathbb{B}}, j, \lambda) = \Omega, \mathbb{B}, j^*, \lambda^*$ and $\sigma' \restriction u = \sigma^0 \restriction u$.

Clearly

(5) $\sigma'(\bar{\xi}_i) = \sigma_j(\bar{\xi}_i) = \xi_i$ for $j \leq n_i$

Hence:

(6) $\sup \sigma'{}'' \lambda = \eta$,

since $\sigma'(\nu) = \sigma_i(\nu) < \eta$ for some $i$, whenever $\nu < \lambda$.

(7) $\sigma'{}''\bar{G}_\nu \subset G'_{\sigma'(\nu)}$ for $\nu < \lambda$

proof.

It suffices to show $\sigma'{}''\bar{G}_{\bar{\xi}_i} \subset G'_{\xi_i}$.

Let $b \in \bar{G}_{\bar{\xi}_i}$. For sufficiently large $j \geq i$ we have: $\sigma'(b) = \sigma_j(b) \in G_{\sigma_j(\bar{\xi}_i)} = G_{\xi_i}$.

QED (7)

However, we need a stronger version of (7):

Set: $\langle e_\nu, \hat{N}_\nu, \hat{G}_\nu \rangle$ = the completion of $\bar{N}, \bar{G}_\nu$ for $\nu \leq \lambda$.

$e_{\nu,\tau} = \mathrm{int}(\bar{G}_\nu, \bar{G}_\tau)$ for $\nu \leq \tau \leq \lambda$.

Since $\bar{G} = \bar{G}_\lambda \in S_\lambda$, we know that:

(8) $\hat{N}_\lambda, \langle e_{\nu,\lambda} \mid \nu \leq \lambda \rangle$ = the direct limit of $\langle \hat{N}_\nu \mid \nu < \lambda \rangle, \langle e_{\nu\tau} \mid \nu \leq \tau < \lambda \rangle$, where $\hat{G}_\lambda$ is $e_\lambda(\bar{\mathbb{B}}_\lambda)$-generic over $\hat{N}_\lambda$.

Set: $\hat{\lambda} = e_\lambda(\lambda)$, $\hat{\mathbb{B}} = e_\lambda(\bar{\mathbb{B}})$, $G^* = \hat{G}_\lambda$,

$G^*_\nu = G \cap \hat{\mathbb{B}}_\nu$ for $\nu \leq \hat{\lambda}$. Then $G^*_\nu$ is $\hat{\mathbb{B}}_\nu$-generic over $\hat{N} =: \hat{N}_\lambda$ for $\nu \leq \hat{\lambda}$.

For $i < \omega$ set: $\sigma^*_i = \mathrm{int}(\sigma_i, G_{\xi_i})$.

Then $\sigma^*_i : \hat{N}_{\bar{\xi}_i} \prec N$, $\sigma^*_i \circ e_{\bar{\xi}_i} = \sigma_i$.

We show:

(9) Let $x \in \hat{N} =: \hat{N}_\lambda$. Let $x_i = e^{-1}_{\bar{\xi}_i,\lambda}(x)$ be defined for $i \geq i_0$. There is $i \geq i_0$ s.t. $\sigma^*_i(x_i) = \sigma^*_j(x_j)$ for all $j \geq i$.

proof of (9)

Let $x \in \hat{N}$, $x = e_{\bar{\xi}_i, \lambda}(x_i)$ for $i \geq i_0$.

Let $x_0 = e_{\bar{\xi}_{i_0}}(\mathring{x})^{\hat{G}_{\bar{\xi}_{i_0}}}$.

Set: $\tilde{G}_i = \hat{G}_{\bar{\xi}_i} \cap e_{\bar{\xi}_i}(\mathbb{B}_{\bar{\xi}_{i_0}})$ for $i \geq i_0$.

Then $\tilde{G}_i$ is $e_{\bar{\xi}_{i_0}}(\mathbb{B}_{\bar{\xi}_{i_0}})$ - generic over $\hat{N}_{\bar{\xi}_i}$

and $e_{\bar{\xi}_i \bar{\xi}_j}{}''\tilde{G}_i \subset \tilde{G}_j$ for $i_0 \leq i \leq j \leq \lambda$,

Hence $e_{\bar{\xi}_i, \bar{\xi}_j}$ extends uniquely to $e_{i,j}$

with: $e_{ij}: \hat{N}_i[\tilde{G}_i] \prec \hat{N}_j[\tilde{G}_j]$ and

$e_{ij}(\tilde{G}_i) = \tilde{G}_j$.

Clearly $x_i = e_{\bar{\xi}_0 \bar{\xi}_i}(x_{i_0}) = \tilde{e}_{i_0 i}(\mathring{x}^{\hat{G}_{\bar{\xi}_{i_0}}}) =$

$= e_{\bar{\xi}_0, \bar{\xi}_i}(\mathring{x})^{\tilde{G}_i}$, for $i_0 \leq i$.

Assume w.l.o.g. that $i_0$ was chosen big enough that $\sigma_i(\mathring{x}) = \sigma_{i_0}(x)$ for $i \geq i_0$.

Then $\sigma_i^*(x_i) = \sigma_i^*(e_{\bar{\xi}_{i_0} \bar{\xi}_i}(\mathring{x})^{\tilde{G}_{\bar{\xi}_i}}) =$

$= \sigma_i(\mathring{x})^{G_{\bar{\xi}_{i_0}}}$ for $i \geq i_0$. QED (9)

Letting $\hat{N} = \hat{N}_\lambda$, we can define

$\sigma^*: \hat{N} \prec N$ by:

Def Let $x \in \hat{N}$, $x = e_{i,\lambda}(x_i)$ for sufficient $i$,

$\sigma^*(x) =: \sigma_i^*(x_i)$ if $\sigma_i^*(x_i) = \sigma_j^*(x_j)$ for

all $j \geq i$,

Then:

(10) $\sigma^* \circ e_\lambda = \sigma'$.

proof.

Let $x = e_{i\lambda}(x_i)$ for sufficient $i$. Then

$\sigma'(x) = \sigma_i^* e_i(x_i) = \sigma^*(x)$ for sufficient $i$.

QED (10)

Recalling our definition;

$G_\nu^* = (\hat{G}_\lambda) \cap \hat{\mathbb{B}}_\nu \quad (\hat{\mathbb{B}} = e_\lambda(\mathbb{B}))$ for $\nu \leq \lambda$,

we have:

(11) $\sigma^{*}{}'' G_\nu^* \subset G'_{\sigma^*(\nu)}$ for $\nu < \hat{\lambda}$.

proof.

Let $b \in \hat{\mathbb{B}}_\nu$, $b, \nu = e_{i,\lambda}(b_i, \nu_i)$ for sufficient $i$.

By Lemma 3.6 and (8) it follows that

$e_\lambda(\lambda) = \sup e_\lambda{}'' \lambda$. Hence $\nu < e_\lambda(\xi_k^i)$

for a sufficient $i < \omega$. But the

$\sigma^*(b) = \sigma_i^*(b_i) \in G_i' \cap \mathbb{B}_{\sigma_i^*(\nu_i)} = G_i' \cap \mathbb{B}_{\sigma^*(\nu)} =$

$= G'_{\sigma^*(\nu)}$. QED (11)

By the fact that we are using an RSC iteration we then get:

(12) $\sigma^{*}{}''\hat{G} \subset G'$

proof.

Let $\mu = \hat{\lambda} = e_\lambda(\lambda)$. Since $\langle \hat{\mathbb{B}}_\nu \mid \nu \leq \mu \rangle$ is an RSC iteration in $\hat{N}$, the set $D$ is dense in $\hat{\mathbb{B}} =: e_\lambda(\bar{\mathbb{B}}_\lambda)$ where:

$a \in D$ iff either $a \in \bigcup_{\nu<\mu} \hat{\mathbb{B}}_\nu$ or else $a = \bigcap_{\nu<\mu} b_\nu(a)$ in $\hat{\mathbb{B}}$ and there is $\nu < \mu$ s.t. $b_\nu(a) \Vdash_{\hat{\mathbb{B}}_\nu} \mathrm{cf}(\check{\mu}) = \omega$.

Since $\sigma^{*}{}''\hat{G}_\nu \subset G'_{\sigma'(\nu)}$, the map $\sigma^{*}$ extends to $\sigma^{**}: \hat{N}[\hat{G}_\nu] \prec N[G'_{\sigma'(\nu)}]$ s.t. $\sigma^{**}(\hat{G}_\nu) = G'_{\sigma'(\nu)}$. Let $f \in \hat{N}[\hat{G}_\nu]$ s.t. $f: \omega \to \mu$ is monotone and cofinal. Hence $\sigma^{**}(f): \omega \to \lambda^{*}$ is monotone and cofinal in $\lambda^{*}$.

Hence $\lambda^{*} = \sup_{i<\omega} \sigma^{**}(f)(i) =$

$= \sup \sigma^{*}{}''\mu$. But then

$\sigma^{*}(b) = \bigcap_{\nu<\mu} h_{\sigma^{*}(\nu)}(\sigma^{*}(b))$, where

$h_{\sigma^{*}(\nu)}(\sigma^{*}(b)) = \sigma^{*}(h_\nu(b)) \in G_{\sigma^{*}(\nu)}$.

Since $\lambda^* = \sup \sigma^{*}{}'' \mu$, we have
$\sigma^*(b) = \bigwedge_{\nu < \lambda^*} h_\nu(\sigma^*(b))$ where
$h_\nu(\sigma^*(b)) \in G_\nu$ for $\nu < \lambda^*$. By
genericity it follows that
$\sigma^*(b) = \bigwedge_{\nu < \lambda^*} h_\nu(\sigma^*(b)) \in G'$.

QED (12)

~~But then:~~

(13) $\langle \sigma', G' \rangle$ witnesses $\bar{G}$

proof:
By (4) it is enough to show:
$\sigma'{}''\bar{G} \subset G'$. But
$\sigma'{}''\bar{G} = \sigma^* \cdot e_\lambda{}''\bar{G} \subset \sigma^*{}''\hat{G}_\lambda \subset G'$.

QED (13)

But then:

(14) $\langle \sigma', G' \rangle$ is strong over $\langle \sigma^0, G^0 \rangle$

proof.

The only thing left to show is:

$C^N_\delta(\mathrm{rng}\,\sigma^0) = C^N_\delta(\mathrm{rng}\,\sigma')$, where $\delta = \delta(\mathbb{B}_\lambda)$.

($\le$) We show $\mathrm{rng}\,\sigma^0 \subset C^N_\delta(\mathrm{rng}\,\sigma')$

Let $x \in \mathrm{rng}\,\sigma^0$. Then $x = \sigma^0(x_i)$ for some $i$. Hence $x \in \sigma_i(w_i)$ and $\sigma_j(w_i) = \sigma_i(w_i)$ for $j \ge i$. Hence $x \in \sigma'(w_i)$ where $\bar{\bar{w}}_i \le \delta(\bar{\mathbb{B}}_{\bar{\xi}_i}) \le \delta(\bar{\mathbb{B}}_\lambda)$ in $\bar{N}$. Hence there is $f \in \bar{N}$ s.t. $f : \bar{\delta} \xrightarrow{\text{onto}} w_i$, where $\bar{\delta} = \delta(\bar{\mathbb{B}}_\lambda)$ (hence $\sigma'(\bar{\delta}) = \delta = \delta(\mathbb{B}_{\lambda^*})$.) Then:

$x = \sigma'(f)(\mu)$ for some $\mu < \delta$.

QED ($\le$)

($\geq$) We show: $\operatorname{rng}\sigma' \subset C^N_\delta(\operatorname{rng}\sigma^0)$.
Let $x = \sigma'(\vec{x})$. Then $x = \sigma_i(\vec{x}) \in$
$\in C^N_{\delta_i}(\operatorname{rng}\sigma_i) = C^N_{\delta_i}(\operatorname{rng}\sigma^0_i) \subset$
$\subset C^N_\delta(\operatorname{rng}\sigma^0)$ where $\delta_i = \delta(\mathbb{B}_{\xi_i})$.

QED (8)

This proves the Claim on the assumption that $\langle a_i\rangle, \langle \mathring{\tau}_i\rangle, \langle \mathring{\Gamma}_i\rangle$ are defined in $V[G^0]$ and satisfy I–V. We now define these sequences and verify I–V. As stated, the definition of each $\langle a_i, \mathring{\tau}_i, \mathring{\Gamma}_i\rangle$ takes place in $V$ and the definition of the full sequence takes place in $V[\langle \eta_i \mid i<\omega\rangle] \subset V[G^0]$. We define $a_i, \mathring{\tau}_i, \mathring{\Gamma}_i$ by induction on $i$.

Case 1 $i=0$

Set $a_0 = a$, $\dot{\tau}_0 = \dot{\sigma}$. Then I is trivial.
But then II holds with $n_0 = n_0^+ = 1$.
We now construct $\dot{\Gamma}_0$ and verify III - V
at 0. We set $\dot{\Gamma}_0 = \check{\Gamma}$ where:

$$\Gamma = \langle\langle \xi_i, \dot{\sigma}_i, b_i, \dot{u}_i \rangle \mid i \leq 1 \rangle$$

with $\quad \xi_i = \eta_i \quad (i = 0, 1)$

We set $b_0 = a_0 = a$, $\dot{u}_0 = \check{u}_0$ where $u_0 = u \cup \{\xi_0, \xi_1\}$,
and $\dot{\sigma}_0 = \dot{\sigma}$. If $G \ni a_0$ is $\mathbb{B}_{j^*}$ - generic, then
$\langle G, \sigma \rangle$ witnesses $\bar{G}_j$ where $\sigma = \dot{\sigma}_0^G$.
But then we can apply II of the
induction hypothesis to $\langle \eta_0, \eta_1 \rangle$,
where $j^* = \eta_0 < \eta_1 = \sigma(\xi_1)$ and to $u_0$.
That gives us $b \in \mathbb{B}_{\eta_1}$ s.t. $h_{\eta_0}(b) = a$
and whenever $G' \ni b$ is $\mathbb{B}_{\eta_1}$ - generic, then,
letting: $G = G' \cap \mathbb{B}_{\xi_0}$, $\sigma' = \dot{\sigma}_1^{G'}$, we have:

- $\langle \sigma', G' \rangle$ witnesses $\bar{G}_{\xi_1}$
- $\langle \sigma', G' \rangle$ is strong over $\langle \sigma, G \rangle$
- $\sigma \upharpoonright u_0 = \sigma' \upharpoonright u_0$.

Set: $b_1 = b$. Then there is $\dot{\sigma}' \in V^{\mathbb{B}_{\eta_1}}$
s.t. $b$ forces the above to hold
- i.e.

$b \Vdash_{\mathbb{B}_{\eta_1}} (\langle \ddot{\sigma}', \dot{G} \rangle$ witnesses $\check{G}_{\bar{\xi}_1} \wedge \langle \dot{\sigma}, \dot{G} \rangle$ is strong over

$\langle \tilde{\sigma}, \tilde{G} \rangle$ where $\hat{G} = \dot{G} \cap \check{\mathbb{B}}_{\eta_0} \wedge \tilde{\sigma} = \check{\mathring{\sigma}}_0^{\,G} \wedge$

$\wedge\ \dot{\sigma} \upharpoonright \check{u}_0 = \tilde{\sigma} \upharpoonright \check{u}_0 )$

($\dot{G}$ being the canonical generic name).

Finally we note that there is $\dot{u}' \in V^{\mathbb{B}_{\eta_1}}$ with the property that whenever $G' \ni b$ is $\mathbb{B}_{\eta_1}$-generic, then

$$\dot{u}'^{G'} = u \cup \{\bar{\xi}_0, \ldots, \bar{\xi}_n\} \cup \{x_0, w_0\}$$

where $w_0$ is least s.t. $\sigma(x) \in \sigma'(w_0)$ and $\bar{\bar{w}}_0 = \delta(\bar{\mathbb{B}}_{\xi_0})$ and $n$ is least s.t. $\sigma'(\bar{\xi}_n) \geq \eta_1$.

(where $\langle \sigma, G \rangle, \langle \sigma', G' \rangle$ are as above).

We then set: $b_1 = b$, $\xi_1 = \eta_1$, $\dot{\sigma}_1 = \mathring{\sigma}'$, $\dot{u}_1 = \dot{u}'$.

Then I – V hold at $i = 0$ (V(b) being vacuous).

Case 2 $i = k+1$

Let $\mathring{\xi}, \mathring{\sigma}, \mathring{b}, \ddot{u}, \mathring{n}_k$ be terms s.t.

$$a_k \Vdash_{\mathbb{B}_{\eta_k}} \mathring{\Gamma}_k = \langle\langle \mathring{\xi}_l, \ddot{\sigma}_l, \mathring{b}_l, \ddot{u}_l \rangle \mid l \le \mathring{n}_k \rangle$$

Since I – V hold at $k$, we know that

$$a_k \Vdash_{\mathbb{B}_{\eta_i}} (\mathring{\xi}_{n_k} \le \check{\eta}_i \wedge \mathring{b}_{n_i} \in \check{\mathbb{B}}_{\eta_i})$$

For $b \in \mathbb{B}_{\eta_i}$ set: $a(b) =: a_k \cap [\![ \mathring{b}_{n_i} = \check{b} ]\!]_{\mathbb{B}_{\eta_i}}$

Then $a(b) \cap a(b') = 0$ for $b \neq b'$ and

$a_k = \bigcup_b a(b)$. Set: $a_i =: \bigcup_{a(b) \neq 0} b$.

If $a(b) \neq 0$, then

$$h_{\eta_k}(b) = [\![ \check{b}/\mathring{G} \neq 0 ]\!]_{\mathbb{B}_{\eta_k}} = a(b),$$

since if $G \ni a_k$ is $\mathbb{B}_{\eta_i}$ – generic, then

$b/G \neq 0 \longleftrightarrow a(b) \in G$,

since $a_k \Vdash_{\mathbb{B}_{\eta_k}} \mathring{b}_{n_k} \in \mathring{G}$,

Thus $h_{\eta_k}(a_i) = \bigcup_{a(b) \neq 0} a(b) = a_k$.

We then ~~let~~ $a_k \Vdash_{\mathbb{B}_{\eta_k}} t = \ddot{\sigma}_{n_k}$ and

let $\mathring{\tau} \in \mathbb{B}_{\eta_i}$ s.t. $a_k \Vdash_{\mathbb{B}_{\eta_i}} \mathring{\tau} = \check{t}^{\mathring{G} \cap \check{\mathbb{B}}_{\mathring{\xi}_{n_k}}}$.

Then $a_i, \dot{\tau}_i$ satisfy (I), (III). To see II note that if $\tilde{G} \ni a_i$ is $\mathbb{B}_{\eta_i}$ - generic, then $G' = \tilde{G} \cap \mathbb{B}_{\xi_{n_i}}$ extends $G = \tilde{G} \cap \mathbb{B}_{\eta_k}$, where $\Gamma_k = \dot{\Gamma}_k^G =$

$= \{ \langle \xi_j, \dot{\sigma}_j, b_j, \dot{u}_j \rangle \mid j \le n_k \}$ and $\tau_i = \dot{\tau}_i^G =$

$= \dot{\sigma}_{n_k}^{G'}$. Since $\xi_0, \ldots, \xi_{n_k^+} \in u_k = \dot{u}_{n_k}$,

we have (e) at $i$. (a) - (d) are straightforward. V (b) is also immediate.

We now define $\dot{\Gamma}_i$ and verify III, IV and V (a) at $i$.

Let $G \ni a_i$ be $\mathbb{B}_{\eta_i}$ - generic. Inside $V^G$ we define

$$\Gamma = \Gamma^G = \langle \langle \xi_j, \dot{\sigma}_j, b_j, \dot{u}_j \rangle \mid j \le n_i \rangle$$

where $n_i, n_i^+$ are defined as usual from $\tau_i = \dot{\tau}_i^G$, $\Gamma_k = \dot{\Gamma}_k^{G \cap \eta_k}$ is given and $\Gamma$ will be a continuation of it.

Then $\tau_i = \sigma_{\xi_{n_k}} = \dot{\sigma}_{\xi_{n_k}}^{G \cap \xi_{n_k}}$. For $n_k < j \le n_i$ we have:

$\xi_j = \tau_i(\xi_j)$. (At $n_i = n_k$ - i.e. $\tau_i(\xi_{n_k+1}) > \eta_i$, then $\Gamma_i = \Gamma_k$.)

By induction on $j$ we define $\dot\sigma_j, b_j, \dot u_j$ for $n_k < j \le n_i$. We closely imitate our procedure for defining $\dot\sigma_1, b_1, \dot u_1$ in Case 1.

<u>Case A</u> $j = n_k + 1$.

Then $j > \eta_i$ and $j \ge n_k^+$. We apply the induction hypothesis to $\xi_{n_k}, \xi_j$.

That gives us $b \in \mathbb{B}_{\xi_j}$ s.t. $h_{\xi_{n_k}}(b) = b_{n_k}$ and whenever $G' \ni b$ is $\mathbb{B}_{\xi_j}$ - generic, then, letting $G = G' \cap \mathbb{B}_{\xi_{n_k}}$, $\sigma = \dot\sigma_{n_k}^G$, we have:

- $\langle \sigma', G' \rangle$ witnesses $\bar G_{\bar\xi_j}$
- $\langle \sigma', G' \rangle$ is strong over $\langle \sigma, G \rangle$ where $\langle \sigma, G \rangle$ witnesses $\bar G_{\bar\xi_{n_k}}$
- $\sigma \upharpoonright u_{n_k} = \sigma' \upharpoonright u_{n_k}$, where $u_{n_k} = \dot u_{n_k}^G$.

Set: $b_j = b$. Then there is $\dot\sigma_j \in V^{\mathbb{B}_{\xi_j}}$ s.t. $b$ forces the above to hold with $\sigma' = \dot\sigma_j^{G'}$. - i.e.

$b \Vdash_{\mathbb{B}_{\xi_j}} (\langle \dot\sigma_j, \dot G \rangle$ witnesses $\check G_{\bar\xi_j}$ and $\langle \dot\sigma_j, \dot G \rangle$ is strong over $\langle \tilde\sigma, \tilde G \rangle$ where $\tilde G = \dot G \cap \check{\mathbb{B}}_{\xi_{n_k}} \wedge \tilde\sigma = (\overbrace{\dot\sigma_{n_k}})^{\tilde G} \wedge$
$\wedge\ \dot\sigma_j \upharpoonright \check u_{n_k} = \tilde\sigma \upharpoonright \check u_{n_k}$)

This gives us $\dot{\sigma}_j$. Finally we note that there is $\dot{u} \in V^{\mathbb{B}_{\xi_j}}$ with the property that whenever $G'$ is $\mathbb{B}_{\xi_j}$ - generic, then for $G = G' \cap \mathbb{B}_{\xi_{m_k}}$ we have:

$$\dot{u}^{G'} = u_{m_k}^{G} \cup \{\bar{\xi}_0, \dots, \bar{\xi}_n\} \cup \{x_j, w\}$$

where $n$ is least s.t. $\dot{\sigma}_j(\bar{\xi}_n) \geq \eta_{i+1}$ and $w =$ = the $\bar{N}$-least $w$ s.t. $\bar{\bar{w}} \leq \sigma(\mathbb{B}_{\bar{\xi}_{m_k}})$ in $\bar{N}$ and $\sigma_0(x_{m_k}) \in \sigma_{m_k}(w)$, where $\sigma_{m_k} = \dot{\sigma}_{m_k}^{G}$, $\sigma_j = \dot{\sigma}_{m_j}^{G'}$ and $u = \dot{u}_{m_k}^{G}$.

We set: $\dot{u}_j = \dot{u}$.

This completes Case A.

<u>Case B</u>: $j > m_k + 1$. Let $h = j - 1$. We define $b_j, \dot{\sigma}_j$ by exactly the same procedure using $\langle \xi_h, \xi_j \rangle$ in place of $\langle \xi_{m_k}, \xi_{m_k+1} \rangle$.

We then set $\dot{u}_j = \dot{u}$, where $\dot{u}$ has the property that whenever $G'$ is $\mathbb{B}_{\xi_j}$ - generic, then: $\dot{u}^{G'} = \dot{u}_h^{G} \cup \{w_h, x_h\}$, $w_h$ being defined as before. I, II follow as before.

This completes the construction of $\Gamma = \Gamma^G$ in $V[G]$. But since $\Gamma^G$ is uniformly $V[G]$ - definable in the parameters: $\bar{N}, N, \Omega, \mathbb{B}, \bar{\Omega}, \bar{\mathbb{B}}$, $\bar{\xi}_0, \dots, \bar{\xi}_i$ and $\eta_0, \dots, \eta_i$, it follows that there is a term $\dot{\Gamma}_i \in V^{\mathbb{B}_{\eta_i}}$ s.t. $\dot{\Gamma}_i^G = \Gamma^G$ whenever $G$ is $\mathbb{B}_{\eta_i}$ - generic. This completes the construction. The verification of III - V is straightforward.

This completes the proof of II in Case 2, and with it the main theorem.