§2 α-subproper forcing

α-subproper forcing generalizes the notion of α-proper forcing, invented by Shelah and lucidly exposited by Avraham in [PF]. In this forcing we deal not with a single embedding $\sigma : \bar{N} \prec N$ from a countable to an uncountable structure, but rather with an entire tower of embeddings.

<u>Def</u> Let $N = L_\tau^A$ be a ZFC^- model. Let $\Gamma = [\alpha, \beta]$ where $\alpha \leq \beta < \omega_1$. By an Γ-<u>pretower</u> <u>for</u> N we mean a $\pi = \langle \pi^i \mid i \in \Gamma \rangle$ s.t.

- $\pi^i : N^i \prec N$ where N^i is countable and full
- $\text{rng}(\pi^i) \subset \text{rng}(\pi^j)$ and $\omega_1^{N^i} < \omega_1^{N^j}$ for $i < j$
- If λ is a limit point of Γ, then $\text{rng}(\pi^\lambda) = \bigcup_{\nu \in \Gamma \cap \lambda} \text{rng}(\pi^\nu)$.

We also set: $\pi^{ij} = (\pi^j)^{-1} \circ \pi^i$ for $i \leq j$, $i, j \in \Gamma$.

Since N^i is determined by N, π^i, we also denote it by N^{π^i}. We write: $\pi^{ij} = (\pi^j)^{-1} \circ \pi^i$.

By an α-pretower we mean a $[0,\alpha]$-pretower $\langle \pi^i \mid i \le \alpha \rangle$. We shall generally state our definitions for α-towers, leaving it to the reader to work out the $[\alpha,\beta]$-version.

Def An α-pretower $\pi = \langle \pi^i \mid i \le \alpha \rangle$ for N is a tower iff $\pi \in N$ and $\langle \pi^{h,i+1} \mid h \le i \rangle \in N^{i+1}$ for $i < \alpha$.

Note If π is an α-tower, then $\pi^{i+1}(\pi^{h,i+1}) = \pi^h$ for $h \le i < \alpha$, since $(\pi^{i+1}(\pi^{h,i+1}))(x) =$
$= (\pi^{i+1}(\pi^{h,i+1}))(\pi^{i+1}(x)) = \pi^{i+1}(\pi^{h,i+1}(x)) = \pi^h(x)$

Hence, if $\xi \in N^h$, we have:

$$\pi^{i+1}(\sup \pi^{h,i+1}{}''\xi) = \sup \pi^h{}''\xi$$

Def Let π, σ be α-pretowers. σ is a revision of π iff

- $N^{\pi^i} = N^{\sigma^i}$ for $i \le \alpha$
- $\bigcup \operatorname{rng} \pi^\alpha = \bigcup \operatorname{rng} \sigma^\alpha$
- $\bigcup \operatorname{rng} \pi^{i,i+1} = \bigcup \operatorname{rng} \sigma^{i,i+1}$ for $i < \alpha$.

Def Let π be an α-pretower for N. Let $x \in N$. π absorbs x iff $(\pi^i)^{-1}(x)$ exists for all $i \le \alpha$.

Note If π absorbs x, we often write: x^i for $(\pi^i)^{-1}(x)$.

Def Let σ be a revision of π. σ respects π at x iff σ absorbs x and $(\sigma^i)^{-1}(x) = (\pi^i)^{-1}(x)$ for $i \leq \alpha$.

Thus x^i has the same meaning for σ, π if σ respects π at x.

We also say: σ is an x-revision of π to mean that σ respects π at x.

Def Let σ be a revision of π. Let $x \in N^\alpha$ (where π is an α-pretower). σ coincides with π at x iff

- $\sigma^\alpha(x) = \pi^\alpha(x)$
- Whenever $i < \alpha$ and $\pi^{i,\alpha}(\bar{x}) = x$, then $\sigma^{i,\alpha}(\bar{x}) = x$.

Def σ coincides with π on $u \subset N^\alpha$ iff σ coincides with π at each $x \in u$.

(In other words, $\pi^\alpha \restriction u = \sigma^\alpha \restriction u$ and $(\pi^i)^{-1} \restriction u \subset (\sigma^i)^{-1} \restriction u$ for $i < \alpha$.)

Def Let $\mathbb{B} \in N$ be a complete BA in N.
σ is an $x, \mathbb{B}$-revision of π iff
- σ is an x-revision of π and absorbs $\mathbb{B}$
- If σ respects π at $\mathbb{B}$ and $\delta = \delta(\mathbb{B})$,
- Then: $C_\delta^N(\mathrm{rng}\,\sigma^\alpha) = C_\delta^N(\mathrm{rng}\,\pi^\alpha)$ and
$$C_{\delta^{i+1}}^{N^{i+1}}(\mathrm{rng}\,\sigma^{i,i+1}) = C_{\delta^{i+1}}^{N^{i+1}}(\mathrm{rng}\,\pi^{i,i+1})$$
for all $i < \alpha$.

(Hence, being an $x, \mathbb{B}$-revision says little if σ does not respect π at $\mathbb{B}$.)

Def Let σ be an $x, \mathbb{B}$-revision of π. Let G be $\mathbb{B}$-generic over N. We say that x is an $x, \mathbb{B}, G$-revision of π iff
- $G^i =_{\text{Df}} (\sigma^i)^{-1}{}''G$ is $\mathbb{B}^i$-generic over N^i for $i \leq \alpha$, where $\mathbb{B}^i = (\sigma^i)^{-1}(\mathbb{B})$.
- $\sigma \in N[G]$
- $\langle \sigma^{h,i+1} \mid h \leq i \rangle \in N^{i+1}[G^{i+1}]$ for $i < \alpha$.

(Thus, being an $x, \mathbb{B}, G$-revision is also a much stronger statement if σ respects π at $\mathbb{B}$.)

Def Let σ be an $x, \mathbb{B}, G$ - revision of π,
The canonical completion $\tilde{\sigma} = \langle \tilde{\sigma}^i \mid i \leq \alpha \rangle$
of σ is defined by:

$\sigma^i : N^i[G^i] \prec N[G]$; $\sigma^i(G^i) = G$

for $i \leq \alpha$.

Then, setting $N^G = L_\tau^{A,G}$ where $N = L_\tau^A$,
we have: $\tilde{\sigma}$ is a tower for N^G.
Moreover $(N^G)^{\tilde{\sigma}^i} = L_{\tau_i}^{A^i, G^i}$, where
$N^i = L_{\tau_i}^{A^i}$.

This fact has many consequences for σ - e.g. $\sigma^{i+1}(\sup \sigma^{h,i+1}{}''\xi) = \sup \sigma^h{}''\xi$
for $\xi \in N^h$, $h \leq i < \alpha$.

Def Let $\mathbb{A}, \mathbb{B}$ be complete BA's in N
with $\mathbb{A} \subseteq \mathbb{B}$. Let σ be an $x, \mathbb{A}, A$ - revision of π and σ' an $x, \mathbb{B}, B$ - revision of π. σ' coheres with σ wrt $\mathbb{A}$
iff

- σ' is an $\langle x, \mathbb{A} \rangle, \mathbb{B}, B$ - revision of σ'
- $A = B \cap \mathbb{A}$
- $A^i = B^i \cap \mathbb{A}^i$ for $i \leq \alpha$, where $A^i = (\sigma^i)^{-1}{}''A$ and $B^i = (\sigma'^i)^{-1}{}''B$ (and $\mathbb{A}^i = (\sigma^i)^{-1}(\mathbb{A})$).

We are now ready to define the concept of α-subproperness:

<u>Def</u> Let $\mathbb{B}$ be a complete BA, Let $\alpha < \omega_1$. $\mathbb{B}$ is α-<u>subproper as witnessed by</u> the cardinal $\theta > \omega_1$ iff $\mathbb{B} \in H_\theta$ and the following holds:

Let $N = L_\tau^A$ be a ZFC$^-$ model, where $H_\theta \subset N$ and $\theta < \tau$. Let $\beta \leq \alpha$ and let π be a β-tower for N which absorbs $\theta, \mathbb{B}$. Let $u \subset N^\beta$ be finite. Let $\bar{b} \in \mathbb{B}^0 \setminus \{0\}$ Then there is $b \in \mathbb{B} \setminus \{0\}$ s.t. whenever $G \ni b$ is $\mathbb{B}$-generic, then there is $\sigma \in V[G]$ s.t. σ is a $\{\theta, \mathbb{B}\}, \mathbb{B}, G$-revision of π coinciding with π on u and s.t. $\bar{b} \in G^0$.

<u>Def</u> $\mathbb{B}$ is α-subproper iff it is α-subproper as verified by some θ.

<u>Def</u> $\mathbb{B}$ is ω_1-subproper iff it is α-subproper for all $\alpha < \omega_1$.

<u>Note</u> Clearly $\mathbb{B}$ is subproper iff it is 0-subproper.

Def θ verifies the α-subproperness of $\mathbb{B}$ iff every $\theta' \geq \theta$ witnesses the α-subproperness of $\mathbb{B}$.

Just as before, we can relativize the notion of α-subproperness to a fixed parameter p: We obtain the notion "$\mathbb{B}$ is α-subproper as witnessed by $\langle \theta, p \rangle$" by altering the above definition to require that $p \in H_\theta$, π absorbs p, and σ is a $\{p, \theta, \mathbb{B}\}$, $\mathbb{B}, G$-revision. It is again easily seen that this apparently weaker notion implies full α-subproperness — a fact that we shall often employ tacitly.

The two step iteration theorem reads:

<u>Thm 1</u> Let $\mathbb{A} \subseteq \mathbb{B}$ where $\mathbb{A}$ is α-subproper and $\Vdash_{\mathbb{A}} (\check{\mathbb{B}}/\dot{G}$ is α-subproper). Then $\mathbb{B}$ is α-subproper.

proof

Let θ be big enough that it verifies the α-subproperness of $\mathbb{A}$ and $\Vdash_{\mathbb{A}} (\check{\theta}$ verifies the $\check{\alpha}$-subproperness of $\check{\mathbb{B}}/\dot{G})$

Let $N = L_\tau^A$ be a ZFC^- model with $H_\theta \subset N$, $\theta < \tau$,

Let $\pi = \langle \pi_i \mid i \leq \alpha \rangle$ be an α-tower with $\pi^i : N^i \prec N$, $\pi^i(\theta^i, \mathbb{A}^i, \mathbb{B}^i) = \theta, \mathbb{A}, \mathbb{B}$ for $i \leq \alpha$.

Let $u \subset N^\alpha$ be finite. Let $e \in \mathbb{B}^0 \setminus \{0\}$.

Then there is $a \in \mathbb{A} \setminus \{0\}$ s.t. if $A \ni a$ is $\mathbb{A}$-generic, then there is $\sigma \in V[A]$ which is a $\{\theta, \mathbb{A}, \mathbb{B}\}, \mathbb{A}, A$ - revision of π coinciding with π on u and s.t.: $h_{\mathbb{A}^0}(e) \in A$. Let A, σ be given.

Let $\tilde{\sigma}$ be the canonical completion of σ, Set $\tilde{N} = L_\tau^{D, A}$, where $N = L_\tau^D$ and $\tilde{N}^i = L_{\tau^i}^{D^i, A^i}$ where $N^i = L_{\tau^i}^{D^i}$. Then $\tilde{\sigma}$ is a tower for $\tilde{N}$ with $\tilde{N}^i = \tilde{N}^{\tilde{\sigma}^i}$. Moreover, $\tilde{\sigma}$ absorbs $\theta, \mathbb{A}, \mathbb{B}, G$. Set: $\tilde{\mathbb{B}} = \mathbb{B}/A$. $\tilde{\mathbb{B}}$ is α-subproper in $V[A]$. Set:

$\tilde{e} = e/A$. Then $\tilde{e} \neq 0$ since $h_{A^0}(e) \in A^0$. Hence there is $\tilde{b} \in \tilde{\mathbb{B}} \setminus \{0\}$ which forces that, if $\tilde{B} \ni \tilde{b}$ is $\tilde{\mathbb{B}}$-generic, then there is $\sigma^* \in V[A][\tilde{B}]$ which is a $\{\theta, \mathbb{A}, \mathbb{B}, A\}, \tilde{\mathbb{B}}, \tilde{B}$ - revision of $\tilde{\sigma}$ coinciding on u and s.t. $e \in \tilde{B}^0$. Let $\tilde{B}, \sigma^*$ be given and work in $V[A][\tilde{B}]$. Then $B = A * \tilde{B} =_{Df}$ $=_{Df} \{b \in \mathbb{B} \mid b/A \in \tilde{B}\}$ is $\mathbb{B}$-generic over V and $V[A][\tilde{B}] = V[B]$. Define:

$\sigma' = \langle \sigma'^i \mid i \leq \alpha \rangle$, where $\sigma'^i = \sigma^{*i} \restriction N^i$.

<u>Claim</u> σ' is a $\{\theta, \mathbb{A}, \mathbb{B}\}, \mathbb{B}, B$ - revision of π which coincides with π on u, and s.t. $e \in B^0$.

proof. Clearly:

(a) $e \in B^0$ since $e/A^0 \in \tilde{B}^0$.

(b) $\sigma'^h : N^h \prec N$, $h \leq i \leq \alpha \rightarrow \mathrm{rng}(\sigma'^h) \subset \mathrm{rng}(\sigma'^i)$, $\mathrm{rng}(\sigma'^\lambda) = \bigcup_{i<\lambda} \mathrm{rng}(\sigma'^i)$ for limit $\lambda \leq \alpha$

(c) $B^h = \sigma^{h\,\prime\prime} B$ is $\mathbb{B}^h$-generic over N^h, since $B^h = A^h * \tilde{B}^h$.

(d) $\sigma' \in N[B]$, $\langle \sigma'^{h,i+1} \mid h \leq i \rangle \in N^{i+1}[B^{i+1}]$

(e) σ' coincides with σ (hence with π) on u,

(a)-(e) are immediate. It remains only to show:

(f) $C^N_\delta(\mathrm{rng}\,\sigma'^{\alpha}) = C^N_\delta(\mathrm{rng}\,\pi^\alpha)$ $(\delta = \delta(\mathbb{B}))$.

(g) $C^{N^{i+1}}_{\delta^{i+1}}(\mathrm{rng}\,\sigma'^{i,i+1}) = C^{N^{i+1}}_{\delta^{i+1}}(\mathrm{rng}\,\pi^{i,i+1})$

where $\delta^l = \delta(\mathbb{B}^l)$

We prove (f), the proof of (g) being virtually identical.

Since $\delta(\mathbb{B}) \geq \delta(\tilde{\mathbb{B}})$, we have:

(1) $C^{\tilde N}_\delta(\mathrm{rng}\,\tilde\sigma^\alpha) = C^{\tilde N}_\delta(\mathrm{rng}\,\sigma^{*\alpha})$

But:

(2) $N \cap C^{\tilde N}_\delta(\mathrm{rng}\,\sigma^{*\alpha}) = C^N_\delta(\mathrm{rng}\,\sigma'^\alpha)$

prf. $(\supset)$ is trivial. We show $(\subset)$.

Let $x \in N \cap C^{\tilde N}_\delta(\mathrm{rng}\,\sigma^{*\alpha})$. Then $x = \sigma^*(f)(\xi)$ for a $\xi < \delta$, where $f \in \tilde N^\alpha$ maps $\delta_\alpha = \delta(\mathbb{B}^\alpha)$ into N^α. Let $f = \mathring{f}^{A^\alpha}$, $\mathring{f} \in (\tilde N^\alpha)^{A^\alpha}$.

Then there is $a \in \mathbb{A}$ s.t. $a \Vdash_{\mathbb{A}} \check x = \sigma'(\mathring f)(\check\xi)$. But since $\delta \geq \delta(\mathbb{A})$, there is a dense set Δ in $\mathbb{A}$ s.t. $\bar{\bar\Delta} \leq \delta$. Hence there is such a $\Delta \in C = C^N_\delta(\mathrm{rng}\,\sigma'^\alpha)$ s.t. $\Delta \subset C$. We may assume $a \in \Delta$. Hence x is C-definable in $a, \sigma'(\mathring f), \xi$.

Hence $x \in C$.

Since $\sigma'^{\alpha} = \tilde{\sigma}^{\alpha} \upharpoonright N$, the same proof shows:

(3) $N \cap C^{\tilde{N}}_{\sigma}(\mathrm{rng}\,\tilde{\sigma}^{\alpha}) = C^{N}_{\sigma}(\mathrm{rng}\,\sigma^{\alpha})$.

Hence $C^{N}_{\sigma}(\mathrm{rng}\,\sigma'^{\alpha}) = C^{N}_{\sigma}(\mathrm{rng}\,\sigma^{\alpha}) = C^{N}_{\sigma}(\mathrm{rng}\,\pi^{\alpha})$.

QED (Thm 1)

The proof of Thm 1 contains much more information than we have stated. We can drop the assumption that $\mathbb{A}$ is α-subproper, merely assuming:

$\Vdash_{\mathbb{A}} (\check{\theta}$ verifies the $\check{\alpha}$-subproperness of $\check{\mathbb{B}}/\mathring{A})$,

$\mathring{A}$ being the canonical $\mathbb{A}$-generic name.

We assume that $a \in \mathbb{A} \setminus \{0\}$ forces the existence of a $\theta, \mathbb{A}, \mathring{A}$-revision of π (but not necessarily a $\langle \theta, \mathbb{A}, \mathbb{B} \rangle, \mathbb{A}, \mathring{A}$ revision). But then there is $\dot{\sigma} \in V^{\mathbb{A}}$ s.t. a forces $\dot{\sigma}^{A}$ to be a $\theta, \mathbb{A}, \mathring{A}$-revision of π, when $A \ni a$ is $\mathbb{A}$-generic. We can replace our fixed $u \subset N^{\alpha}$ by $\dot{u}^{A}$, where $a \Vdash_{\mathbb{A}} (\dot{u} \subset \check{N}^{\alpha}$ is finite). Similarly we can replace $e \in \mathbb{B}^{\circ} \setminus \{0\}$ by $\dot{e}^{A}$, where $a \Vdash_{\mathbb{A}} \dot{e} \in (\dot{\sigma}^{0})^{-1}(\check{\mathbb{B}})$ and $a \Vdash_{\mathbb{A}} h_{\check{\mathbb{A}}}(\dot{e}) \in \mathring{A}^{\circ}$ ($\mathring{A}^{\circ}$ being an abbreviation for $(\dot{\sigma}^{0})^{-1}$ "$\mathring{A}$.) We then let $\tilde{b}$ force the existence of σ^{*} s.t. which is a $\langle \theta, \mathbb{A}, \mathbb{B} \rangle, \tilde{\mathbb{B}}, \tilde{B}$-revision of $\tilde{\sigma}$, coinciding with $\tilde{\sigma}$ on $u = i^A$ and s.t. $\tilde{e} = e/A \in \tilde{\mathbb{B}}^\circ$, where $e = \dot{e}^A$. Since for every $A \ni a$ there is such a $\tilde{b}$, we may assume $\tilde{b} = \mathring{b}^A$, where a forces $\mathring{b}$ to have these properties. We may also assume w.l.o.g. that But then $\Vdash_{/A} \mathring{b} \in \check{\mathbb{B}}/\mathring{A}$ and $[\![\mathring{b} \neq 0]\!] = a$. Hence there is $b \in \mathbb{B}$ s.t. $\Vdash_{A} \check{b}/\mathring{A} = \mathring{b}$. Letting $h_{A}(b) = [\![\check{b}/\mathring{A} \neq 0]\!] = a$, $\sigma'\dot{\imath} =_{Df} \sigma^*\dot{\imath} \upharpoonright N'$ as before, it is forced by b that σ' is a $\langle \theta, A, \mathbb{B} \rangle, \mathbb{B}, B$ - revision of σ for generic $B \ni b$.

(We must replace (f) by: $C^N_\sigma(\text{rng}\,\sigma'\alpha) = C^N_\sigma(\text{rng}\,\sigma^\alpha)$; similarly for (g), since σ' will not necessarily be a $\langle \theta, A, \mathbb{B} \rangle$ revision of π.)

σ' then coincides with σ on $u = i^A$ and is s.t. $e = \dot{e}^A \in B^\circ$. Since $\sigma^*(A^i) = A$ for $i \leq \alpha$, it follows easily that σ' coheres with σ w.r.t. A.

Putting all of this together, we get:

<u>Lemma 2</u> Let $\mathbb{A} \subseteq \mathbb{B}$ be complete BA's. Let $\Vdash_{\mathbb{A}}$ ($\check{\theta}$ verifies the $\check{\alpha}$-subproperness of $\check{\mathbb{B}}/\mathring{A}$), where $\mathring{A}$ is the canonical $\mathbb{A}$-generic name.

Assume also:

Let $N = L_\tau^A$ be a ZFC^- model s.t. $H_\theta \subset N$ and $\theta < \tau$. Let π be a tower for N with $\pi : N' \prec N$, $\pi'(\theta') = \theta$. Let $a \in \mathbb{A} \setminus \{0\}$ force that, whenever $A \ni a$ is $\mathbb{A}$-generic, then $\dot{\sigma}^A$ is an $\langle x, \theta \rangle, \mathbb{A}, A$-revision of π. $\dot{u}^A \subset N^\alpha$ is finite $\dot{e}^A \in \mathbb{B}^0 = (\sigma^0)^{-1}(\mathbb{B})$ and $h_{\mathbb{A}}(\dot{e}^A) \in A$.

Then there is $b \in \mathbb{B} \setminus \{0\}$ s.t. $a = h_{\mathbb{A}}(b)$ and whenever $B \ni b$ is $\mathbb{B}$-generic, $A = B \wedge \mathbb{A}$, $\sigma = \dot{\sigma}^A$, $u = \dot{u}^A$, $e = \dot{e}^A$, then there is $\sigma' \in V[B]$ which is an $\langle x, \theta, \mathbb{A}, \mathbb{B} \rangle, \mathbb{B}, B$-revision of σ cohering with σ w.r.t. $\mathbb{A}$ and coinciding with σ on u. Moreover, $e \in B^0$.

<u>Note</u> To show that $\mathbb{B}$ does not collapse ω_1, we must assume that $\mathbb{A}$ does not do so. To show that $\mathbb{B}$ is α-subproper we must assume that $\mathbb{A}$ has the property.

Thm 3 Let $\mathbb{B} = \langle \mathbb{B}_\nu \mid \nu < \gamma \rangle$ be an RCS-iteration. Let $\alpha < \omega_1$. Assume that for all $i+1 < \alpha$:

(a) $\mathbb{B}_i \neq \mathbb{B}_{i+1}$

(b) $\Vdash_i (\check{\mathbb{B}}_{i+1} / \dot{G})$ is $\check{\alpha}$-subproper

(c) $\Vdash_{i+1} (\delta(\check{\mathbb{B}}_i)$ has cardinality $\leq \omega_1$).

Then every $\mathbb{B}_i$ is α-subproper

proof

Set $\delta_i = \delta(\mathbb{B}_i)$. As before we get:

(1) $\delta_i \leq \delta_j$ for $i \leq j < \gamma$

(2) $\bar{\bar{\nu}} \leq \delta_\nu$ for $\nu < \gamma$

By induction on $i < \gamma$ we prove:

Claim Let $h \leq i$, Let G_h be $\mathbb{B}_h$-generic. Then $\mathbb{B}_i / G_h$ is α-subproper in $V[G_h]$.

The cases $i = 0$, $i = h$, $i = j+1$ follow exactly as before in §1 Thm 3, using the two step theorem.

There remains the case that $i = \lambda$ is a limit ordinal. By our induction hypothesis $\mathbb{B}_j / G_h$ is α-subproper in $V[G_h]$ for $h \leq j < \lambda$. We consider two cases, as before:

Case 1 $cf(\lambda) \leq \delta_i$ for an $i < \lambda$.

It again suffices to prove the claim for $h \geq i$, since it will then hold for smaller ordinals by the two step thm. But then $cf(\lambda) \leq \omega_1$ in $V[G_n]$. We display the proof in the special case:

$$cf(\lambda) \leq \omega_1 \text{ in } V,$$

showing that $\mathbb{B}_\lambda$ is α-subproper in V, since we can then repeat the proof in $V[G_n]$ to show that $\mathbb{B}_\lambda / G_n$ is α-subproper. (The induction hypothesis holds in $V[G_n]$ just as before.)

Now let $N = L^A_\tau$ be a ZFC^- model s.t. $H_\theta \subset N$, $\theta < \tau$. We shall prove:

<u>Main Claim</u> Let $\beta \leq \alpha$. Let π be a β-tower for N which absorbs $\langle \theta, \mathbb{B}, \lambda \rangle$. Let $e \in \mathbb{B}^0 \setminus \{0\}$. Then let $u \subset N^\beta$ be finite. Let $e \in \mathbb{B}^0 \setminus \{0\}$. Then there is $b \in \mathbb{B}_\lambda \setminus \{0\}$ which forces that if $G \ni b$ is $\mathbb{B}_\lambda$-generic, then in $V[G]$ there is a $\langle \theta, \mathbb{B}, \lambda \rangle$, $\mathbb{B}_\lambda$, G-revision of π which which coincides with π on u and s.t. $e \in G^0$.

The proof will be by induction on β, but we shall need a stronger induction hypothesis. We split into two subcases.

<u>Case 1.1</u> $cf(\lambda) = \omega$

Let $f = \langle \xi_i \mid i<\omega \rangle$ be the N-least ω-sequence which is monotone and cofinal in λ and s.t. $\xi_0 = 0$. Set $f^h = \langle \xi_i^h \mid i<\omega \rangle = (\pi^h)^{-1}(f)$ for $h \leq \beta$.

Clearly, if σ is any $\langle \theta, \mathbb{B}, \lambda \rangle$-revision of π, then $\sigma^h(f^h) = f$ and $\sigma^h(\xi_i^h) = \xi_i$ for $i<\omega$.

We now refer back to the definitions of $\mathbb{B}_{<\lambda}$, $\mathbb{B}^+_{<\lambda}$ that we gave at the end of §1. Recall that $G \subset \mathbb{B}_{<\lambda}$ is called $\mathbb{B}_{<\lambda}$-<u>generic</u> iff $G \cap \mathbb{B}_\nu$ is $\mathbb{B}_\nu$-generic for all $\nu<\lambda$. G is then called $\mathbb{B}^+_{<\lambda}$-generic iff the set G^+ of $b \in \mathbb{B}^+_{<\lambda}$ s.t. $h_\nu(b) \in G$ for $\nu<\lambda$ meets every dense $\bigvee$-subset of $\mathbb{B}^+_{<\lambda}$. Setting $G^{++} = \{ b \in \mathbb{B}_\lambda \mid \bigvee a \in \mathbb{B}^+_\lambda \; a \subset b \}$, this is equivalent to saying that G^{++} is $\mathbb{B}_\lambda$-generic.

We now define:

Def Let $G \in \mathbb{B}_{<}$, where $\xi \leq \lambda$ is a limit ordinal. $\langle \theta, \mathbb{B}, \lambda \rangle, \mathbb{B}_{<\lambda}, G$ - revision of π iff

- σ is a $\langle \theta, \mathbb{B}, \lambda \rangle, \mathbb{B}_{\xi}$ - revision of π
- G^i is $(\mathbb{B}^i_{<\xi^i})^+$ - generic over N^i for $i \leq \beta$, where $G^i =_{Df} (\sigma^i)^{-1}$ "G
- $\sigma \in N[G]$ and $\langle \sigma^{h, i+1} \mid h \leq i \rangle \in N^{i+1}[G^{i+1}]$ for $i < \beta$.

Note It follows that, if σ respects π at $\mathbb{B}_{\xi}$, then $C^N(\operatorname{rng} \sigma^{\alpha}) = C^N_{\delta}(\operatorname{rng} \pi^{\alpha})$ and $C^{N^{i+1}}_{\delta^i}(\operatorname{rng} \sigma^{i, i+1}) = C^{N^{i+1}}_{\delta^i}(\operatorname{rng} \pi^{i, i+1})$, where $\delta = \delta(\mathbb{B}_{\xi})$ and $\delta^i = \delta(\mathbb{B}^i_{\xi^i})$.

Note that this definition makes no assumption about the genericity of G.

Clearly it suffices to show:

At $\beta \leq \alpha$ and π; u, e are as in the Main Claim, then there is $b \in (\mathbb{B}_{<\lambda})^+$ s.t. whenever G is $\mathbb{B}_{<\lambda}$ generic and $b \in G^+$, then there is $\sigma \in V[G]$ which is a $\langle \theta, \mathbb{B}, \lambda \rangle, \mathbb{B}_{<\lambda}, G$ - revision of π which coincides with π on u and is s.t. $e \in (G^0)^+$.

Def Let $G \subset \mathbb{B}_{<\lambda}$ be $\mathbb{B}_{<\lambda}$ – generic. Set: $G_i = G \cap \mathbb{B}_{\xi_i}$. By a good matrix for G w.r.t. π we mean a sequence $\langle \sigma_i \mid i<\omega \rangle$ s.t.

(a) $\sigma_i = \langle \sigma_i^h \mid h<\beta \rangle$ is a $\langle\langle \theta, \mathbb{B}, \lambda \rangle, \mathbb{B}_{\xi_i}, G \rangle$ – – revision of π which coheres with σ_l w.r.t. $\mathbb{B}_{\xi_l}$ for $l \leq i$

(b) $G^i = \bigcup_{l<\omega} G_l^i$ is $\mathbb{B}^+_{<\lambda^l}$ – generic over N^i for $i \leq \beta$.

(c) If $x \in N^i$, there is $j<\omega$ s.t. $\sigma_l^i(x) = \sigma_j^i(x)$ for $l \geq j$.

(d) If $x \in N^\lambda$, x being a limit ordinal, then there are $h<\lambda$, $j<\omega$, $\bar{x} \in N^h$ s.t. $x = \sigma_l^{h\lambda}(\bar{x})$ for all $l \geq j$

(e) If $x \in N^h$, there are $j<\omega$, $w \in N^h$ s.t. $\bar{\bar{w}} \leq \delta^h = \sup_i \sigma(\mathbb{B}^h_{\xi_i h})$ in N^h and:

- $\pi^{h,h+1}(x) \in \sigma_j^{h,h+1}(w) = \sigma_l^{h,h+1}(w)$ for $l \geq j$ if $h<\beta$
- $\pi^\beta(x) \in \sigma_j^\beta(w) = \sigma_l^\beta(w)$ for $l \geq j$ if $h=\beta$

(f) $\langle \sigma_i \mid i<\omega \rangle \in N[G]$ and $\langle \sigma_i^{h,j+1} \mid h \leq j, i<\omega \rangle \in N^{j+1}[G^{j+1}]$ for $j<\beta$.

Note By (c), if $i \leq h \leq \beta$, there is $j<\omega$ s.t. $\sigma_l^{ih} = \sigma_j^{ih}$ for all $l \geq j$.

We can then define: $\sigma = \sigma_\omega$ by:

$\sigma^h(x) = \sigma_j^h(x)$ if $\sigma_l^h(x) = \sigma_j^h(x)$ for all $l \geq j$.

Clearly $\sigma^h : N^h \prec N$. If we set $\sigma^{hi} = (\sigma^i)^{-1} \cdot \sigma^h$ for $h \leq i \leq \beta$, we get:

$\sigma^{hi}(x) = \sigma_i^{hi}(x)$ if $\sigma_l^{hi}(x) = \sigma_i^{hi}(x)$ for $l \geq i$.

<u>Sublemma 3.1</u> σ is a $\langle \theta, \mathbb{B}, \lambda \rangle, \mathbb{B}_{<\lambda}, G$ - revision of π which coheres with σ_i wrt $\mathbb{B}_{\xi_i}$ for $i < \omega$.

proof.

(1) σ is a pretower.

We must show: $rng(\sigma^\eta) = \bigcup_{i<\eta} rng(\sigma^i)$ for limit $\eta \leq \beta$. This follows from (d) which says that each $x \in N^\eta$ has the form $\sigma^{i\eta}(\bar{x})$ for an $x \in N^i$, $i < \eta$. QED(1)

(2) $C_\delta^N(rng\, \sigma^\beta) = C_\delta^N(rng\, \pi^\beta)$, where $\delta = \sup_{i<\lambda} \delta(\mathbb{B}_{\xi_i}) \leq \delta(\mathbb{B}_\lambda)$.

proof.

$(\subset)$ $\sigma^\beta(x) = \sigma_i^\beta(x) \in C_\delta^W(rng\, \sigma_i^\beta) = C_\delta^N(rng\, \pi^\beta)$

$(\supset)$ Let $x \in C_\delta^N(rng\, \pi^\beta)$. Then $x = \pi(f)(\xi)$ where $\xi < \delta$, $f \in N^\beta$. But $\pi(f) \in \sigma(w)$ for a $w \in N^\beta$, $\bar{\bar{w}} \leq \delta$.

Hence $\pi(f) = \sigma(g)(\zeta)$ where $\zeta < \delta$. H

Hence $x = (\sigma(g)(\zeta))(\xi) \in C_\delta^N(rng\, \sigma)$,

QED (2)

Similarly:

(3) $C^{N^{i+1}}_{\delta^{i+1}}(\mathrm{rng}\,\sigma^{i,i+1}) = C^{N^{i+1}}_{\delta^{i+1}}(\mathrm{rng}\,\pi^{i,i+1})$

for $i<\beta$, where $\delta^i = \sup\limits_{h<\omega} \sigma(\mathbb{B}^i_{\xi^i_n})$.

By (f) we trivially have:

(4) $\sigma \in N[G]$ and $\langle \sigma^{i,h+1} \mid i \leq h\rangle \in N^{h+1}[G^{h+1}]$

for $h<\beta$

Finally:

(5) σ coheres with σ_i wrt $\mathbb{B}_{\xi_i}$ for $i<\omega$,

proof.

We must show: $\sigma^h{}''G^h_i \subset G_i$ for $h \leq \beta$, $i<\omega$.

Let $b \in G^h_i$. Then $\sigma^h(b) = \sigma^h_j(b) \in$

$G_j \cap \mathbb{B}_{\xi_i} = G_i$ for some $j \geq i$.

QED (Sublemma 3.1)

We note that σ also has the properties:

- Let $u \subset N^\beta$ be finite s.t. each σ_i coincides with π on u. Then σ coincides with π on u.
- Let $b \in (\mathbb{B}^i_{<\lambda^i})^+$ s.t. $h_{\xi_h}(b) \in G^i_h$ for $h<\omega$. Then $b \in G^{i+}$.

Thus it suffices to show for $\beta \leq \alpha$:

(*) Let π be a β-tower for N which absorbs $\Theta, \mathbb{B}, \lambda$. Let $u \subset N^\beta$ be finite.
Let $e \in (\mathbb{B}^0_{<\lambda^0})^+ \setminus \{0\}$. Then there is $b \in \mathbb{B}^+_{<\lambda}$ and a sequence $\langle \dot\sigma_i \mid i<\omega \rangle \in N$ s.t. $\dot\sigma_i \in N^{\mathbb{B}_{\xi_i}}$ for $i<\omega$ and whenever G is $\mathbb{B}_{<\lambda}$-generic, $b \in G^+$, $G_i = G \cap \mathbb{B}_{\xi_i}$ $(i<\omega)$ and $\sigma_i = \dot\sigma_i^{G_i}$ for $i<\omega$, then $\langle \sigma_i \mid i<\omega \rangle$ is a good matrix which coincides with π on u and is s.t. $e \in (G^0)^+$, where:
$G_i^h = (\sigma_i^h)^{-1} {}'' G_i$, $G^h = \bigcup_i G_i^h$.

We prove this by induction on β. However, as induction hypothesis we need the even stronger statement:

(**) Let $i<\omega$ and let G_i be $\mathbb{B}_{\xi_i}$-generic.
Then (*) holds in $V[G_i]$ with $\mathbb{B}/G_i$ in place of $\mathbb{B}$ and $N^{G_i} = L^{A, G_i}_\tau$ in place of N (where $N = L^A_\tau$) and $\langle \xi_j \mid i \leq j < \omega \rangle$ in place of $\langle \xi_j \mid j<\omega \rangle$.
(Recall that $\mathbb{B}/G_i = \langle \mathbb{B}_\nu / G_i \mid \nu \geq i \rangle$.)

It will suffice at each stage of the induction to display the proof of $(*)$, since the same proof can then be repeated in $V[G_i]$.

<u>Case 1.1.1</u> $\beta = 0$. The construction of b, $\langle \dot{\sigma}_i \mid i<\omega\rangle$ and the verification that $\langle \sigma_i \mid i<\omega\rangle$ is a good matrix for G whenever G is $\mathbb{B}_{<\lambda}$ - generic, $b \in G^+$, and $\sigma_i = \dot{\sigma}_i^{G_i}$ $(i<\omega)$ is given in Case 1 of the proof of §1 Lemma 3. To see that we can have $\langle \dot{\sigma}_i \mid i<\omega\rangle \in N$, note that $S \in N$, where $S = \{\langle b, i, y, x\rangle \mid b \Vdash_{\xi_i} \dot{\sigma}(\check{x}) = \check{y}\}$.
To see this note that $S \subset \mathbb{B}_\lambda \times C$, where $C = C_\delta^N(\mathrm{rng}\,\pi)$, $\delta = \sup_i \delta(\mathbb{B}_{\xi_i})$.
Let $\tilde{\tau} = \sup \pi^\beta{}''\mathrm{On}_{N^\beta}$, $\tilde{N} = L^A_{\tilde{\tau}}$, where $N = L^A_\tau$.
Then $\tilde{N} \prec N$ and $\pi^\beta : N^\beta \prec \tilde{N}$ cofinally.
But then $C \subset \tilde{N}$; hence $C = C_\delta^{\tilde{N}}(\mathrm{rng}\,\pi) \in N$ and $\overline{\overline{C}} = \delta < \theta$ in N. Let $f \in N$ map a $\gamma < \theta$ onto $\mathbb{B}_\lambda \times C$. Let $\bar{S} = f^{-1}{}''S$. Then $\bar{S} \in \mathcal{P}(\gamma) \subset H_\theta \subset N$. Hence $S = f''\bar{S} \in N$

QED (Case 1.1.1)

Case 1.1.2 $\beta = \nu + 1$

Then (*) holds at ν. We use:

Fact The statement "(*) holds at ν" is uniformly expressible over N in parameters from $\text{rng}(\pi^\beta)$.

proof.

(*) says that if $u \subset N^\nu$ is finite and $e \in \mathbb{B}^{\circ +}_{<\lambda^\circ} \setminus \{0\}$, then there are $b \in \mathbb{B}^+_{<\lambda} \setminus \{0\}$, $\langle \dot\sigma_i \mid i<\omega \rangle \in N$ s.t. if G is $\mathbb{B}_{<\lambda}$ - generic and $\sigma_i = \dot\sigma_i^{G_i}$ $(i<\omega)$, then $\langle \sigma_i \mid i<\omega \rangle$ is a good matrix with certain properties. (The quantification over the nonexistent G can be replaced by the statement that the above holds in $N^{\text{coll}(\omega, \bar{\bar{\mathbb{B}}}_\lambda)}$.) Hence it suffices to show that, if G is $\mathbb{B}_{<\lambda}$ - generic and $\langle \sigma_i \mid i<\omega \rangle \in N[G]$, then the statement:

"$\langle \sigma_i \mid i<\omega \rangle$ is a good matrix for G"

is uniformly expressible over $N[G]$ in parameters from $\text{rng}\,\pi^\beta$;

...

This at first glance seems dubious, since the statement involves clauses of the form:

$\pi^h : N^h \prec N$ $(h \leq \nu)$ and

$C^N_\sigma(\text{rng}\,\sigma^\nu) = \text{rng}\, C^N_\sigma(\text{rng}\,\pi^\nu)$.

However $\pi^\beta(\langle \pi^h \upharpoonright \beta \mid h \leq \nu \rangle) = \pi \upharpoonright \beta$ and, letting $\tilde{\tau} = \sup \pi^\nu{}'' On_{N^\nu}$, we have: $\pi^\beta(\tilde{\tau}') = \tilde{\tau}$, where $\tilde{\tau}' = \sup \pi^{\nu\beta}{}'' On_{N^\nu}$.

Ab $\tilde{N} = L^A_{\tilde{\tau}}$ (where $N = L^A_\tau$), then $\tilde{N} \in \text{rng}(\pi^\beta)$, $\text{rng}(\pi^\nu) \subset \tilde{N}$, and $\tilde{N} \prec N$. Thus we can replace the questionable clauses by:

$\pi^h : N^h \prec \tilde{N}$ $(h \leq \nu)$ and

$C^{\tilde{N}}_\sigma(\text{rng}\,\sigma^\nu) = \text{rng}\, C^{\tilde{N}}_\sigma(\text{rng}\,\pi^\nu)$.

QED (Fact)

But since $\sigma^\beta : N^\beta \prec N$, the corresponding statement holds over N^β. Thus, letting $\bar{u} = (\sigma^{\nu\beta})^{-1}{}''u$ (where $u \subset N^\beta$ is finite), there are $\bar{b} \in (\mathbb{B}^\beta_{<\lambda^\beta})^+$, $\langle \dot{\bar{\sigma}}_i \mid i<\omega \rangle \in N^\beta$ s.t. if $\bar{G}$ is $\mathbb{B}^\beta_{<\lambda^\beta}$-generic, $\bar{b} \in \bar{G}^+$, and $\bar{\sigma}_i = \dot{\bar{\sigma}}_i^{\bar{G}}$ $(i<\omega)$, then $\langle \bar{\sigma}_i \mid i<\omega \rangle$ is a good matrix for $\bar{G}$ wrt. $\bar{\pi} = \langle \pi^{h\beta} \mid h \leq \nu \rangle$ which coincides with π on $\bar{u}$ and is s.t. $e \in (\bar{G}^0)^+$.

Set $\bar{b}_i = h_{\xi_i\beta}(\bar{b})$. Note that if $\bar{G}_i \ni \bar{b}_i$ is any $\mathbb{B}^{\beta}_{\xi_i\beta}$ - generic set, then $\dot{\bar{\sigma}}_i^{\bar{G}_i}$ is a revision of $\bar{\pi}$. By Case 1.1.1, however, there are $b \in (\mathbb{B}_{<\omega})^+ \setminus \{0\}$ and $\langle \dot{\sigma}_i' \mid i<\omega \rangle \in N$ s.t. $\sigma_i' \in N^{\mathbb{B}_{\xi_i}}$ and whenever G is $\mathbb{B}_{<\lambda}$ - generic and $\sigma_i' = \dot{\sigma}_i'^G$ $(i<\omega)$, then $\langle \sigma_i' \mid i<\omega \rangle$ is a good matrix for G wrt. the tower $\pi \upharpoonright \{\beta\}$ ~~of~~ length 0 s.t. σ_i' coincides with $\pi \upharpoonright \{\beta\}$ on u and $e_i \in G_i^\beta = \sigma_i'{}''G_i$ for $i<\omega$, (where $e_i = h_{\xi_i^\beta}(e)$), just as above, letting $b_i = h_{\xi_i}(b)$, we have:

$b_i \Vdash_{\xi_i} (\dot{\sigma}_i'$ is a revision of $\pi \upharpoonright \{\beta\}$).

But then there is obviously a term $\dot{\sigma}_i \in N^{\mathbb{B}_{\xi_i}}$ s.t. if $G_i \ni b_i$ is $\mathbb{B}_{\xi_i}$ - generic, then $\sigma_i = \dot{\sigma}_i^{G_i}$ is the revision of π defined by: $\sigma_i^\beta = \sigma_i'^\beta$ and $\sigma_i^h = \sigma_i^\beta \circ \bar{\sigma}_i^h$ for $h<\beta$. Then $\langle \dot{\sigma}_i \mid i<\omega \rangle$ has the desired properties. QED (Case 1.1.2)

Case 1.1.3 β is a limit ordinal

Fix a sequence $\langle \beta_i \mid i<\omega \rangle$ which is monotone and cofinal in β with $\beta_0 = 0$ and β_i a successor ordinal for $i>0$. We also write: $\tilde{\beta}_i = (\beta_{i+1} - 1)$. Set $\langle x_i^h \mid i<\omega \rangle$ = the N-least enumeration of of N^h for $h \leq \beta$. Then $\langle x_i^h \mid h \leq j \rangle \in N^{j+1}$ for $j < \beta$. In order to simplify our notation we also write: $\hat{\mathbb{B}}_i^h = \mathbb{B}_{\xi_i}^h$.

We must produce a good matrix $\langle \sigma_i \mid i<\omega \rangle$. To do this we essentially define σ_i by induction on i. To make sure this works, however, we also anticipate the matrix "from below", simultaneously constructing $\mu(i)$ s.t. $\mu(i)$ is a good matrix for $N^{\beta_{i+1}}$ with $\mu(i)_l^h = \sigma_l^{h,\beta_{i+1}}$ for $l \leq i$.

Of course, we are working in V, and will not directly construct $\langle \sigma_i \mid i<\omega \rangle$ but rather an $a \in \mathbb{B}^+_{<\lambda}$ and $\langle \dot{\sigma}_i \mid i<\omega \rangle$ s.t. $\dot{\sigma}_i \in N^{\mathbb{B}_{\xi_i}}$ and whenever G is $\mathbb{B}_{<\lambda}$-generic and $a \in G^+$, then, letting $\dot{\sigma}_i^{G_i} = \sigma_i$, $\langle \sigma_i \mid i<\omega \rangle$ will be the desired good matrix. Thus, we inductively construct $a_i = h_{\xi_i}(a)$ and $\dot{\sigma}_i$.

We construct $a_i, \dot{\sigma}_i, \dot{u}_i$ s.t.

(I) (a) $a_i \in \mathbb{B}_{\xi_i}$

(b) $\dot{\sigma}_i, \dot{u}_i \in N^{\mathbb{B}_{\xi_i}}$

(c) $a_l = h_{\xi_l}(a_i)$ for $l \le i$.

(II) Let $G \ni a_i$ be $\mathbb{B}_{\xi_i}$-generic. Set:

$G_l = G \cap \mathbb{B}_{\xi_l}$, $\sigma_l = \dot{\sigma}_l^{G_l}$, $u_l = \dot{u}_l^{G_l}$ ($l \le i$).

Then:

(a) σ_i is a $\langle\langle \theta, \mathbb{B}, \lambda\rangle, \mathbb{B}_{\xi_i}, G_i\rangle$ revision of π. Moreover $\sigma_0 = \pi$

(b) σ_i coheres with σ_l wrt. $\mathbb{B}_{\xi_l}$ for $l \le i$

(c) $u_i \subset N^\beta$ is finite

(d) $\sigma_i \upharpoonright [\beta_l, \beta]$ coincides with $\sigma_l \upharpoonright [\beta_l, \beta]$ on u_l for $l \le i$

(e) $u \cup \bigcup_{l<i} u_l \subset u_i$.

(f) $x_h^\beta, w_h \in u_i$ for $h < i$, where w = the N^β-least w s.t. $\bar{\bar{w}} \le \delta^\beta$ in N^β and $\pi(x^\beta_i) \in \sigma_i^\beta(w)$ (where $\delta^\beta = \sup_{h<\omega} \delta(\mathbb{B}^\beta_{\xi^\beta_h})$).

(g) $\sigma_i^{\beta_h}(x_i^{\beta_h}) \in u_i$ for $h < i$.

Simultaneously we construct $\dot{b}^i$, $\ddot{\mu}(i) \in N^{\mathbb{B}_{\xi_i}}$ s.t.

<u>III</u> Let G be as in <u>II</u>. Set: $b^i = (\dot{b}^i)^G$ and $\dot{\mu}(i) = \ddot{\mu}(i)^G$. Then:

(a) $b^i \in (\mathbb{B}^{\beta_{i+1}}_{<\lambda^{\beta_{i+1}}})^+$. Set $b^i_\ell = h_{\xi_\ell \beta_{i+1}}(b^i)$.

(b) $\dot{\mu}(i) = \langle \dot{\mu}(i)_\ell \mid \ell < \omega \rangle$, where

$$\dot{\mu}(i)_\ell \in (N^{\beta_{i+1}})^{\hat{\mathbb{B}}^{\beta_{i+1}}_\ell} \text{ for } \ell < \omega$$

(c) $b^i_i = 1$; $b^k_i \in G_i^{\beta_{k+1}}$ for $k < i$

<u>IV</u> Let $G, b^i, \dot{\mu}(i)$ be as in <u>III</u>. Let $H \supset G_i^{\beta_{i+1}}$ be $\mathbb{B}^{\beta_{i+1}}_{\lambda^{\beta_{i+1}}}$ - generic over $N^{\beta_{i+1}}$ s.t. $b^i \in H^+$.

Set: $H_\ell = H \cap \hat{\mathbb{B}}^{\beta_{i+1}}_\ell$ for $\ell < \omega$. (Hence $H_\ell = G_\ell^{\beta_{i+1}}$ for $\ell \leq i$.) Set:

$$\mu(i)_\ell = (\dot{\mu}(i)_\ell)^{H_\ell} \quad (\ell < \omega). \text{ Then:}$$

(a) $\mu(i) = \langle \mu(i)_\ell \mid \ell < \omega \rangle$ is a good matrix for $N^{\beta_{i+1}}$ w.r.t. H

(b) $\mu(i)_\ell = \langle \sigma_\ell^{h,\beta_{i+1}} \mid h \leq \tilde{\beta}_i \rangle$ for $\ell \leq i$.

(c) $\mu(i)_\ell \upharpoonright [\beta_h, \tilde{\beta}_i]$ coincides with $\mu(i)_h \upharpoonright [\beta_h, \tilde{\beta}_i]$ on $(\sigma^{\tilde{\beta}_h})^{-1}$ " u_h for $h \leq i$, $h \leq \ell < \omega$.

We of course set: $H^{j}_{l} = {}^{-}(\mu(i)^{j}_{l})^{-1}\text{''}\underline{H}_{l}$ for $l \leq \omega$, $j \leq \tilde{\beta}_{i}$. But then $H^{j} = \bigcup_{l} H^{j}_{l}$ in $(\mathbb{B}^{d^{j}}_{<\lambda^{j}})^{+}$ - generic over N^{j} for $j \leq \tilde{\beta}_{i}$. Hence, for $h < i$, we can form $b^{h} = (b^{\cdot h})^{H^{\beta_{h+1}}}$, $\mu(h) = \ddot{\mu}(h)^{H^{\beta_{h+1}}}$.

We shall ensure that:

(d) $b^{h} \in (H^{\beta_{h+1}})^{+}$ for $h < i$; Moreover $e \in (H^{0})^{+}$.

But then $\mu(h)_{l} = \dot{\mu}(h)_{l}^{H^{\beta_{h+1}}}$ is defined for $l < \omega$ and satisfies the above conditions.

We ensure:

(e) $\mu(h)^{j}_{l} = \mu(i)^{j,\beta_{h+1}}_{l}$ for $j \leq \tilde{\beta}_{h}$, $l < \omega$.

In this context it is useful to write: $\mu(i)^{j,\beta_{i+1}}_{l} =_{\mathrm{Df}} \mu(i)^{j}_{l}$ for $j \leq \tilde{\beta}_{i}$, $l < \omega$,

With this convention we have:

$\mu(h)^{j,k}_{l} = \mu(i)^{j,k}_{l}$ for $l < \omega$, $j \leq k \leq \beta_{h+1}$ for $h \leq i$, and we can, without confusion, write $\mu^{i,k}_{l}$.

We note that $\underline{\text{IV}}$ (a) – (e) hold "locally" – i.e. from $\underline{\text{IV}}$ we can derive:

$\underline{\text{V}}$ Let G, b^i, $\mu(i)$ be as in $\underline{\text{III}}$. Let $j \geq i$ and let $H \ni b^i_j$ be $\hat{\mathbb{B}}^{\beta_i+1}_j$ – generic over N^{β_i+1} s.t. $H \supset G^{\beta_i+1}_i$. Set: $H_l = H \cap \hat{\mathbb{B}}^{\beta_i+1}_l$ for $l \leq j$

Set: $\mu(i)_l = (\mu(i)_l)^{H_l}$ $(l \leq j)$. Then

(a) $\mu(i)_j$ is a $\langle\langle \theta^{\beta_i+1}, \mathbb{B}^{\beta_i+1}, \lambda^{\beta_i+1}\rangle, \hat{\mathbb{B}}^{\beta_i+1}_j, H\rangle$ – – revision of π cohering with $\mu(i)_l$ w.r.t. $\hat{\mathbb{B}}^{\beta_i+1}_l$ for $l \leq j$

(b) $\mu(i)_l = \langle \sigma^{h,\beta_i+1}_l \mid h \leq \tilde{\beta}_i \rangle$ for $l \leq i$

(c) $\mu(i)_j \upharpoonright [\beta_h, \tilde{\beta}_i]$ coincides with $\mu(i)_h \upharpoonright [\beta_h, \tilde{\beta}_i]$ on $(\sigma^{\tilde{\beta}_h})^{-1}$ " u_h for $h \leq i$

(d) $b^h_j \in H^{\beta_h+1}_j$ for $h < i$

(e) $\mu(h)^k_l = \mu(i)^{k,\beta_h+1}_l$ for $k \leq \tilde{\beta}_h$, $l \leq j$.

This is because H can be extended to a $\mathbb{B}^{\beta_i+1}_{<\lambda^{\beta_i+1}}$ – generic H' s.t. $b^i \in H'^+$, since $b^i_j = h_{\xi_j^{\beta_i+1}}(b^i) \in H$.

We are now ready to prove (*). Let $a = \bigcap_i a_i$. Let G be $\mathbb{B}_{<\lambda}$ - generic, where $a \in G^+$. Set: $G_i = G \cap \mathbb{B}_{\xi_i}$; $\sigma_i = \dot{\sigma}_i^{G_i}$.

<u>Claim</u>: $\langle \sigma_i \mid i<\omega \rangle$ is a good matrix.

Set: $G_i^h = (\sigma_i^h)^{-1} {}'' G_i$. Then $G_i^h \subset G_{i'}^h$ for $i \leq i'$, by coherence. Set $G^h = \bigcup_{i<\omega} G_i^h$. Then

(1) G^h is $\mathbb{B}^h_{<\lambda^h}$ - generic over N^h.

(2) $b^i \in (G^{\beta_{i+1}})^+$ for $i<\omega$, where $b^i = (\dot{b}^i)^{G_i}$

proof
$b^i_h \in G_h^{\beta_{i+1}}$ for $i<h$ by III (c). But
$b^i = \bigcap_{i<h} b^i_h$.
QED (2)

Thus, letting $\mu(i) = \dot{\mu}(i)^{G_i}$, we have:

(3) $\mu(i) = \dot{\mu}(i)^{G^{\beta_{i+1}}}$ exists and satisfies IV (a) - (e) with $H = G^{\beta_{i+1}}$.

Thus we can write $\mu_l^{hj} = \mu(i)_l^{hj}$ for $l<\omega$, $h \leq j \leq \beta_{i+1}$, the choice of $i<\omega$ being irrelevant. By IV (b):

(4) $\mu_l^{hj} = \sigma_l^{hj}$ for $l<\omega$, $h \leq j < \beta$.

We now verify (a) - (f) in the definition of good matrix:

(a) is immediate

We prove (b): (b) is proven for $i < \beta$, so let $i = \beta$. Let $\Delta \in N^\beta$ be strongly dense in $\mathbb{B}^\beta_{\lambda^\beta}$. We must find $c \in (G^\beta)^+$ s.t. $c \in \Delta$. Let $\Delta \in U_{i_0}$. Let $j > i_0$ s.t. $\sigma_j^{\beta_j, \beta}(\bar{\Delta}) = \Delta$. Then $\sigma_\ell^{\beta_j, \beta}(\bar{\Delta}) = \Delta$ for all $\ell \geq j$. $\bar{\Delta}$ is strongly dense in $\mathbb{B}^{\beta_j}_{\lambda^{\beta_j}}$. Hence there is $\bar{c} \in (G^{\beta_j})^+$ s.t. $\bar{c} \in \bar{\Delta}$. Let $\bar{c} = x_k^{\beta_j}$. Assume w.l.o.g. that $j > k$. Let $c = \sigma_j^{\beta_j \beta}(\bar{c})$. Then $c = \sigma_\ell^{\beta_j \beta}(\bar{c})$ for all $\ell \geq j$.

Hence, since $h_{\xi_\ell^{\beta_j}}(\bar{c}) \in G_\ell^{\beta_j}$,

we have $\sigma_\ell^{\beta_j \beta}(h_{\xi_\ell^{\beta_j}}(\bar{c})) = h_{\xi_\ell^\beta}(c) \in$
$\in G_\ell^\beta$. Hence $c = \bigcap_{\ell \geq j} h_{\xi_\ell^\beta}(c) \in$
$\in (G^\beta)^+$. But $c = \sigma_j^{\beta_j, \beta}(\bar{c}) \in$
$\in \sigma_j^{\beta_j \beta}(\bar{\Delta}) = \Delta$. QED (b).

We now prove (c).

Let $i \leq \beta_n$ and let $j_0 < \omega$ s.t.
$\sigma_\ell^{i \beta_n}(x) = \sigma_{j_0}^{i \beta_0}(x)$ for $\ell \geq j$.

(j_0 exists because $\sigma_l^{i, \beta_n} = \mu_l^{i, \beta_n}$ and $\mu(h)$ is a good matrix for N^{β_h+1}.)

Let $\sigma_{j_0}^{i, \beta_n}(x) = x_k^{\beta_n}$. Let $j_1 > j_0, k$.

For $l \geq j_1$ we have:

$\sigma_l^{i}(x) = \sigma_l^{\beta_n}(\sigma_l^{i, \beta_n}(x)) =$

$= \sigma_{j_1}^{\beta_n}(\sigma_{j_1}^{i, \beta_n}(x)) = \sigma_{j_1}^{i}(x)$. QED (c)

(d) holds for $\lambda < \beta$, since $\mu(i)$ is a good matrix, where $\beta_i > \lambda$. (d) holds at β by II (d).

(e) holds at $h < \beta$ because $\mu(i)$ is a good matrix, where $h < \beta_i$. (e) holds at β by II (f).

We prove (f). We can assume w.l.o.g. that: $\langle \sigma_i \mid i < \omega \rangle \in N$.

This follows by the fact that:

$S = \{\langle b, i, y, x \rangle \mid i < \omega \wedge b \in \mathbb{B}_{\xi_i} \wedge b \Vdash \dot\sigma(\check x) = \check y\} \in$

$\in N$.

(To see this, note that $S \subset \mathbb{B}_\lambda \times C$, where $C = C_\delta^N(\operatorname{rng} \pi)$, where $\delta = \sup_i \delta(\mathbb{B}_{\xi_i})$. But, just as in the proof of Case 1.1.2, we have $C \in N$, $\bar{\bar{C}} < \theta$ in N. Thus, setting $\bar{S} = f^{-1}{}''S$, where $f \in N$ maps a $\delta < \theta$ onto $\mathbb{B}_\lambda \times C$, we have $\bar{S} \subset \mathcal{P}(\delta) \subset H_\theta \subset N$ + hence $S = f''\bar{S} \in N$,)

But then $\langle \sigma_i \mid i<\omega \rangle = \langle \dot{\sigma}_i^{G_i} \mid i<\omega \rangle \in$ $\in N[G]$.

For $j<\beta$, $\langle \sigma_i^{h,j+1} \mid h \le j, i<\omega \rangle \in N^{j+1}$ because $\mu(k)$ is a good matrix, taking $j \le \beta_k$. QED (Claim)

It remains only to note:

- σ_i coincides with π on u by (d), since $\sigma_0 = \pi$ and $u \subset u_0$
- $e \in (G^0)^+$ by IV (d)

This completes the proof of (*).

All that remains is to define a_i, σ_i, u_i, b^i and $\mu(i)$ and verify I - IV.

We proceed by induction on i:

Case 1 $i=0$. Set $a_0 = 1$, $\dot{\sigma}_0 = \check{\pi}$. By the induction hypothesis there are b, $\langle \dot{\mu}_i \mid i<\omega\rangle$ satisfying (*) at $\tilde{\beta}_0$. But, just as in the proof of Case 1.1.2, this fact is expressible over N in parameters from $\mathrm{rng}\,\pi^{\beta_1}$. Since $\pi^{\beta_1} : N^{\beta_1} \prec N$, the corresponding statements hold in N^{β_1}. This gives us $b^0 \in (\mathbb{B}_{<\lambda^0})^+$, $\dot{\mu}(0) = \langle \dot{\mu}(0)_i \mid i<\omega\rangle \in N^{\beta_1}$ satisfying (*) for N^{β_1} wrt $\langle \pi^{h\beta_1} \mid h \le \tilde{\beta}_0\rangle$.
Set: $\dot{b}^0 = \check{b}^0$, $\ddot{\mu}(0) = \check{\dot{\mu}}(0)$.
The verifications are straightforward.

Case 2 $i = k+1$. We find construct a_i, σ_i.
By Lemma 2, there are a, $\dot{\sigma}$ s.t. $a \in \mathbb{B}_{\xi_i}$ and $a_k = h_{\xi_k}(a)$, $\dot{\sigma} \in V^{\mathbb{B}_{\xi_i}}$ s.t. whenever $G \ni a$ is $\mathbb{B}_{\xi_i}$-generic, then $\sigma = \dot{\sigma}^G$ is a $\langle\langle \theta, \mathbb{B}, \lambda\rangle, \mathbb{B}_{\xi_i}, G\rangle$-revision of $\pi \restriction [\beta_i, \beta]$ coinciding with $\dot{\sigma}_k^{G_{k+1}} \restriction [\beta_i, \beta]$ on $u_k = \dot{u}_k^{G_k}$, and s.t. $b_i^k \in G^{\beta_i}$. (Here $b_i^k = \dot{b}^{G_k}$, where $a_k \Vdash_{\xi_k} \dot{b} = h_{\mathbb{B}_i^{\beta_i}}(\dot{b}^k)$. Thus $h_{\xi_k \beta_i}(b_i^k) = b_k^k = 1$.) Set: $a_i = a$.

Since $b_i^k \in G^{\beta_i}$ and $G_k^{\beta_i} \subset G^{\beta_i}$, we can form $\mu(k)_i = (\dot{\mu}(k)_i)^{G^{\beta_i}}$, where $\dot{\mu}(k) = \ddot{\mu}(k)^{G_k}$. Set:

$$\sigma_i^h = \begin{cases} \sigma^h & \text{if } \beta_i \leq h \leq \beta \\ \sigma^{\beta_i} \circ \mu(k)_i^h & \text{if } h < \beta_i \end{cases}$$

Let $\dot{\sigma}_i$ be the $\dot{\sigma}_i \in V^{\mathbb{B}_{\xi_i}}$ s.t. $\sigma_i = \dot{\sigma}_i^G$ satisfies the above definition for all $\mathbb{B}_{\xi_i}$ - generic $G \ni a$. We can assume w.l.o.g. that $\dot{\sigma}_i \in N$, since, arguing as in Case 1.1.2, we have $S \in N$, where:

$$S = \{\langle b, h, y, x\rangle \mid h \leq \beta \wedge b \Vdash_{\xi_i} \dot{\sigma}_i(\check{x}) = \check{y}\},$$

Noting that $\mu(k)_i \upharpoonright [\beta_\ell, \tilde{\beta}_k]$ coincides with $\mu(k)_\ell \upharpoonright [\beta_\ell, \tilde{\beta}_h] = \langle \sigma_\ell^{h,\beta_i} \mid \beta_\ell \leq h \leq \tilde{\beta}_h\rangle$ on $(\sigma_\ell^{\beta_i})^{-1}$" u_ℓ for $\ell \leq k$, we get:

$\sigma_i \upharpoonright [\beta_\ell, \tilde{\beta}_k]$ coincides with $\sigma_\ell \upharpoonright [\beta_\ell, \tilde{\beta}_k]$ on u_ℓ for $\ell \leq k$.

Finally set:

$$u = \dot{u}^{n\,G} = u_k \cup \{x_i^\beta, w_i\} \cup \{\sigma_i^{\beta_h}(x_i^{\beta_h}) \mid h < \omega\}.$$

The verification of I, II is straightforward.

We now construct $\dot{b}, \dot{\mu}(i)$. Let $G \subset \mathbb{B}_{\xi_i}$ be as above. Work in $V[G]$. Set:

$N^* = N^G = L_\tau^{A,G}$ (where $N = L_\tau^A$)

$N^{*h} = N^{h,G^h} = L_{\tau^h}^{A^h, G^h}$ (where $N^h = L_{\tau^h}^{A^h}$).

Let σ^* be the canonical completion of σ_i. Then σ^* is a tower in $V[G]$ with $\sigma^{*h} : N^{*h} \prec N^*$, $\sigma^{*h}(\theta^h, \mathbb{B}^h, \lambda^h, G^h) = \theta, \mathbb{B}, \lambda, G$.
Set $\mathbb{B}^* = \mathbb{B}/G = \langle \mathbb{B}_\ell / G \mid \ell \geq i \rangle$. Then $\sigma^*(\mathbb{B}^{*h}) = \mathbb{B}^*$ where $\mathbb{B}^{*h} = \mathbb{B}^h / G^h$.

In $V[G]$ the principle $(*)$ holds for $\nu < \beta$. In particular it holds at $\tilde{\beta}_i = \beta_{i+1} - 1$.

Hence there are $b, \mu \in N^*$ s.t. $b \in (\mathbb{B}^*_{<\lambda})^+$, $\dot{\mu} = \langle \dot{\mu}_\ell \mid i \leq \ell < \omega \rangle$ with $\dot{\mu}_\ell \in N^{\mathbb{B}^*_{\xi_\ell}}$, s.t whenever H^* is $\mathbb{B}^*_{<\lambda}$-generic over N^* s.t. $b \in (H^*)^+$, then $\mu = \langle \dot{\mu}_\ell^{H^*_\ell} \mid i \leq \ell < \omega \rangle$ is a good matrix for N^* w.r.t. the tower $\sigma^* \upharpoonright [\beta_i, \tilde{\beta}_i]$ s.t. μ_ℓ coincides with $\sigma^* \upharpoonright [\beta_i, \tilde{\beta}_i]$ on u_i and $b^{*k} \in (H^{\beta_i})^+$, where $b^{*k} = b^k / G$.

Moreover $\mu_i = \sigma^*$.

Just as in Case 1.1.2, this statement is expressible in N^* in parameters from $\mathrm{rng}(\sigma^{*\beta_{i+1}})$. Since $\sigma^{*\beta_{i+1}} : N^{*\beta_{i+1}} \prec N^*$, the corresponding statement holds in $N^{*\beta_{i+1}}$. This gives us: $b^*, \mu^* \in N^{*\beta_{i+1}}$ s.t. $b^* \in (\mathbb{B}^{*\beta_{i+1}}_{<\lambda^{\beta_{i+1}}})^+$,

$\dot{\mu}^*_l = \langle \dot{\mu}^*_l \mid i \le l < \omega \rangle$ w.r.t.

$\dot{\mu}^*_l \in (N^{\beta_{i+1}})^{\hat{\mathbb{B}}^{*\beta_{i+1}}_l}$ s.t. whenever H^* is $\mathbb{B}^{\beta_{i+1}}_{<\lambda^{\beta_{i+1}}}$-generic and $b^* \in (H^*)^+$, then $\mu^* = \langle (\dot{\mu}^*_l)^{H^*_l} \mid i \le l < \omega \rangle$ is a good matrix w.r.t. $\sigma^*(i) =_{\mathrm{Df}}$ $=_{\mathrm{Df}} \langle \sigma^{*h,\beta_{i+1}} \mid i \le h \le \tilde{\beta}_i \rangle$ for the model $N^{*\beta_{i+1}}$. Moreover, μ^*_l coincides with $\sigma^*(i)$ on u_i for $i \le l < \omega$, and $b^k / G^{\beta_i} \in H^{*\beta_i}$.

Moreover $\mu^*_i = \sigma^*(i)$,

Since $b^* \in N^{\beta_{i+1}}[G^{\beta_{i+1}}]$

$\dot{b} \in (N^{\beta_{i+1}})^{\mathbb{B}_i^{\beta_{i+1}}}$ and $b^* = \dot{b}^{G^{\beta_{i+1}}}$.

We can assume w.l.o.g. that

$\Vdash_{\xi_i^{\beta_{i+1}}} \dot{b} \in (\check{\mathbb{B}}'/\dot{G}) \setminus \{0\}$,

where $\mathbb{B}' = (\mathbb{B}^{\beta_{i+1}}_{<\lambda^{\beta_{i+1}}})^+$ and $\dot{G}$ is the canonical generic name. But then there is a unique $b^i \in \mathbb{B}^{\beta_{i+1}}$ s.t.

$\Vdash_{\xi_i^{\beta_{i+1}}} \check{b}^i/\dot{G} = \dot{b}$. Thus we have:

$h_{\xi_i^{\beta_{i+1}}}(b^i) = [\![\check{b}^i/\dot{G} \neq 0]\!] = 1$.

Clearly $b^i/G = b^*$.

Now let $H \supset G^{\beta_{i+1}}$ be $\mathbb{B}^{\beta_{i+1}}_{<\lambda^{\beta_{i+1}}}$ - generic s.t. $b^i \in (H)^+$. Set:

$$H^* = H/G^{\beta_{i+1}} =_{\mathrm{df}} \{b/G^{\beta_{i+1}} \mid b \in H\}$$

Then $b^* \in (H^*)^+$ and H^* is $\mathbb{B}^{*\beta_{i+1}}_{<\lambda^{\beta_{i+1}}}$ - generic over $N^{*\beta_{i+1}}$. Letting $\mu^* = \dot{\mu}^{*H^*}$, as above, we define $\langle \mu_l \mid i \leq l < \omega \rangle$ by:

$\mu_\ell^h = \mu_\ell^{*h} \upharpoonright N^h$ for $\beta_i \leq h \leq \tilde{\beta}_i$.

Then $\mu = \langle \mu_\ell \mid i \leq \ell < \omega \rangle \in N^{\beta_{i+1}}[H]$,

$\mu_\ell^h : N^h \prec N^{\beta_{i+1}}$, and

(1) μ_ℓ is a $\langle \theta^{\beta_{i+1}}, \mathbb{B}^{\beta_{i+1}}, \lambda^{\beta_{i+1}} \rangle, \hat{\mathbb{B}}_\ell^{\beta_{i+1}}; H_\ell$ - revision of $\sigma(i) =_{\mathrm{Df}} \langle \sigma_i^{h, \beta_{i+1}} \mid \beta_i \leq h \leq \tilde{\beta}_i \rangle$.

The proof of this is straightforward, noting that μ_ℓ is coherent with $\sigma(i)$ wrt. $\hat{\mathbb{B}}_i^{\beta_{i+1}}$ since $\mu_\ell^{*h}(G^h) = \sigma^{*h, \beta_{i+1}}(G^h) = G^{\beta_{i+1}} = H \cap \hat{\mathbb{B}}^{\beta_{i+1}}$. We also need:

$C_\delta^{N^{h+1}}(\mathrm{rng}\, \mu_\ell^{h, h+1}) = C_\delta^{N^{h+1}}(\mathrm{rng}\, \sigma_i^{h, h+1})$

for $\delta = \delta(\hat{\mathbb{B}}_\ell^{h+1})$, $\beta_i \leq h \leq \tilde{\beta}_i$ (where $\mu^{h, \beta_{i+1}} =_{\mathrm{Df}} \mu^h$). This follows exactly like the corresponding step in the proof of Theorem 2.

We also have:

(2) μ_ℓ coincides with $\sigma(i)$ on $\bar{u} = (\sigma^{\tilde{\beta}_i} \upharpoonright \mathbb{B})^{-1} `` u_i$.

(3) $b^k \in (H^{\beta_i})^+$, since $b_\ell^k / G^{\beta_i} \in H_\ell^{*\beta_i}$ for $i \leq \ell < \omega$.

(4) μ_ℓ coheres with μ_i w.r.t. $\hat{\mathbb{B}}_i^{\beta_{i+1}}$ for $i \le \ell$.

proof.
This follows by the coherence of μ_ℓ^* with μ_i^* and the fact that:
$b \in H_\ell^h \leftrightarrow b/G_\ell^h \in H_\ell^{*h}$ and
$\mu_\ell^{*h}(b/G_\ell^h) = \mu_\ell^h(b)/G_\ell^{\beta_{i+1}}$
for $\beta_i \le h \le \tilde{\beta}_i$, $i \le \ell < \omega$.
But it is then easily seen that:
(5) $\langle \mu_\ell \mid i \le \ell < \omega \rangle$ is a good matrix for $\langle \pi^{h,\beta_{i+1}} \mid \beta_i \le h \le \tilde{\beta}_i \rangle$ w.r.t. H

But we have:
(6) There is $\langle \dot{\mu}_\ell \mid i \le \ell < \omega \rangle \in N^{\beta_{i+1}}$ s.t. $\dot{\mu}_\ell \in (N^{\beta_{i+1}})^{\hat{\mathbb{B}}_\ell^{\beta_{i+1}}}$ and whenever $H \supset G^{\beta_{i+1}}$ is $\mathbb{B}^{\beta_{i+1}}_{<\lambda^{\beta_{i+1}}}$-generic over $N^{\beta_{i+1}}$ with $b' \in H^+$, then $\dot{\mu}_\ell^H = \mu_\ell$, where μ_ℓ satisfies the above definition.

proof.
Since $\mu^* = \langle \mu_\ell^* \mid i \le \ell < \omega \rangle \in N^{\beta_{i+1}}[G^{\beta_{i+1}}]$
there is $\ddot{\mu} \in N^{\beta_{i+1}}$ s.t. $\ddot{\mu} = \langle \ddot{\mu}_\ell \mid i \le \ell < \omega \rangle$
and $\ddot{\mu}_\ell \in (N^{\beta_{i+1}})^{\hat{\mathbb{B}}_i^{\beta_{i+1}}}$ with:

$\Vdash_{\mathbb{B}_i^{\beta_{i+1}}} (\ddot{\mu}_\ell \in V^{\check{\mathbb{B}}_\ell^*} / \mathring{G})$, $\dot{G}$ being the canonical generic name and $\ddot{\mu}_\ell^{H^*} = \mu_\ell^*$ for $i \leq \ell < \omega$.

But then there is $\mu_\ell \in (N^{\beta_{i+1}})^{\mathbb{B}_\ell^{\beta_{i+1}}}$ s.t. $\Vdash_{\mathbb{B}_\ell^{\beta_{i+1}}} (\mathrm{At}(\check{\ddot{\mu}}_\ell^{\dot{G}})^{\dot{H}/\dot{G}}$ is a function defined on $[\check{\beta_i, \tilde{\beta}_i}]$, then so is $\dot{\mu}_\ell$ and

$$\dot{\mu}_\ell^h = (\check{\ddot{\mu}}_\ell^{\dot{G}})^{\dot{H}/\dot{G}}(h) \upharpoonright \check{\vec{N}}(h) \text{ for}$$

$$h \in [\beta_i, \tilde{\beta}_i] \,)$$

where $\vec{N} = \langle N^h \mid \beta_i \leq h \leq \tilde{\beta}_i \rangle$, $\dot{H}$ is the canonical $\mathbb{B}_\ell^{\beta_{i+1}}$ - generic name, and $\Vdash_{\mathbb{B}_\ell^{\beta_{i+1}}} \dot{G} = \dot{H} \cap \check{\hat{\mathbb{B}}}_i^{\beta_{i+1}}$. Clearly $\dot{\mu}_\ell$ has the desired property and $\dot{\mu} = \langle \dot{\mu}_\ell \mid i \leq \ell < \omega \rangle \in N^{\beta_{i+1}}$. QED (6)

Since $b^k \in (H^{\beta_i})^+$, the sequence $\mu(k) = \langle \mu(k)_\ell \mid \ell < \omega \rangle$ is given, where

$$\mu(k)_\ell = (\dot{\mu}(k)_\ell)^{H_\ell^{\beta_i}} \text{ and}$$

$$\ddot{\mu}(k) = \langle \dot{\mu}(k)_\ell \mid \ell < \omega \rangle = \ddot{\mu}(k)^{G_k}.$$

(Recall that $b^k = (\dot{b}^k)^{G_k}$.)

Define $\mu(i) = \langle \mu(i)_l \mid l < \omega \rangle$ by:

$\mu(i)_l = \langle \sigma^{h,\beta_{i+1}} \mid h \leq \tilde{\beta}_i \rangle$ for $l \leq i$;

$$\mu(i)^h_l = \begin{cases} \mu^h_l & \text{if } \beta_i \leq h \leq \tilde{\beta}_i \\ \mu^{h,\beta_i}_l \circ \mu(k)^h_l & \text{if } h < \beta_i . \end{cases}$$

It is straightforward to see that

(7) $\mu(i) = \langle \mu(i)_l \mid l < \omega \rangle$ satisfies IV (a) – (e).

But, imitating the proof of (6) we easily have:

(8) There is $\dot{\mu}(i) = \langle \dot{\mu}(i)_l \mid l < \omega \rangle \in N^{\beta_{i+1}}$ s.t. $\dot{\mu}(i)_l \in (N^{\beta_{i+1}})^{\hat{\mathbb{B}}^{\beta_{i+1}}_l}$ for $\beta_i \leq l \leq \tilde{\beta}_i$ and <u>whenever</u> $H \supset G^{\beta_{i+1}}$ is $\mathbb{B}^{\beta_{i+1}}_{<\lambda^{\beta_{i+1}}}$-generic s.t. $b' \in H^+$, then $\dot{\mu}(i)^{H_l}_l = \mu(i)_l$ satisfying the above definition.

For $l \leq i$, we know that $\langle \sigma_l^{h\beta_{i+1}} \mid h \leq \tilde{\beta}_i \rangle \in N^{\beta_{i+1}}[G_l]$; hence there must be $t_l \in (N^{\beta_{i+1}})^{\hat{\mathbb{B}}^{\beta_{i+1}}_l}$ s.t. $t_l^{G_l} = \langle \sigma_l^{h\,\beta_{i+1}} \mid h \leq \tilde{\beta}_i \rangle$. Set:

<u>Def</u> $\dot{\mu}(i)_l = t_l$ for $l \leq i$.

Then:

(9) $b^i, \mu(i)$ satisfy III (a)–(c)

By (7):

(10) $b^i, \mu(i)$ satisfy IV

But $b^i, \mu(i)$ with these properties exist for every $G \ni a$ which is $\mathbb{B}_{\xi_i}$-generic.

This gives us $\dot{b}^i, \dot{\mu}(i) \in N^{\mathbb{B}_{\xi_i}}$ satisfying III–IV.

This completes the construction.

QED (Case 1.1).

Case 1.2 $cf(\lambda) = \omega_1$

By induction on $\beta \leq \alpha$ we prove:

(*) Let π be a β-tower for N s.t. $\pi^i : N^i \prec N$ and $\pi^i(\theta^i, \mathbb{B}^i, \lambda^i) = \theta, \mathbb{B}, \lambda$ for $i \leq \beta$. Let $e \in (\mathbb{B}^0_{<\lambda^0})^+$. Let $u \subset N^\beta$ be finite. Set: $\lambda_\beta = \sup \pi''\lambda^\beta$. Then there is $b \in \mathbb{B}^+_{<\lambda_\beta}$ s.t. whenever G is $\mathbb{B}^+_{<\lambda_\beta}$ - generic with $b \in G^+$, then there is $\sigma \in V[G]$ s.t. σ is a $\langle \theta, \mathbb{B}, \lambda \rangle, \mathbb{B}_{<\lambda}, G$ - revision of π coinciding with π on u and s.t. $e \in (G^0)^+$.

Our actual induction hypothesis will be the stronger statement:

(**) Let $\nu < \lambda$ and let G be $\mathbb{B}_\nu$ - generic. Then (*) holds in $V[G]$ with N^G, $\mathbb{B}/G$ in place of N, $\mathbb{B}$, where $N^G = L^{A,G}_\tau$ (with $N = L^A_\tau$).

At each step of the iteration it will suffice to display the proof of (*), since we can repeat the same proof in $V[G]$.

Case 1.2.1 $\beta = 0$. This is immediate by §1 Lemma 3.1.

Case 1.2.2 $\beta = \gamma + 1$.

Then (*) holds at γ. This say that, given finite $u \subset N^\gamma$ and $e \in (\mathbb{B}^0_{<\lambda^0})^+$, there are $b \in \mathbb{B}^+_{<\lambda_\gamma}$ and $\dot{\sigma} \in V^{\mathbb{B}_{\lambda_\gamma}}$ s.t., whenever G is $\mathbb{B}^+_{<\lambda_\gamma}$ - generic and $b \in G^+$, then $\sigma = \dot{\sigma}^G$ has the properties listed in (*). Arguing as in Case 1.1.1, we can assume: $\dot{\sigma} \in N$. But then, arguing as in Case 1.1.2, we see that this statement is uniformly expressible over N in parameters from $\mathrm{rng}(\pi^\beta)$. Since $\pi^\beta : N^\beta \prec N$, the corresponding statement holds in N^β. Hence there are $\bar{b} \in \mathbb{B}^\beta_{<\lambda^\beta_\gamma}$ and $\dot{\sigma} \in V^{\mathbb{B}^\beta_{\lambda^\beta_\gamma}}$ s.t. whenever $\bar{G}$ is $(\mathbb{B}^\beta_{<\lambda^\beta_\gamma})^+$ - generic over N^β and $\bar{b} \in \bar{G}^+$, then, letting $\bar{\sigma} = \dot{\sigma}^{\bar{G}}$, we have:

$\bar\sigma$ is a $\langle \theta^\beta, \mathbb{B}^\beta, \lambda^\beta \rangle, \mathbb{B}_{<\lambda^\beta_\beta}, \bar{G}$ – revision of $\bar\pi = \langle \pi^{h\beta} \mid h \leq \gamma \rangle$ wrt. N^β which coheres with $\bar\pi$ on $\bar{u} = (\bar\pi^{\gamma\beta})^{-1} {}''u$ and is s.t. $e \in (\bar{G}^\circ)^+$.

But by Case 1.2.1 there is $b \in \mathbb{B}^+_{<\lambda_\beta}$ s.t. whenever G is $\mathbb{B}^+_{<\lambda_\beta}$ generic with $b \in G^+$, then there is $\sigma' \in V$ which is a $\langle \theta, \mathbb{B}, \lambda \rangle, \mathbb{B}_{<\lambda_\beta}, G$ – revision of $\pi \upharpoonright \{\beta\}$ coinciding with $\bar\pi \upharpoonright \{\beta\}$ on u and s.t. $\bar{b} \in (G^\beta)^+$. But then G^β is $\mathbb{B}^\beta_{<\lambda^\beta}$ – generic and we can set: $\bar\sigma = \dot\sigma^G$. Set:

$$\sigma^i = \begin{cases} \sigma'^\beta & \text{if } i = \beta \\ \sigma'^\beta \bar\sigma^i & \text{if } i < \beta \end{cases}$$

σ is easily seen to have the desired properties. QED (Case 1.2.2).

<u>Case 1.2.3</u> β is a limit ordinal.

As before, let $\langle \beta_i \mid i < \omega \rangle$ be monotone and cofinal in β s.t. $\beta_0 = 0$ and β_i is a successor ordinal for $i > 0$.

Set: $\tilde{\beta}_i = \beta_{i+1} \dot{-} 1$.

Set: $\lambda_h = \sup \pi^{h} {}'' \lambda^h$ for $h < \lambda$.

Then $\langle \lambda_h \mid h \leq \lambda \rangle$ is a normal fcn.

Set: $\lambda^j_h = \sup \pi^{hj} {}'' \lambda^h$ for $h \leq j$.

Then $\langle \lambda^j_h \mid h \leq j \rangle$ is normal with $\lambda^j_j = \lambda^j$.

<u>Fact</u> Let σ be any λ-revision of ..

(a) $\lambda_h = \sup \sigma^{h} {}'' \lambda^h$ $(h \leq \beta)$

(b) $\lambda^j_h = \sup \sigma^{hj} {}'' \lambda^h$ $(h \leq j \leq \beta)$

(c) $\sigma^j(\lambda^j_h) = \lambda_h$ for $h < j \leq \beta$.

proof.

Let f = the N-least normal function on ω_1 converging to λ. Then $\pi^i(f^i) = \sigma^i(f^i) = f$, where f^i has the corresponding def. in N^i.

Hence:

$\lambda_h = f(\omega_1 N^h) = \sup \sigma^{h}{}'' \lambda^h \quad (h < \beta)$

$\lambda_h^{i} = f^{i}(\omega_1 N^h) = \sup \sigma^{hi}{}'' \lambda^h \quad (h < i \leq \beta)$,

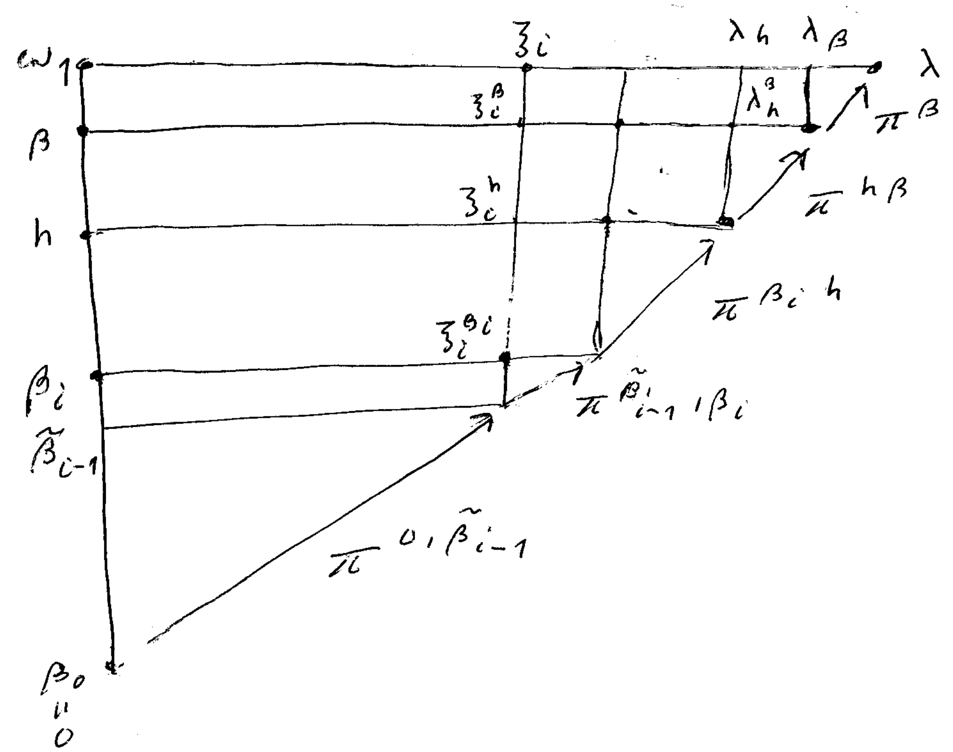
$\sigma^{i}(\lambda_h^{i}) = f(\omega_1 N^h) = \lambda_h \quad (h < i \leq \beta)$

QED (Fact)

Now set: $\xi_0 = 0$, $\xi_i = \lambda_{\tilde{\beta}_{i-1}}$ for $i > 0$

Then $\langle \xi_i \mid i < \omega \rangle$ is monotone and cofinal in λ_β. Set $\xi_i^{i} = (\pi^{i})^{-1}(\xi_i)$ for $\beta_i \leq i < \beta$. (Hence $\xi_i^{i} = \lambda^{i}_{\tilde{\beta}_{i-1}}$ for $i > 0$.)

Then $\sigma^{i}(\xi_i^{i}) = \xi_i$ for any λ-revision σ of π.



We define b_i, $\dot{\sigma}_i$, $\dot{u}_i$ $(i<\omega)$ s.t.

(I) (a) $b_i \in \mathbb{B}^+_{<\xi_i}$ for $i>0$; $b_0=1$

(b) $b_\ell = h_{\xi_\ell}(b_i)$ for $\ell \le i$

(c) $\dot{\sigma}_i$, $\dot{u}_i \in V^{\mathbb{B}_{\xi_i}}$

(II) Let G be $\mathbb{B}^+_{<\xi_i}$ - generic s.t. $b \in G^+$. Set:

$G_h = G \cap \mathbb{B}_{<\xi_h}$, $\sigma_h = \dot{\sigma}_h^{G_h}$ for $h \le i$. Then:

(a) σ_i is a $\langle \theta, \mathbb{B}, \lambda\rangle$ - revision of π.

Moreover, $\sigma_0 = \bar{\pi}$,

(b) $\sigma_i \upharpoonright [\beta_i, \beta]$ is a $\langle \theta, \mathbb{B}, \lambda\rangle$, $\mathbb{B}_{<\xi_i}$, G - revision of $\pi \upharpoonright [\beta_i, \beta]$ which coheres with $\sigma_h \upharpoonright [\beta_i, \beta]$ w.r.t. $\mathbb{B}_{\xi_h}$ for $h<i$.

(c) $u_i \in N^\beta$ is finite.

(d) σ_i coincides with σ_h on u_h for $h \le i$.

(e) $u \cup \bigcup_{h<i} (u_h \cup \{x^\beta_h, w_h\}) \subset u_i$, where, as before $\langle x^h_\ell \mid \ell<\omega\rangle$ is the N-least enumeration of N^h and $w_i =$

$=$ the N^β-least w s.t. $\bar{\bar{w}} \le \delta = \bigcup_{i<\omega} \delta(\mathbb{B}^\beta_{\xi^\beta_i})$ in N^β and $\pi^\beta(x^\beta_i) \in \sigma_i^\beta(w)$.

(f) $\sigma_i^{\ell,j} = \sigma_h^{\ell,j}$ for $h \le i$, $\ell \le j \le \beta_h$

(g) $G_h^\ell = G_i^\ell$ for $h \le i$, $\ell < \beta_h$

where $G_i^\ell = (\sigma_i^\ell)^{-1} {}'' G_h$.

<u>Note</u> $G_i^{\beta_i} \subset \mathbb{B}^{\beta_i}_{< \xi_i^{\beta_i}} \subset \mathbb{B}^{\beta_i}_{<\lambda^{\beta_i}}$. Hence $G_i^\ell \subset \mathbb{B}^\ell_{<\lambda^\ell}$ for $\ell < \beta_i$, since $\sigma_i^{\ell,\beta_i}(\lambda^\ell) = \lambda^{\beta_i}$.

Set: $\tilde{\sigma}_i = \langle \sigma_i^{\ell,\beta_i} \mid \ell \le \tilde{\beta}_i \rangle$. (hence $\sigma_0 = \phi$)

(h) $\tilde{\sigma}_i$ is a $\langle \theta^{\beta_i}, \mathbb{B}^{\beta_i}, \lambda^{\beta_i} \rangle$, $\mathbb{B}^{\beta_i}_{<\lambda^{\beta_i}}$, G - revision of $\tilde{\pi}_i = \langle \pi^{\ell,\beta_i} \mid \ell \le \tilde{\beta}_i \rangle$ wrt N^{β_i} s.t. $e \in G^0$.

(i) $\tilde{\sigma}_i \upharpoonright [\beta_h, \tilde{\beta}_i]$ coheres with $\langle \sigma_h^{\ell,\beta_i} \mid \beta_h \le \ell \le \tilde{\beta}_i \rangle$ wrt $\hat{\mathbb{B}}^{\beta_i}_{\beta_h} = \mathbb{B}^{\beta_i}_{\xi^{\beta_i}_{\beta_h}}$.

Now let $\langle \nu_i, z_i \rangle \mid i < \omega \rangle$ be an enumeration of $\{\langle \nu, z \rangle \mid \nu \le \beta \wedge z \in N^\nu\}$ with infinitely many enumerations of each point.

(j) If $\nu_i \le \beta_i$, then $\sigma_i^{\nu_i,\beta}(\pi_i) \in u_i$ for $i > 0$.

We now prove (*) from (I), (II). Set: $b = \bigcap_i b_i$. Then $b \in \mathbb{B}^+_{<\lambda_\beta}$. Let G be $\mathbb{B}^+_{<\lambda_\beta}$ - generic s.t. $b \in G^+$. For $h < \beta$ set: $G^h = G_i^{h,\beta_i}$, where $h < \beta_i$. For $h \leq j < \beta$ set $\sigma^{hj} = h_i^{hj}$ where $h \leq j < \beta_i$. These definitions are unique by II (f), (g). But ~~by~~ II (h) we have:

G^h is $\mathbb{B}^+_{<\lambda^h}$ - generic. Note that: $\sigma^{hj}: N^h \prec N^j$, $\sigma^{hj}(\lambda^h) = \lambda^j$, $\sigma^{hj}{}''G^h \subset G^j$ for $h \leq j < \beta$. Then

(1) $\langle N^h \mid h < \beta \rangle, \langle \sigma^{hj} \mid h \leq j < \beta \rangle$ has a direct limit of the form:

$$N^\beta, \langle \sigma^h \mid h < \beta \rangle.$$

proof.

Let $\mathfrak{A}, \langle f^h \mid h < \beta \rangle$ be a direct limit. We first define a $k: N^\beta \prec \mathfrak{A}$ as follows: $k(x) = f^h(\bar{x})$ where $\bar{x} \in N^h$ and for sufficiently large $i < \omega$ we have $\sigma_i^h(\bar{x}) = x$ and $\sigma^{hj}(\bar{x}) = \sigma_i^{hj}(\bar{x})$ for all $j \in [h, \beta)$.

The choice of $\bar{x}, h$ clearly doesn't matter. We now show that k is elementary. Let $x_1, \ldots, x_n \in N^\beta$. Let $x_\ell = \sigma_i^h(\bar{x}_\ell)$ s.t. $\sigma_i^{hj}(\bar{x}_\ell) = \sigma_i^{hj}(\bar{x}_\ell)$ for all $j \in [h, \beta)$ for suff. large i. Then

$$N^\beta \models \varphi(x_1, \ldots, x_n) \longleftrightarrow N^h \models \varphi(\bar{x}_1, \ldots, \bar{x}_n) \longleftrightarrow$$

$$\longleftrightarrow \mathfrak{A} \models \varphi(k(x_1), \ldots, k(x_n)).$$

Finally, we show that k is onto $\mathfrak{A}$. Let $x \in \mathfrak{A}$, $x = f^h(\bar{x})$. Let $\langle \nu_i, z_i \rangle = \langle h, \bar{x} \rangle$ for an i_0 large enough that $h < \beta_{i_0}$. Then $x' = \sigma_i^h(\bar{x}) \in U_i$. Hence for $i \geq i_0$, we have $x' = \sigma_i^h(\bar{x})$ and $\sigma^{hj}(\bar{x}) = \sigma_i^{hj}(\bar{x})$ for all $j \in [h, \beta]$. Hence $x = k(x')$. QED (1)

Clearly, $\sigma^{hj} {}'' (G^h)^+ \subset (G^j)^+$. Hence, letting $G^\beta = \bigcup \sigma^{h} {}'' G^h$, we have:

(2) G^β is $(\mathbb{B}^\beta_{<\lambda^\beta})^+$ - generic over N^β.

proof.
Let $\Delta \in N^\beta$ be strongly dense in $\mathbb{B}^\beta_{\lambda^\beta}$. Let $\sigma^h(\Delta^h) = \Delta$. Then Δ^h is strongly dense in $\mathbb{B}^h_{\lambda^h}$. Hence there is $c \in (G^h)^+$ s.t. $c \in \Delta^h$. Hence $\sigma^h(c) \in (G^\beta)^+ \cap \Delta$. QED (2)

By the proof of (1) we note that:

(3) Let $x \in N^\beta$ and $\bar{x} \in N^h$ ($h < \beta$) s.t. for sufficiently large i we have: $\sigma_i^h(\bar{x}) = x$ and $\sigma^{hj}(\bar{x}) = \sigma_i^{hj}(\bar{x})$ for all $j \in [h, \beta)$. Then $x = \sigma^h(\bar{x})$

proof.
$N^\beta = k^{-1}{}''\Omega$, $\sigma^h = k^{-1} \circ f^h$ and $k(x) = \tilde{\sigma}^h(\bar{x})$; where Ω, f, k are as in the proof of (1). QED (3).

But then:

(4) $G^\beta = \bigcup_{i<\omega} G_i^\beta$.

proof.
By genericity it suffices to show ($\supset$). Let $b \in G_i^\beta$ with i large enough.

that $b \in U_i$. Then, letting $\sigma_i^h(\bar{b}) = b$, we have $\bar{b} \in G_i^h$ and $b = \sigma^h(\bar{b}) \in G^\beta$, by (3). QED(4)

Now define $\sigma' : N^\beta \prec N$ by:

$\sigma'(x) = \sigma_i^\beta(x)$ if $\sigma_i^\beta(x) = \sigma_j^\beta(x)$ for $j \geq i$.

The usual proof shows that

$\sigma' `` G^\beta \subset G$, $C_\delta^N(\text{rng}\,\sigma') = C_\delta^N(\text{rng}\,\bar{\pi})$, and $\sigma' \restriction u = \pi \restriction u$, where $\delta = \sup_{h<\lambda_\beta} \delta(\mathbb{B}_h)$. Hence, setting:

$$\sigma^h = \begin{cases} \sigma' & \text{if } h = \beta \\ \sigma'\sigma^h & \text{if } h < \beta, \end{cases}$$

we see that σ^h has the desired properties. This proves (*)

All that remains is to define b_i, σ_i and verify (I), (II), Simultaneously we define c_i, $\mathring{u}(i)$ s.t. the following holds:

(III) Let G be as in II. Set $c^l = (\dot{c}^l)^{G_l}$, $\mu(l) = \dot{\mu}(l)^{G_l}$ for $l \le i$. Then:

(a) $c_i \in (\mathbb{B}^{\beta_{i+1}}_{<\xi^{\beta_{i+1}}_{i+1}})^+$ s.t. $h_{\xi_i^{\beta_{i+1}}}(c_i) = 1$

(b) $\mu(i) \in (N^{\beta_{i+1}})^{\mathbb{B}^{\beta_{i+1}}_{\xi^{\beta_{i+1}}_{i+1}}}$.

Note: $\xi^{\beta_{i+1}}_{i+1} = \lambda^{\beta_{i+1}}_{\tilde{\beta}_i}$.

(Note It will again be convenient to use the notation: $\hat{\mathbb{B}}^h_l = \mathbb{B}^h_{\xi^h_l}$. Hence $\mathbb{B}^{\beta_{i+1}}_{\xi^{\beta_{i+1}}_{i+1}} = \hat{\mathbb{B}}^{\beta_{i+1}}_{i+1}$.)

(IV) Let $G, c^l, \mu(l)$ $(l \le i)$ be as in III. Let H be $(\mathbb{B}^{\beta_{i+1}}_{<\xi^{\beta_{i+1}}_{i+1}})^+$-generic over $N^{\beta_{i+1}}$ s.t.

$$G_i^{\beta_{i+1}} \subset H \text{ and } c_i \in H^+.$$

Set: $\mu(i) = \dot{\mu}(i)^H$. Then:

(a) $\mu(i)$ is a $\langle \theta^{\beta_{i+1}}, \mathbb{B}^{\beta_{i+1}}, \lambda^{\beta_{i+1}} \rangle$, $\mathbb{B}^{\beta_{i+1}}_{<\lambda^{\beta_{i+1}}}$, H-revision of $\langle \pi^{h,\beta_{i+1}} \mid h \le \tilde{\beta}_i \rangle$.

(b) $\mu(i) \upharpoonright [\beta_i, \tilde{\beta}_i]$ coheres with $\langle \sigma_i^{h,\beta_{i+1}} \mid \beta_i \le h \le \tilde{\beta}_i \rangle$ wrt $\hat{\mathbb{B}}^{\beta_{i+1}}_i$

(c) $\mu(i)$ coincides with $\langle \sigma_i^{h,\beta_{i+1}} \mid h \le \tilde{\beta}_i \rangle$ on $(\sigma_i^{\tilde{\beta}_i})^{-1}$ " u_i

Set: $H^j = (\mu(i)_j)^{-1}$ " H for $j \le \tilde{\beta}_i$.

(d) $e \in H^0$ if $i = 0$

(e) Let $i = k+1$. Then $c^k \in (G^{\beta_i})^+$ and $\mu(k)^{i'} = \mu(i)^{i'\beta_i}$ for $i' \leq \tilde{\beta}_k$ where $\mu(k) = \dot{\mu}(k)^{G^{\beta_i}}$, $\dot{\mu}(k) = \ddot{\mu}(k)^{G_k}$

(Note that $G^{\beta_i} \subset H^{\beta_i}$.)

We construct $\mu(i)$ preparatory to constructing σ_{i+1} with:

$$\sigma_{i+1}^{i'\beta_{i+1}} = \mu(i)^{i'} \text{ for } i' \leq \tilde{\beta}_{i+1},$$

where $\mu(i) = \dot{\mu}(i)^{G_{i+1}^{\beta_{i+1}}}$.

We construct $b_i, \sigma_i, u_i, \dot{c}_i, \ddot{\mu}(i)$ and verify (I) – (IV) by induction on i.

<u>Case 1</u> $i = 0$. $b_0 = 1$, $\sigma_0 = \check{\pi}$, $u_0 = \check{u}$.

We construct $\dot{c}_0, \ddot{\mu}(0)$ as follows. Since $(*)$ holds at $\tilde{\beta}_0$, there are $c, \dot{\mu}$ s.t. $c \in \mathbb{B}_{\xi_1}$, $\dot{\mu} \in V^{\mathbb{B}_{\xi_1}}$ and whenever $G \ni c$ is $\mathbb{B}_{\xi_1}$-generic, then $\mu = \dot{\mu}^G$ is a $\langle \theta, \mathbb{B}, \lambda \rangle, \mathbb{B}_{\xi_1}, G$-revision of π coinciding with π on $\underset{u_0}{u}$ and s.t. $e \in G$. By the usual argument this fact is expressible in N uniformly in parameters from $rng(\pi^{\beta_1})$. Hence the corresponding fact holds in N^{β_1}, giving c_0, μ_0 s.t. $c_0 \in \hat{\mathbb{B}}_1^{\beta_1}$; $\check{\mu}_0 \in (N^{\beta_1})^{\hat{\mathbb{B}}_1^{\beta_1}}$ and whenever $H \ni c_0$ is $\hat{\mathbb{B}}_1^{\beta_1}$-generic over N^{β_1}, then $\mu_0 = \check{\mu}_0^H$ is a $\langle \theta^{\beta_1}, \mathbb{B}^{\beta_1}, \lambda^{\beta_1} \rangle, \hat{\mathbb{B}}_1^{\beta_1}, H$-revision of $\bar{\bar{\pi}} = \langle \pi^{l\beta_1} \mid l \leq \tilde{\beta}_0 \rangle$ coinciding with $\bar{\pi}$ on $\bar{u} = (\sigma^{\beta_1})^{-1}{}''u$, and s.t. $e \in H^0$. We set:

$$\dot{c}_0 = \check{c}_0 \; ; \; \ddot{\mu}(0) = \check{\mu}_0 .$$

The verifications are straightforward.

<u>Case</u> 2 $i = k+1$.

Then $\dot{c}_k, \ddot{\mu}(k)$ are given.

By Lemma 2 there are $b_i, \dot{\sigma}'$ s.t. $b_i \in \mathbb{B}_{\xi_i}$, $\dot{\sigma}' \in V^{\mathbb{B}_{\xi_i}}$ and whenever $G \ni b_i$ is $\mathbb{B}_{\xi_i}$-generic, then $\dot{\sigma}'$ is a $\langle \theta, \mathbb{B}, \lambda \rangle, \mathbb{B}_{\xi_i}, G$-revision of $\pi \upharpoonright [\beta_k, \beta]$ coinciding with

$\sigma_k \upharpoonright [\beta_k, \beta]$ on $u_k = \dot{u}_k^G$ and s.t.

$c_k = \dot{c}_k^G \in G^{\beta_k}$. Moreover, σ' cohere

with $\sigma_k \upharpoonright [\beta_k, \beta]$ w.r.t. $\mathbb{B}_{\xi_k}$.

We then set:

$$\sigma_i^j = \begin{cases} \sigma' j & \text{if } \beta_k \leq j \leq \beta \\ (\sigma''^{\beta_k} \circ \mu(k))^j & \text{if } j < \beta_k \end{cases}$$

where $\mu(k) = \dot{\mu}(k)^{G^{\beta_k}}$; $\dot{\mu}(k) = \ddot{\mu}(k)^G$.

(Note we employ the convention that $V^{\mathbb{B}_{\xi_k}} \subseteq V^{\mathbb{B}_{\xi_i}}$; hence if $t \in V^{\mathbb{B}_{\xi_k}}$, then $t \in V^{\mathbb{B}_{\xi_i}}$ and $t^G = t^{G \cap \mathbb{B}_{\xi_k}}$.)

It is easily seen that there is a term $\dot{\sigma}_i \in V^{\mathbb{B}_{\xi_i}}$ s.t. $\sigma_i = \dot{\sigma}_i^G$ is forced by b_i to have the above definition. Finally we set:

$$u_i = u_k \cup \{x_k^\beta, w_k\} \cup \{z\}, \text{ where}$$

$u_k = \dot{u}_k^G$ and

$$z = \begin{cases} \sigma_i^{\gamma_i}(z_i) & \text{if } \gamma_i < \beta_i \\ \emptyset & \text{if not.} \end{cases}$$

There is obviously a $\dot{u}_i \in V^{\mathbb{B}_{\xi_i}}$

s.t. $u_i = \check{u}_i^G$ satisfies this definition for $G^+ \ni b_i$. This gives us b_i, σ_i', u_i'. The verification of (I), (II) is straightforward, as is $i\, c_k \in (G^{\beta_i})^+$.

We now define $\check{c}_i, \ddot{\mu}(i)$.

Let G be as above. Work in $V[G]$. Let σ^* be the canonical extension of $\sigma_i \upharpoonright [\beta_i, \beta]$ defined by:

$$\sigma^{*i} : N^i[G^i] \prec N[G], \quad \sigma^{*i}(G^i) = G,$$

Set $N^* = N^G =_{df} L_\tau^{A,G}$, where $N = L_\tau^A$.

Let $\sigma^{*i} : N^{*i} \prec N^*$ (hence $N^{*i} = L_{\tau^i}^{A^i, G^i}$).

Then $(*)$ holds at $\tilde{\beta}_i$ in $V[G]$ w.r.t. the tower σ^* (in place of π). This fact is expressible, as usual, in N^* in parameters from $\mathrm{rng}(\sigma^{*\beta_{i+1}})$. Hence the corresponding fact holds in $N^{*\beta_{i+1}}$, and we have:

Set: $\bar{N} = N^{\beta_{i+1}}$, $\bar{N}^* = N^{*\beta_{i+1}}$, $\bar{\mathbb{B}} = \mathbb{B}^{\beta_{i+1}}$,
$\bar{\mathbb{B}}^* = \bar{\mathbb{B}}/\bar{G} = \mathbb{B}^{*\beta_{i+1}}$, where $\bar{G} = G^{\beta_{i+1}}$,
We also set: $\bar{\theta} = \theta^{\beta_{i+1}}$, $\bar{\lambda} = \lambda^{\beta_{i+1}}$,
$\bar{\xi}_\ell = \xi_\ell^{\beta_{i+1}}$. Then there is $\bar{c}^* \in (\bar{\mathbb{B}}^*_{<\bar{\xi}_{i+1}})^+$
and a name $\underset{\sim}{\bar{u}}^* \in \bar{N}^{*\bar{\mathbb{B}}^*_{\bar{\xi}_{i+1}}}$ s.t.,
whenever H^* is $(\bar{\mathbb{B}}^*_{<\bar{\xi}_{i+1}})^+$ - generic,
and $\bar{c} \in (H^*)^+$, then
$\bar{u}^* = (\underset{\sim}{\bar{u}}^*)^{H^*}$ is a
$\langle \bar{\theta}, \bar{\mathbb{B}}, \bar{\lambda} \rangle$, $\bar{\mathbb{B}}^*_{<\bar{\lambda}}$, $\bar{H}^*$ - revision
of $\bar{\sigma}^* = \langle \sigma^{*i,\beta_{i+1}} \mid \beta_i \leq i \leq \tilde{\beta}_i \rangle$ coinciding with
$\bar{\sigma}^*$ on $\bar{u} = (\sigma^{\tilde{\beta}_i})^{-1} {}'' u_i$.

Since $\bar{c}^* \in \bar{N}[\bar{G}]$, there is $\mathring{c} \in \bar{N}^{\bar{\mathbb{B}}_{\bar{\xi}_{i+1}}}$
s.t. $\bar{c}^* = \mathring{c}^{\bar{G}}$. We can assume
w.l.o.g. that $\Vdash^{\bar{N}}_{\bar{\mathbb{B}}_{\bar{\xi}_{i+1}}} \mathring{c} \in (\check{\bar{\mathbb{B}}}^+_{<\bar{\xi}_{i+1}}) \setminus \{0\}$.

Let $c \in \bar{\mathbb{B}}_{\bar{\xi}_{i+1}}$ s.t. $\Vdash^{\bar{N}}_{\bar{\mathbb{B}}_{\bar{\xi}_{i+1}}} \check{c}/\mathring{G} = \mathring{c}$,
$\mathring{G}$ being the canonical generic
name.

It follows easily that $c \in \bar{\mathbb{B}}^+_{<\bar{\xi}_{i+1}}$ and $h_{\bar{\xi}_i}(c) = [\![\mathring{c} \neq 0]\!] = 1$.

Now suppose that $H \supset G^{\beta_{i+1}}$ is $\bar{\mathbb{B}}^+_{<\bar{\xi}_{i+1}}$ - generic with $c \in H^+$.

Then $H^* = H/\bar{G} =_{Df} \{ b/\bar{G} \mid b \in H \}$ satisfies the above assumptions.

Letting $\bar{\mu}^* = (\mathring{\bar{\mu}}^*)^{H^*}$, set:

$$\bar{\mu}^j =_{Df} \bar{\mu}^{*j} \cap N^j \text{ for } j \in [\beta_i, \tilde{\beta}_i].$$

Imitating the corresponding step in the proof of Thm 1 we get:

(1) $\bar{\mu}$ is a $\langle \bar{\theta}, \bar{\mathbb{B}}, \bar{\lambda} \rangle$, $\mathbb{B}_{<\bar{\lambda}}$, H - revision of $\bar{\pi} = \langle \pi^j \restriction \beta_{i+1} \mid \beta_i \le j \le \tilde{\beta}_i \rangle$ cohering with $\bar{\sigma} = \langle \sigma^j \restriction \beta_{i+1} \mid \beta_i \le j \le \tilde{\beta}_i \rangle$ w.r.t. $\bar{\sigma}$ and coinciding with $\bar{\sigma}$ on $\bar{u}$, $\bar{\mathbb{B}}_{\bar{\xi}_i}$.

But then

(2) There is $\dot{\bar{\mu}} \in \bar{N}^{\bar{\mathbb{B}}_{\bar{\xi}_{i+1}}}$ s.t. whenever $\bar{H} \supset \bar{G}$ is $\bar{\mathbb{B}}^+_{<\bar{\xi}_{i+1}}$ - generic with $c \in \bar{H}^+$, then $\bar{\mu} = \dot{\bar{\mu}}^{\bar{H}}$ is defined as above from $\bar{\mu}^* = \mathring{\dot{\bar{\mu}}}^{*\,\bar{H}/\bar{G}}$,

proof of (2)

Since $\dot{\bar{\mu}}^* \in \bar{N}[\bar{G}]$ there is $t \in \bar{N}^{\bar{\mathbb{B}}_{\bar{\xi}_i}}$ s.t. $t^{\bar{G}} = \dot{\bar{\mu}}^*$. But then there is $\dot{\bar{\mu}} \in \bar{N}^{\bar{\mathbb{B}}_{\bar{\xi}_{i+1}}}$ s.t.

$\Vdash_{\bar{\mathbb{B}}_{\bar{\xi}_{i+1}}}$ (If $\check{t}^{\dot{G}}$ is a function defined on $[\beta_i, \check{\tilde{\beta}}_i]$, then so is $\dot{\bar{\mu}}$ and for all $j \in [\beta_i, \check{\tilde{\beta}}_i]$ we have:

$$\dot{\bar{\mu}}(j) = (\check{t}^{\dot{G}})^{\dot{H}}\,)$$

where $\dot{H}$ is the generic name and $\dot{G}$ is an abbreviation for: $\dot{H} \cap \bar{\mathbb{B}}_{\bar{\xi}_i}$.

QED (2)

Now define μ with domain $[0, \tilde{\beta}_i]$ by:

$$\mu^j = \begin{cases} \bar{\mu}^j & \text{if } \beta_i \leq j \\ \bar{\mu}^{\beta_i} \circ \mu(k)^j & \text{if not} \end{cases}$$

where $\mu(k) = \dot{\mu}(k)^{\bar{G}^{\beta_i}} = \dot{\mu}(k)^{\bar{H}^{\beta_i}}$ and $\dot{\mu}(k) = \ddot{\mu}(k)^{G_k} = \ddot{\mu}(k)^{G}$.

Then $\underline{\mathrm{IV}}$ (a) – (e) hold with μ in place of $\mu(i)$.

But it is easily seen that there is $\dot{\mu} \in \bar{N}^{\bar{\mathbb{B}}_{\bar{\xi}_{i+1}}}$ s.t. whenever $H \supset \bar{G}$ is $\bar{\mathbb{B}}^+_{<\bar{\xi}_{i+1}}$ - generic and $c \in H$, then $\dot{\mu}^H = \mu$ is defined as above and hence satisfies IV (a), (e). This means that III, IV are satisfied with $c, \dot{\mu}$ in place of $c_i, \dot{\mu}(i)$. But for any $\mathbb{B}^+_{\leq \xi_i}$ - generic G with $b_i \in G^+$, there is such a pair $c, \dot{\mu}$. This gives us $\dot{c}_i, \ddot{\mu}(i)$ satisfying III, IV.

This completes the construction.

QED (Case 1)

<u>Case 2</u> Case 1 fails.

Then λ is regular and $\lambda > \delta_i = \delta(\mathbb{B}_i)$ for $i < \lambda$. Clearly λ remains regular in $V[G]$ whenever G is $\mathbb{B}_i$-generic and $i < \lambda$. $\mathbb{B}_\lambda$ is the direct limit of $\langle \mathbb{B}_i \mid i < \lambda \rangle$. Moreover, we took the direct limit at all $\tau \in S$, where $S =$ the set of $\tau < \lambda$ s.t. $cf(\tau) = \omega_1$ in V. Since S is stationary in λ, it follows easily that $\mathbb{B}_\lambda$ satisfies the λ-chain condition.

In some ways this case resembles Case 1.2. We again have: $\sup \pi^{i} {}'' \lambda^i < \lambda$ for $i \leq \beta$. Our proof will, in fact, be a modification of that given in Case 1.2. In one important respect, however, our present situation is quite different: In Case 1.2 we defined $\lambda_i = \sup \pi^{i} {}'' \lambda^i$ and observed that $\lambda_i = \sup \sigma^{i} {}'' \lambda^i$ whenever σ is a λ-revision of π. This is no longer true, nor do we know that $\sigma^{i+1}(\lambda_i^{i+1}) = \lambda^i$ for $i < \beta$. Hence we shall abandon our earlier notation and instead define:

<u>Def</u> Let σ be a β-pretower with $\sigma^i(\lambda^i) = \lambda$ for $i \leq \beta$.

$$\lambda_\beta = \lambda_\beta^\sigma =_{Df} \sup \sigma^{\beta}{}''\lambda^\beta$$

$$\lambda_i = \lambda_i^\sigma =_{Df} \sup \sigma^{i,i+1}{}''\lambda^i \text{ for } i < \beta.$$

Happily, we shall be able to work with revisions of π which are <u>good</u> in the following sense:

<u>Def</u> σ is a <u>good</u> revision of π iff σ is a λ-revision and $\lambda_i^\sigma = \lambda_i^\pi$ for $i \leq \beta$

We are able to work with good revisions because of the following:

<u>Fact</u> Let σ be as above. Then

(a) $\lambda_\beta = \sup \lambda \cap C_\nu^N(\mathrm{rng}\,\sigma^\beta)$ if $\nu \leq \lambda_\beta$

(b) $\lambda_i = \sup \lambda^{i+1} \cap C_\nu^{N^{i+1}}(\mathrm{rng}\,\sigma^{i,i+1})$ if $\nu \leq \lambda_i$

proof. We prove (a).

($\leq$) is trivial. We prove ($\geq$). Let $\xi \in \lambda \cap C_\nu^N(\mathrm{rng}\,\sigma^\beta)$. Then $\xi = \sigma^\beta(f)(\eta)$ where $\eta < \nu$ and $f \in N^\beta$ is a map to λ^β.

But then $\eta < \sigma^\beta(\tau)$ for a $\tau < \lambda^\beta$. Set $\mu = \sup f''\tau$. Then $\mu < \lambda^\beta$ by regularity and $\xi < \sigma^\beta(\tau) \in \sigma^{\beta\,''}\lambda^\beta$.

QED (Fact)

It follows that, if σ' is a $\langle \theta, \mathbb{B}, \lambda \rangle, \mathbb{B}_\nu, G$ - revision of a good σ, where $\nu < \lambda_\beta$ and σ' respects σ at ν, then σ' is also good. (Ultimately we shall construct a sequence σ_i where $\sigma_0 = \pi$, σ_{i+1} is a $\langle \theta, \mathbb{B}, \lambda \rangle, \mathbb{B}_{\xi_i}, G_i$ revision of σ_i where σ_{i+1} respects σ_i at $\xi_i = \sigma_i^{\beta_i}(\bar{\xi}_i)$.)

Since we are able to work with good revisions, we write $\lambda_i = \lambda_i^\pi$ for $i \leq \beta$.

. . . .

By induction on $\beta \leq \alpha$ we then prove (*)$_\alpha$, formulated exactly as before, but with the additional stipulation that the revision $\sigma \in V[G]$ be <u>good</u>. The induction hypothesis (**) is formulated exactly as before.

Case 2.1 $\beta = 0$

This is again immediate by §1 Lemma 3.1. $\{\langle\sigma, 0\rangle\}$ is a good revision of $\{\langle\pi, 0\rangle\}$ since $\lambda_0^\sigma = \lambda_0^\pi = \tilde{\lambda}$.

Case 2.2 $\beta = \gamma + 1$.

The proof is again exactly as before; We constructed σ as an "amalgamation" of $\sigma', \bar{\sigma}$, where σ' is a revision of $\pi \upharpoonright \{\dot{\beta}\}$ and $\bar{\sigma}$ is a revision of $\bar{\pi} = \langle \pi^{i\beta} \mid i \leq \gamma \rangle$ w.r.t. N^β. As $\sigma', \bar{\sigma}$ are both good, it follows that σ is good.

There remains the case:

Case 2.3 β is a limit ordinal.

We must take some care in handling this case, since - even when σ is a good revision of π - we do not know that $\pi^{i+1}(\lambda_i) = \sigma^{i+1}(\lambda_i)$, which was an essential property used in Case 1.2.3.

We shall modify the proof in Case 1.2.3 the way the proof in Case 1 of §1 Thm 3 was modified in Case 2.

We again let $\langle \beta_i \mid i<\omega \rangle$ be monotone and cofinal in β with $\beta_0 = 0$ and $\beta_{i+1} = (\tilde{\beta}_i + 1)$. Set:

$$\bar{\xi}_0 = 0 \ ; \ \bar{\xi}_{i+1} = \lambda \tilde{\beta}_i .$$

In the proof of Case 1.2.3 we worked with $\xi_i = \pi^{\beta_i}(\bar{\xi}_i)$, but we no longer know that $\pi^{\beta_i}(\bar{\xi}_i) = \sigma^{\beta_i}(\bar{\xi}_i)$ for arbitrary λ-revisions σ of π.

We define:

Def Let σ_0, σ_1 be good revisions of π.

σ_1 is <u>true to</u> σ_0 <u>at</u> $i<\omega$ iff σ_1 coincides with σ_0 at $\sigma_0^{\beta_i,\beta}(\bar{\xi}_i)$.

σ_1 is <u>true to</u> σ_0 <u>on</u> $u<\omega$ iff σ_1 is true to σ_0 at each $i \in u$.

Thus if σ_1 is true to σ_0 at i, we have: $\sigma_1^{\beta_i}(\bar{\xi}_i) = \sigma_0^{\beta_i}(\bar{\xi}_i)$ and $\sigma_1^{\beta_i h}(\bar{\xi}_i) = \sigma_0^{\beta_i h}(\bar{\xi}_i)$ for $\beta_i \leq h \leq \beta$.

But this means that

$\sup(\sigma_1{}^{\tilde{\beta}_{i-1}}{}''\lambda^{\tilde{\beta}_{i-1}}) = \sup(\sigma_0{}^{\hat{\beta}_{i-1}}{}''\lambda^{\hat{\beta}_{i-1}})$
for $i > 0$, since if σ is any good revision, then σ^{β_i} takes $\bar{\xi}_i$ cofinally to $\sigma^{\beta_i}(\bar{\xi}_i)$, since $\bar{\xi}_i$ is ω-cofinal in N^{β_i}.

Fix an arbitrary sequence $\langle \eta_i \mid i<\omega \rangle$ which is monotone and cofinal in λ_β with $\eta_0 = 0$.

We construct $a_i, \dot{\pi}_i$ $(i<\omega)$ s.t.

(I) (a) $a_i \in \mathbb{B}_{\eta_i}$, $a_0 = 1$

(b) $h_{\eta_l}(a_i) = a_l$ for $l \leq i$

(c) $\dot{\pi}_i \in V^{\mathbb{B}_{\eta_i}}$

(II) Let $G \ni a_i$ be $\mathbb{B}_{\eta_i}$-generic. Set; $G(\nu) = G \cap \mathbb{B}_\nu$ $(\nu \leq \eta_i)$; $\pi_h = \dot{\pi}_h^G$ $(h \leq i)$.
(Hence $\pi_h = \dot{\pi}_h^{G(\eta_h)}$.)

(a) π_i is a good $\langle \theta, \mathbb{B}, \lambda \rangle$-revision of π and $\pi_0 = \pi$

(b) Let n_h = the least n s.t. $\pi_h^{\beta_n}(\bar{\xi}_h) > \eta_{h+1}$. Then π_i is true to π_h on $n_h + 1$ for $h \leq i$.

We again let $\langle x_\ell^i \mid \ell < \omega \rangle$ be the N-least enumeration of N^i for $i \leq \beta$. We also let $\langle\langle \nu_i, z_i \rangle \mid i < \omega \rangle$ be an enumeration of $\{\langle \nu, z \rangle \mid \nu \leq \beta \wedge z \in N^\nu\}$ with infinite repetitions of each point.

In addition to a_i, π_i, we define $\dot{\Gamma}_i^0, \dot{\Gamma}_i^1$ $(i < \omega)$ s.t. $\dot{\Gamma}_i^\ell \in V^{\mathbb{B}_{\gamma_i}}$.

III Let G be as in II.
Set: $\Gamma_h^\ell = \dot{\Gamma}_h^{\ell G}$ for $h \leq i$, $\ell < 2$. Then
$\Gamma_i^0 = \langle\langle \xi_j, b_j, \dot{\sigma}_j \rangle \mid j \leq m_i \rangle$; $\Gamma_i^1 = \langle \dot{u}_j \mid j < m_i \rangle$
where:

(a) $\Gamma_h^\ell \subset \Gamma_i^\ell$ for $h \leq i$, $\ell < 2$.

(b) $\xi_j = \pi_i^{\beta_j}(\overline{\xi}_j)$ for $j \leq m_i$

(c) $b_0 = 1$ and $b_j \in \mathbb{B}^+_{<\xi_j}$ for $0 < j \leq m_i$

Note: We let $\mathbb{B}^+_{<0} =_{\text{Df}} \mathbb{B}_0 = 2$.

(d) $b_\ell = h_{\xi_\ell}(b_j)$ for $\ell \leq j \leq m_i$

(e) $\dot{\sigma}_j \in V^{\mathbb{B}_{\xi_j}}$ for $j \leq m_i$, $\dot{u}_j \in V^{\mathbb{B}_{\xi_j}}$ for $j < m_i$

Note We also write: $\xi_j^\ell = \pi_i^{\beta_j \ell}(\overline{\xi}_j)$ for $\beta_j \leq \ell \leq \beta$. Then $\xi_j^\ell = \pi_h^{\beta_j \ell}(\overline{\xi}_j)$ for $j \leq m_h$ by II (b).

Since $\eta_i < \xi_{n_i}$, we can extend our generic $G \subset \mathbb{B}_{\eta_i}$ to a $\mathbb{B}^+_{<\xi_{n_i}}$ - generic G' s.t. $b_{n_i} \in G'^+$.

$\underline{\mathrm{IV}}$ Let G, Γ^l_h $(h \le i, l = 0, 1)$ be as in $\underline{\mathrm{III}}$

Let G' be $\mathbb{B}^+_{<\xi_{n_i}}$ - generic s.t. $G' \supset G$ and $b_{n_i} \in G'^+$. Set:

$G'_j = G' \cap \mathbb{B}_{<\xi_j}$, $\sigma_j = \dot{\sigma}_j^{G'}$ for $j \le n_i$

$u_j = \dot{u}_j^{G'}$ for $j < n_i$.

Then:

(a) $\underline{\mathrm{II}}$ (a), (b) and (f) - (i) of Case 1.2.3 hold at $j \le n_i$ (i.e. they hold with "i" replaced by "j" and "G" replaced by "G'").

(b) $\underline{\mathrm{II}}$ (c) - (e) and (j) of Case 1.2.3 hold at $j < n_i$

(c) Let $j = 0 = h$ or $j = n_h - 1$, where $h < i$. Then $\{\xi_l \mid l \le n_h\} \subset u_j$.

Note (c) guarantees that for all $j \le m_i$, σ_j is true to π on m_0+1 and true to π_h on m_h+1 if $h < i$ and $j \ge m_h - 1$.

(d) Let $j = m_i - 1$. Set: $m_i^+ =$ the least m s.t. $\sigma_j^{\beta_m}(\overline{\xi}_m) > \eta_h$, where h is least s.t. $\eta_h > \xi_{m_i}$ (hence $h > i+1$). Then $\{\sigma_j^{\beta_\ell}(\overline{\xi}_\ell) \mid \ell \le m_i^+\} < u_j$.

Note (d) will enable us to continue beyond i while preserving (c).

Finally:

V Let $i = k+1$, where G is as in II. Then

(a) $h_{\eta_i}(b_{m_k}) \in G$

(b) $\pi_i = (\dot{\sigma}_{m_k - 1})^G$

Note $\xi_{m_k - 1} \le \eta_k$; hence $b_{m_k - 1} \in G$ and $(\dot{\sigma}_{m_k-1})^G = (\dot{\sigma}_{m_k - 1})^{G_{m_k - 1}}$ is defined.

We now verify (*) at β, given $a_i, \dot{\pi}_i, \dot{\Gamma}_i^0, \dot{\Gamma}_i^1$ $(i<\omega)$ satisfying I–V. Set $a = \bigcap_{i<\omega} a_i$. Then $a \in \mathbb{B}^+_{<\lambda_\beta}$. Let G be $\mathbb{B}_{<\lambda_\beta}$-generic s.t. $a \in G$. Set: $G(\nu) = G \cap \mathbb{B}_\nu$ for $\nu < \lambda_\beta$. Then $a_i \in G(\eta_i)$. Let $\Gamma_i^\ell = (\dot{\Gamma}_i^\ell)^{G_{\eta_i}}$ for $\ell = 0, 1$. Set: $\Gamma^\ell = \bigcup_{i<\omega} \Gamma_i^\ell$. Then $\Gamma^0 = \langle\langle \xi_i, b_i, \dot{\sigma}_i\rangle \mid i<\omega\rangle$ and $\Gamma^1 = \langle u_i \mid i<\omega\rangle$. Set:

$G_i = G \cap (\mathbb{B}_{<\xi_i})$ for $i<\omega$. Then:

$h_{\eta_i}(b_{m_k}) \in G(\eta_i)$ and $b_{m_i - 1} = h_{\xi_{m_i-1}}(b_{m_i}) \in G^+_{m_i-1}$.

Hence $b_i \in G_i^+$ for $i<\omega$ and $b \in G^+$, where $b = \bigcap_i b_i$. We can then form $\sigma_i = \dot{\sigma}_i^{G_i}$ for $i<\omega$. The rest of the proof is exactly like Case 1.2.3.

This proves (*) at β.

It remains only to construct $c_i, \dot{\pi}_i, \dot{\Gamma}_i^0, \dot{\Gamma}_i^1$ and verify I–V.

In addition to a_i, π_i, $\dot{\Gamma}_i^0$, $\dot{\Gamma}_i^1$ we shall also construct $\dot{\Gamma}_i^2$ s.t.

<u>VI</u> Let G be as in <u>II</u>. Set $\Gamma_i^2 = \dot{\Gamma}_i^{2G}$.

Then $\Gamma_i^2 = \langle\langle \mathring{\sigma}_j \; \ddot{\mu}(j) \rangle \mid j < n_i \rangle$.

If $G' \supset G$ is as in <u>IV</u> and $\sigma_j = \dot{\sigma}_j^{G'}$, $\dot{\mu}(j) = \ddot{\mu}(j)^{G'}$ for $j < n_i$, then <u>III</u> (a)–(c) of Case 1, 2, 3 and <u>IV</u> of Case 1, 2, 3 hold for $j < n_i$ (with "j" in place of "i" and "G'" in place of "G").

<u>Note</u> $\dot{\mu}(j)$ was a step toward constructing a_{j+1}, $\dot{\sigma}_{j+1}$.

Much of the construction - and the verification - will be a repetition of Case 1, 2, 3 and we shall omit the details here.

Case 1 $i=0$ Set: $a_0 = 1$, $\dot{\pi}_0 = \check{\pi}$

I, II are then trivial.

We then define Γ_0^l $(l=0,1,2)$ so as to satisfy III (b)-(e), IV, and VI and set: $\dot{\Gamma}_0^l = \check{\Gamma}_0^l$.

We set: $\xi_j = \pi^{B_i}(\bar{\xi}_j)$ for $j \leq n_0$. By induction on $j \leq n_0$ we then define $b_j, \dot{\sigma}_j$. For $j < n_0$ we also define $\dot{u}_j, \dot{c}^j, \dot{\mu}(j)$. By induction on j we verify:

(1) Let G' be $\mathbb{B}^+_{<\xi_j}$ - generic s.t. $a_j \in G'$. Set: $G_j' = G' \cap \mathbb{B}_{<\xi_j}$, $\sigma_j = \dot{\sigma}_j^{G'}$ for $j \leq n_i$, $u_j = \dot{u}_j^{G'}$; $c_j = (\dot{c}_j)^{G'}$, $\mu(j) = \dot{\mu}(j)^{G'}$ for $j < n_i$. Then IV (a)-(d) hold at j, as does VI.

The only difference to the construction in Case 1.2.3 is the definition of u_j. We must ensure that $\xi_l \in u_j$ for $l \leq n_i$ and that IV (d) is satisfied. We shall, in fact, first define a $\dot{u}_j$ and then -

if necessary - expand it to $\mathring{u}_i$ so that IV (d) holds. $\mathring{u}_i$ is then a term set.

(2) Let G be as in (1), $u_i' = \mathring{u}_i'^{G}$, $u_i = \mathring{u}_i^{G}$. Then $u_i = u_i'$ if $i < n_i - 1$. Otherwise

$u_i = u_i' \cup \{\dot{\sigma}_i^{\beta_\ell, \beta}(\xi_\ell) \mid \ell \leq m_i^+\}$, where

m_i^+ is defined as in IV (d).

This defines $\mathring{u}_i$ from $\mathring{u}_i'$.

For $i = 0$ we set: $b_0 = 1$, $\dot{\sigma}_0 = \check{\pi}$.

We then set $\mathring{u}_0' = \check{u}_0'$, where

$u_0' = u \cup \{\xi_\ell^\beta \mid \ell \leq n_i\}$. (This differs from Case 1, 2, 3, where we set: $\mathring{u}_0 = \check{u}$.)

$\mathring{u}_0$ is then defined by (2). We then define c_0, μ_0 exactly as in Case 1, 2, 3 (but now using our new u_0) and set $\dot{c}^0 = \check{c}^0$, $\ddot{\mu}(0) = \check{\mu}_0$.

Now let $j = k+1$. Using b_k, $\dot{\sigma}_k$, $\mathring{u}_k$, $\dot{c}^k$, $\ddot{\mu}(k)$ we define b_j, $\dot{\sigma}_j$ exactly as in Case 1, 2, 3. The verifications are the same. We then define $\mathring{u}_j'$ exactly the way $\mathring{u}_j$ was defined in Case 1, 2, 3, if $j < n_i$. This gives us $\mathring{u}_j$ by (2). We then define $\dot{c}_i$, $\dot{\mu}(j)$ exactly as in Case 1, 2, 3. The verification are the same. QED (Case 1)

<u>Case 2</u> $i = n+1$.

We first define a_i. $\dot{\Gamma}^l_k$ $(l = 0,1)$ give us Terms $\dot{\xi}, \ddot{\sigma}, \dot{b}, \ddot{u}', \dot{m}_k$ ...

$a_k \Vdash_{\eta_k} (\dot{m}_k < \omega \wedge \dot{\xi}, \ddot{\sigma},$ are function on $\dot{m}_k + 1 \wedge$
$\wedge\ \ddot{u}$ is a function on $\dot{m}_k)$;

$a_k \Vdash_{\eta_k} \dot{\Gamma}^0_k = \langle\langle \dot{\xi}(l), \dot{b}(l), \ddot{\sigma}(l)\rangle \mid l \leq \dot{m}_k\rangle$

where $a_k \Vdash_{\eta_k} (\dot{\xi}(l) \leq \check{\eta}_i < \dot{\xi}(\dot{m}_k)$ for $l < \dot{m}_k)$.

Let $\dot{a} \in V^{\mathbb{B}_{\eta_k}}$ s.t.

$\Vdash_{\eta_k} (\check{a}_k \in \dot{G} \wedge \dot{a} = \check{h}_{\eta_i}(\dot{b}(\dot{m}_k))/\dot{G}) \vee (\check{a}_k \notin \dot{G} \wedge \dot{a} = 0)$,

$\dot{G}$ being the canonical generic name.

Set: $a_i =$ the unique $a \in \mathbb{B}_{\eta_i}$ s.t. $\Vdash_{\eta_k} \check{a}/\dot{G} = \dot{a}$.

Then $a_i \in \mathbb{B}_{\eta_i}$ and $h_{\eta_k}(a_i) = [\![\check{a}/\dot{G} \neq 0]\!] = a_k$.

Hence I (a) holds, as does V (a), since if $G \ni a_i$ in $\mathbb{B}_{\eta_i}$ - generic and $\bar{G} = G(\eta_k)$, then $a_i/\bar{G} = h_{\eta_i}(b_{m_k})/\bar{G} \in G/\bar{G} =$
$= \{b/\bar{G} \mid b \in G\}$. Hence $h_\eta(b_{m_k}) \in G$.

We then let $\dot{\pi}_i$ be a term s.t. $a_k \Vdash_{\eta_k} \dot{\pi}_i = \ddot{\sigma}^{\dot{G}}(\dot{m}_k+1)$.

If $G \ni a_i$ is $\mathbb{B}_{\eta_i}$ - generic and $\overline{G} = G(\eta_i)$, we then have: $\dot{\pi}_i^G = (\ddot{\sigma}^{\overline{G}}(m_k-1))^G = (\dot{\sigma}_{m_k-1})^G$.

Hence V(b) holds, I(b) is trivial. II follows by the fact that IV holds at k.

We now turn to the construction of $\dot{\Gamma}_i^l$ $(l=0,1,2)$. Let S be the set of sequences $s = \langle s_0, s_1, s_2, \xi_0, \ldots, \xi_n \rangle$ s.t. $\xi_0 < \cdots < \xi_{n-1} \leq \eta_{i+1} < \xi_n$ and $a_s \neq 0$, where $a_s = a_i \cap [\![\bigwedge_{l\leq 2} \check{s}_l = \dot{\Gamma}_k^l \wedge \bigwedge_{l=1}^{n} \dot{\pi}_i^{\check{\beta}_l}(\check{\xi}_l) = \check{\xi}_l]\!]_{\eta_i}$.

Fix $s \in S$. Let $\bar{a} = a_s$. Suppose that:

(3) $G \ni \bar{a}$ is $\mathbb{B}_{\eta_i}$ - generic, $\pi_h = \dot{\pi}_h^G$ for $h \leq i$, and $\Gamma_k^l = \dot{\Gamma}_k^{lG}$ for $l = 0,1,2$.

Let $\Gamma_k^0 = s_0 = \langle\langle \xi_l, b_l, \dot{\sigma}_l \rangle \mid l \leq m_k \rangle$

$\Gamma_k^1 = s_1 = \langle \dot{u}_l \mid l < m_k \rangle$

$\Gamma_k^2 = s_2 = \langle \langle \dot{c}_l, \ddot{\mu}(l) \rangle \mid l < m_k \rangle$

Clearly $m_k \leq n$, and ξ_l as given by Γ_k^0 is the same as ξ_l given by s for $l \leq m_k$.

We define $\Gamma_i^l \supseteq \Gamma_k^l$ $(l=0,1,2)$ satisfying III (a) – (e), IV and VI for all G as in (3).

If $n = n_k$, we simply set: $\Gamma_i^l = \Gamma_k^l$. All verifications are trivial.

Now let $n > n_k$. We shall, of course, have: $n_i = n$ and the $\xi_l = \pi_i^{\beta_l}(\bar{\xi}_l)$ are given by s. Note that, since $\bar{a} \in G$ and $h_{\gamma_i}(b_{n_k}) \in G$, we have: $\bar{a} \wedge b_{n_k} \neq 0$. Set: $\bar{b}_{n_k} = \bar{a} \wedge b_{n_k}$.

We successively define $\bar{b}_j$, $\dot{\sigma}_j$ $(n_k \leq j \leq n_i)$ s.t. $\bar{b}_h = h_{\xi_h}(\bar{b}_j)$ for $n_k \leq h \leq j \leq n_i$, and whenever $G' \supset G$ in $\mathbb{B}^+_{<\xi_j}$ - generic with $\bar{b}_j \in G'^+$, then IV (a) holds at j.

At $j < n_i$, we also construct $\dot{u}_j$ s.t. IV (b), (c), (d) hold at j.

For $j < n_i$ we also construct $\dot{c}_j$, $\ddot{u}(j)$ s.t. VI holds at j.

We proceed exactly as in Case 1, but with $\mathring{\sigma}_{m_k}, \vec{b}$ in place of $\mathring{\sigma}_0 = \check{\pi}, b_0 = 1$. As before we first construct $\ddot{u}_j'$ by imitating the proof in Case 1.2.3 and then expand it to $\ddot{u}_j$ according to (2). We first define $\ddot{u}_{m_k}'$ exactly as $\ddot{u}_{m_k}$ was defined in Case 1.2.3. Applying (2) we get $\dot{u}_{m_k}$ and we then define $\dot{c}_{m_k}, \ddot{\mu}(m_k)$ exactly as in Case 1.2.3. For $j = h+1$, $h \geq m_k$, we use Lemma 2 to get $\bar{b}_j, \dot{\sigma}_j$ with the desired properties, where $h_{\bar{\xi}_h}(\bar{b}_j) = \bar{b}_h$. The verification are straightforward. When we are done, we define $b_j = (b_{m_k} \setminus \bar{b}_{m_k}) \cup \bar{b}_j$ for $m_k \leq j \leq m_i$. Then $b_{\bar{\xi}_\ell} = h_{\bar{\xi}_\ell}(b_j)$ for all $\ell \leq j \leq m_i$, and III, IV (a)-(d) and V hold at j whenever $G \ni \bar{a}$ in $\mathbb{B}_{\eta_i}$-generic (since then for $G' \supset G$ which is $\mathbb{B}_{<\bar{\xi}_j}$-generic and $b_j \in G'^+$, we have: $b_i \cap \bar{a} = \bar{b}_i \in G'^{+}$.) Set:

$\Gamma_i^0 = \langle\langle \xi_l, b_l, \dot{\sigma}_l \rangle \mid l \leq m_i \rangle$,

$\Gamma_i^1 = \langle \dot{u}_l \mid l < m_i \rangle$, $\Gamma_i^2 = \langle\langle \dot{c}_l, \ddot{u}(l) \rangle \mid l < m_i \rangle$.

Then all conditions are satisfied.

Γ_i^l depends on s, of course, and we shall denote it by $\Gamma_i^l(s)$.

Since $s \neq s' \to a_s \cap a_{s'} = 0$ for $s, s' \in S$, there is $\dot{\Gamma}_i^l \in V^{\mathbb{B}_{\eta_i}}$

s.t. $[\![\dot{\Gamma}_i^l = \check{\Gamma}_i^l(s)]\!] = a_s$

for $s \in S$. $\dot{\Gamma}_i^l$ $(l = 0, 1, 2)$ then have the desired properties.

This completes the construction.

QED (Thm 3)

A glance at the definition of "α-proper" in [PF] shows:

<u>Thm 4</u> Every α-proper forcing is α-subproper.

We now show that every subcomplete forcing is ω_1-subproper. In fact, we show a slightly stronger result. We define:

Let $\alpha < \omega_1$

<u>Def</u> $\mathbb{B}$ is α-<u>subcomplete</u> <u>as</u> <u>witnessed</u> by by θ iff $\mathbb{B} \in H_\theta$ and the following holds: Let $N = L^A_\tau$ be a ZFC^- model s.t. $H_\theta \subset N$ and $\theta < \tau$. Let $\pi = \langle \pi^i \mid i \leq \alpha \rangle$ be an α-tower for N with $\pi^i : N^i \prec N$ and $\pi^i(\theta^i, \mathbb{B}^i) = \theta, \mathbb{B}$ for $i \leq \alpha$. Let $u \subset N$ be finite. Let $\bar{G}$ be $\mathbb{B}^0$-generic over N^0. Then there is $b \in \mathbb{B} \setminus \{0\}$ s.t. whenever $G \ni b$ is $\mathbb{B}$-generic, then there is a $\sigma \in V[G]$ which is a $\langle \theta, \mathbb{B} \rangle, \mathbb{B}, G$-revision of π coinciding with π on u and s.t. $G^0 = \bar{G}$.

$\mathbb{B}$ is α-<u>subcomplete</u> iff it is α-subcomplete as witnessed by some θ.

θ <u>verifies</u> the α-subcompleteness of $\mathbb{B}$ iff every cardinal $\theta' \geq \theta$ witnesses the α-subcompleteness of $\mathbb{B}$.

It again follows that if θ witnesses the α-subcompleteness of $\mathbb{B}$, then $(2^\theta)^+$ verifies the α-subcompleteness of $\mathbb{B}$. We can, of course, also introduce a parameter p into the definition to get the notion: $\mathbb{B}$ is α-subcomplete as witnessed by $\langle p, \theta\rangle$. It again turns out that every such $\mathbb{B}$ is, in fact, fully α-subcomplete — a fact which is used tacitly in verifying α-subcompleteness.

We shall take "subcomplete" as meaning "0-subcomplete". This is a slight change from our earlier definition, and is analogous to the changes made to the definition of "subproper" in §1. "ω_1-subcomplete" of course means "α-subcomplete for all $\alpha < \omega$". We then get:

Thm 5 Let $\mathbb{B}$ be subcomplete. Then $\mathbb{B}$ is ω_1-subcomplete.

proof:

We prove by induction on $\alpha < \omega_1$ that $\mathbb{B}$ is α-subcomplete. The case $\alpha = 0$ is immediate. Now let $\alpha = \beta + 1$.

Then there is $b \in \mathbb{B}$ s.t. whenever $G \ni b$ is $\mathbb{B}$-generic, there is $\sigma \in V[G]$ s.t. σ is a $\langle \theta, \mathbb{B} \rangle, \mathbb{B}, G$-revision of π coinciding with $\pi \upharpoonright \alpha$ on $(\pi^{\beta,\alpha})^{-1}{}''u$ and s.t. $G^{\circ} = \bar{G}$. Arguing as before, this fact is expressible in N uniformly in parameters from $\mathrm{rng}(\pi^{\alpha})$. Since $\pi^{\alpha}: N^{\alpha} \prec N$, the corresponding statement holds in N^{α}. (cf. Case 1.1.1 and 1.1.2 of Thm 3). Hence there are G', σ' s.t. G' is $\mathbb{B}^{\alpha}$-generic over N^{α}, $\sigma' \in N[G']$ is a $\langle \theta^{\alpha}, \mathbb{B}^{\alpha} \rangle, \mathbb{B}^{\alpha}, G'$-revision of $\pi' = \langle \pi^{h\alpha} \mid h \leq \beta \rangle$ coinciding with π' on $(\pi^{\beta})^{-1}{}''u$ and s.t. $G'^{\circ} = \bar{G}$. But then there is $b \in \mathbb{B}$ s.t. if $G \ni b$ is $\mathbb{B}$-generic, there is $\sigma'' \in V[G]$ which is a $\langle \theta, \mathbb{B} \rangle, \mathbb{B}, G$-revision of $\pi \upharpoonright \{\alpha\}$ coinciding with $\pi \upharpoonright \{\alpha\}$ on u and s.t. $G^{\alpha} = G'$. Define $\sigma \in V[G]$ by:

$$\sigma^{i} = \begin{cases} \sigma''^{\alpha} & \text{if } i = \alpha \\ \sigma''^{\alpha} \circ \sigma'^{i} & \text{if } i \leq \beta. \end{cases}$$

Then σ has the desired properties.

QED (Case $\alpha = \beta + 1$).

We now turn to the case that α is a limit ordinal. We again let $\langle \beta_i \mid i<\omega \rangle$ be monotone and cofinal in α s.t. $\beta_0 = 0$ and $\beta_{i+1} = \tilde{\beta}_i + 1$. Let $\langle x_i \mid i<\omega \rangle$ be an enumeration of N^α. Let $\langle \langle \nu_i, z_i \rangle \mid i<\omega \rangle$ be an enumeration of $\{ \langle \nu, z \rangle \mid \nu < \alpha \wedge z \in N^\nu \}$ with infinite repetitions of each point.

Let $u \subset N^\alpha$ be finite and $\bar{G} \subset \mathbb{B}^0$ be $\mathbb{B}^0$-generic over N^0.

We inductively construct $\tilde{G}_i, \tilde{\sigma}_i$ $(i<\omega)$ s.t.

- $\tilde{G}_i$ is $\mathbb{B}^{\beta_{i+1}}$-generic over $N^{\beta_{i+1}}$
- $\tilde{\sigma}_i \in N^{\beta_{i+1}}[\tilde{G}_i]$ is a $\langle \theta^{\beta_{i+1}}, \mathbb{B}^{\beta_{i+1}} \rangle, \mathbb{B}^{\beta_{i+1}}, \tilde{G}_i$ - revision of $\tilde{\pi}^i = \langle \pi^{h, \beta_{i+1}} \mid h \leq \tilde{\beta}_i \rangle$ coinciding with $\tilde{\pi}^i$ on $u^i = (\pi^{\tilde{\beta}_i, \alpha})^{-1}{}'' u_i$ and s.t. $\bar{G} = (\tilde{G}_i)^0$ (i.e. $\tilde{\sigma}_i^0{}'' \bar{G} \subset \tilde{G}_i$).
- $u_i \subset N^\alpha$ is finite s.t. $u \cup \bigcup_{h<i} (u_i \cup \{x_h\}) \subset u_i$
- If $\nu_i \leq \tilde{\beta}_i$, then $\pi^{\beta_{i+1}, \alpha} \tilde{\sigma}_i^{\nu_i}(z_i) \in u_{i+1}$.
- $\sigma_h^j = \sigma_i^{j, \beta_{h+1}}$ for $h<i$, $j \leq \tilde{\beta}_h$.

$\tilde{G}_0, \tilde{\sigma}_0$ are given by the induction hypothesis relativized to N^{β_1}

Now let $i = k+1$, where $\tilde{G}^k, \tilde{\sigma}^k, u_k$ are given. Set: $u_i = u_k \cup \{x_k\} \cup \{z\}$, where:

$$z = \begin{cases} \pi^{\beta_i, \alpha} \tilde{\sigma}_k^{\nu_k}(z_k) & \text{if } \nu_k \leq \tilde{\beta}_k \\ \emptyset \text{ if not.} \end{cases}$$

Set: $u' = (\sigma^{\beta_i, \alpha})^{-1} u_i$. Using the induction hypothesis and relativizing to $N^{\beta_{i+1}}$, we get $\tilde{G}_i, \sigma'$ s.t.

- $\tilde{G}_i$ is $\mathbb{B}^{\beta_{i+1}}$-generic over $N^{\beta_{i+1}}$
- σ' is a $\langle \theta^{\beta_{i+1}}, \mathbb{B}^{\beta_{i+1}} \rangle, \mathbb{B}^{\beta_{i+1}}, \tilde{G}_i$ - revision of $\pi' \doteq \langle \pi^{h, \beta_{i+1}} \mid \beta_i \leq h \leq \tilde{\beta}_i \rangle$.
- σ' coincides with π' on u' and is s.t. $(\tilde{G}_i)^{\beta_i} = \tilde{G}_k$.

We then amalgamate $\tilde{\sigma}_k, \sigma'$ by setting:

$$\tilde{\sigma}_i^h = \begin{cases} \sigma'^h & \text{if } \beta_i \leq h \leq \tilde{\beta}_i \\ \sigma'^{\beta_i} \tilde{\sigma}_k^h & \text{if } h < \beta_i. \end{cases}$$

$\tilde{\sigma}_i$ then has the desired properties.

We can unambiguously define:

Def $\tilde{\sigma}^{hi} = \tilde{\sigma}_i^{hi}$ for $h \leq i \leq \tilde{\beta}_i$.

(Hence $\tilde{\sigma}^{h, \beta_{i+1}} = \tilde{\sigma}_i^h$.)

By a familiar method we prove:

Claim $\langle N^h \mid h < \alpha \rangle$, $\langle \tilde{\sigma}^{hi} \mid h \leq i < \alpha \rangle$ has a direct limit of the form:

N^α, $\langle \tilde{\sigma}^h \mid h < \alpha \rangle$.

proof.

Let $\mathfrak{A}$, $\langle f_i^h \mid h < \alpha \rangle$ be a direct limit.

Since every $x \in N^\alpha$ lies in a U_i, we can define an embedding $k : N^\alpha \to \mathfrak{A}$ by:

$k(x) = \tilde{\sigma}^h(\bar{x})$, where $\tilde{\sigma}^{hi}(\bar{x}) = (\tilde{\pi}^{i\alpha})^{-1}(x)$ for all $i \geq h$. The choice of $h, \bar{x}$ obviously doesn't matter. It follows easily that $k : N^\alpha \prec \mathfrak{A}$. But k is onto, since if $z = \tilde{\sigma}^h(\bar{z}) \in \mathfrak{A}$, then $\langle h, \bar{z} \rangle = \langle \nu_i, z_i \rangle$ for a $i \geq h$ s.t. $\nu_i \leq \tilde{\beta}_i$.

Thus, if $\tilde{z} = \sigma_i^{\nu_i}(\bar{z})$; then, letting $x = \pi^{\beta_i+1, \alpha}(\tilde{z})$, we have $\sigma^{\beta_{i+1}, l}(\tilde{z}) = x \in U_{i+1}$ and hence $= (\tilde{\pi}^{l,\alpha})^{-1}(x)$ for $l \geq i+1$. Hence $z = k(x)$. QED (Claim)

Set: $\tilde{G} = \bigcup_{i<\omega} \tilde{\sigma}^{\beta_i}{}''\tilde{G}_i$. Since $\tilde{\sigma}^{\beta_i, \beta_i}{}''\tilde{G}_i \subset \tilde{G}_i$ and $\tilde{G}_i$ is $\mathbb{B}^{\beta_i}$-generic over N^{β_i}, it follows

easily that $\tilde{G}$ is $\mathbb{B}^\alpha$ - generic over N^α.
By subcompleteness, there is then $b \in \mathbb{B}$ s.t. whenever $G \ni b$ is $\mathbb{B}$ - generic, then there is $\sigma' \in V[G]$ s.t. σ' is a $\langle \theta, \mathbb{B} \rangle, \mathbb{B}, G$ - revision of $\pi \upharpoonright \{\alpha\}$, coinciding with $\pi \upharpoonright \{\alpha\}$ on u and s.t. $G^\alpha = \tilde{G}$. But then we can form $\sigma \in V[G]$ by amalgamating $\langle \tilde{\sigma}^h \mid h < \alpha \rangle$ and σ', setting

$$\sigma^h = \begin{cases} \sigma'^\alpha & \text{if } h = \alpha \\ \sigma'^\alpha \circ \tilde{\sigma}^h & \text{if } h < \alpha . \end{cases}$$

σ has the desired properties.

QED (Thm 5)