§4 Dee-Forcing

The Theory of α-proper forcing and Dee-proper forcing was developed in [S] and is lucidly exposited in [A], We refer to [A] for all definitions. We write "Dee-proper" to mean "simply Dee-proper" in the sense of [A] §5.3 (i.e. Dee-properness is witnessed by a simple completeness system), since this is the version which lends itself to forcing axioms. The Dee-proper forcing axiom (DPFA) is a strengthening of CPFA which says that every forcing which is ω_1-proper and Dee-proper satisfies Martin's axiom. Its consistency relative to a supercompact cardinal is proven the usual way: The concepts "ω_1-proper" and "Dee-proper" are both locally based in the sense of our earlier definition. Moreover,

- All complete forcings are Dee-proper and ω_1-proper.
- If $\mathbb{A}$ is Dee-proper and $\Vdash_{\mathbb{A}} \mathring{\mathbb{B}}$ is complete, then $\mathbb{A} * \mathring{\mathbb{B}}$ is Dee-proper.

- Any countable support iteration of ω_1-proper forcings is ω_1-proper.
- Any countable support iteration of forcings which are ω_1-proper and Dee-proper will add no reals.

Given this, we can do the usual construction over a supercompact cardinal, getting the "natural model" for $DPFA^+ + CH$. If we make GCH true by a prior application of Silver forcing, the model will satisfy GCH as well. Shelah has shown that if T is any Aronszajn tree, then there is a forcing which converts T into a special Aronszajn tree and is both ω_1-proper and Dee-proper. Suppose w.l.o.g. that $T \subset \omega_1$. By DPFA there is $X \prec H_{\omega_2}$ s.t. $\omega_1 \cup \{T\} \subset X$ and $X \models$ "T is special Aronszajn". Hence T is really special Aronszajn.

Thus DPFA implies that every Aronszajn tree is special, and is therefore not consistent with $\diamondsuit$.

DPFA posits Martin's axiom for a class of forcings which are proper, hence do not change cofinalities. In [DSP] we generalized the notions "α-proper" and "Dee-proper" to "α-subproper" and "Dee-subproper". We refer to [DSP] for the definitions. Both these concepts are locally based. Moreover, we proved:

(a) All subcomplete forcings are Dee-subproper and ω_1-subproper.

(b) All α-proper forcings are α-subproper

(c) All Dee-proper forcings are Dee subproper

Forcings which are Dee-subproper and ω_1-subproper add no reals

(d) If $\mathbb{A}$ is α-subproper and $\Vdash_{\mathbb{A}} \mathring{\mathbb{B}}$ is $\check{\alpha}$-subproper, then $\mathbb{A} * \mathring{\mathbb{B}}$ is α-subproper

(e) If $\mathbb{A}$ is Dee-subproper and $\Vdash_{\mathbb{A}} \mathring{\mathbb{B}}$ is subcomplete, then $\mathbb{A} \circ \mathring{\mathbb{B}}$ is Dee-subproper.

(f) An RCS iteration of α-subproper forcings subject to the usual restraints (as in §3 Theorem 2) yields α-subproper forcings.

(g) An RCS iteration of forcings which are both ω_1-subproper and Dee-subproper, subject to the usual restraints, does not add reals.

Note In our original version of [DSP] we neglected to prove (e). §3 of [DSP] has now been amended accordingly. We expect the amended version to be on our website within a few weeks (dated 29 Feb. 2012).

Putting all of this together, we can do the usual construction, iterating forcings which are ω_1-subproper and Dee-subproper up to a supercompact cardinal. This gives us the "natural model" of the Dee-subproper forcing axiom (DSbPFA), which says that all forcings which are both ω_1-subproper and Dee-subproper satisfy Martin's axiom. The natural model, in fact, satisfies DSbPFA$^+$ + CH, and it is consistent to suppose that it satisfies GCH.

DSbPFA strengthens both DPFA and SCFA. Its natural model is, again, very different from that of SCFA, since DSbPFA implies that all Aronszajn trees are special.