V A Remark on $\square$ in L^E

Consider a model $K = L^E$ with the property that each $K\|\eta$ is a weak mouse in the sense of I, Schimmerling and Zeman have shown that $\square_\lambda$ holds in K for many cardinals λ. Let S_λ = the set of $\nu \in (\lambda, \lambda^+)$ s.t. E_ν is a supercompact extender (i.e. $E_\nu \neq \emptyset$ and $\lambda = lh(E_\nu)$). By the methods of Schimmerling + Zeman $\square_\lambda$ holds if S_λ is not stationary in λ^+ (in K). We now show that this result is best possible:

<u>Thm</u> Let S_λ be stationary in λ^+. Then $\square_\lambda$ fails.

proof. Suppose not.

Set: $H = L^E_{\lambda^{++}}$. Set $D =$

$$= \{\alpha \in (\lambda, \lambda^+) \mid \alpha = \lambda^+ \cap h_H(\alpha)\}.$$

Then D is cub in λ^+. Let $\alpha \in D \cap S_\lambda$. Set $X = h'_H(\alpha)$ and let $\bar{H} = L^{\bar{E}}_\beta$, σ be defined by $\sigma : \bar{H} \stackrel{\sim}{\longleftrightarrow} H|X$. Then $\sigma : \bar{H} \longrightarrow_{\Sigma_1} H$, $\bar{H}$ is a premouse which is sound above α and $\omega\rho^1_{\bar{H}} = \alpha$. We apply Lemma 4' in IV (which generalized Lemma 4 of §8 [NFS]). Clearly $\bar{H} \neq core_\alpha(H)$ since $\rho^1_{\bar{H}} = \alpha < \lambda^{++} = \rho^\omega_H$, hence (a) in Lemma 4' fails. Moreover $\bar{H}$ is not an initial segment of H, since $E^H_\alpha \neq \emptyset$, $E^{\bar{H}}_\alpha = \emptyset$. Hence (b) fails. (c) also fails, since α = the largest cardinal in $\bar{H}$, where $\alpha = crit(\sigma)$. Thus (d) must hold, and $\bar{H}$ is a segment of $L^{E'}$, where $\pi : L^E \longrightarrow_{E_\nu} L^{E'}$.

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We are assuming that L^E has a $\Box_\lambda$ sequence. Let

$$C = \langle C_\eta \mid \lambda < \eta < \lambda^+ \wedge \lim(\eta) \rangle$$

be the $<_{L^E}$ - least such. Since σ is Σ_1 preserving, there is $\bar{C} \in \bar{H}$ s.t. $\sigma(\bar{C}) = C$. Hence $\bar{H}$ thinks that $\bar{C}$ is a $\Box_\lambda$ - sequence. Since $\alpha = \lambda^{+L^{E'}}$ and H is an initial segment of $L^{E'}$, it follows that $\bar{C}$ is the $<_{L^{E'}}$ - least $\Box_\lambda$ sequence in $L^{E'}$.

Since $\pi(\tau) = \alpha$, where $\kappa = \mathrm{crit}(E_\alpha)$ and $\tau = \kappa^+$ in L^E, it follows that there is $C' = \pi^{-1}(\bar{C})$ which is a $\Box_\kappa$ sequence in L^E.

Set: $\bar{\pi} = \pi \restriction L^E_\tau$. Then

(1) $\bar{\pi} : \langle L^E_\tau, C' \rangle \prec \langle L^E_\alpha, \bar{C} \rangle$

(2) $\langle L^E_\alpha, \bar{C} \rangle \prec_{\Sigma_1} \langle L^E_{\lambda^+}, C \rangle$,

Hence:

(3) $\overline{\pi} : \langle L^E_\tau , C' \rangle \longrightarrow_{\Sigma_1} \langle L^E_{\lambda^+} , C \rangle$,

where C' is a $\square_\kappa$ sequence.

This is known to yield a contradiction: Consider C_α. Since $\alpha = \sup \overline{\pi}'' \tau$ and τ is regular, the set $C^*_\alpha \cap rng(\pi)$ is unbounded in α.

where C^*_α = the limit pts. of C_α

Suppose $\eta \in C^*_\alpha \cap rng(\pi)$, $\pi(\eta') = \eta$. Then $\pi(C'_{\eta'}) = C_\eta = \eta \cap C_\alpha$. Since C_η is a proper segment of C_α, we have $otp(C_\eta) < \lambda$. Hence $otp(C'_{\eta'}) < \kappa$. Hence $otp(C_\eta) = otp(C'_{\eta'}) < \kappa$, since $\pi \upharpoonright \kappa = id$. Thus $otp(C_\alpha) = \sup\limits_{\eta \in C^*_\alpha \cap rng(\pi)} otp(C_\eta) \le \kappa$,

Contr! since $cf(\alpha) = \tau > \kappa$ in L^E.

QED