II. A Recasting of §7 of [NFS]: Dealing with Superstrong Extenders

In the appendix to §7 we noted that the proof of solidity made use of some consequences of 1-smallness. We then suggested an amendment to the proof which we thought would give the result in full generality. Unfortunately, that didn't work. As it turns out, we are unable to get the full result without Steel's device of k-iterations. Here we develop that machinery and redo the proof.

The difficulties are caused by the presence of <u>superstrong</u> <u>extenders</u>. An extender F on $\mathbb{P}(\kappa)$ is called <u>superstrong</u> iff $V_{\pi(\kappa)} \subset H'$, where $\pi : H_{\kappa^+} \longrightarrow_F H'$. If $M = \langle J_\alpha^E, E_{\omega\alpha} \rangle$ is a premouse and $\nu < \alpha$, we say that ν is a <u>superstrong</u> <u>index</u> in M iff

$E_\nu \neq \emptyset$ and $\lambda = lh(E_\nu)$ is a cardinal in M. This is equivalent to: E_ν is a superstrong extender in M. It is not hard to check that the proofs in §7 (and most of §8) go through on the assumption that the premice in question have no superstrong indices. The machinery developed below is intended to deal with the presence of such indices. (As noted in the appendix to §7, one problem arises in Case 2.2.4 of Lemma 6, when $i = 0$, $\nu_0 = \alpha$. (Hence E_α is superstrong, since $\lambda = lh(E_\alpha)$ is a cardinal in M, by the cardinal absoluteness of ...) Since $\nu_i > \alpha$ for $i > 0$ we can argue as before that $A \in Q_1$, where $\pi : M \xrightarrow[E_\alpha]{*} Q_1$. I don't, however, see how to get from this to $A \in M$. One approach is to redo the argum with n-iterations, where $\omega\rho^{n+1} \leq \alpha < \omega\rho^n$ in M. Then $\pi_{01} : M \xrightarrow[E_\alpha]{} Q_1$ will be an n-

ultrapower rather than a $*$-ultrapower. Thus, in particular, if $\tilde{\tau} = \tau_0^{+M} = \kappa_0^{++M}$ and $\tilde{\nu} = \nu_0^{+Q_1} = \lambda_0^{++Q_1}$, we have: $\pi : J_{\tilde{\tau}}^{E^M} \xrightarrow[E_\alpha]{} J_{\tilde{\nu}}^{E^{Q_1}}$. Hence $A \in J_{\tilde{\nu}}^{E^{Q_1}} \subset$ $\subset M$.

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<u>Def</u> Let $k \leq \omega$. Let M be acceptable. $\Gamma^m(\kappa, M) =$ the set of $f : \kappa \to M$ s.t. $f \in M$ or f is a good $\Sigma_1^{(h)}(M)$ function for an $h < k$ s.t. $\omega\rho_M^{h+1} > \kappa$. (Thus $\Gamma^*(\kappa, M) = \Gamma^\omega(\kappa, M)$, where Γ^* is the set of functions used in forming the $*$-ultraproduct $\mathrm{Ult}^*(M, F)$, where F is an extender on M.) Note that if $\sigma : M \xrightarrow[\Sigma_0^{(k)}]{} M'$, then $\sigma(f)$ is uniquely defined for such f, since if f_0', f_1' are two good, functionally absolute $\Sigma_1^{(h)}$ definitions, then the statement $\bigwedge_{\xi<\kappa} f_0'(\xi) = f_1'(\xi)$ is $\Sigma_0^{(h}$,

We then define $Ult^k(M,F)$ exactly like $Ult^*(M,F)$, using Γ^k in place of Γ^*. (Hence we first construct a "term model" $\mathbb{D} = \mathbb{D}^k(M,F)$ whose points consist of pairs $\langle \alpha, f \rangle$ s.t. $\alpha < lh(F)$ and $f \in \Gamma^k(\kappa, M)$ where $\kappa = crit(F)$.) We also write: $\pi : M \longrightarrow_F^k M'$ if $M' = Ult^k(M,F)$ and π is the canonical embedding.

Repeating arguments of §2 we get:

<u>Fact 1</u> Let $\pi : N \longrightarrow_F^k N'$, $\kappa = crit(F)$

(a) $\pi : N \longrightarrow_{\Sigma_1} N'$

(b) $\pi : N \longrightarrow_{\Sigma_0^{(k)}} N'$ if $\omega\rho_N^k > \kappa$

(c) $\pi : N \longrightarrow_{\Sigma_1^{(k)}} N'$ and $\omega\rho_{N'}^k = \sup \pi''\omega\rho_N^k$ and $\pi''\mathbb{R}_N^k \subset \mathbb{R}_{N'}^k$ if $\mathbb{R}_N^k \neq \emptyset$.

(d) $\pi : N \longrightarrow_{\Sigma_1^{(k)}} N'$ if F is close to N.

((a) is immediate by cofinality if $k = 0$; otherwise (a) follows by (b). (b) follows by the proof of §2 Lemma 3 of [NFS]. (c) is by the proof of §2 Lemma 4.4. (d) follows by observing that N' is a $*$-ultrapower if $\omega\rho^k \leq \kappa$.)

<u>Def</u> M is k-<u>extendible by</u> F iff $Ult^k(M,F)$ exists.
Repeating the proof of §3 Lemma 1, we get:

<u>Fact 2</u> Let $\langle\sigma, g\rangle : \langle\bar{M}, \bar{F}\rangle \longrightarrow \langle M, F\rangle$, where $\sigma : \bar{M} \longrightarrow_{\Sigma_0^{(h)}} M$ for all $h < k$ s.t. $\omega\rho^h_{\bar{M}} > \bar{\kappa} = \text{crit}(\bar{F})$. Let M be l-extendable by F, where $k \leq l \leq \omega$. Then $\bar{M}$ is k-extendible by $\bar{F}$. Moreover if $\bar{\pi} : \bar{M} \longrightarrow^{k}_{\bar{F}} \bar{M}'$, $\pi : M \longrightarrow^{l}_{F} M'$, then there is a unique σ' s.t.

(a) $\sigma' : \bar{M}' \longrightarrow_{\Sigma_0^{(h)}} M'$ for all $h \leq k$ s.t. $\bar{\kappa} < \omega\rho^h_{\bar{M}}$

(b) $\sigma'\bar{\pi} = \pi\sigma$; $\sigma' \upharpoonright lh(\bar{F}) = g$.

σ' is defined by ; $\sigma'(\bar{\pi}(f)(\alpha)) = \pi\sigma(f)(g(\alpha))$ for $\alpha < lh(\bar{F})$, $f \in \Gamma^k(\bar{\kappa}, \bar{M})$

We can then repeat the proof of §3 Lemma 2 to get;

<u>Fact 3</u> Assume:

(a) $\langle\sigma, g\rangle : \langle\bar{M}, \bar{F}\rangle \longrightarrow^{*} \langle M, F\rangle$

(b) $\sigma : \bar{M} \longrightarrow_{\Sigma_0^{(k)}} M$

(c) $\bar{F}$, F are weakly amenable.

Let $l, \bar{M}', M', \sigma'$ be as above.

Then $\sigma' : \bar{M}' \longrightarrow_{\Sigma_0^{(k)}} M'$.

The notion of a *generalized k-iteration* is defined like that of a generalized $*$-iteration in §4, except that clause (f) is replaced by:

(f') Let $i \in D$. Then $i+1 < \theta$ and $E_{\nu_i}^{M_i} \neq \emptyset$. Let $\kappa_i = \mathrm{crit}(E_{\nu_i}^{M_i})$, $\tau_i = \kappa_i^{+M_i \| \nu_i}$, $\xi = T(i+1)$, $M_i^* = M_\xi \| \eta_i$. If $i+1$ is simple in $\mathcal{I}$, then

$$\pi_{\xi, i+1} : M_i^* \xrightarrow[E_{\nu_i}]{k} M_{i+1}.$$

Otherwise $\pi_{\xi, i+1} : M_i^* \xrightarrow[E_{\nu_i}]{*} M_{i+1}$.

(Recall that i is called *simple* in $\mathcal{I}$ iff there is no truncation point $j+1 \leq_T i$.)

The notions of *direct*, *standard*, *normal*, *smooth*, *good* k-iteration are defined correspondingly.

The proof of §4 Lemma 1 shows:

<u>Fact 4</u> Let $\mathcal{I} = \langle\langle M_i \rangle, \ldots, \rangle$ be a normal k-iteration. If $i \in D$, then $E_{\nu_i}^{M_i}$ is close to M_i^*.

(Hence, letting $i+1 \leq_T i'$, $\xi = T(i+1)$, we see that $\pi_{\xi i'} : M_i^* \longrightarrow M_{i'}$ is $\Sigma_0^{(k)}$-preserving and Σ^*-preserving if $i+1$ is not simple. In particular, if $\pi_{hi'}$ is total on M_h, then $\pi_{hi'}$ is $\Sigma_0^{(k)}$-preserving and Σ^*-preserving if h is not simple.)

The notions of (normal, smooth, good) k-iterability and k-iteration strategy are defined as before.

Def Let $\sigma : \bar{M} \xrightarrow[\Sigma_0]{} M$ where $\bar{M}, M$ are premi[ce]. Let $\bar{M}$ have a generalized k-iteration $\bar{\mathcal{J}} = \langle\langle \bar{M}_i \rangle, \ldots\rangle$ of length θ. Let $k \leq l \leq$ [...]. The notion of the $\langle k, l\rangle$ copy $\mathcal{J} = \sigma(\bar{\mathcal{J}})$ of $\mathcal{J}$ onto M with copying maps $\langle \sigma_i \mid i < \theta\rangle$ is defined exactly as before, except that (v) is replaced by:

(v)′ Let $\xi = T(i+1)$, $M_i^* = M_\xi \| \eta_i$, $\sigma_i^* = \sigma_\xi \upharpoonright M_i^*$. Then $\sigma_i(\bar{\pi}_{\xi\, i+1}(f)(\alpha)) = \pi_{\xi\, i+1}\sigma^*(f)(\sigma_i(\alpha))$, where $\alpha < \lambda_i$, $f \in \Gamma$ and

$$\Gamma = \begin{cases} \Gamma^k(\varkappa_i, M_i^*) & \text{if } i+1 \text{ is simple;} \\ \Gamma^*(\varkappa_i, M_i^*) & \text{if not.} \end{cases}$$

By the proof of §5 Lemma 1 we get:

<u>Fact 5</u> Let $\sigma : \bar{M} \longrightarrow_{\Sigma_0^{(k)}} M$. Let $\bar{\mathcal{I}}$ be a normal k-iteration of $\bar{M}$. Let $k \leq l \leq \omega$ and suppose the $\langle k, l \rangle$-copy $\mathcal{I} = \sigma(\bar{\mathcal{I}})$ with copying maps to exist. Then

(a) $\sigma_i : \bar{M}_i \longrightarrow_{\Sigma_0^{(h)}} M_i$ for all $h \leq k$ s.t. $\omega\rho^h_{\bar{M}_i} \geq \sup\limits_{\xi \in D \cap i} \bar{\lambda}_\xi$

(b) If i is not simple in $\mathcal{I}$, then (a) holds without the restriction $h \leq k$,

It follows exactly as before that if S is a successful normal l-iteration strategy for M, then if $\sigma : \bar{M} \longrightarrow_{\Sigma_0^{(k)}} M$ and $k \leq l$, there is a successful <u>derived</u> strategy $\bar{S}$ for $\bar{M}$ defined by: Let $\bar{\mathcal{I}}$ be a normal k iteration of $\bar{M}$ of limit length. If the $\langle k, l \rangle$ copy $\mathcal{I} = \sigma(\bar{\mathcal{I}})$ exists, we set: $\bar{S}(\bar{\mathcal{I}}) \simeq S(\mathcal{I})$. Otherwise $\bar{S}(\mathcal{I})$ is undefined.

By the proof of §5 Lemma 3.1 we get:

<u>Fact 6</u> Let $\sigma : \bar{M} \longrightarrow_{\Sigma_0^{(k)}} M$. Let $\bar{\mathcal{I}}$ be a normal k-iteration of $\bar{M}$ and let $\mathcal{I} = \sigma(\bar{\mathcal{I}})$ be the $\langle k, l \rangle$ copy with copying maps $\langle \sigma_i \rangle$. Let $i \in D$, $\xi = T(i+1)$, $\sigma^* = \sigma_\xi \upharpoonright M_i^*$, $\bar{F} = E_{\bar{\nu}_i}^{\bar{M}_i}$, $F = E_{\nu_i}^{M_i}$.

Suppose that $k > 0$ or $k = 0$ and i is not simple in $\bar{\mathcal{I}}$. Then

$$\langle \sigma_i^*, \sigma_i \upharpoonright \bar{\lambda}_i \rangle : \langle \bar{M}_i^*, \bar{F} \rangle \longrightarrow^* \langle M_i^*, F \rangle.$$

If $k > 0$, the proof is a virtual repetition. If $k = 0$, we prove Lemma 3.2 only for non simple i by ind. on i. We need some extra argument in the proof of (3), since we can only apply the ind. hyp. to h (where $i = h+1$) if we know that h is also not simple. We prove the following Claim, which holds for all normal iterations $\mathcal{I} = \langle\langle M_i \rangle, \dots, T \rangle$.

Claim Let i be simple in $\mathcal{I}$. Define $\kappa = \hat{\kappa}_i$, $\tau = \hat{\tau}_i$, $\delta = \delta_i$, $\hat{\eta} = \hat{\eta}_i$ as before. Then M_δ is simple and $\hat{\eta} = ht(M_\delta)$.

proof. Suppose not.
Let i be the least counterexample.
Then $i = h+1$ and $\delta < i$. Let $\zeta = T(i)$.
Then $\kappa_\zeta > \kappa$. (Otherwise, letting $\bar{\kappa} = \pi_{\zeta, i}^{-1}(\kappa) = crit(E_{ht}^{M_h^*})$, we get $\kappa = \pi_{\zeta i}(\bar{\kappa}) \geq \pi_{\zeta i}(\kappa_\zeta) = \lambda_i$. Hence $\delta = i$. Contr!) But then $M_h^* = M_\zeta$ and h is simple, since i is simple. Hence $\hat{\kappa}_h = \kappa$, $\delta_h = \delta$. Moreover $\hat{\tau}_h = \tau$, since $\tau = \kappa^{+M_i} =$
$= \kappa^{+J_{\lambda_h}^{EM_i}} = \kappa^{+J_{\lambda_h}^{EM_\zeta}} = \kappa^{+M_i^*} =$
$= \kappa^{+M_\zeta}$. But then $\eta = \hat{\eta}_h$. The conclusion follows by minimality of i. QED (Claim)

Applying this to h in (3), where $\delta_h = \zeta$, we see that, if h were simple, then ζ is simple and $\eta_h = ht(M_\zeta)$. Hence i is simple. Contr!

The rest of the proof of Fact 6 is exactly as before. Fact 6 then gives:

<u>Fact 7</u> Let $\bar{M}, M, \sigma, \bar{y}, k, l, y = \sigma(\bar{y})$, $\langle \sigma_i \rangle$ be as above. Then

(a) σ_i is $\Sigma_0^{(k)}$-preserving for all i

(b) " " Σ^*- " if i is not simple.

It again follows that if $\sigma : \bar{M} \xrightarrow{}_{\Sigma_0^{(k)}} M$, $k \leq l \leq \omega$, and M has a successful iteration strategy for good l-iterations, then the derived strategy is successful for good k-iterations of $\bar{M}$.

We now define:

<u>Def</u> M is a <u>weak mouse</u> iff whenever $\overline{M}$ is countable and $\sigma : \overline{M} \longrightarrow_{\Sigma^*} M$, then $\overline{M}$ is $\omega_1 + 1$ $*$-iterable.

<u>Note</u> If M is a weak mouse, $\overline{M}$ is a premouse, and $\sigma : \overline{M} \longrightarrow_{\Sigma_0^{(k)}} M$, then $\overline{M}$ is $\omega_1 + 1$ k-iterable. This follows from the foregoing by the fact that there is a countable $\overline{M}'$ and a $\sigma' : \overline{M}' \longrightarrow_{\Sigma^*} M$ s.t. $\sigma'^{-1}\sigma : \overline{M} \longrightarrow_{\Sigma_0^{(k)}} \overline{M}'$.

<u>Note</u> A countable weak mouse is the same as a $*$-iterable countable mouse. By Löwenheim-Skolem the solidity lemma will hold for weak mice iff it holds for countable weak mice. The same is true of the condensation lemmas in §8.

We now turn to the <u>Neeman - Steel</u> lemma for countable weak mice.

<u>Def</u> Let M be a countable premouse. Let $e = \langle e_i \mid i < \omega \rangle$ enumerate $On \cap M$. A k-iteration strategy S for M is called e-<u>minimal</u> iff whenever $\mathcal{I}$ is an S-iteration, $M' = M_i$, $\pi = \pi^{\mathcal{I}}_{0i}$, and $\sigma : M \xrightarrow[\Sigma_0^{(k)}]{} M'$, then i is simple in $\mathcal{I}$ and $\pi(e) \leq \sigma(e)$ lexicographically.

(Here $\sigma(e) =_{\text{df}} \langle \sigma(e_i) \mid i < \omega \rangle$).

Obviously:

<u>Fact 8</u> Let S be e-minimal and let M' be an S-iterate of M. Then

(a) M' cannot be both a simple and non simple S-iterate of M.

(b) If M' is a simple S-iterate, then the iteration map $\pi = \pi^S_{MM'}$ is uniquely determined. (That is, if $\mathcal{I}^0, \mathcal{I}^1$ are S-iterations giving iteration maps π^0, π^1, then $\pi^0 = \pi^1$.)

The Neeman-Steel lemma says:

Fact 9 Let $\sigma : \bar{M} \longrightarrow_{\Sigma_0^{(k)}} M$, where $\bar{M}$ is a premouse and M is a weak mouse. Let $e = \langle e_i \mid i < \omega \rangle$ enumerate $On \cap \bar{M}$. Then $\bar{M}$ has an e-minimal k-iteration strategy which is successful up to $\omega_1 + 1$.

proof.
We follow the proof sketch given in the appendix to §7 for the special case $\bar{M} = M$, $k = \omega$. By Löwenheim-Skolem we may assume w.l.o.g. that M is countable. Fix a successful iteration strategy S for M. Let P = the set of pairs $\langle M', \mathcal{J} \rangle$ s.t. $\mathcal{J} = \langle \langle M_i \mid i \leq \theta \rangle, \ldots \rangle$ is an S-iteration from M to $M' = M_\theta$. If $a = \langle M_a, \mathcal{J}_a \rangle \in P$, then M_a has a successful iteration strategy S_a defined by letting $S_a(\mathcal{J}) \simeq$ the branch determined by $S(\mathcal{J}_a \circ \mathcal{J})$, where $\mathcal{J}_a \circ \mathcal{J}$ is the composite iteration.

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Our intention is to find an $a = \langle M_a, \mathcal{T}_a \rangle \in P$ and a $\sigma : M \longrightarrow_{\Sigma_1^{(k)}} M_a$ s.t., whenever $\mathcal{T}$ is an S_a - iteration of M_a to M' and $\sigma' : M \longrightarrow_{\Sigma_0^{(k)}} M'$, then M' is simple in $\mathcal{T}$ and $\pi\sigma(e) \leq \sigma'(e)$ lexicographically where $\pi = \pi_{\mathcal{T}}$ is the iteration map. We then let $\bar{S}$ be the derived k-iteration strategy for $\bar{M}$:

$$\bar{S}(\mathcal{T}') \simeq S_a(\sigma(\mathcal{T}')) \text{ if } \sigma(\mathcal{T}')$$

is the $\langle k, \omega \rangle$ - copy of $\mathcal{T}'$.

$\bar{S}$ clearly has the desired property. It remains only to construct a, σ. Define a partial ordering $\leq_P$ on P by:

Def Let $a = \langle M_a, \mathcal{T}_a \rangle$, $b = \langle M_b, \mathcal{T}_b \rangle$ s.t. $a, b \in P$.

$$a \leq_P b \longleftrightarrow_{\text{Df}} \mathcal{T}_b = \mathcal{T}_a \circ \mathcal{T}_{ab} \text{ for some } \mathcal{T}_{ab}.$$

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Then $J_{ab} \in S_a$ is uniquely determined. Let π_{ab} be the iteration map given by J_{ab}. Then π_{ab} is a partial map from M_a to M_b and is total iff M_b is simple in J_{ab}.

By ind. on $i < \omega$ we define a_i, σ_i s.t. $a_j \leq_P a_i$ for $i \leq j$ and $\sigma_i : \bar{M} \longrightarrow_{\Sigma_0^{(k)}} M_a$. We first define a_0. Let R be the relation on P defined by:

$$bRa \longleftrightarrow_{\text{df}} (a \leq_P b \text{ and } M_a \text{ is not simple in } J_{ab}).$$

Clearly R is well founded. Let a_0 be R-minimal in the set of $a \in P$ s.t. there is $\sigma : \bar{M} \longrightarrow_{\Sigma_0^{(k)}} M_a$.

Pick $\sigma_0 : \bar{M} \longrightarrow_{\Sigma_0^{(k)}} M_{a_0}$.

Now let a_i, σ_i be given. Define a relation R' on the set of $\langle \xi, a \rangle$ s.t. $\xi \in M_a$ by:

$\langle \rho, b \rangle R' \langle \xi, a \rangle$ iff

iff ($a \leq_p b$, π_{ab} is total on M_a, and $\rho < \pi_{ba}(\xi)$).

Then R' is well founded. Let $\langle \xi, a_{i+1} \rangle$ be R'-minimal in the set of $\langle \xi, a \rangle$ s.t. $a_i \leq_p a$, and there is $\sigma : \bar{M} \xrightarrow[\Sigma_0^{(k)}]{} M_a$ s.t. $\pi_{a_i a} \sigma_i (e_h) = \sigma(e_h)$ for $h < i$ and $\sigma(e_i) = \xi$. Pick $\sigma_{i+1} = \sigma$ satisfying the above for $a = a_{i+1}$.

Clearly we have: $\pi_{a_i a_j} \sigma_i (e_h) = \sigma_j (e_h)$ for $h < i \leq j$. Clearly $\mathcal{T}' = \bigcup_i \mathcal{T}_{a_i}$ is an S-iteration. Let $\mathcal{T}$ = the result of completing $\mathcal{T}'$ by the branch $S(\mathcal{T}')$. Let $a = \langle M_a, \mathcal{T} \rangle$, where M_a is the last model of $\mathcal{Y}$. Then

$$M_a, \langle \pi_{a_i, a} \rangle = \lim_{i \leq j} (M_{a_i}, \pi_{a_i a_j}).$$

Define $\sigma : \bar{M} \xrightarrow[\Sigma_0^{(k)}]{} M_a$ by

$\sigma(e_h) = \pi_{a_i a} \sigma_i(e_h)$ for $h < i$.

Then a, σ have the desired properties. QED (Fact 9)

We now turn to the theory of double rooted iterations developed in §7. We modify it so as to get a theory of double rooted k-iterations. The notion of a <u>generalized</u> <u>double</u> <u>rooted</u> k-<u>iteration</u> of a pair of premice $\langle N, M \rangle$ (with $N = M_{-1}$, $M = M_0$) is defined exactly as before. Similarly for "direct" and "standard". Similarly for the notion "normal k-iteration on $\langle N, M, \lambda \rangle$". The <u>good</u> triples $\langle N, M, \lambda \rangle$ are defined as before. In practice, we always pick k s.t. $k \geq n$ whenever $\omega\rho_M^n > \lambda$. We define:

Def $k \leq \omega$ is adequate for the good triple $\langle N, M, \lambda \rangle$ iff $k \geq n$ whenever $\omega\rho_M^n > \lambda$.

In place of Cor 5.1.1 we then have:

Cor 5.1.1′ Let k be adequate for $\langle N, M,$ Let $\mathcal{J} = \langle\langle M_i \rangle, \ldots \rangle$ be a k-iteration of $\langle N, M, \lambda \rangle$. If $i \leq_T j$ and π_{ij} is a total function on M_i, then π_{ij} is $\Sigma_0^{(k)}$-preserving. Moreover, π_{ij} is Σ^*-preserving if $i \geq 0$ in T or i is not simple in $\mathcal{J}$

(For $0 \leq \xi = T(i+1)$, $i+1$ simple, we have $\omega\rho_M^m = \omega\rho_{M_\xi}^m$ for $\omega\rho_M^m \leq \lambda$ and $\kappa_i \geq \lambda$; hence $\pi_{\xi, i+1}$ is $*$-preserving.)

(Note If $\xi + 1 \leq_T i$ is an anomaly, then by definition i is not simple.)

Corollary 5.1.2 must be similarly modified.

The definition of $\langle N, M, \lambda \rangle$ is witnessed by σ is exactly as before.

Now let $\langle N, M, \lambda\rangle$ be witnessed by σ and let $l \geq k \geq n$ for all n s.t. $\omega\rho^n_M > \lambda$. If $\mathcal{I} = \langle\langle M_i\rangle, \dots, T\rangle$ is a normal k-iteration of $\langle N, M, \lambda\rangle$, we define the $\langle k, l\rangle$-copy $\sigma(\mathcal{I}) = \langle\langle N_i\rangle, \dots, T'\rangle$ with copying maps $\langle\sigma_i\rangle$ exactly as before. It follows again that if S is a successful normal l-iteration strategy for N, then it induces a successful k-iteration strategy $\bar{S}$ for $\langle N, M, \lambda\rangle$ s.t. if $\mathcal{I}$ is an $\bar{S}$-iteration of $\langle N, M, \lambda\rangle$, then the $\langle k, l\rangle$ copy $\sigma(\mathcal{I})$ exists and is an S-iteration of N. Lemma 5.6 of §7 must be modified to:

<u>Lemma 5.6'</u> Let σ witness $\langle N, M, \lambda\rangle$ and let $\mathcal{I}' = \sigma(\mathcal{I})$ be the $\langle k, l\rangle$ copy of the normal k-iteration $\mathcal{I}$ of $\langle N, M, \lambda\rangle$. Let $i \in D$, $\xi = T(i+1)$, where $i+1$ is not an anomaly. Set $\sigma_i^* = \sigma_\xi \upharpoonright M_i^*$, $F = E^{M_i}_{\nu_i}$, $\bar{F} = E^{N_i}_{\sigma_i(\nu_i)}$. Then:

$$\langle\sigma_i^*, \sigma_i \upharpoonright \lambda_i\rangle : \langle M_i^*, F\rangle \xrightarrow{*} \langle N_i^*, F'\rangle.$$

As a consequence we again get:

<u>Lemma 5.8'</u> Let σ, $\mathcal{T}$, $\mathcal{T}' = \sigma(\mathcal{T})$, h, l be as above. Then

(a) If $i \geq_T -1$ and no $h \leq_T i$ is anomalous, then σ_i is $\Sigma_0^{(k)}$-preserving

(b) If some non anomalous $h+1 \leq_T i$ is a truncation pt., then σ_i is Σ^*-preserving

(c) If $i \geq_T 0$ is simple, then σ_i is $\Sigma_0^{(n)}$ preserving whenever $\omega\rho_M^n > \lambda$.

(d) Let $\xi+1 \leq_T i$ s.t. $\xi+1$ is an anomaly and $\pi_{\xi+1,i}$ is total. Then σ_i is $\Sigma_0^{(n)}$ preserving whenever $\omega\rho_M^n \geq \lambda$.

(a)–(c) are proven as before. (d) results from a more careful analysis of the anomalous case than we made before.

Let $M^* = M \| \eta_\xi$. Then $\pi_{-1,\xi+1} : M^* \xrightarrow[E_{\nu_\xi}]{*} M_{\xi+1}$.

Let $\omega\rho^{n+1} < \lambda \leq \omega\rho^n$ in M^*. Then $\omega\rho^{n+1} \leq \kappa_\xi < \omega\rho^n$, since $\lambda = \kappa_\xi^{+M^*}$.

By §2 Cor 4.2 we know that in $M_{\xi+1}$:

$\omega\rho^{n+1} < \lambda$, $\omega\rho^{n+1} \geq \nu_\xi = \pi_{-1,\xi+1}(\kappa_\xi)$

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Moreover $\sigma_{\xi+1} : M_{\xi+1} \longrightarrow \pi'_{0,\xi+1}(M^*)$ is $\Sigma_0^{(n)}$-preserving by our construction.

Now let $\xi+1 = T(j+1)$, where $j+1$ is not a truncation point.

Then, as we have seen, $N_j^* = \pi'_{0,\xi+1}(M^*) = N_{\xi+1} \| \eta_j$ where $\eta_j < ht(N_{\xi+1})$. (Thus $j+1$ is a truncation point in $\mathcal{J}'$ but not in $\mathcal{J}$.) We have

(1) $\pi_{\xi+1,j+1} : M_{\xi+1} \xrightarrow[E_{\nu_j}]{*} M_{j+1}$, since $\xi+1$ was a truncation pt. and $\pi'_{\xi+1,j+1} : N_j^* \xrightarrow[E'_{\nu_j}]{*} N_{j+1}$, since $j+1$ is a truncation point. But σ_{j+1} is defined by:

$$\sigma_{j+1}(\pi_{\xi+1,j+1}(f)(\alpha)) = \pi'_{\xi+1,j+1}\sigma_\xi(f)(\sigma_j(\alpha)),$$

where $f \in \Gamma^*(\kappa_j, M_{\xi+1})$. But $\Gamma^*(\kappa_j, M_{\xi+1}) \subset \Gamma^n(\kappa_j, M_{\xi+1})$, since $\omega\rho^{n+1}_{M_{\xi+1}} < \lambda < \lambda_\xi \leq \omega\rho^n_{M_{\xi+1}}$, where $\lambda < \kappa_j < \lambda_j$.

By Lemma 5.6′ and Fact 7 we get that σ_{i+1} is $\Sigma_0^{(n)}$-preserving. But $\omega\rho^{n+1} \le \lambda < \omega\rho^n$ in M_{i+1} by (*).

Proceeding inductively in this fashion we get: Let $\xi+1 \le_T i$ s.t. $\pi_{\xi+1,i}$ is total. Then $\omega\rho^{n+1} \le \lambda < \omega\rho^n$ in M_i and σ_i is $\Sigma_0^{(n)}$-preserving, which proves (d).

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From now on let M be a countable weak mouse, let $\sigma : W \longrightarrow M$ witness $\langle M, W, \delta \rangle$ and let $n \le k \le \omega$ whenever $\omega\rho^n_W > \delta$. Let $e = \langle e_i \mid i < \omega \rangle$ enumerate $On \cap M$. Fix an e-minimal k-iteration strategy S for M. Let $\bar{S}$ be the derived k-iteration strategy for $\langle M, W, \delta \rangle$. Assume that M is presolid (i.e. $M \| \eta$ is solid for all $\eta < ht(M)$). Then W is also presolid. Coiterate

$\langle M, W, \gamma\rangle$ against M, using $\bar{S}$ on the W-side and S on the M-side. Let $\langle \mathcal{I}^W, \mathcal{I}^Q\rangle$ be the coiteration with $\mathcal{I}^W = \langle\langle W_i\rangle, \ldots, \langle \pi^W_{ij}\rangle, T^W\rangle$ $\mathcal{I}^Q = \langle\langle Q_i\rangle, \ldots, \langle \pi^Q_{ij}\rangle, T^Q\rangle$. Let $\langle \nu_i\rangle$ be the coiteration indices. Let $\mathcal{I}^M = \sigma(\mathcal{I}^W) = \langle\langle M_i\rangle, \ldots, \langle \pi^M_{ij}\rangle, T^M\rangle$ be the $\langle k, k\rangle$-copy of $\mathcal{I}^W$. (Thus $\mathcal{I}^M$ and $\mathcal{I}^Q$ are both S-iterations of M.) Let $\langle \sigma_i\rangle$ be the copying maps. It is now time to point out a peculiarity of these iterations: Some of the models W_i may not be premice, since the initial segment condition may fail. This will occur if $i = \xi + 1$ is an anomaly and $\nu = \gamma$ is a superstrong index in M. Set: $\kappa = \mathrm{crit}(E^M_\nu)$, $\lambda = lh(E^M_\nu)$. Then $\kappa_\xi = \lambda$, $\pi_{-1,i} : M\|\nu \xrightarrow[E_\nu]{*} M_i$. Thus $M_i = \langle J^E_\nu, F\rangle$, where $\nu = \lambda^{+M_i}$ and $M\|\nu = \langle J_{\bar\nu}, F|\lambda\rangle$ is a premouse.

This violates the initial segment condition. We call such $\xi+1$ a <u>radical anomaly</u>. By I we know that whenever $\pi: Q \xrightarrow[F]{k} Q'$, Q is a premouse and $\mathrm{dom}(F) \in Q$, then Q' is a premouse. Hence W_i can only fail to be a premouse if some $\xi+1 \leq_T^W i$ is a radical anomaly and $\pi_{\xi+1,i}$ is total. In this case the top extender of W_i does not satisfy the initial segment condition but all others do – i.e. $W_i || \eta$ is a premouse for all $\eta < ht(W_i)$. Clearly W_i is a prepremouse (ppm). Since the basic iteration and copying lemmas go through for ppms, the iterations $\mathcal{I}^W, \mathcal{I}^M, \mathcal{I}^Q$ and the copying maps $\langle \sigma_i \rangle$ still exist and have the preservation properties stated above. The initial segment was used to prove that coiterations terminate. Thus we must still prove:

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<u>Lemma 1</u> The coiteration $\langle \mathcal{T}^W, \mathcal{T}^Q \rangle$ terminates below ω_1.

proof Suppose not.

Since S is an ω_1+1 strategy, we have $lh(\mathcal{T}^W) = lh(\mathcal{T}^Q) = \omega_1+1$. Let $X \prec H_{\omega_2}$ s.t. $\mathcal{T}^W, \mathcal{T}^Q \in X$. Let $\sigma : \bar{H} \xrightarrow{\sim} X$, where $\bar{H}$ is transitive. Set $\bar{\mathcal{T}}^W, \bar{\mathcal{T}}^Q = \sigma^{-1}(\mathcal{T}^W, \mathcal{T}^Q)$, $\alpha = \sigma^{-1}(\omega_1)$. Then $\alpha = crit(\sigma)$. By the usual methods we get: $\bar{\mathcal{T}}^W = \mathcal{T}^W | \alpha+1$, $\sigma(\pi_\alpha^W) = \pi_{\omega_1}^W$, and $\pi_{\alpha,\omega_1}^W = \sigma \upharpoonright W_\alpha$. Similarly for $\bar{\mathcal{T}}^Q$. Now let $i_0+1 \leq_{T^W} \omega_1$ s.t. $\alpha = T^W(i_0+1)$. Let $i_1+1 \leq_{T^Q} \omega_1$ s.t. $\alpha = T^Q(i_1+1)$. Then $crit(E^W_{\nu_{i_0}}) = crit(\pi_{\alpha,\omega_1}) = crit(\sigma) = \alpha$ and $E^W_{i_0}(X) = \sigma(X) \cap X^W_{i_0}$ for $X \in \mathcal{P}(\alpha) \cap W_\alpha$. Similarly for $E^Q_{\nu_{i_1}}$. If $i_0 = i_1 = i$, then $J^{E^{W_i}}_{\nu_i} = J^{E^{Q_i}}_{\nu_i}$

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and hence $E_{\nu_i}^{W_i} = E_{\nu_i}^{Q_i}$. Contr!
Hence $i_0 \neq i_1$. Suppose, first, that $i_0 < i_1$. Then $E_{\nu_{i_0}}^{W_{i_0}} =$
$= E_{\nu_1}^{Q_{i_1}} | \lambda_{i_0}^{W}$ and $\nu_{i_0}^{W} = (\lambda_{i_0}^{W+})^{W_{i_1}}$,
where $\langle J_{\nu_{i_0}}^{E^{Q_{i_1}}}, E_{\nu_1}^{Q_{i_1}} \rangle = W_{i_0} || \nu_{i_0}$ is a ppm. Contr!, since Q_{i_1} satisfies the initial segment condition. Now let $i_1 < i_0$. By the same argument, $E_{\nu_{i_0}}^{W_{i_0}}$ fails the initial segment condition. Hence $F = E_{\nu_{i_0}}^{W_{i_0}}$ is the top extender of W_{i_0}. Moreover there is a radical anomaly $\xi + 1 \leq_T i_0$ and $\pi_{\xi+1, i_0}$ is total. But then $\mathrm{crit}(E_{\nu_{i_0}}^{W_{i_0}}) = \mathrm{crit}(E_\gamma^M) < \gamma$ where E_γ^M is superstrong in M. But $\mathrm{crit}(E_{\nu_{i_0}}^{W_{i_0}}) = \alpha \geq \sup_{i<\alpha} \lambda_i^W > \gamma$. Contr! QED (Lemma 1)

We now prove:

Lemma 2 Let $\mathcal{J}^W$ have length θ. Then

(a) $0 \leq_T \theta$ in $\mathcal{J}^W$.

(b) θ is simple in $\mathcal{J}^W$

(c) W_θ is a segment of Q_θ.

proof.

We consider several cases.

Case 1 $\theta \geq -1$ in T^W

Case 1.1 Q_θ is a proper segment of W_θ

Then $\sigma_\theta \pi^Q_{0\theta} : M \xrightarrow[\Sigma_0^{(k)}]{} \sigma_\theta(Q_\theta)$, where $\sigma_\theta(Q_\theta)$ is a non simple iterate of M.

Case 1.2 $W_\theta = Q_\theta$ and there is a non anomalous truncation pt $i+1 \leq \theta$ in T^W. Then σ_θ is Σ^*-preserving. Hence $\sigma_\theta \pi^Q_{0\theta} : M \xrightarrow[\Sigma_0^{(k)}]{} M_\theta$, where M_θ is a non simple iterate of M.

Case 1.3 W_θ is a proper segment of Q_θ or $Q_\theta = W_\theta$ is a non simple iterate of M. Then $\pi^W_{0\theta} : M \xrightarrow[\Sigma^*]{} W_\theta$ where W_θ is a non simple iterate of M.

<u>Case 1.4</u> The above fail and some $i+1 \leq \theta$ in T^W is an anomaly.
Then $\pi^W_{i+1,\theta}$ is total on W_{i+1}. Set: $M^* = M^*_i = M \| \eta_i$. Then $M^* \in M$ is sound and there is n s.t. $\omega\rho^{n+1} \leq \kappa < \omega\rho^n$ in M^*, where $\kappa = \kappa_i$ (i.e. $\gamma = \kappa + M^*$). But then $\omega\rho^{n+1} = \kappa$ in M^*, since κ is a cardinal in M and $M^* \in M$. As we have seen above, it follows that σ_θ is $\Sigma_0^{(n)}$-preserving and that $\omega\rho^{n+1}_{W_\theta} \leq \kappa$, $\omega\rho^n_{W_\theta} \geq \nu_i > \gamma$. We distinguish two subcases:

<u>Case 1.4.1</u> $k > n$.

We have $\pi^W_{-1,i+1} : M^* \longrightarrow^*_F W_{i+1}$, where $F = E^{W_i}_{\nu_i}$. But by Lemma 5.1 we have $F_\alpha \in M$ for $\alpha < \lambda^W_i$. Hence the proof of §2 Lemma 5.2 shows: $\mathcal{P}(\kappa) \cap \underline{\Sigma}_1^{(n)}(W_{i+1}) \subset M$. Since $E^{W_l}_{\nu_l}$ is close to $M_{T(l+1)}$ and $\kappa < \gamma < \lambda_l \leq \kappa_l$ for $i+1 \leq_T l+1 \leq_T \theta$, it follows that:

(1) $\underline{\Sigma}_1^{(n)}(W_\theta) \cap \mathcal{P}(\kappa) \subset M$.

We know that Q_θ is a simple iterate of M in $\mathcal{J}^Q$, since W_θ is a non simple iterate of $\langle M, w, \gamma \rangle$. Hence $\kappa_i^Q \geq \omega\rho_M^{n+1}$ for all i, since otherwise

$\omega\rho^{n+1}_{W_\theta} = \omega\rho^{n+1}_{Q_\theta} \geq \lambda_i > \kappa$, since

$\mathcal{J}^Q$ is a k-iteration and $k > n$.

But then, $\rho = \omega\rho^{n+1}$, we have $\rho = \omega\rho_M^{n+1} \leq \kappa$, and

(2) $\underset{\sim}{\Sigma}_1(M) \cap \mathbb{P}(\rho) = \underset{\sim}{\Sigma}_1^{(n)}(W_\theta) \cap \mathbb{P}(\rho) \subset M$.

Contr! QED (1.4.1)

<u>Case 1.4.2</u> $k \leq n$.

Then $\sigma_\theta \pi_{-1,\theta} : M \longrightarrow_{\Sigma^{(k)}} N_\theta$,

where N_θ is a non simple iterate of M. Contr! by Neeman-Steel.

QED (1.4)

<u>Case 1.1.5</u> The above fail. Then

(1) $\pi^W_{-1,\theta} = \pi^Q_{0,\theta}$

prf. $\pi^M_{0\theta}(e) \leq \sigma_\theta \pi^Q_{0\theta}(e)$, since

$\sigma_\theta \pi_{0\theta} : M \longrightarrow_{\Sigma_0^{(k)}} M_\theta$, by Neeman-Steel

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Hence $\pi^{W_\theta}_{-1\theta}(e) = \sigma_\theta^{-1}\pi^{M}_{0\theta}(e) \leq \pi^{Q}_{0,\theta}(e)$ by Neeman-Steel. But

$\pi^{Q}_{0\theta}(e) \leq \pi^{W}_{-1,\theta}(e)$ since $\pi^{W}_{-1,\theta} : M \to_{\Sigma_0^{(k)}} Q_\theta$

QED(1)

Let $\pi = \pi^{W}_{-1,\theta} = \pi^{Q}_{0,\theta}$; $n = \mathrm{crit}(\pi)$.

Let $i+1$ = the least $i+1 \leq_{T^W} \theta$;
$j+1$ = " " $j+1 \leq_{T^Q} \theta$.

Then $\kappa_i^W = \kappa_j^Q = \mathrm{crit}(\pi)$. Moreover;

(2) $E^{W_i}_{\nu_i}(X) = \pi(X) \cap \lambda_i$

$E^{Q_j}_{\nu_j}(X) = \pi(X) \cap \lambda_j$.

Hence $i \neq j$, since otherwise $E^{W_i}_{\nu_i} = E^{Q_i}_{\nu_i}$.

Let e.g. $i < j$. Then $E^{W_j}_{\nu_i} = E^{Q_j}_{\nu_i} = \emptyset$.

But $E^{Q_j}_{\nu_i} | \nu_i = E^{W_i}_{\nu_i}$ by (2). Hence

$\langle J^{EQ_j}_{\nu_i}, E^{Q_j}_{\nu_j} | \lambda_i \rangle = W_i \| \nu_i$, where

$\nu_i = \lambda_i + Q_j$. This violates the initial segment condition. Contr!

QED(Case 1)

There remains only

<u>Case 2</u> $i \geq 0$ in T^W.

But then Q_θ cannot be an initial segment of W_θ, nor can $Q_\theta = W_\theta$ where θ is non simple in $\mathcal{J}^W$, since then $\pi^Q_{0\theta} : M \xrightarrow[\Sigma_0^{(k)}]{} Q_\theta$, where Q_θ is a non simple S-iterate of M. Contr! by Neeman-Steel.

QED (Lemma 2)

We now prove the solidity lemma in full generality:

<u>Lemma 3</u> Let M be a weak mouse. Then M is solid.

proof.

It suffices to prove the assertion for countable premice by Löwenheim-Skolem. We proceed by induction on ht(M). Hence it suffices to show:

<u>Claim</u> If M is a presolid countable weak mouse, then M is solid.

Suppose not. Then there is a max. maximal $\alpha \in P = P_M$ s.t. $W^\alpha_M \notin P$.

Set: $W = W$ and let $\sigma : W \to M$

be the canonical map. Let $\omega\rho^{n+1} \le \nu < \omega\rho^n$ in M. We can assume w.l.o.g. that:

(*) $R^n_M \neq \emptyset$.

To see this, suppose not. Then $n > 0$. Set: $r = \langle p_M, \langle W^\xi_M \mid \xi \in p_M \setminus \omega\rho^n_M \rangle\rangle$. Let $r^* = \langle r, 0, \dots, 0 \rangle$ and let $\tilde{M}, \tilde{r}$ be s.t. $\tilde{M}^{n,\tilde{r}} = M^{n,r}$, $\tilde{r} \in R^n_{\tilde{M}}$. Let $\tilde{\sigma} : \tilde{M} \xrightarrow{\Sigma_1^{(n)}} M$ be the induced map with $\tilde{\sigma} \restriction M^{n,r} = id$, $\tilde{\sigma}(\tilde{r}) = r$. Then $\tilde{\sigma}(p_{\tilde{M}} \setminus \omega\rho^n) = p_M \setminus \omega\rho^n$, since $\tilde{\sigma}^{-1}(W^\xi_M)$ is a generalized witness for ξ for $\xi \in p_M \setminus \omega\rho^n$. It follows easily that $\tilde{\sigma}$ is Σ^*-preserving and $\tilde{\sigma}(p_{\tilde{M}}) = p_M$. Hence $W^\alpha_{\tilde{M}} = W^\alpha_M = W$. Thus it suffices to prove the assertion for $\tilde{M}$, since then $W \in J^{E^{\tilde{M}}}_{\rho^n} = J^{E^M}_{\rho^n} \subset M$, since $\omega\rho^n_W = \nu < \omega\rho^n$. QED

Set $q = p \setminus (\alpha+1)$, $\bar{M} = M^{n,q\restriction n} = M^{n,p\restriction n}$ and let $\bar{\sigma} : \bar{W} \overset{\sim}{\longleftrightarrow} h_{\bar{M}}(\alpha \cup q^n)$. Then $\bar{W} = W^{n,\bar{q}\restriction n}$, $\bar{\sigma} = \sigma \restriction \bar{W}$ where $\sigma(\bar{q}) = q$.

Hence $\omega\rho^{n+1}_{W} = \omega\rho^{1}_{\bar{W}} \leq \alpha$, since $\bar{q}\restriction n \in R^n_W$. Fix an enumeration $e = \langle e_i \mid i < \omega \rangle$ of $On \cap M$ and an e-minimal n-iteration strategy S for M which is succesful up to $\omega_1 + 1$. Let Q = the e_0-th element of M in $<_M$. Note that $\langle M, W, \alpha \rangle$ is a good triple as witnessed by σ. Let $\bar{S}$ be the derived n-iteration strategy for $\langle M, W, \alpha \rangle$. Let $\langle \mathcal{J}^Q, \mathcal{J}^W \rangle$ be the coiteration of M, $\langle M, W, \alpha \rangle$, using $\bar{S}$, S respectively. Let $\mathcal{J}^M = \sigma(\mathcal{J}^W)$ be the $\langle n, n \rangle$-copy of $\mathcal{J}^W$ with copying maps $\langle \sigma_i \rangle$. Thus $\mathcal{J}^Q, \mathcal{J}^M$ are both S-iterations of M. Let

$$\mathcal{J}^Q = \langle \langle Q_i \rangle, \langle \nu_i^Q \rangle, \langle \eta_i^Q \rangle, \langle \pi_{ij}^Q \rangle, T^Q \rangle$$

$$\mathcal{J}^W = \langle \langle W_i \rangle, \langle \nu_i^W \rangle, \ldots, T^W \rangle$$

$$\mathcal{J}^M = \langle \langle M_i \rangle, \langle \nu_i^M \rangle, \ldots, T^M \rangle.$$

~~Let~~ θ be the length of the coiteration. We know that

W_θ is a simple iterate of W and a segment of Q_θ. Moreover $\theta \geq 0$ in T^W. Hence $\pi^W_{0i} : W \xrightarrow[\Sigma^*]{} W_\theta$ and $\pi^W_{0i} \restriction \alpha = id$, and $\omega\rho^{n+1}_{W_i} = \omega\rho^{n+1}_{W}$ for $i \leq \theta$ in T^W.
(This follows by induction on i, noting that if $i = T(j+1)$, $j+1 \leq \theta$ in T, then $\pi_{i,j+1} : W_i \xrightarrow[E_{\nu_j}]{*} W_{j+1}$ is a $*$-ultraproduct, since it is an n-ultraproduct and $\kappa_j \geq \alpha \geq \omega\rho^{n+1}_{W_i}$.)
Similarly $\pi^M_{0i} : M \xrightarrow[\Sigma^*]{} M_i$ and $\pi^M_{0i} \restriction \sigma(\alpha) = id$, $\omega\rho^{n+1}_{M_i} = \omega\rho^{n+1}_{M}$ for $i \leq \theta$ in T^M. (If $\alpha = ht(W)$ we take $\sigma(\alpha) = ht(M)$. In this case $W = W_\theta$, $M = M_\theta$).

We consider several cases:

<u>Case 1</u> $W_\theta = Q_\theta$ is a simple iterate of M.

For $i+1$ on the main branch of $\mathcal{J}^W$ we have: $\omega\rho^{n+1}_{W_{T(i+1)}} = \omega\rho^{n+1}_{W}$ and W_{i+1} is a $*$-ultraproduct of $W_{T(i+1)}$. This follows by ind. on i, since $\kappa_i \geq \nu \geq \omega\rho^{n+1}_{W}$. Thus:

Set: $\pi^W = \pi^W_{0\theta}$, $\pi^Q = \pi^Q_{0\theta}$, $\pi^M = \pi^M_{0\theta}$.

(1) (a) $\pi^W : W \longrightarrow_{\Sigma^*} W_\theta$, $\mathrm{crit}(\pi^W) \geq \alpha$

(b) $\omega\rho^{n+1}_W = \omega\rho^{n+1}_{W_\theta} \geq \alpha$

(c) $\underline{\Sigma}^*(W_\theta) \cap \alpha \subset \underline{\Sigma}^*(W)$

(d) $\pi^W{}'' P^*_W \subset P^*_{W_\theta}$

(e) $\omega\rho^n_{W_\theta} = \sup \pi^W{}'' \omega\rho^n_W$.

But $R^n_M \neq \emptyset$. For $i+1$ on the main branch of $\mathcal{T}^Q$ we then inductively get:

Let $\xi = T(i+1)$. Then $R^n_{Q_\xi} \neq \emptyset$,

$\pi^Q_{\xi, i+1} : Q_\xi \xrightarrow[E_{\nu_i}]{n} Q_{i+1}$; hence $\omega\rho^n_{Q_{i+1}} =$

$= \sup \pi^Q_{\xi, i+1}{}'' \omega\rho^n_{Q_\xi}$, $\pi^Q_{\xi, i+1}{}'' R^n_{Q_i} \subset R^n_{Q_{i+1}}$,

and $\pi^Q_{\xi, i+1} : Q_\xi \longrightarrow_{\Sigma_1^{(n)}} Q_{i+1}$, by Fact 1. Hence:

(2) (a) $\pi^Q : M \longrightarrow_{\Sigma_1^{(n)}} Q_\theta$

(b) $\omega\rho^n_{Q_\theta} = \sup \pi^Q{}'' \omega\rho^n_M$.

(3) $\pi^W(\bar{q}) \leq \pi^Q(q)$ in M_θ.

proof:

$\sigma_\theta \pi^Q(q) \geq \pi^M(q) = \pi^M \sigma(\bar{q}) = \sigma_\theta \pi^W(\bar{q})$, by the e-minimality of $\mathcal{J}^M$ and our choice of e_0, since $\delta_\theta \pi^Q$ is $\Sigma_0^{(n)}$-preserving. Hence $\pi^Q(q) \geq \pi^W(\bar{q})$.

QED (3)

(4) $\pi^Q(q) = \pi^W(\bar{q})$ = the top k elements of P_{W_θ}

proof.

$\pi^W(\bar{q}) \leq_* \pi^Q(q)$ by (3). Let $u \subset \alpha$ s.t. $u \cup \bar{q} \in P^*_W$. Then $\pi^W(u \cup \bar{q}) = u \cup \pi^W(\bar{q}) \in P^*_{W_\theta}$. Using (2) it follows by ind. on $\nu \in q$ in descending order that $\pi^Q(\nu) \in \pi^W(\bar{q})$ and $\pi^Q(W^\nu_M$ is a witness for $\pi^Q(\nu)$ in W_θ.

QED (4)

(5) $\kappa_i^Q \geq \omega\rho^\omega_M$ for $i+1 \leq \theta$ in T^Q.

Suppose not. Let i be the first counterexample. Then $T(i+1) = 0$. Let $A = \{\langle i, \xi\rangle \mid i < \omega \wedge \bar{W} \models \varphi_i[\xi, q]\}$ where $\bar{W} = W^{m, \bar{q}, m}$ and $\langle \varphi_i \rangle$ is a recursive enumeration of the Σ_1 fmlae. Then $A \cap \alpha \notin W$ by a diagonal argument. But $a = a_\mu \in W$ for all $\mu < \omega\rho_W^{m+1}$. Hence $a \in J_\alpha^{E^W} = J_\alpha^{E^M} \subset M$, since α is a cardinal in W. Now let $\delta : u \longleftrightarrow \overline{W} | h_{\overline{W}}(\mu \cup \bar{q}^n)$, where $u = u_\mu$ is transitive. Let $r = r_\mu = \delta^{-1}(\bar{q}^n)$. u is then coded by $a \in J_\alpha^{E^M}$. Hence $u \in J_\alpha^{E^M}$. We have:

$\bigwedge i \bigwedge \xi < \mu (\overline{M} \models \varphi_i[\xi, q^n] \rightarrow u \models \varphi_i[\xi, r])$.

This is $\Pi_1^{(n)}(M)$ in q, u, r, μ. Hence the same statement holds over $W_\theta = Q_\theta$ in $\tilde{q}, \tilde{u}, \tilde{r}, \tilde{\mu} = \pi^Q(q, u, r, \mu)$:

$\bigwedge i \bigwedge \xi < \tilde{\mu} (\overline{W}_\theta \models \varphi_i[\xi, \tilde{q}^n] \rightarrow \tilde{u} \models \varphi_i[\xi, \tilde{r}])$,

where $\overline{W}_\theta = W_\theta^{n, \tilde{q}rn}$. We then imitate the proof that a witness can be constructed from a generalized witness to see that there is $\eta \leq ht(\tilde{u})$ s.t. setting $u' = u'_\mu =_{\mathrm{Df}} \tilde{u}|\eta = \langle J_\eta^{E^u}, A^u \cap J_\eta^{E^u}$ we have:

$\bigwedge i \bigwedge \xi < \tilde{\mu} (\overline{W}_\theta \models \varphi_i[\xi, \tilde{q}^n] \longleftrightarrow u' \models \varphi_i[\xi, \tilde{r}])$

But $\tilde{q} = \pi^W(\bar{q})$. Set:

$\tilde{A} = \{\langle i, \xi \rangle | \overline{W}_\theta \models \varphi_i[\xi, \tilde{q}^n]\}$.

Then $\tilde{A} \cap \tilde{\mu} \in W_\theta$ whenever $\mu < \omega\rho_W^{n+1}$.

If we can choose $\tilde{\mu}$ s.t. $\tilde{\mu} = \pi^Q(\mu) \geq \alpha$, then $A \cap \alpha = \tilde{A} \cap \alpha \in$

$\mathfrak{P}(\alpha) \cap W_\theta = \mathfrak{P}(\alpha) \cap W$. Contr! But if there is no such α we must have $\nu_i = \alpha$, λ_i = the largest cardinal $< \alpha$ in M. Clearly E_{ν_i} is a superstrong extender in M. Since $\kappa_i < \omega\rho_W^{n+1}$ but $\pi_{01}^Q(\mu) \leq \pi^Q(\mu) = \tilde{\mu} < \nu_i$ for all $\mu < \omega\rho_W^{n+1}$, we conclude: $\omega\rho_W^{n+1} =$ $= \kappa_i^{+M} < \alpha$. But $\sup \pi_{01}^Q{}'' \kappa^{+M} =$ $= \alpha$. Thus $A \cap \tilde{\mu} \in \tilde{A} \cap \tilde{\mu} \in \mathfrak{P}(\alpha) \cap W_\theta =$ $= \mathfrak{P}(\alpha) \cap W$ for arbitrary large $\tilde{\mu} < \alpha$. Since $h_{\bar{W}}(\alpha \cup \bar{q}^n) = \bar{W}$, it follows easily that $\alpha = \omega\rho_W^{n+1}$. Contradiction! QED(5)

We know by the construction of W that $\omega\rho_M^{n+1} < \omega\rho_W^{n+1}$. But then $\kappa_i > \omega\rho_M^{n+1}$ for all $i+1$ on the main branch of $\mathcal{J}^Q$. It follows that we formed $*$-ultrapower and that $\omega\rho_M^{n+1} = \omega\rho_{W_\theta}^{n+1} = \omega\rho_W^{n+1}$. Contr!

QED (Case 1)

Case 2 W_θ is a proper segment of Q_θ. Then W_θ is sound. It follows that $E^{W_i}_{\nu_i} = \emptyset$ for all $i < \theta$, since otherwise ~~there~~ would be a first $i+1 \leq \theta$ in T^W s.t. $E^{W_i}_{\nu_i} \neq \emptyset$. But $\kappa_i \geq \alpha \geq \omega\rho^{n+1}_W$. Hence W_θ would be unsound. Contr! Hence $E^Q_{\nu_i} \neq \emptyset$ for all i. Set:

$$A = \{\langle i, \xi\rangle \mid i < \omega,\ \xi < \alpha,\ \bar{W} \models \varphi_i[\xi, \bar{q}^n]\}$$

(where $\bar{q}, \bar{W}, q$ are defined as above). Then A codes $\bar{W}$ and it suffices to prove:

Claim $A \in M$.

We consider two cases:

Case 2.1 $\nu_i > \alpha$ for all $i < \theta$.

If $Q_\theta = M$, there is nothing to prove. Otherwise $A \in J^{E^{Q_\theta}}_{\nu_0} = J^{E^M}_{\nu_0} \subset M$ since ν_0 is a cardinal in Q_θ.

Case 2.2 Case 2.1 fails.

Then $\nu_0 = \alpha$. This means that λ_0 is a cardinal in M + hence E_{ν_0} is a superstrong extender in M.

Clearly, if $0 < i < \theta$, then $A \in J_{\nu_i}^{E^{Q_\theta}} = J_{\nu_i}^{E^{Q_1}}$.
Hence $A \in Q_1$, where $\pi : M \xrightarrow[E_\alpha]{n} Q_1$.
But this is an n-ultrapower, where $\omega\rho_M^n > \tau_0^{+M}$ ($\tau_0 = \kappa_0^{+M}$). Set $\hat{\tau} = \tau_0^{+M}$, $\tilde{\alpha} = \alpha^{+Q_1}$. Then $\overline{\overline{\pi}} : J_{\tilde{\tau}}^{E^M} \xrightarrow[E_\alpha]{} J_{\tilde{\alpha}}^{E^{Q_1}}$.
Hence $A \in J_{\tilde{\alpha}}^{E^{Q_1}} \in M$. QED (Case 2)

<u>Case 3</u> $W_\theta = Q_\theta$ is a non-simple iterate of Q.
Let $i+1 \leq \theta$ in T^Q be the maximal truncation pt. If $\kappa_i < \alpha$, then $i = 0$ and κ_0 is a cardinal in M by the cardinal absoluteness of α. But then $Q_0 = M$ and $M_0^*, E_{\nu_0} \in M$, where $\pi : M_0^* \xrightarrow[E_{\nu_0}]{*} Q_1$. Hence $A \in Q_1 \in M$.
Now let $\kappa_i \geq \alpha$. Then $\pi : M_i^* \xrightarrow[E_{\nu_i}]{*} Q_{i+1}$ and $A \in \mathbb{P}(\alpha) \cap \underline{\Sigma}^*(Q_\theta) \subset \mathbb{P}(\alpha) \cap \underline{\Sigma}^*(Q_{i+1}) \subset \mathbb{P}(\alpha) \cap \underline{\Sigma}^*(M_i^*) \subset Q_\xi$, where $\xi = T(i+1)$. (We use the fact that $*$-ultrapowers are taken at $j+1$ for all j s.t. $i+1 \leq j+1 \leq \theta$ in T^Q.)
We now repeat the proof in Case 2 with Q_ξ in place of Q_θ to get: $A \in M$. QED (Lemma 3)