I A Correction to §4 of [NFS]: The Initial Segment Condition

The initial segment condition stated in §4 is equivalent to:

(i) Let $E_\nu \neq \emptyset$, $\kappa = \mathrm{crit}(E_\nu)$, $\lambda = lh(E_\nu) = E_\nu(\kappa)$, $\kappa^+ < \lambda' < \lambda$ in J^E_ν. Set: $\nu' = \lambda'^{+J^E_\nu}$. Then $\langle J^E_{\nu'}, E_\nu|\lambda' \rangle$ is not a ppm, where $(E_\nu|\lambda')(X) =_{Df} E_\nu(X) \cap \lambda'$.

(The original condition says that if $\langle J^E_{\nu'}, E_\nu|\lambda' \rangle$ is a ppm, then $E_{\nu'} \neq \emptyset$. But then $\lambda' = lh(E_{\nu'})$ and $\omega\rho^1_{M\|\nu'} \leq \lambda'$. Hence ν' is not a cardinal in J^E_ν. Contradiction!)

We shall refer to (i) as the <u>minimal initial segment condition</u> (MIS). We have shown that the coiteration of two premice satisfying MIS will terminate, <u>provided</u> <u>that</u> <u>each</u> <u>of</u> <u>the</u> <u>iterates</u> <u>also</u> <u>satisfies</u> <u>MIS</u>. Itay Neeman pointed out that we have not verified this condition, and I see no way to do so. We shall therefore propose a new initial segment condition (IS) which has the desired preservation properties. The properties we want are:

(ii) If M satisfies IS and M' is a normal iterate of M, then M' satisfies IS.

(iii) If M is a ppm satisfying IS and $\pi : \bar{M} \xrightarrow[\Sigma_1]{} M$, then $\bar{M}$ satisfies IS.

(iii) was used in two places. In the proof of solidity in §6 we used it to show:

(iv) If M is a pm, $P = P_M$ is the standard parameter, $\nu \in p$, and N is the solidity witness for ν, then N is a pm (i.e. N satisfies IS, since the other properties are trivial).

We also argued that if M is a mouse and $\pi : \bar{M} \xrightarrow[\Sigma^*]{} M$, then $\bar{M}$ is a mouse. For this we need:

(v) If M is a ppm satisfying IS and $\pi : \bar{M} \longrightarrow_{\Sigma^*} M$, then $\bar{M}$ satisfies IS.

(iii) will not hold fully for our condition IS, but (iv) and (v) will.

As a prelude to formulating IS we define:

<u>Def</u> Let $\nu \leq ht(M)$, $E_\nu^M \neq \emptyset$, $\kappa = \text{crit}(E_\nu)$, $\lambda = lh(E_\nu)$, $\tau = \kappa^{+J_\nu^E}$ in M.

$C = C_\nu^M = C_{M\|\nu} =_{\text{Df}}$ the set of $\lambda' \in (\kappa, \lambda)$ s.t. $E_\nu|\lambda'$ is its own trivial completion (i.e. if $\alpha < \lambda'$, $f \in ({}^\kappa\kappa)^{M\|\nu}$, $\pi : J_\tau^E \xrightarrow[E_\nu]{} J_\nu^E$, then $\pi(f)(\alpha) < \lambda'$).

Note that if M is a ppm, $\nu \leq ht(M)$ and $\lambda' \in C_\nu^M$, then there is a unique ppm $N = N_{\nu,\lambda'}^M = N_{M\|\nu,\lambda'}$ defined by $N = \langle J_\nu^E, E_\nu|\lambda' \rangle$, where $\pi' : J_\tau^E \xrightarrow[E_\nu|\lambda']{} J_{\nu'}^{E'}$.

Moreover there is $\sigma_{\lambda',\nu} = \sigma_{\lambda',\nu}^M = \sigma_{\lambda', M\|\nu}$ defined by $\sigma(\pi'(f)(\alpha)) = \pi(f)(\alpha)$ for $\alpha < \lambda'$, $f \in ({}^\kappa\tau)^{M\|\nu}$.

Clearly $\sigma : N \xrightarrow[\Sigma_0]{} M \| \nu$ cofinally.

Since $E_\nu^{N'} = E_\nu | \lambda'$ is an initial segment of E_ν, we certainly expect N to satisfy the initial segment condition if M does. Thus our minimal requirements for IS are:

(vii) IS $\to$ MIS

(viii) If $\lambda' \in C_\nu^M$ and M satisfies IS, then $N = N^M_{\lambda', \nu}$ satisfies IS.

In addition, IS should have the preservation properties (ii), (iv), (v). For this reason IS should - presumably be stronger than MIS. We do not, however, want it to be so strong as to restrict the class of mice. We note that, by our proofs, any class of mice satisfying a condition IS with the above properties (vii), (viii), (ii), (iv), (v) will satisfy the condensation lemmas in [NFS] §8 (more precisely, it will satisfy the generalized condensation lemmas in)

Thus, a version of IS which itself follows from the condensation lemmas will not be unduly restrictive. Thus (with a nod to John Steel) we adopt the following initial segment condition:

(IS) Let $\nu \leq ht(M)$, $E_\nu^M \neq \emptyset$, $M\|\nu = \langle J_\nu^E, E_\nu \rangle$. If $\lambda' \in C = C_{M\|\nu}$, then $E_\nu | \lambda' \in J_\nu^E$.

Clearly, IS satisfies (vi), (vii). We shall show that it satisfies (iii), (iv), (v) as well. Before doing so, however, we show that if IS satisfies (v) and (iii) (for double rooted iteration as well), then it follows from the condensation lemmas and §8 Lemma that IS must hold for all mice.

It suffices to prove IS for the top extender of mice $M = \langle J_\nu^E, F \rangle$

s.t. $F \neq \emptyset$. We can also assume w.l.o.g. that M is sound. (Otherwise let $\bar{M} = \text{core}(M)$. Let σ be the core map with critical point δ. Coiterate $\langle M, \bar{M}, \delta \rangle$, M to a common M'. Then M' satisfies IS, since M' is an iterate of $\bar{M}$. Hence so does M, since $\pi_{M,M'}: M \xrightarrow[\Sigma^*]{} M'$.) Let $\lambda' \in C_M$

$N = N_{\lambda', M}$, $\sigma = \sigma_{\lambda, M}$. Then

$\sigma: N \xrightarrow[\Sigma_0]{} M$ cofinally, $\lambda' = \text{crit}(\sigma)$.

Clearly λ' does not generate F. Hence $F | \lambda' \in M$ by §8 Lemma 4.2 of [NFS]

. . . . .

We now verify (ii), (iv), (v). In place of (ii) we prove:

(ix) Let G be a weakly amenable extender on M, where M is a ppm satisfying IS. Let $\text{dom}(G) = \mathbb{P}(\kappa) \cap M \in M$ and suppose $\pi: M \xrightarrow[G]{*} M^*$. Then M^* satisfies IS

If (ix) holds for all M, it follows easily that (ii) holds. In fact, every iterate of $\langle M, \bar{M}, \delta \rangle$ satisfies IS, where $M, \bar{M}$ satisfy IS and $\langle M, \bar{M}, \sigma \rangle$ is good. (This is needed for the result in §6, §7 of [NFS].) (The same will, of course, be true of other phalanx iterations.)

We note first that if M satisfies IS and $\sigma : \bar{M} \xrightarrow[\Sigma_1]{} M$ or $\sigma : M \xrightarrow[\Sigma_1]{} \bar{M}$, then $\bar{M} \| \bar{\nu}$ satisfies IS whenever $\bar{\nu} \in \bar{M}$ s.t. $E_\nu^{\bar{M}} \neq \emptyset$. Hence it suffices to prove (ix), (iv), (v) for ppms of the form $M = \langle J_\nu^E, F \rangle$, where $F \neq \emptyset$. Moreover, IS need only be verified for the top extender.

Let $M = \langle J_\nu^E, F \rangle$ be given with $\kappa = \mathrm{crit}(F)$, $\tau = \kappa^{+M}$, $\lambda = F(\kappa) = lh(F)$. Set $C = C_\nu^M = C_M$.

We prove (ix), (iv), (v) by cases (again with a nod to Steel).

<u>Case A</u> $C = \emptyset$. (We then call M a type A mouse.) The fact that M is type A is expressed by a $\Pi_2(M)$ statement:

Let $\mathbb{D} = \langle D, \in_{\mathbb{D}}, =_{\mathbb{D}}, E_{\mathbb{D}} \rangle$ be the term model for the Σ_0 ultraproduct of J_ν^E by F. Thus
$D = \{\langle \alpha, f \rangle \mid \alpha < \lambda \wedge f : \kappa \to M \wedge f \in M\}$.

For any Σ_0 formula $\varphi(x_1, \ldots, x_n)$, the set

$$\{\langle t_1, \ldots, t_n \rangle \in D^n \mid \mathbb{D} \models \varphi(t_1, \ldots, t_n)\}$$

is uniformly $\Delta_1(M)$. Moreover $\tilde{\alpha} = \langle \alpha, \mathrm{id} \rangle$ denotes α in $\mathbb{D}$ for $\alpha < \lambda$. Set $D_{\lambda'} = \{\langle \alpha, f \rangle \in D \mid \alpha < \lambda' \wedge f \in {}^\kappa\kappa\}$

The statement $\lambda' \in C$ is expressed by: $\lambda' \in (\kappa, \lambda) \wedge \bigwedge t \in D_{\lambda'}\ \mathbb{D} \models t < \tilde{\lambda}'$.
Hence C is $\Pi_1(M)$. Hence the statement:

$$\bigwedge \lambda' \in (\kappa, \lambda)\ \lambda' \notin C$$

is $\Pi_2(M)$. We know that if G is as in (ix) and $\pi : M \xrightarrow[G]{*} M^*$ with $\mathrm{crit}(G) < \omega\rho^1_M$, then π is Π_2-preserving. Hence $C_{M'} = \emptyset$ and M' satisfies IS. Now

let $\omega\rho^1_M \le \mathrm{crit}(G)$. Then $\pi : M \to_\sigma M^*$ is a Σ_0-ultrapower. Hence π is cofinal in $M^* = \langle J^{E^*}_{\nu^*}, F^*\rangle$ and λ is taken cofinally to $\lambda^* = lh(F^*) = \sigma(\lambda)$. Clearly if $\lambda' \in (\kappa, \lambda)$, then $\lambda' \notin C$ and hence $\pi(\lambda') \notin C_{M^*}$ since $\lambda' \notin C$ is a $\Sigma_2(M)$ statement about λ'. Now suppose $\delta \in (\kappa^*, \lambda^*)$ s.t. $\delta \notin rng(\pi)$. Let $\lambda' =$
$= \sup\{\xi \mid \pi(\xi) < \delta\}$. ~~Let~~

$\bar\lambda = \sup \pi''\lambda'$. Then $\bar\lambda \le \delta < \pi(\lambda')$. Let $\alpha < \lambda'$, $f \in ({}^\kappa\kappa)$ s.t. $\langle f, \alpha\rangle \ge \tilde\lambda'$ in $\mathbb{D}$. Then $\pi(\alpha) < \bar\lambda \le \delta$ and $\langle \pi(\alpha), \pi(f)\rangle \ge \widetilde{\pi(\lambda')} > \tilde\delta$ in $\mathbb{D}_{M^*}$. Hence $\delta \notin C_{M^*}$. This proves (ix). (iii) + hence (iv), (v) follow by the fact that $C_M = \emptyset$ is uniformly Π_2 over M.

Note that the application of (ii) or (iii) to a type A ppm always yields a type A ppm.

QED (Case A)

Note that C is closed in λ. Thus, if Case A fails, C either has a maximal element or is unbounded in λ.

Case B C has a maximal element λ'.

We first prove (ix). Let $\pi : M \xrightarrow{*}_{G} M^*$.

Then $C_{M^*} \setminus \pi(\lambda')+1 = \emptyset$ as before.

Let $e = E_\nu | \lambda'$. Then e is characterized by the $\Pi_1(M)$ condition on e, λ', τ

$$\mathrm{Funk}(e) \wedge \mathrm{dom}(e) = \mathbb{P}(\kappa) \cap J^E_\tau \wedge$$

$$\wedge \bigwedge x \in \mathrm{dom}(E_\nu) \; F(x) \cap \lambda' = e(x).$$

Thus $\pi(e) = E^{M^*}_{\nu^*} | \pi(\lambda')$.

Let $N = N_{\lambda', M}$. Then N is characterized by:

N is a ppm $\wedge$ e is the top extender of N $\wedge$

$\wedge \; J^{E^N}_{\lambda'} = J^E_{\lambda}$

Thus $\pi(N) = N_{\pi(\lambda'), M^*}$. The fact that IS holds below λ' in M is expressed by:

N satisfies IS.

Hence $\pi(N)$ satisfies IS and IS holds for M^*.

Moreover, M^* is a type B ppm,

This proves (ix). A similar proof gives us a weaker version of (iii) – to wit:

(iii)′ Let $\sigma : \bar{M} \xrightarrow[\Sigma_1]{} M$ s.t. $e = E_\nu \cap \lambda' \in \mathrm{rng}(\sigma)$. Then $\bar{M}$ satisfies IS. and is a type B ppm (Moreover, $\bar{\lambda}' = \sigma^{-1}(\lambda') = \max C_{\bar{M}}$ and $\sigma^{-1}(e) = E_\nu^{\bar{M}} \cap \bar{\lambda}'$.)

Thus, the most convenient way of handling Case B is to make $e = E_\nu^M \cap \lambda'$ a part of the structure, replacing M by $\langle M, \{e\} \rangle$. If we use $\langle M, \{e\} \rangle$ in defining the $\Sigma_i^{(n)}$ – hierarchy, the standard codes, reducts, and the standard parameter, then everything goes through exactly as before.

<u>Case C</u> The above cases fail. Then $\sup C = \lambda$. Then

(1) $\lambda = \omega\rho_M^1$

proof

Suppose not. Let $\rho = \omega\rho_M^1 < \lambda$. Let $A \subset \rho$ be $\Sigma_1(M)$ s.t. $A \notin M$.

Let A be $\Sigma_1(M)$ in p. Clearly $M = \bigcup_{\lambda' \in C} \mathrm{rng}(\sigma_{\lambda', M})$, where $\sigma_{\lambda', M}$ is defined as above. Let $\sigma = \sigma_{\lambda', M}$ where λ' is big enough that $p \in \mathrm{rng}(\sigma)$. Set $\bar{p} = \sigma^{-1}(p)$, $N = N_{\lambda', M}$. Then $N \in M$ and A is $\Sigma_1(N)$ in $\bar{p}$. Hence $A \in M$. Contr! QED(1)

Since $M = h_M(\lambda)$, we also know:

(2) $P_M \cap [\lambda, \nu) = \emptyset$ $\quad (\nu = ht(M))$.

We now recall that the predicates

$\lambda' \in C$ and $(\lambda' \in C \wedge e = E_\nu \cap \lambda')$

are uniformly $\Pi_1(M)$ in τ, hence they are $\Sigma_0^{(1)}$ in τ. But then the statement:

(3) There are arbitrarily large $\lambda' \in C$ s.t. $E_\nu \cap \lambda' \in M$

is $Q^{(1)}(M)$ in τ.

If $\pi : M \xrightarrow[G]{*} M^*$ is as in (xi), then $\mathrm{crit}(G) < \lambda$ + hence π is

π is $\ddot{Q}^{(1)}$ – preserving. Hence the same statement holds for M^*. By an argument in Case B, it follows easily that M^* satisfies IS and is a type C ppm. Thus (xi) holds.
In place of (iii) we get the weaker version:
(iii)′ If $\sigma : \overline{M} \longrightarrow_{\Sigma_1^{(1)}} M$, then $\overline{M}$ satisfies IS and is a type C ppm.
This clearly suffices to give (v).
It also gives (iv) by (2). QED