Appendix

In §4 we showed that, if CH holds, then $\mathbb{N}'$ is Dee-subproper. For the sake of completeness we now prove the companion theorem:

Thm Assume CH, Then $\mathbb{N}$ is subcomplete.

(In [SPSC] we proved this on the additional assumption that $2^{\omega_1} = \omega_2$, using the fact, proven in [LF], that $\mathbb{N}$ is then equivalent to a certain $\mathcal{L}$-forcing. Our present proof is more direct.)

We use the following definition of $\mathbb{N}$:

Def $\Omega = (\omega_2)^{<\omega}$ = the set of all monotone $s : m \to \omega_2$, where $m \in \omega$, Ω is then a tree, its tree relation $\leq_\Omega$ being inclusion on Ω.

For $s \in \Omega$ we set $|s| = \mathrm{dom}(s)$, $|s|$ is called the height of s. We write $s|t$ to mean that s, t are incomparable in Ω.

We then set:

<u>Def</u> $\mathbb{N}$ = the set of all subtrees T of Ω s.t.

- $s \leq t \in T \rightarrow s \in T$
- If $s \in T$, then $\{t \in T \mid s \leq t\} = \omega_2$.

$\mathbb{N}$ is the set of <u>Namba conditions</u>, partially ordered by inclusion. We list some facts about $\mathbb{N}$:

<u>Fact 1</u> If CH holds, then $\mathbb{N}$ adds no new reals.

<u>Fact 2</u> $\mathbb{N}$ adds a cofinal ω-sequence to ω_1. In fact, if G is $\mathbb{N}$-generic, then $c^G = \bigcup \bigcap G$ is such a sequence. G is then recoverable from $c = c^G$ by:
$G = \{T \in \mathbb{N} \mid \bigwedge n \; c \restriction n \in T\}$.

<u>Def</u> We call $c : \omega \to \omega_2^V$ $\mathbb{N}$-<u>generic</u> iff $G = \{T \in \mathbb{N} \mid \bigwedge n \; c \restriction n \in T\}$ is $\mathbb{N}$-generic (hence $c = c^G$).

Although we shall not use it, we also mention the fact:

<u>Fact 3</u> $\mathbb{N}$ does not collapse ω_1 (in fact it preserves stationary subsets of ω_1).

It would be possible to prove the Theorem by a - somewhat simpler - imitation of the proof that $\mathbb{N}'$ is Dee-subproper, redefining "good sequence" to mean simply a sequence monotone and cofinal in ω_2.

Instead, however, we shall derive the Theorem from the following lemma, which is of interest in its own right and is provable in ZFC without use of CH:

<u>Lemma 1</u> Let G be $\mathbb{N}$-generic. Let $b \in V[G]$ s.t. $b : \omega \to \omega_2$ is monotone and cofinal in ω_2. Then b is an $\mathbb{N}$-generic sequence and $V[G] = V[b]$.

For now we defer the proof of Lemma 1, and show first that it implies the Theorem.

Assume CH. Let $\theta > 2^{2^{\omega_2}}$.

We prove:

<u>Main Claim</u> $\langle \theta, \mathbb{D} \rangle$ witnesses the subcompleteness of $\mathbb{N}$.

Let $N = L_\tau^A$ be a ZFC$^-$ model s.t. $H_\theta \subset N$ and $\theta < \tau$. Let $\pi : \bar{N} \prec N$ s.t. $\bar{N}$ is countable and full, where $\pi \in N$. Let $\pi(\bar{\theta}, \bar{\mathbb{N}}, \bar{s}) = \theta, \mathbb{N}, s$. Let $\bar{G}$ be $\bar{\mathbb{N}}$-generic over $\bar{N}$.

<u>Claim</u> There is $T \in \mathbb{N}$ s.t. whenever $G \ni T$ is $\mathbb{N}$-generic, there is $\sigma \in V[G]$ s.t.

(a) $\sigma : \bar{N} \prec N$

(b) $\sigma(\bar{\theta}, \bar{\mathbb{N}}, \bar{s}) = \theta, \mathbb{N}, s$

(c) $C^{N}_{\omega_2}(\mathrm{rng}\,\sigma) = C^{N}_{\omega_2}(\mathrm{rng}\,\pi)$

(d) $\sigma''\bar{G} \subset G$.

We work in $V[G^*]$, where G^* is any $\mathbb{N}$-generic set. Let $c^* = c^{G^*}$.

Set: $N' = \langle N, \theta, \mathbb{N}, s, \pi \rangle$. Set: $\sigma^* : \tilde{N}' \overset{\sim}{\longleftrightarrow} Z$, where Z is the smallest $Z \prec N'$ s.t. $c^* \subset Z$. Let $\tilde{N}' = \langle \tilde{N}, \tilde{\theta}, \tilde{\mathbb{N}}, \tilde{s}, \tilde{\pi} \rangle$.

Exactly as in §4 Lemma 4.6 we show:

(1) There is $\tilde{\sigma} : \bar{N} \prec \tilde{N}$ in V s.t.

- $\tilde{\sigma}$ takes $\bar{\omega}_2 = \omega_2^{\bar{N}}$ cofinally to $\tilde{\omega}_2 = \omega_2^{\tilde{N}}$
- $C^{\tilde{N}}_{\tilde{\omega}_2}(\mathrm{rng}\,\tilde{\sigma}) = C^{\tilde{N}}_{\tilde{\omega}_2}(\mathrm{rng}\,\tilde{\pi})$.
- $\tilde{\sigma}(\bar{\theta}, \bar{\mathbb{N}}, \bar{s}) = \tilde{\theta}, \tilde{\mathbb{N}}, \tilde{s}$.

Let $\tilde{c} = \tilde{\sigma}''\bar{c}$, where $\bar{c} = c^{\bar{G}}$.

Let $c = \sigma^{*}{}''\tilde{c}$. Then c is an $\mathbb{N}$-generic sequence and $V[c] = V[G^*]$ by Cor 2. Set $G = G^c = \{T \mid \bigwedge n\ c \restriction n \in T\}$. Then G is $\mathbb{N}$-generic and $V[G] = V[G^*]$

Set: $\sigma = \sigma^* \circ \tilde{\sigma}$. Then $\sigma \in V[G^*] = V[G]$ and

(a) $\sigma : \bar{N} \prec N$

(b) $\sigma(\bar{\theta}, \bar{N}, \bar{\lambda}) = \theta, N, \lambda$

(c) $C^N_{\omega_2}(\mathrm{rng}\,\sigma) = C^N_{\omega_2}(\mathrm{rng}\,\pi)$

(d) $\sigma''\bar{G} \subset G$.

(a), (b) are trivial. To see (c) note that, letting $\lambda = \sup \pi''On_{\bar{N}}$, $\tilde{\lambda} = (\sigma^*)^{-1}(\lambda)$, we have: $\sigma^*(C^{\tilde{N}}_{\tilde{\omega}_2}(\mathrm{rng}\,\tilde{\pi})) = C^N_{\omega_2}(\mathrm{rng}\,\pi)$.

Hence

$$C^N_{\omega_2}(\mathrm{rng}(\sigma)) = \sigma^*(C^{\tilde{N}}_{\tilde{\omega}_2}(\mathrm{rng}\,\tilde{\sigma})) =$$
$$= \sigma^*(C^{\tilde{N}}_{\tilde{\omega}_2}(\mathrm{rng}\,\tilde{\pi})) = C^N_{\omega_2}(\mathrm{rng}\,\pi).$$

(d) follows from $\sigma''\bar{C} = C$.

But then there is $T \in G$ which forces the existence of $\sigma \in V[G]$ satisfying (a)–(d).

QED (Theorem)

<u>Note</u> This proof actually shows:

Let G be $\mathbb{N}$-generic. There is $\sigma \in V[G]$ s.t.

- $\sigma : \bar{N} \prec N$
- $\sigma(\bar{\theta}, \bar{\mathbb{N}}, \bar{s}) = \theta, \mathbb{N}, s$
- $C^N_{\omega_2}(\text{rng}\,\sigma) = C^N_{\omega_1}(\text{rng}\,\pi)$
- Let $\bar{G}$ be $\bar{\mathbb{N}}$-generic over $\bar{N}$. There is a unique G' s.t. G' is $\mathbb{N}$-generic and $\sigma''\bar{G} \subset G'$.

The proof used CH to show that $\bar{N} \in H_{\omega_1} \in N$.

If we dropped CH we could still get:

Let $G, \bar{N}, \pi$ be as above. Let $\bar{N} = L^{\bar{A}}_{\bar{\tau}}$, $\pi(\bar{\xi}) = \xi$.

There are $\bar{N}', \sigma \in V[G]$ s.t. $\bar{N}' = L^{\bar{A}'}_{\bar{\tau}}$ and

- $\sigma : \bar{N}' \prec N$
- $\sigma(\bar{\xi}) = \xi$
- $\theta, \mathbb{N}, s \in \text{rng}(\sigma)$
- $C^N_{\omega_2}(\text{rng}\,\sigma) = C^N_{\omega_2}(\text{rng}\,\pi)$
- Let $\sigma(\bar{\mathbb{N}}') = \mathbb{N}$. Let $\bar{G}$ be $\bar{\mathbb{N}}'$-generic over $\bar{N}'$. There is a unique G' s.t. G' is $\mathbb{N}$-generic and $\sigma''\bar{G} \subset G'$.

This does not imply subcompleteness, but is enough to give Fact 3 above.

We now turn to the proof of Lemma 1. We first define:

Def $X \subset \Omega$ is a <u>front</u> iff

- X is an antichain in Ω
- $\{s \mid \forall u \in X \; s \leq u\}$ contains no infinite branch.

Note that any antichain whose elements have bounded height is a front. Moreover, if X is a front and for each $u \in X$, Y_u is a front of points which properly extend u, then $Y = \bigcup_{u \in X} Y_u$ is a front.

Def By an <u>array</u> we mean $S = \langle s_u \mid u \in \Omega \rangle$ s.t.

(a) $s_u \in \Omega$

(b) ~~If~~ $u < v$ in Ω, then $s_u < s_v$ in Ω

(c) If $u \mid v$ in Ω, then $s_u \mid s_v$ in Ω

(d) $\{s_u \mid |u| = n\}$ is a front.

Note that (b), (c), (d) can be equivalently replaced by:

(b') $s_u < s_{u^\frown\langle i\rangle}$ in Ω for $i<\omega_2$

(c') $i \neq j \rightarrow s_{u^\frown\langle i\rangle} \mid s_{u^\frown\langle j\rangle}$ for $i,j<\omega_2$

(d') $\{s_{u^\frown\langle i\rangle} \mid i<\omega_2\}$ is a front,

Def Af S is an array, then

$S^* =_{Df} \{t \mid \forall u \in \Omega \; t \leq s_u\}$.

For $T \in \mathbb{N}$ we set:

$S^*_T = \{t \mid \forall u \in T \; t \leq s_u\}$.

We leave it to the reader to show:

(1) $S^*_T \in \mathbb{N}$ for $T \in \mathbb{N}$,

Note that the property of being a front is absolute in generic extensions, by the absoluteness of well founded-ness.

(2) Let b be an infinite branch in S^*_T. Then $\forall u\,(|u|=n \wedge s_u \in b)$ for all n.

proof.

Set:

$S' = \{t \mid \forall u\,(|u|=n \wedge (t \leq s_u \vee s_u \leq t))\}$.

Then $S^*_T \subset S'$. But then $\forall u\; s_u \in b$, since $\{s_u \mid |u|=n\}$ is a maximal antichain in S' and we would otherwise have: $b \subset \{t \mid \forall u\ (|u| = n \wedge t \leq s_u\}$, where $\{s_u \mid |u| = n\}$ is a front. Contr! QED (2)

Obviously (2) also holds in generic extensions.

By (2) we get:

(3) Let $T \subset S^*$, $T \in \mathbb{N}$. Set $T' = \{u \in \Omega \mid s_u \in T\}$. Then $T' \in \mathbb{N}$ and $T = S^*_{T'}$.

proof.

$S^*_{T'} \subset T$ is trivial.

<u>Claim</u> $T \subset S^*_{T'}$.

Let $t \in T$. Let $b \ni t$ be an infinite branch in T. Let $|t| = n$. By (2) there is $u \in \Omega$ s.t. $s_u \in b$. But $|u| \leq |s_u|$ by ind. on $|u|$. Hence $t \leq s_u \in T$. QED (Claim)

It remains only to show

<u>Claim</u> $T' \in \mathbb{N}$.

Let $u \in T'$. For each $t \in T$

there is $t' \geq t$ s.t. t' has ω_2 many immediate successors in T (otherwise we could show by induction on m that $\mathrm{card}\{t' \in T \mid t \leq t' \wedge |t'| = m\} < \omega_2$, hence $\mathrm{card}\{t' \in T \mid t \leq t'\} < \omega_2$. Contradiction!) Let $t' \geq s_u$ s.t. t' has ω_2 many imm. successors. Let $\langle s_i \mid i < \omega_2 \rangle$ be an enumeration of the successors and pick u_i s.t. $s_{u_i} \geq s_i$ $(i < \omega_2)$. Then $u \leq u_i$ in T' and $\{u_i \mid i < \omega_2\}$ is a set of size ω_2. QED (3)

<u>Def</u> Let $T \in \mathbb{N}$.

$\mathbb{N}|T = \{T' \mid T' \leq T \text{ in } \mathbb{N}\}$.

We have shown that:

(4) Let S be an array. There is $\sigma : \mathbb{N} \longleftrightarrow \mathbb{N}|S^*$ defined by: $\sigma(T) = S^*_T$. Moreover $\sigma^{-1}(T) = \{u \mid s_u \in T\}$ for $T \leq S^*$ in $\mathbb{N}$.

It is then apparent that:

(5) $T \leq T'$ in $\mathbb{N}$ iff $\sigma(T) \leq \sigma(T')$ in $\mathbb{N}$,

We also denote σ by σ_S.

Using this machinery we prove Lemma 1.

Let $\Vdash (\dot{b}$ is a monotone cofinal sequence in $\check{\omega}_2)$

<u>Claim</u> $\Vdash \dot{b}$ is $\check{\mathbb{N}}$-generic over $\check{V}$.

It suffices to show that the set of T' s.t. $T' \Vdash \dot{b}$ is $\check{\mathbb{N}}$-generic over $\check{V}$ is dense in $\mathbb{N}$, so fix $T \in \mathbb{N}$. We show that there is such a $T' \leq T$.

Define an array $S_0 = \langle s^0_u \mid u \in \Omega \rangle$ and an auxiliary sequence $\langle T_u \mid u \in \Omega \rangle$ s.t. $s^0_u \in T_u \in \mathbb{N}$ and $u \leq v \to T_v \subset T_u$:

$s^0_\emptyset = \emptyset$, $T_\emptyset = T$,

given s^0_u, T_u, let $t \geq s_u$ have ω_2 many immediate successors in T_u. Let $\langle t_i \mid i < \omega_2 \rangle$ be a 1-1 enumeration of these successors.

Set $\mathring{s}_{u^\frown\langle i\rangle} = t_i$ for $i < \omega_2$. Letting $T_{(t)} = \{s \mid s \leq t \vee t \leq s \text{ in } T\}$, we then pick $T_{u^\frown\langle i\rangle} \leq (T_u)_{(t_i)}$ s.t. $T_{u^\frown\langle i\rangle}$ fixes the value of $\mathring{b}(|\check{u}|)$.

(i.e. $\bigvee_\xi\ T_{u^\frown\langle i\rangle} \Vdash \mathring{b}(|\check{u}|) = \check{\xi}$).

Set: $T^\circ_0 = S^*_0$. Then $T^\circ \leq T$ and

(6) $T^\circ_{(s^\circ_u)}$ fixes the value of $\mathring{b} \restriction |u|$, for $u \in \Omega$.

Def Let $u \in \Omega$, $n > |u|$.

X^u_n = the set of ν s.t. for some $v > u$ with $|v| = n$, we have: $T^\circ_{(s_v)} \Vdash \mathring{b}(\check{n-1}) = \check{\nu}$

(7) $T^\circ_{(s_u)} \Vdash \mathring{b}(\check{n-1}) \in \check{X}^u_n$ for $n \geq |u|$

prf.

Let $T^\circ_{(s_u)} \in G$ where G is $\mathbb{N}$-generic.

Let $c = c^G$. Then $\tilde{c} = \{c \restriction n \mid n < \omega\}$ is an infinite branch in $T^\circ_{(s_u)}$, where $Y = \{s_v \mid u < v \wedge |v| = n\}$ is a front.

Hence there is $s_v \in C$ s.t. $v \in Y$

Let $T^\circ_{s_v} \Vdash \mathring{b}(\check{n-1}) = \check{\nu}$. Then

$b(n-1) = \nu \in X^u_n$, where $b = \mathring{b}^G$. QED (7)

(8) $\forall n > |u|\ \overline{\overline{X^u_n}} = \omega_2$

prf.
If not, then $\overline{\overline{X}} < \omega_2$ where $X \subset \bigcup_{n \in \omega} X^u_n$.
Let G be $\mathbb{N}$-generic, $b = \dot{b}^G$. Then
$\mathrm{rng}(b) \subset X$ is bounded in ω_2. Contr!
QED (8)

But then:

(9) Let $t \in T^\circ$. There is a front Y in T°
s.t. $\overline{\overline{Y}} = \omega_2$, and for some $n > |t|$
we have:

- If $s \in Y$, then $s > t$ and there is b_s s.t. $T^\circ_{(s)} \Vdash \dot{b} \upharpoonright \check{n} = \check{b}_s$
- If $s, s' \in Y$ are distinct, then $b_s \neq b_{s'}$.

proof
Assume w.l.o.g. $t = s_u$. Let n be as in (8). Then there is a subset Y of $\{s_v \mid v > u \wedge |v| = n\}$ with the desired properties. QED (9)

We then construct a new array $S_1 = \langle s^1_u \mid u \in \Omega \rangle$ and an associated sequence $\langle m_u \mid u \in \Omega \rangle$ by:

Call $\langle m, Y\rangle$ a <u>good pair</u> for $t \in T^0$ if

(9) holds.

$s^1_\emptyset = \emptyset$

Let s^1_u be given. Pick a good pair $\langle m, Y\rangle$ for s^1_u and let $\langle s^1_{u^\frown\langle i\rangle} \mid i < \omega_1\rangle$ be a 1-1 enumeration of Y. Set $m_u = m$.

Set: $T^1 = S_1^*$. Then $T^1 \subset T^0$.

Let $\sigma^1 = \sigma_{S_1}$.

Now define an array S_2 by:

$s^2_\emptyset = \emptyset$

$s^2_{u^\frown\langle i\rangle} = b$ where $T^0_{(s^2_{u^\frown\langle i\rangle})} \Vdash \mathring{b} \restriction \check{m}_u = \check{b}$.

S_2 is easily seen to be an array. ($\{s^2_{u^\frown\langle i\rangle} \mid i < \omega_2\}$ is a front, since it is an antichain of bounded height).

Set $T^2 = S_2^*$, $\sigma^2 = \sigma_{S_2}$.

Set: $\sigma = \sigma^2 \circ (\sigma^1)^{-1}$. Then

(10) $\sigma : \mathbb{N}/T^1 \longleftrightarrow \mathbb{N}/T^2$ and

$T \le T' \longleftrightarrow \sigma(T) \le \sigma(T')$ in $\mathbb{N}$

for $T, T' \in \mathbb{N}/T^1$.

However:

(11) Let $T \leq T^1$ in $\mathbb{N}$. Then
$T \Vdash \bigwedge n \; \dot{b} \upharpoonright n \in \sigma(\check{T})$.

proof.
Let G be $\mathbb{N}$-generic with $T \in G$.
Let $c = \dot{c}^G$. Let $s^1_{u^\frown\langle i\rangle} \in c$. Then
$T_{(s^1_u)} \in G$ and hence $b \upharpoonright n_u = s^2_{u^\frown\langle i\rangle}$,
where $b = \dot{b}^G$. Hence $b \upharpoonright n_u \in \sigma(T)$
since $u^\frown\langle i\rangle \in (\sigma_{S_1})^{-1}(T)$. But by
(2), for every $m < \omega$ there is
$u^\frown\langle i\rangle$ s.t. $|u| = m$ and $s^1_{u^\frown\langle i\rangle} \in c$
(hence $n_u \geq m$). QED (11)

Hence:
(12) $T^1 \Vdash (\dot{b}$ is an $\check{\mathbb{N}}$ generic sequence over $\check{V})$

proof.
Let $G \ni T^1$ be $\mathbb{N}$-generic. Let
$b = \dot{b}^G$. Set $G' = G^b = \{T \in \mathbb{N} \mid \bigwedge n \; b \upharpoonright n \in T\}$.
Let Δ be dense in $\mathbb{N}$.
<u>Claim</u> $G' \cap \Delta \neq \emptyset$.

Set $\Delta' = \{T \leq T^1 \mid \sigma(T) \in \Delta\}$. Then Δ' is dense below T^1. Hence $\Delta' \cap G \neq \emptyset$. Let $T \in \Delta' \cap G$. Then $\sigma(T) \in \Delta$ and $\Lambda_n b \Gamma_n \in T$ by (11). Hence $\sigma(T) \in \Delta \cap G'$. QED (12)

Since $T^1 \leq T$, this completes the proof the Claim.

It remains only to show that $V[G] = V[b]$, where $b = \overset{\circ}{b}{}^G$.

We have seen that $G' =$ the set of T' s.t. $\sigma(T) \leq T'$ for a $T \in G$. Doing the argument in reverse, we get $G =$ the set of T' s.t. $\sigma^{-1}(T) \leq T'$ for a $T \in G'$. Hence $G \in V[G']$.

Hence $V[G] = V[G']$. QED