IV. Some Amendments to §8 of [NFS]

As stated in the appendix to §8, the proofs given hold only on the assumption that $\omega\rho^1_{M\|\nu} \le \lambda$ whenever $\lambda = lh(E^M_\nu)$. We now generalize these results, dropping that assumption altogether. As stated in the appendix we again use the Neeman–Steel lemma in place of Dodd–Jensen, employ a Löwenheim–Skolem argument to reduce to the case of countable mice. In the theorems of §8 the word "mouse" should be understood as "weak mouse" as defined in II – i.e. M is a weak mouse iff whenever Q is a countable premouse and $\sigma : Q \longrightarrow_{\Sigma^*} M$, then Q is $\omega_1 + 1$ iterable.

Lemma 1 goes through as before, but in fact we shall make use of the generalization which was stated as Lemma 2 of II. (This permits $\mathcal{J}^W, \mathcal{J}^Q, \mathcal{J}^{\prime}$ to be k-iterations, where $k \le \omega$ and $k \ge n$ whenever $\omega\rho^n \ge \alpha$. We suppose $\mathcal{J}^Q, \mathcal{J}^M$ to be ^{W} S-iterations, where

S is a fixed e-minimal k-strategy for M (with e an ω-enumeration of $On \cap M$) and $\mathcal{J}^W$ uses the derived k-strategy $\bar{S}$ for $\langle M, W, \alpha \rangle$.) A slight verbal change is needed in the proof of Lemma 2: In Case 1 of the proof of the Claim (p.3) we can justify $\nu_i > \alpha$ by observing that $M_i = M$ and α is a cardinal in M. Lemma 3.1 goes through as before, as does the revised proof of Lemma 3.2 in III of these notes.

Lemma 4 must be reformulated by adding a further disjunctive clause to (a), (b), (c):

__Lemma 4'__ Let M be a mouse and let $\nu \leq On \cap M$ be cardinal preserving in M (i.e. if $\tau < \nu$ is a cardinal in $M \| \nu$, then τ is a cardinal in M). Let $\sigma : \bar{M} \xrightarrow{\Sigma_0} M$ s.t. $\nu = \max\{\xi \mid \sigma \restriction \xi = id\}$ and σ is $\Sigma_0^{(n)}$-preserving whenever $\omega\rho^n_{\bar{M}} > \nu$. Then $\bar{M}$ is a mouse. Moreover, if $\omega\rho^\omega_{\bar{M}} \leq \nu$ and $\bar{M}$ is sound above ν, then one of the following holds:

(a) $\bar{M} = \mathrm{core}_\nu(M)$ and σ is the core map

(b) $\bar{M} = M\|\eta$ for an $\eta < ht(M)$

(c) $\pi : M\|\eta \xrightarrow[E^M_\mu]{*} \bar{M}$, where

(i) $\nu \le \eta < ht(M)$ and $\omega\rho^\omega_{M\|\eta} < \nu$

(ii) $\mu \le \omega\eta$

(iii) $\nu = \kappa^{+M\|\eta}$ where $\kappa = \mathrm{crit}(E^M_\mu)$

(iv) E^M_μ is generated by $\{\kappa\}$.

(d) $\bar{M} = M_1\|\eta$ where $\eta < ht(M_1)$ and

$\pi : M \xrightarrow[E^M_\nu]{} M_1$.

Note In cases (b), (c), (d) we have $\bar{M} \in M$

Note In case (d) E^M_ν is superstrong in M. Moreover $\nu = \lambda^{+\bar{M}} = \lambda^{+M^1}$, where $ht(\bar{M}) < \kappa^{+1}$ since $\omega\rho^\omega_{\bar{M}} \le \nu$ and $\bar{M}$ is sound. Hence $\nu = \omega\rho^\omega_{\bar{M}}$, since ν is a cardinal in M_1.

Note In the original version of Lemma 4 we did not assume that ν is cardinally absolute in M. In the proof, however, we reduced to that case as follows: Suppose ν is not cardinally absolute. Then $\omega\rho^1_{\bar{M}} \le \nu$ and $\bar{M} = \langle J^{\bar{E}}_{\bar{\alpha}}, \emptyset\rangle$. Let $\alpha = \mathrm{lub}\, \sigma``\bar{\alpha}$, $\tilde{M} = \langle J^{E^M}_\alpha, \emptyset\rangle$. Then

$\sigma : \bar{M} \longrightarrow \tilde{M}$ satisfies all assumptions, including the cardinal absoluteness of ν. But the conclusions of the old lemma 4 will hold for M if they hold for $\tilde{M}$. In the present version we added the assumption of cardinal absoluteness, since otherwise a more complicated formulation of (d) would be needed. The above reduction to that case is always available.

Note In cases (c), (d) we have: ν is a successor cardinal in $\bar{M}$. Hence if ν is a limit cardinal in $\bar{M}$, case (a) or (b) must hold.

Note In case (c) we have $\omega\rho^{\omega}_{\bar{M}} < \nu$. Hence $M = \mathrm{core}(\bar{M})$ or (b) or (d) holds if

$$\omega\rho^{\omega}_{\bar{M}} = \nu.$$

Note Cases (b), (c) are impossible if E_ν is superstrong in M.

Note The proof of Lemma 4 given in §8 goes through virtually unchanged if E_ν is not superstrong in M. The changes are needed solely to handle this case.

We now sketch the proof of Lemma 4', referring in some cases to the old proof of Lemma 4. If $\omega\rho^{\omega}_{\bar{M}} > \nu$, then $\sigma : \bar{M} \longrightarrow_{\Sigma^*} M$ and $\bar{M}$ is a mouse, which is all there is to prove. Now let $\omega\rho^{\omega}_{\bar{M}} \leq \nu$. If $\bar{M}$ is not sound above ν, we only need to show that $\bar{M}$ is a mouse and we simply repeat the old proof: Fix an enumeration $e = \langle e_i \mid i < \omega \rangle$ of $\mathrm{On} \cap M$ and let S be an e-minimal $*$-iteration strategy for M. Let $\bar{S}$ be the derived strategy for $\langle M, \bar{M}, \nu \rangle$. Since σ witnesses the goodness of $\langle M, \bar{M}, \nu \rangle$, we can coiterate $\langle M, \bar{M}, \nu \rangle$ and M to $\bar{M}_\theta$, M_θ. We know that $\theta \geq 0$ and that $\bar{M}_\theta$ is a segment of M_θ. Thus $\pi^{\bar{M}}_{0\theta} : \bar{M} \longrightarrow_{\Sigma^*} \bar{M}_\theta$ and $\bar{M}_\theta$ is a mouse. Hence $\bar{M}$ is a mouse. QED.

Now let $\bar{M}$ be sound above ν. If E_ν is not superstrong in M, we could again repeat the old proof. However, in order to handle the possibility that E_ν is superstrong, we must

work with n-iterations instead of $*$-
-iterations, where $\omega\rho^{n+1} < \nu < \omega\rho^n$ in $\bar{M}$.
We can also assume w.l.o.g. that:

(*) $R^n_M \neq \emptyset$.

To see this let $r = \langle \sigma(p_{\bar{M}}), p_M, \langle W^\xi_M \mid \xi \in p_M \rangle\rangle$.
Let $\tilde{M}^* = M^{n,\tilde{r}} \restriction m$, where $\tilde{r} = \langle r, 0, -, 0\rangle$
and let M^*, r^* be s.t. $r^* \in R^n_{M^*}$,
$M^{*\,m,r^*} = \tilde{M}^*$. Let $\sigma^* : M^* \xrightarrow[\Sigma_1^{(n)}]{} M$
be the induced map with
$\sigma^* \restriction \tilde{M}^* = id$, $\sigma^*(r^*) = r$. It follows
easily that $\sigma^* : M^* \xrightarrow[\Sigma^*]{} M$ and
$\sigma^*(p_{M^*}) = p_M$. Hence M^* is a mouse.
Set $\sigma' = \sigma^{*-1}\sigma : \bar{M} \xrightarrow[\Sigma_0^{(n)}]{} M^*$. It
suffices to prove the assertion for σ',
M^* in place of σ, M: Suppose first
that (a) holds. Then $\sigma \restriction \nu = id$ and
$\sigma(p_{\bar{M}}) = \sigma^*(p_{M^*}) = p_M$. Hence $\bar{M} =$
$= core_\nu(M)$. Now let (b) hold. Then
$\bar{M}$ is a segment of M^*, hence of
$J^{E^{M^*}}_{\omega\rho^n} = J^{E^M}_{\omega\rho^n}$, since $\omega\rho^\omega_{\bar{M}} \leq \nu < \omega\rho^n$.

Now let (c) hold. Then $M^*\|\eta$ is a segment of $J^{E^{M^*}}_{\omega\rho^n} = J^{E^M}_{\omega\rho^n}$, since $\omega\rho^\omega_{M\|\eta} < \nu < \omega\rho^n$. Hence (c) holds with M in place of M^*. Now let (d) hold. Then $\bar{M}$ is a segment of M_1^*, where $\pi^* : M^* \xrightarrow[E_\nu]{} M_1^*$ and $E_\nu^{M^*} = E_\nu^M$, since $\nu < \omega\rho^n$. Let $\pi : M \xrightarrow[E_\nu]{} M_1$ and let $\bar{\pi} : J^{E^M}_{\omega\rho^n} \xrightarrow[E_\nu]{} J^{E^{M_1}}_\gamma$. Then $\bar{\pi} = \pi^* \upharpoonright J^{E^{M^*}}_{\omega\rho^n} = \pi \upharpoonright J^{E^M}_{\omega\rho^n}$ and $J^{E^{M_1}}_\gamma = J^{E^{M_1^*}}_\gamma$. But γ is a cardinal in M_1 and M_1^* (or $\gamma = ht(M_1) = ht(M_1^*)$). Hence $\bar{M}$ is a segment of $J^{E^{M_1}}_\gamma$, since $\omega\rho^\omega_{\bar{M}} \leq \nu < \gamma$. QED

We choose our enumeration $e = \langle e_i \mid i < \omega \rangle$ of $On \cap M$ in such a way that $\sigma(p_{\bar{M}})$ is the e_0-th element of M. We then let S be an e-minimal n-iteration strategy for M and let $\bar{S}$ be the derived strategy for $\langle M, \bar{M}, \nu \rangle$. We again let $\langle \bar{J}, J \rangle$ be the coiteration of $\langle M, \bar{M}, \nu\rangle$ against M, where $\bar{\mathcal{J}} = \langle\langle \bar{M}_i\rangle, \ldots, \langle \bar{\pi}_{ij}\rangle, \bar{T}\rangle$, $\mathcal{J} = \langle\langle M_i\rangle, \ldots, \langle \pi_{ij}\rangle, T\rangle$. We also let $\mathcal{J}' = \sigma(\bar{\mathcal{J}}) = \langle\langle M'_i\rangle, \ldots, \langle \pi'_{ij}\rangle, T'\rangle$ be the $\langle m, n\rangle$ copy of $\bar{\mathcal{J}}$ with copying map $\langle \sigma_i\rangle$. Let the length of the coiteration be θ. Then $\theta \geq 0$; $\bar{M}_\theta$ is a simple iterate of $\bar{M}$ in $\bar{\mathcal{J}}$, and $\bar{M}_\theta$ is a segment of M_θ by II Lemma 2. Since $\kappa_i \geq \nu$ for $i+1$ on the main branch of $\bar{\mathcal{J}}$ and $\nu \geq \omega\rho^{n+1}_{\bar{M}}$, each M_{i+1} is a $*$-ultraproduct of M_ξ, where $\xi = T(i+1)$, and $\omega\rho^{n+1}_{M_\xi} = \omega\rho^{n+1}_{M}$. Hence, letting $\bar{\pi} = \bar{\pi}_{0\theta}$, $\pi = \pi_{0\theta}$, $\pi' = \pi'_{0\theta}$, we have;

(1) (a) $\bar{\pi} : \bar{M} \longrightarrow_{\Sigma^*} \bar{M}_\theta$, $\mathrm{crit}(\bar{\pi}) \geq \nu$

(b) $\omega\rho^{n+1}_{\bar{M}} = \omega\rho^{n+1}_{\bar{M}_\theta} \geq \nu$

(c) $\underline{\Sigma}^*(\bar{M}_\theta) \cap \nu \subset \Sigma^*(\bar{M})$

(d) $\omega\rho^n_{\bar{M}_\theta} = \sup \bar{\pi}''\omega\rho^n_{\bar{M}}$

(e) $\bar{\pi}(p_{\bar{M}}) = p_{\bar{M}_\theta}$

(To prove (e) note that $\bar{M}$ is a mouse by the earlier argumentation.)

We consider three cases:

Case 1 $\bar{M}_\theta = M_\theta$ is a simple iterate of M.

We imitate the proof of this case in the solidity lemma (Lemma 3) of II

Since $R^n_M \neq \emptyset$ and for each $i+1$ on the main branch M_{i+1} is an n-ultraproduct of M_ξ ($\xi = T(i+1)$), we get:

(2) (a) $\pi : M \longrightarrow_{\Sigma_1^{(n)}} M_\theta$

(b) $\omega\rho^n_{M_\theta} = \sup \pi''\omega\rho^n_M$.

(3) $\pi\sigma(p_{\bar{M}}) \geq \bar{\pi}(p_{\bar{M}}) = p_{M_\theta}$ in M_θ

prf

$\sigma_\theta \pi\sigma(p_{\bar{M}}) \geq \pi'\sigma(p_{\bar{M}}) = \sigma_\theta\bar{\pi}(p_{\bar{M}})$ in M'_θ by the determination of e_0. Hence $\pi\sigma(p_{\bar{M}}) \geq \bar{\pi}(p_{\bar{M}})$. QED(3)

(4) $\omega\rho^n_M = \sup \sigma''\omega\rho^n_{\bar{M}}$ (hence $\sigma : \bar{M} \longrightarrow_{\Sigma_1^{(n)}} M$)

proof.

Suppose not. Let $\lambda = \sup \sigma''\omega\rho^n_{\bar{M}}$. Set $A = A^{n,\sigma(p_{\bar{M}})\restriction n}_M$, $\tilde{M} = \langle J^{E^M}_\lambda, A \cap J^{E^M}_\lambda \rangle$. Then $\sigma : \bar{M}^{n,p_{\bar{M}}\restriction n} \longrightarrow_{\Sigma_1^{(n)}} \tilde{M}$ cofinally.

But $\tilde{M} \in M^{n,\sigma(p_{\bar{M}})\restriction m} \subset M$. Set:

$$B = \{\langle \xi, i\rangle \mid i<\omega \wedge \xi < \nu \wedge \bar{M}^{n,p_{\bar{M}}\restriction m} \models \varphi_i[\xi, p^n_{\bar{M}}]\},$$

where $\langle \varphi_i \rangle$ is a rec. enumeration of the Σ_1 fmlae. Then $B \notin \bar{M}$ by a diagonal argument. But $B \in \underline{\Sigma}_1(\tilde{M})$. Hence $B \in \mathbb{P}(\nu) \cap M = \mathbb{P}(\nu) \cap M_\theta =$
$= \mathbb{P}(\nu) \cap \bar{M}_\theta = \mathbb{P}(\nu) \cap \bar{M}$. Contr!

QED (4)

(5) $\pi\sigma(p_{\bar{M}} \setminus \nu) = p_{M_\theta} \setminus \nu = \bar{\pi}(p_{\bar{M}} \setminus \nu)$

pf.

$\pi\sigma(p_{\bar{M}}) \geq_* p_{M_\theta}$ by (3). But $\pi\sigma$ is $\Sigma_1^{(n)}$-preserving by (2), (4). By descending induction on $\xi \in p_{\bar{M}} \setminus \nu$ it follows that $\pi\sigma(\xi) \in p_{M_\theta}$ and $\pi\sigma(W^{\xi}_{\bar{M}})$ is a witness for $\pi\sigma(\xi)$. Thus $\pi\sigma(p_{\bar{M}} \setminus \nu) =$ the top k elements of $p_{M_\theta} = \bar{\pi}(p_{\bar{M}})$. But $p_{M_\theta} \setminus \nu = \bar{\pi}(p_{\bar{M}} \setminus \nu)$ by (1), where $\overline{\overline{p_M \setminus \nu}} = k$, so these are all the elements of $\bar{\pi}(p_{\bar{M}} \setminus \nu)$.

(6) $\kappa_i \geq \omega\rho_{\bar{M}}^{n+1}$ for $i+1 \leq \Theta$ in $\mathcal{I}$

prf. We imitate (5) in the proof of the solidity lemma (Lemma 3) in II.

Suppose not. Let $\kappa = \kappa_i$ where $i+1 \leq \Theta$ and $\kappa_i < \omega\rho_{\bar{M}}^{n+1}$. Hence $T(i+1) = c$

Set $\bar{N} = \bar{M}^{n, p_{\bar{M}} \upharpoonright n}$, $N = M^{n, \sigma(p_{\bar{M}} \upharpoonright n)}$, $\bar{q} = p_{\bar{M}} \setminus \nu$, $q = \sigma(\bar{q}) = \sigma(p_{\bar{M}}) \setminus \nu$. Set:

$A = \{\langle i, \xi\rangle \mid i < \omega \wedge \bar{N} \models \varphi_i[\xi, \bar{q}^n]\}$. Then $A \cap \nu \notin \bar{M}$ by a diagonal argument.

Now let $\mu < \omega\rho_{\bar{M}}^{n+1}$. Then $A \cap \mu \in \bar{M}$.

Set $a = a_\mu =_{\text{Df}} A \cap \mu$. Then $a \in$ $\in J_\nu^{E^{\bar{M}}} = J_\nu^{E^M} \subset M$, since $\nu \geq \omega\rho_{\bar{M}}^{n+1}$ is a cardinal in $\bar{M}$. Now let $\bar{u}$ be the transitivization of $\bar{N} | h_{\bar{N}}(\mu \cup \bar{q}^n)$. Then a codes u & hence $u \in M$. But u is the transitivization of $N | h_N(\mu \cup q^n)$

Hence we have:

$\bigwedge i \bigwedge \xi < \mu \, (N \models \varphi_i[\xi, q^n] \longleftrightarrow u \models \varphi_i[\xi, r])$

where $r =$ the image of q^n in the transitivization. This statement is $\Pi_2^{(n)}(M)$ in q, r, μ, u. But the direction ($\rightarrow$) is only $\Pi_1^{(n)}$ and therefore holds over M_θ in $\tilde{q}, \tilde{r}, \tilde{\mu}, \tilde{u} = \pi(q, r, \mu, u)$:

$\bigwedge i \bigwedge \xi < \tilde{\mu} \,(N_\theta \models \varphi_i[\xi, \tilde{q}^n] \rightarrow \tilde{u} \models \varphi_i[\xi, \tilde{r}])$,

where $N_\theta = M_\theta^{n, \tilde{q} \upharpoonright n}$. But then we can imitate the proof that the witness is reconstructible from a generalized witness to get an $\eta \leq ht(\tilde{u})$ s.t., setting $u' = u'_\mu = \tilde{u} | \eta$, we have:

$\bigwedge i \bigwedge \xi < \tilde{\mu} \,(N_\theta \models \varphi_i[\xi, \tilde{q}^n] \longleftrightarrow u' \models \varphi_i[\xi, \tilde{r}])$.

Set: $\tilde{A} = \{\langle i, \xi \rangle \mid i < \omega \wedge N_\theta \models \varphi_i[\xi, \tilde{q}^n]\}$.

Then $\tilde{A} \cap \nu = A \cap \nu$, since $\tilde{q} = \pi(\bar{q})$ and $N_\theta = \bar{M}_\theta^{n, \tilde{q} \upharpoonright n}$. If we can choose μ s.t. $\tilde{\mu} = \pi(\mu) \geq \nu$, then $A \cap \nu \in \mathbb{P}(\nu) \cap \bar{M}_\theta = \mathbb{P}(\nu) \cap \bar{M}$. Contr! Since $\pi(\kappa_i) \geq \pi_{0,i+1}(\kappa_i) = \lambda_i$ and $\nu_0 \geq \nu$, the only remaining possibility is: $i = 0$, $\nu_0 = \nu$. Then E_ν is a superstrong extender, since λ_0 is a cardinal in M. Since $\pi_{01}(\kappa_i^{+M}) = \nu$, we have: $\omega\rho_{\bar{M}}^{n+1} = \kappa_i^{+M}$. But the points $\tilde{\mu} = \pi(\mu)$ s.t. $\mu < \omega\rho_{\bar{M}}^n$ are then cofinal in ν. Hence $A \cap \tilde{\mu} \in \bar{M}$ for arbitrarily large $\tilde{\mu} < \nu$. Since $\bar{N} = h_{\bar{N}}(\nu \cup \bar{q})$, it follows easily that $\omega\rho_{\bar{M}}^n = \nu > \kappa_i^{+M}$. Contr! QED(6)

(7) $\omega\rho_M^{n+1} \leq \omega\rho_{\bar{M}}^{n+1}$

proof.

If $\omega\rho_{\bar{M}}^{n+1} = \nu$, this is trivial, since $\omega\rho_M^{n+1} \leq \nu$. If ν is the $\bar{M}$ successor of $\lambda = \omega\rho_{\bar{M}}^{n+1}$, then $\omega\rho_M^{n+1} \leq \lambda$, since ν is not a cardinal in M. Otherwise $\lambda^{+\bar{M}} = \lambda^{+M}$ and $\mathbb{P}(\lambda) \cap \bar{M} = \mathbb{P}(\lambda) \cap M$. Hence $A_{\bar{M}}^{n, p_{\bar{M}} \upharpoonright m} = A_M^{n, \sigma(p_{\bar{M}}) \upharpoonright m} \notin M$ and $\omega\rho_M^{n+1} \leq \lambda$. QED(7)

But then for each $i+1$ on the main branch of $\mathcal{J}$ we have taken a $*$-ultraproduct, since $\kappa_i \geq \omega\rho_M^{n+1}$.

It follows that $\pi : M \underset{\Sigma^*}{\longrightarrow} M_\theta$, $\mathrm{crit}(\pi) \geq \nu$ and $\pi(p_M) = p_{M_\theta} = \bar{\pi}(p_{\bar{M}})$. In particular $q = \pi^{-1}(p_{M_\theta} \setminus \nu) = p_M \setminus \nu$. Hence $\sigma(p_{\bar{M}} \setminus \nu) = p_M \setminus \nu$ and $\sigma \upharpoonright \nu = \mathrm{id}$. Since $\bar{M}$ is sound above ν, it follows that $\bar{M} = \mathrm{core}_\nu(M)$ and σ is the core map. Thus (a) holds. QED (Case 1)

<u>Case 2</u> $\bar{M}_\theta$ is a proper segment of M_θ. Then $\bar{M}_\theta = \bar{M}$, since $\bar{M}_\theta$ is sound and the iteration ~~from~~ $\bar{M}$ to $\bar{M}_\theta$ is above $\nu \geq \omega\rho^{n+1}_{\bar{M}}$. Hence $E^{\bar{M}}_{\nu_i} = \emptyset$ for all i.

<u>Case 2.1</u> $\nu_i > \nu$ for all i.

If $\theta = 0$, then $M_\theta = M$ and there is nothing to prove. Otherwise $\nu_0 > \nu$ is a cardinal in M_θ. Since $\omega\rho^\omega_{\bar{M}} \leq \nu$, we have: $\bar{M}$ is a segment of $J^{E^{M_\theta}}_\nu = J^{E^M}_\nu$. Thus (b) holds.

Case 2.2 Case 2.1 fails.
Then $\nu_0 = \nu$. Hence E_ν^M is a superstrong extender, since λ_0 is a cardinal in M. By the above argument, $\bar{M}$ is a segment of $J_{\nu_1}^{E^{M_\theta}} = J_{\nu_1}^{E^{M_1}}$, since $\nu_1 > \nu$ is a cardinal in M_θ. Thus $\bar{M}$ is a segment of M_1, where $\pi : M \xrightarrow[E_\nu]{n} M_1$. Let $\gamma = \kappa^{++M}$ and let $\bar{\pi} : J_\gamma^{E^M} \xrightarrow[E_\nu]{} J_{\tilde{\gamma}}^{E'}$ be a Σ_0 ultraproduct. Then $\bar{\pi} = \pi \upharpoonright J_\gamma^{E^M}$ and $J_{\tilde{\gamma}}^{E'} = J_{\tilde{\gamma}}^{E^{M_1}}$, since $\gamma \leq \omega\rho_M^n$ and M_1 is a Σ_n ultraproduct. But $\bar{\pi} = \pi' \upharpoonright J_\gamma^{E^M}$, $J_{\tilde{\gamma}}^{E'} = J_{\tilde{\gamma}}^{E^{M_1'}}$, where $\pi' : M \xrightarrow[E_\nu]{} M_1'$ is the Σ_1 ultraproduct. Thus (d) holds. QED (Case 2)

Case 3 The above fail.
Then $\bar{M}_\theta = M_\theta$ is a non simple iterate of M. Hence there is a maximal truncation pt $j+1 \leq \theta$ in $\mathcal{T}$.

Hence $M_i^* = \mathrm{core}(\bar{M}_\theta)$ is a proper segment of M_η, where $\eta = T(i+1)$. If $\kappa_i \geq \nu$, then $M_i^* = \bar{M}$. Hence the coiteration terminates at $\eta < \theta$. Contr! Thus $\kappa_i < \nu$. Hence $\eta = 0$, $M_\eta = M$. But $\pi_i = \kappa_i^{+ J_{\nu_i}^{E^{M_i}}}$ is not a cardinal in M, since we truncate. Hence $\tau_i = \nu < \kappa_i^{+M}$. Since we truncate at $i+1$, we take $*$-ultrapowers at each $i+1$ on the main branch. We can then repeat the argument of Case 2.3 in the proof of Lemma 4 in §8 to get (c). QED (Lemma 4')

Now let M be a mouse s.t. $M = \langle J_\nu^E, F \rangle$, $F \neq \emptyset$. Let $\lambda = lh(F)$, $\kappa = crit(F)$, $\tau = \kappa^{+M}$. Let $k : J_\tau^E \xrightarrow[F]{} J_\nu^E$. As noted in the appendix, the proof of Lemma 4.1 goes through only if we assume $\omega\rho^1_M < \lambda$ for all such M. However, the first few lines of that proof yield an interesting analysis of the strength of the conclusion of Lemma 4.1. Let C_M be defined as in I. (That is C_M = the set of

$\bar{\lambda} \in (\kappa, \lambda)$ s.t. $k(f)(\alpha) < \bar{\lambda}$ for all $\alpha < \bar{\lambda}$, $f \in ({}^{\kappa}\kappa)^M$.)

Then:

<u>Lemma 4.1′</u> Let M be as above. The following are equivalent:

(a) $C_M = \emptyset$

(b) $\lambda = \sup\{k(f)(\kappa) \mid f \in ({}^{\kappa}\kappa)^M\}$

proof.

(b) $\to$ (a) is trivial. (a) $\to$ (b) is proven in (2) of the prf. of Lemma 4.1 QED

(<u>Note</u> This holds even for premice M.)

The proof of Cor 4.2 goes through unchanged:

<u>Cor 4.2</u> Let M as above be sound. Let $\beta < \lambda$. Either β generates F or $F|\beta \in M$.

The proof of Cor 4.3 then shows:

<u>Cor 4.3′</u> Let M as above be sound. There is $\beta < \lambda$ which generates F iff $\omega\rho^1_M < \lambda$.

The proof of Lemma 5 also goes through as before:

<u>Lemma 5</u> Let M be a mouse. Let $\rho = \omega\rho^n_M \in M$, $\tau = \rho^{+M}$. Let $\bar{M} = \mathrm{core}(M)$. Then $\tau = \rho^{+\bar{M}}$, $J^{E^{\bar{M}}}_\tau = J^{E^M}_\tau$.

. . . .

We can prove a further lemma which gives us more information about the set C_M, where $M = \langle J^E_\nu, F\rangle$, $F = \emptyset$. We recall that if $\bar{\lambda} \in C = C_M$, then $N = N_{\bar{\lambda}, M}$ was defined by: $N = \langle J^{\bar{E}}_{\bar{\nu}}, \bar{F}\rangle$, where $\bar{F} = F|\bar{\lambda}$, $\bar{k} : J^E_\tau \longrightarrow J^{\bar{E}}_{\bar{\nu}}$. (We again let τ, κ, λ, k be defined as above.) There is then $\sigma = \sigma_{\bar{\lambda}, M}$ s.t. $\sigma : N \xrightarrow[\Sigma_0]{} M$ cofinally. σ is defined by: $\sigma(\bar{k}(f)(\alpha)) = k(f)(\alpha)$. Clearly, $\sigma \upharpoonright \bar{\lambda} = \mathrm{id}$, $\sigma(\bar{\lambda}) = \lambda$.

N is sound above $\bar{\lambda}$, since $\omega\rho^1 \leq \bar{\lambda}$.
By Lemma 4', we have: $N = \mathrm{core}_{\bar{\lambda}}(M)$
or N is a proper segment of M,
since $\bar{\lambda}$ is a limit cardinal in M.
But $N = \mathrm{core}_{\bar{\lambda}}(M)$ is impossible by
the initial segment condition in I,
since then $N \in M$, $\rho^1_M = \rho^1$ and
and $A^1_M = A^1_N \in M$. Contr!

Thus:

<u>Lemma 6</u>. Let M be as above,
$\bar{\lambda} \in C_M$ and $N = N_{\bar{\lambda}, M}$. Then
N is a proper segment of M.

<u>Note</u> As noted in I, the proofs
of the generalized versions of Lemma 1
– Lemma 5 do not specifically use
the initial segment condition IS of I
but rather <u>any</u> initial segment
condition with certain minimal
properties. It then follows, however,
as in I that any such condition
must imply IS.