§2 Subcomplete Forcing

We recall the definition:

Def Let $\mathbb{B}$ be a complete BA. $\mathbb{B}$ is subcomplete iff for sufficiently large cardinals θ the following hold

Let $\mathbb{B} \in H_\theta$. Let $\tau > \theta$ be regular s.t. $H_\theta \subset W = L_\tau^A$ $(= \langle L_\tau[A], \in, A\rangle)$. Let $\sigma : \bar{W} \prec W$ where $\bar{W}$ is countable, transitive, and full.*) Let $\sigma(\bar{\theta}, \bar{\mathbb{B}}, \bar{\tau}, \bar{\lambda}_1, \ldots, \bar{\lambda}_n) = \theta, \mathbb{B}, \tau, \lambda_1, \ldots, \lambda_n$, where λ_i is regular s.t. $\overline{\overline{\mathbb{B}}} < \lambda_i$ for $i = 1, \ldots, n$. Let $\bar{G}$ be $\bar{\mathbb{B}}$-generic over $\bar{W}$. There is $a \in \mathbb{B} \setminus \{0\}$ s.t. whenever $G \ni a$ is $\mathbb{B}$-generic, then there is $\sigma_0 \in V[G]$ s.t.

(a) $\sigma_0 : \bar{W} \prec W$

(b) $\sigma_0(\bar{\theta}, \bar{\mathbb{B}}, \bar{\tau}, \bar{\lambda}_i) = \theta, \mathbb{B}, \tau, \lambda_i$ $(i = 1, \ldots, n)$

(c) $\sup \sigma_0 {}'' \bar{\lambda}_i = \sup \sigma {}'' \bar{\lambda}_i$ $(i = 0, \ldots, n)$, where $\bar{\lambda}_0 = \mathrm{On} \cap \bar{W}$

(d) $\sigma_0 {}'' \bar{G} \subset G$.

*) $\bar{W}$ is called full iff, letting μ be least s.t. $L_\mu[\bar{W}] \models ZFC^-$, $\bar{W}$ is regular in $L_\mu[\bar{W}]$ (i.e. if $f \in L_\mu[\bar{W}]$, $x \in \bar{W}$ and $f : x \to \bar{W}$, ~~then~~ $f \in \bar{W}$).

We call $\mathbb{B}$ <u>weakly</u> <u>subcomplete</u> iff there is an x s.t. for sufficiently large θ the above holds whenever $x \in \mathrm{rng}(\sigma)$. It turns out that weak subcompleteness implies subcompleteness.

The <u>two</u> <u>step</u> <u>iteration</u> theorem says that if $\mathbb{B}$ is subcomplete and $\Vdash_{\mathbb{B}} \mathring{\mathbb{C}}$ is subcomplete, then $\mathbb{B} * \mathring{\mathbb{C}}$ is subcomplete. (In other words, if $\mathbb{B} \subseteq \mathbb{C}$ and $\Vdash_{\mathbb{B}} (\check{\mathbb{C}}/\dot{G}$ is subcomplete$)$, then $\mathbb{C}$ is subcomplete, $\dot{G}$ being the canonical generic name.)

We shall now prove the following iteration theorem:

(We, of course, write $\Vdash_i$ for $\Vdash_{\mathbb{B}_i}$)

<u>Thm 1</u> Let $\mathbb{B} = \langle \mathbb{B}_i \mid i < \alpha \rangle$ be an RCS-
- iteration s.t. for all $i+1 < \alpha$:

(a) $\mathbb{B}_i \neq \mathbb{B}_{i+1}$

(b) $\Vdash_i (\check{\mathbb{B}}_{i+1} / \dot{G}$ is subcomplete$)$

(c) $\Vdash_{i+1} (\check{\mathbb{B}}_i$ has cardinality $\leq \omega_1)$

Then every $\mathbb{B}_i$ is subcomplete.

proof

By ind. on i we prove:

<u>Claim</u> Let $h \leq i$. Let G be $\mathbb{B}_h$-generic. Then $\mathbb{B}_i / G$ is subcomplete in $V[G]$

The case $h = i$ is trivial, since then $\mathbb{B}_i / G \simeq \{0, 1\}$. Hence $i = 0$ is trivial. Now let $i = j+1$. Then $\mathbb{B}_j / G \subseteq \mathbb{B}_i / G$. Let $\tilde{G}$ be $\mathbb{B}_j / G$ - generic over $V[G]$. Then $G' = G * \tilde{G} = \{ b \in \mathbb{B}_j \mid b/G \in \tilde{G} \}$ is $\mathbb{B}_j$ - generic over V.

But then $(\mathbb{B}_i / G) / \tilde{G} \simeq \mathbb{B}_i / G'$ is subcomplete in $V[G'] = V[G][\tilde{G}]$. Hence $\mathbb{B}_i / G$ is subcomplete in $V[G]$ by the two step theorem. (Since $\mathbb{B}_j / G$ is subcomplete and $\Vdash_{\mathbb{B}/G} (\check{\mathbb{B}}_i / G / \dot{G}$ is subcomplete.)

There remains the case that $i = \lambda$ is a limit ordinal. By our induction hypothesis, $\mathbb{B}_j / G_h$ is subcomplete in $V[G_h]$ whenever $h \leq j < \lambda$. But then $\langle \mathbb{B}_{h+i} / G_h \mid i < \lambda - h \rangle$ satisfies the same induction hypothesis, since if $i \leq k < \lambda - h$ and $\tilde{G}$ is $\mathbb{B}_{h+k} / G_h$ - generic over $V[G_h]$, then $G = G_h * \tilde{G}$ is $\mathbb{B}_{h+k}$ - - generic over V and $(\mathbb{B}_{h+k} / G_h) / \tilde{G} \simeq \mathbb{B}_{h+k} / G$ is subcomplete in $V[G] = = V[G_h][\tilde{G}]$. We consider two cases:

<u>Case 1</u> $\mathrm{cf}(\lambda) \leq \overline{\overline{\mathbb{B}_i}}$ for an $i < \lambda$.

Then $\mathrm{cf}(\lambda) \leq \omega_1$ in $V[G_j]$ whenever $i < j < \lambda$ and G_j is $\mathbb{B}_j$ - generic. It suffices to prove the claim for such j, since if $h < j$ and G_h is $\mathbb{B}_h$ - generic, we can use the two-step lemma to show - - exactly as in the successor case - that $\mathbb{B}_\lambda / G_h$ is subcomplete in $V[G_h]$.

Thus it will suffice to prove:

<u>Claim</u> Assume $\mathrm{cf}(\lambda) \leq \omega_1$ in V. Then $\mathbb{B}_\lambda$ is subcomplete.

since the same proof can then be carried out in $V[G_j]$ to show that $\mathbb{B}_\lambda/G_j$ is subcomplete. Fix $f:\omega_1\to\lambda$ s.t. $\sup f''\omega_1=\lambda$. Let $\theta>\lambda$ be a cardinal s.t. $\bar{\bar{\mathbb{B}}}<\theta$ and θ is big enough that H_θ ($\check{\theta}$ witnesses the subcompleteness of $\check{\mathbb{B}}_j/\check{G}$) for $i\le j<\lambda$. Let $W=L^A_\tau$, where τ is regular, $\tau>\theta$ and $H_\theta\subset W$. Let $\sigma:\bar{W}\prec W$ s.t. $\bar{W}$ is countable, transitive, and full. Suppose moreover that:

$\sigma(\bar{f},\bar{\theta},\bar{\lambda},\bar{\mathbb{B}},\bar{s},\bar{\lambda}_i)=f,\theta,\lambda,\mathbb{B},s,\lambda_i$ $(i=1,\dots,n)$,

where $\lambda_i<\theta$ is regular s.t. $\bar{\bar{\mathbb{B}}}_\lambda<\lambda_i$ for $i=1,\dots,n$. (Note that $\sigma(\mathbb{B}_{\bar{\lambda}})=\mathbb{B}_\lambda$.) Let $\bar{G}$ be $\mathbb{B}_{\bar{\lambda}}$-generic over $\bar{W}$. It suffices to show:

<u>Claim</u> There is $c\in\mathbb{B}_\lambda\setminus\{0\}$ s.t. whenever $G\ni c$ is $\mathbb{B}_\lambda$-generic, then there is $\sigma'\in V[G]$ s.t.

(a) $\sigma'{}''\bar{W}\prec W$

(b) $\sigma'(\bar{f},\bar{\theta},\bar{\lambda},\bar{\mathbb{B}},\bar{s},\bar{\lambda}_i)=f,\theta,\lambda,\mathbb{B},s,\lambda_i$ $(i=1,\dots,n)$

(c) $\sup\sigma'{}''\bar{\lambda}_i=\tilde{\lambda}_i$ $(i=0,\dots,n)$,

where $\bar{\lambda}_0=\mathrm{On}\cap\bar{W}$, $\tilde{\lambda}_i=\sup\sigma''\bar{\lambda}_i$

(d) $\sigma'{}''\bar{G}\subset G$.

<u>Note</u> The use of the parameters $f, \mathbb{B}$ is justified by the fact that weak subcompleteness implies subcompleteness.

Set: $\tilde{\lambda} = \sup \sigma''\bar{\lambda}$; $\tilde{\lambda}_i = \sup \sigma''\lambda_i$ $(i=0,\dots,n)$.

It is easily verified that there is a sequence $\langle \nu_i \mid i<\omega\rangle$ in $\omega_1^{\bar{W}}$ s.t. setting $\bar{\xi}_i = \bar{f}(\nu_i)$, we have: $\bar{\xi}_0 = 0$ and $\langle \bar{\xi}_i \mid i<\omega\rangle$ is monotone and cofinal in $\bar{\lambda}$. Set: $\xi_i = f(\nu_i)$. Then $\xi_i = \sigma(\bar{\xi}_i)$ and $\langle \xi_i \mid i<\omega\rangle$ is monotone and cofinal in $\tilde{\lambda}$.

Moreover:

(1) $\sigma'(\bar{\xi}_i) = \xi_i$ whenever $\sigma' : \bar{W} \prec W$ s.t. $\sigma'(\bar{f}) = f$.

For each $i = 0,\dots,n$ choose $\langle \bar{\xi}_j^i \mid j<\omega\rangle$ which is monotone and cofinal in λ_i. Set $\xi_j^i = \sigma(\bar{\xi}_j^i)$. Then $\langle \xi_j^i \mid j<\omega\rangle$ is monotone and cofinal in $\tilde{\lambda}_i$. (However, we have nothing corresponding to (1).)

Our strategy will be to construct by induction on $k<\omega$ a $c_k \in \mathbb{B}_{\xi_k}$ and a $\dot{\sigma}_k \in V^{\mathbb{B}_{\xi_k}}$ s.t. $\langle c_k \mid k<\omega\rangle$ is a thread through $\langle \mathbb{B}_{\xi_k} \mid k<\omega\rangle$ and the following holds:

(*) Let $G_k \ni c_k$ be $\mathbb{B}_{\xi_k}$ - generic. Set:

$\sigma_j = \dot{\sigma}_j^{G_k} = \dot{\sigma}_j^{G_j}$ for $j \leq k$, where $G_j = G_k \cap \mathbb{B}_j$. *)

Then:

(a) $\sigma_0 = \sigma$

(b) $\sigma_k : \overline{W} \prec W$

(c) $\sigma_k(\bar{f}, \bar{\theta}, \bar{\lambda}, \overline{\mathbb{B}}, \bar{\imath}, \bar{\lambda}_i) = f, \theta, \lambda, \mathbb{B}, \imath, \lambda_i \quad (i = 1, \ldots, n)$

(d) $\sup \sigma_k{}'' \bar{\lambda}_i = \tilde{\lambda}_i \quad (i = 0, \ldots, n)$

(e) $\sigma_k{}'' \overline{G}_k \subset G_k$ where $\overline{G}_k = \overline{G} \cap \mathbb{B}_{\bar{\xi}_k}$

(f) $\sigma_k(x_l) = \sigma_j(x_l)$ for $l \leq j \leq k$, where $\langle x_l \mid l < \omega \rangle$ is a fixed enumeration of $\overline{W}$.

(g) Let $i = 0, \ldots, n$ and $k = j + 1$ s.t. $\sigma_j(\bar{\xi}^i_m) \leq \xi^i_k < \sigma_j(\bar{\xi}^i_{m+1})$. Then

$\sigma_k(\bar{\xi}^i_l) = \sigma_j(\bar{\xi}^i_l)$ for $l \leq m+1$. **)

*) Note: As in [SPSC] we assume that the "natural injection" of $V^{\mathbb{B}_j}$ into $V^{\mathbb{B}_k}$ is the identity if $j \leq k < \alpha$. Thus $t^{G_j} = t^{G_k}$ if $t \in V^{\mathbb{B}_j}$. This also implies:

$$[\![\varphi(t_1, \ldots, t_n)]\!]_{\mathbb{B}_j} = [\![\varphi(t_1, \ldots, t_n)]\!]_{\mathbb{B}_k}$$

if $t_1, \ldots, t_n \in V^{\mathbb{B}_j}$ and φ is a Σ_0-formula.

**) <u>Note</u> We have $\sigma_k(\bar{\xi}_i) = \xi_i$ for $i < \omega$, since $\sigma_k(\bar{f})$. We do <u>not</u> necessarily have $\sigma_k(\bar{\xi}^i_l) = \bar{\xi}^i_l$, even though σ_h takes $\bar{\lambda}_i$ cofinally to $\tilde{\lambda}_i$.

Before constructing $c_i, \dot{\sigma}_i$ $(i<\omega)$ and verifying $(*)$, we show that $(*)$ will prove the claim. Clearly $cf(\tilde{\lambda}) = \omega$ in V. Hence we can set $c = \bigcap_{i<\omega} c_i$, getting $c \neq 0$ and $c \in \mathbb{B}_{\tilde{\lambda}} \subset \mathbb{B}_\lambda$. Let $G \ni c$ be $\mathbb{B}_\lambda$-generic. Set $G_h = G \cap \mathbb{B}_{\xi_h}$ for $h<\omega$. Set $\sigma_h = \dot{\sigma}_h^G = \dot{\sigma}_h^{G_h}$. Clearly, $(*)$ hold for all $k<\omega$, since $c_k \in G_k$.

But by (f) we can define a new $\sigma' : \bar{W} \prec W$ by: $\sigma'(x) =_{Df}$ that y s.t. $\sigma_i(x) = y$ for sufficiently large i. It is then obvious that (a), (b) of the Claim hold. To prove (c) we note first that $\sigma'{}''\bar{\lambda}_i \subset \tilde{\lambda}_i$, since if $\eta < \bar{\lambda}_i$ there is h s.t. $\sigma'(\eta) = \sigma_h(\eta) < \tilde{\lambda}_i$. To see that $\sigma''\bar{\lambda}_i$ is unbounded in $\tilde{\lambda}_i$, let ξ_j^i be given. Since $\sup \sigma_j{}''\bar{\lambda}_i = \tilde{\lambda}_i$, there is m s.t.

$\sigma_j(\bar{\xi}_m^i) \leq \xi_k^i < \sigma_j(\bar{\xi}_{m+1}^i)$ where $k = j+1$.

Hence $\sigma_h(\bar{\xi}_{m+1}^i) = \sigma_j(\bar{\xi}_{m+1}^i)$ for $h \geq k$.

Hence $\sigma'(\bar{\xi}_{m+1}^i) > \xi_k^i > \xi_j^i$. QED (c)

We now prove (d). We first note that $\sigma' {}'' \bar{G}_h \subset G$ for $h < \omega$, since if $a \in \bar{G}_h$, then $\sigma'(a) = \sigma_l(a) \in G$ for sufficiently large l.

If $cf(\lambda) = \omega_1$, then $cf(\bar{\lambda}) = \omega_1$ in $\bar{W}$ and $\bigcup_{\xi < \bar{\lambda}} \bar{\mathbb{B}}_\xi$ is dense in $\bar{\mathbb{B}}_{\bar{\lambda}}$. Hence $\bigcup_l \bar{G}_l$ is dense in $\bar{G}$ – i.e. if $a \in \bar{G}$ there is $a' \subset a$ s.t. $a' \in \bigcup_l \bar{G}_l$. Hence $\sigma(a) \supset \sigma(a') \in G$.

Now let $cf(\lambda) = \omega$. Then $cf(\bar{\lambda}) = \omega$ in $\bar{W}$. Let $\langle \nu_i \mid i < \omega \rangle \in \bar{W}$ be monotone and cofinal in $\bar{\lambda}$. For any $a \in \bar{G}$ there is a thread $\langle a_i \mid i < \omega \rangle \in \bar{W}$ through $\langle \bar{\mathbb{B}}_{\nu_i} \mid i < \omega \rangle$ s.t. $a_i \in \bar{G}$ and $\bigcap_i a_i \subset a$. Hence $\sigma(a_i) \in G$ and $\bigcap_i \sigma(a_i) = \sigma(\bigcap_i a_i) \subset \sigma(a)$, where $\bigcap_i \sigma(a_i) \in G$. QED (d)

All that remains now is to define c_k, $\dot{\sigma}_k$ for $k < \omega$ and verify (*). We proceed by induction on k. For $k = 0$ we set: $c_0 = 1$, $\dot{\sigma}_0 = \check{\sigma}$. All verifications are trivial. Now let $k = j+1$. The construction of c_k, $\dot{\sigma}_k$ is essentially a repeat of the two step iteration lemma. We give the details. Let $G_j \ni c_j$ be $\mathbb{B}_{\xi_j}$-generic. Set: $\sigma_j = \dot{\sigma}_j^{G_j}$. Then σ_j extends uniquely to a $\sigma_j^* : \bar{W}[\bar{G}_j] \prec W[G_j]$ s.t. $\sigma^*(\bar{G}_j) = G_j$ by (*)(e). But Θ verifies the sub-completeness of $\mathbb{B}' = \mathbb{B}_{\xi_k}/G_j$ in $V[G_j]$. Hence there is $c' \in \mathbb{B}' \setminus \{0\}$ s.t. whenever $G' \ni c'$ is $\mathbb{B}'$-generic, there is $\sigma' \in V[G_j][G']$ with the following properties:

(a) $\sigma' : \bar{W}[\bar{G}_j] \prec W[G_j]$, $\sigma'(\bar{G}_j) = G_j$

(b) $\sigma'(\bar{f}, \bar{\theta}, \bar{\lambda}, \bar{\mathbb{B}}, \bar{s}, \bar{\lambda}_i) = f, \theta, \mathbb{B}, s, \lambda_i$ $\quad (i = 1, \ldots, n)$

(c) $\sup \sigma'{}''\bar{\lambda}_i = \tilde{\lambda}_i$ $\quad (i = 0, \ldots, n)$

(d) $\sigma'{}''\bar{G}' \subset G'$ where $\bar{G}' = \bar{G}_k/\bar{G}_j = \{b/\bar{G}_j \mid b \in \bar{G}_k\}$.

($\bar{G}'$ is easily seen to be $\bar{\mathbb{B}}_{\bar{\xi}_k}/\bar{G}_j$-generic over $\bar{W}$.)

Since we can force any finite amount of pointwise coherence between σ_j^* and σ', we can also impose the requirements:

(e) $\sigma'(x_i) = \sigma_j(x_i)$ for $i \leq k$

(f) Let $i = 0, \ldots, n$ and $\sigma_j(\bar{\xi}^i_m) \leq \xi^i_k < \sigma_j(\bar{\xi}^i_{m+1})$. Then

$$\sigma_k(\bar{\xi}^i_\ell) = \sigma_j(\bar{\xi}^i_\ell) \text{ for } \ell \leq m+1.$$

Note that $G = G_j * G' = \{b \in \mathbb{B}_{\bar{\xi}_k} \mid b/G_j \in G'\}$ is $\mathbb{B}_{\bar{\xi}_k}$-generic. Moreover $\bar{G}_k = \bar{G}_j * \bar{G}' = \{b \in \bar{\mathbb{B}}_{\bar{\xi}_k} \mid b/\bar{G}_j \in \bar{G}'\}$. Hence $\sigma'``\bar{G}_k \subset G$. If we set: $\sigma_k = \sigma' \upharpoonright \bar{W}$, it is clear that σ_k satisfies (*)(b)–(g) with $G = G_j * G'$. But $\sigma_k \in V[G_j][G'] = V[G]$.

We may assume w.l.o.g. that $c' = (\mathring{c}')^{G_j}$, where all of the above is forced by c_j – i.e. the above holds of $c' = (\mathring{c}')^{G_j}$ whenever $c_j \in G_j$ and G_j is $\mathbb{B}_{\bar{\xi}_j}$-generic. We may also suppose w.l.o.g. that $\Vdash_{\bar{\xi}_j} (\check{c}_j \notin \dot{G}_j \longrightarrow \dot{c}' = 0)$, $\dot{G}_j$ being the canonical generic name. Thus $\Vdash_{\bar{\xi}_j} \dot{c}' \in \check{\mathbb{B}}_{\bar{\xi}_k}/\dot{G}_j$. Hence there is $c_k \in \mathbb{B}_{\bar{\xi}_k}$ s.t. $\Vdash_{\bar{\xi}_j} \check{c}_k/\dot{G}_j = \dot{c}'$. But then

$$h_{\bar{\xi}_j}(c_k) = [\![\check{c}_k/\mathring{G}_j \neq 0]\!]_{\bar{\xi}_j} = [\![\mathring{c}' \neq 0]\!]_{\bar{\xi}_j} = c_j.$$

Thus if $G_k \ni c_k$ is $\mathbb{B}_{\bar{\xi}_k}$-generic, then $c_j \in G_k$ and $c_k/G_j = \mathring{c}'^{G_j} \in \mathbb{B}_k/G_j$, which means that there is $\sigma_k \in V[G]$

satisfying (*) (b) – (g). But then we may assume $\sigma_k = \dot{\sigma}_k G_k$, where this fact is forced by c_k. Thus $c_k, \dot{\sigma}_k$ have the desired properties.

QED (Case 1)

<u>Case 2</u> Case 1 fails.

Then λ is regular and $\overline{\overline{\mathbb{B}}}_i < \lambda$ for all $i < \lambda$.

Let $\bar{W}, W, \theta, \sigma$ be as before with $\sigma(\bar{\theta}, \bar{\mathbb{B}}, \bar{\imath}, \bar{\lambda}, \bar{\lambda}_i) = \theta, \mathbb{B}, \imath, \lambda, \lambda_i$ $(i = 1, \ldots, n)$ and $\lambda_1, \ldots, \lambda_n$ are as before. (However, there is nothing corresponding to the function f.) As before set: $\tilde{\lambda}_0 = \mathrm{On} \cap \bar{W}$; $\tilde{\lambda}_i = \sup \sigma'' \bar{\lambda}_i$ for $i = 0, \ldots, n$. We extend the sequence $\lambda_1, \ldots, \lambda_n$ by setting: $\lambda_{n+1} = \lambda$. We also set: $\bar{\lambda}_{n+1} = \bar{\lambda}$, $\tilde{\lambda}_{n+1} = \tilde{\lambda} = \sup \sigma'' \lambda$.

Clearly it will suffice to show:

<u>Claim</u> There is $c \in \mathbb{B}_\lambda$ s.t. whenever $G \ni c$ is $\mathbb{B}_\lambda$-generic, there is $\sigma' \in V[G]$ s.t.

(a) $\sigma' : \bar{W} \prec W$

(b) $\sigma'(\bar{\theta}, \bar{\mathbb{B}}, \bar{\imath}, \bar{\lambda}_i) = \theta, \mathbb{B}, \imath, \lambda_i$ $\quad (i = 1, \ldots, n+1)$

(c) $\sup \sigma'' \bar{\lambda}_i = \tilde{\lambda}_i$ $\quad (i = 0, \ldots, n+1)$

(d) $\sigma'' \bar{G} \subset G$

Just as before we choose for $j = 0, \dots, n+1$ a sequence $\langle \bar{\xi}_i^j \mid i < \omega \rangle$ which is monotone and cofinal in $\bar{\lambda}_j$. We then set $\xi_i^j = \sigma(\bar{\xi}_i^j)$. Hence $\langle \xi_i^j \mid i < \omega \rangle$ is cofinal in $\tilde{\lambda}_j$.

We also set: $\bar{\xi}_i = \bar{\xi}_i^{n+1}$, $\xi_i = \xi_i^{n+1}$.

As before, our strategy is to construct $c_k, \dot{\sigma}_k$ $(k < \omega)$ s.t. $\langle c_k \mid k < \omega \rangle$ is a thread in $\langle \mathbb{B}_{\bar{\xi}_k} \mid k < \omega \rangle$ and c_k forces $\dot{\sigma}_k : \bar{W} \prec W$. The intention is again that if $c = \bigcap_k c_k \in G$ and G is $\mathbb{B}_\lambda$-generic, then we will be able to define the desired embedding in $V[G]$ from $\langle \sigma_k \mid k < \omega \rangle$, where $\sigma_k = \dot{\sigma}_k^G$. However, we no longer had the function $\bar{f}$ available in defining $\langle \bar{\xi}_i \mid i < \omega \rangle$. As a result, we will no longer be able to enforce $\sigma_k(\bar{\xi}_i) = \xi_i$. However, we can still enforce $\sup \sigma_k{}''\bar{\lambda} = \tilde{\lambda}$. We shall have to make do with that. Note that, although $\bar{\bar{\mathbb{B}}}_\lambda \geq \lambda$, we have $\bar{\bar{\mathbb{B}}}_i < \lambda$ for $i < \lambda$. Thus $\lambda = \lambda_{n+1}$ is like the other λ_i wrt. $\langle \mathbb{B}_i \mid i < \lambda \rangle$.

We inductively construct $c_k \in \mathbb{B}_{\xi_k}$, $\dot\sigma_k \in V^{\mathbb{B}_{\xi_k}}$ s.t.

(I) (a) $c_0 = 1$, $\dot\sigma_0 = \check\sigma$

(b) $h_{\xi_j}(c_k) = c_j$ for $k = j+1$

(II) Let $G \ni c_k$ be $\mathbb{B}_{\xi_k}$-generic. Set:

$G_\eta = G \cap \mathbb{B}_\eta$ $(\eta \leq \xi_k)$, $\bar G_\eta = \bar G \cap \bar{\mathbb{B}}_\eta$ $(\eta \leq \bar\xi_k)$ and

$\sigma_j = \dot\sigma_j^G = \dot\sigma_j^{G_{\xi_j}}$ for $j \leq k$. Then:

(a) $\sigma_k : \bar W \prec W$

(b) $\sigma_k(\bar\Theta, \bar{\mathbb{B}}, \bar\tau, \bar\lambda_i) = \Theta, \mathbb{B}, \tau, \lambda_i$ $(i = 1, m, n+1)$

(c) $\sup \sigma_k{}''\bar\lambda_i = \tilde\lambda_i$ $(i = 0, m, m+1)$

(d) Let $\sigma_k(\bar\xi_m) \leq \xi_k < \sigma_k(\bar\xi_{m+1})$. Then

$$\sigma_k{}''\bar G_{\bar\xi_m} \subset G$$

(e) Let $k = j+1$. Then $\sigma_k(x_l) = \sigma_j(x_l)$ for $l \leq j$

(f) Let $k = j+1$, $i = 0, m, m+1$. At $\sigma_j(\bar\xi^i_m) \leq \xi^i_k < \sigma_j(\bar\xi^i_{m+1})$, then $\sigma_k(\bar\xi^i_l) = \sigma_j(\bar\xi^i_l)$ for $l \leq m+1$.

<u>Note</u> By (e) it follows that $\sigma_k(x_i) = \sigma_i(x_i)$ for $i \leq k$.

<u>Note</u> We shall, in fact, arrange that if $\sigma_j(\bar\xi_m) \leq \xi_j < \xi_k < \sigma_j(\bar\xi_{m+1})$, then $\sigma_k = \sigma_j$.

We now show that I, II imply the <u>Claim</u>.

Set $c = \bigcap_k c_k$. Then $c \in \mathbb{B}_{\tilde{\lambda}} \subset \mathbb{B}_\lambda$. Let $G \ni c$ be $\mathbb{B}_\lambda$ - generic. Set:

$$\sigma_k = \dot{\sigma}_k^G = \dot{\sigma}_k^{G_{\xi_k}}.$$

By II (e) there is $\sigma' \in V[G]$ s.t. $\sigma' : \bar{W} \prec W$ defined by: $\sigma'(x) =$ that y s.t. $\sigma_k(x) = y$ for sufficiently large k.

(a), (b) of the Claim are clearly satisfied.

We prove (c). $\sigma'{}'' \bar{\lambda}_i \subset \tilde{\lambda}_i$ since $\sigma'(\nu) = \sigma_k(\nu)$ for some k, where $\sigma_k{}'' \bar{\lambda}_i \subset \tilde{\lambda}$. To see that $\sigma'{}'' \bar{\lambda}_i$ is unbounded in $\tilde{\lambda}_i$, fix ξ_k and let $\sigma_k(\bar{\xi}_m) \leq \xi_k < \sigma_k(\bar{\xi}_{m+1})$. Then $\sigma'(\bar{\xi}_{m+1}) = \sigma_k(\bar{\xi}_{m+1}) > \xi_k$. QED (c)

We now prove (d). Let $p \in \bar{G}_{\bar{\xi}_k}$. Let $l > k$ be large enough that $\sigma'(p) = \sigma_l(p)$. Then $\sigma'(p) \in G$. But $\bigcup_i \bar{G}_{\bar{\xi}_i}$ is dense in $\bar{G}$. QED (d)

All that remains is to inductively define c_k, $\dot{\sigma}_k$ and verify (I), (II). This will be somewhat trickier than the corresponding step in Case 1. We shall, in fact, have to add some further induction hypotheses. Before defining c_k we shall define a $b_k \in \mathbb{B}_{\xi_k}$ s.t.

III (a) $b_0 = 1$

(b) $h_{\bar{\xi}_j}(b_k) = c_j$ if $k = j+1$

(c) II (a)–(f) hold whenever $b_k \in G$.

(d) $c_k \subset b_k$.

$\dot{\sigma}_k$ will be defined simultaneously with b_k, before defining c_k. Our next induction hypothesis states an important property of b_k:

Def Let $\nu \le \xi_k < \mu < \tilde{\lambda}$ s.t. $\xi_i < \nu$ for $i<k$,

$$a^{i\nu\mu} =_{Df} b_k \wedge [\![\dot{\sigma}_k(\check{\bar{\xi}}_i) = \check{\nu} \wedge \dot{\sigma}_k(\check{\bar{\xi}}_{i+1}) = \check{\mu}]\!]_{\xi_k},$$

It is easily seen that:

(1) $a^{i\nu\mu} \wedge a^{i'\nu'\mu'} = 0$ if $\langle i,\nu,\mu\rangle \neq \langle i',\nu',\mu'\rangle$.

proof. Suppose $a^{i\nu\mu} \wedge a^{i'\nu'\mu'} \in G$ where G is $\mathbb{B}_{\xi_k}$-generic. Then $i = i'$ since if e.g. $i < i'$, then $\mu = \sigma_k(\bar{\xi}_{i+1}) \le \sigma_k(\bar{\xi}_{i'}) = \nu' \le \xi_k$ Contr! But then $\nu = \sigma_k(\bar{\xi}_i) = \nu'$ and $\mu = \sigma_k(\bar{\xi}_{i+1}) = \mu'$. Contr! QED (1)

We shall inductively verify:

IV $a^{i\nu\mu} \wedge [\![\dot{\sigma}_k(\check{x}) = \check{y}]\!]_{\xi_k} \in \mathbb{B}_\nu$

for $\sup_{i<k} \xi_i < \nu \le \xi_k < \mu$.

(Hence $a^{i\nu\mu} = a^{i\nu\mu} \wedge [\![\dot{\sigma}_k(\check{0}) = \check{0}]\!] \in \mathbb{B}_\nu$)

Set $A = A_k$ = the set of $a^{i\nu\mu} \neq 0$ s.t. $\sup_{i<k} \xi_i < \nu \leq \xi_k < \mu$. By IV we see that for each $a = a^{i\nu\mu} \in A$ there is $\dot{\sigma}_a \in V^{\mathbb{B}_\nu}$ s.t.

(2) $\dot{\sigma}_a^{G_\nu} = \sigma_k^G$ for $\mathbb{B}_{\xi_k}$ - generic $G \ni a$.

But:

(3) If $G \ni a$ is $\mathbb{B}_\nu$ - generic, then G extends to a $\mathbb{B}_{\xi_k}$ - generic G' s.t. $G = G'_\nu$. Hence

$\dot{\sigma}_a^G = \dot{\sigma}_a^{G'} = \dot{\sigma}_k^{G'}$.

Hence:

(4) Let $G \ni a$ be $\mathbb{B}_\nu$ - generic, where $a = a^{i\nu\mu} \in A_k$. Then II holds with $\sigma_a = \dot{\sigma}_a^G$ in place of σ_k, $\sigma_j = \dot{\sigma}_j^G = \dot{\sigma}_j^{G_{\xi_j}}$ for $j < k$, where $G_\eta = G \cap \mathbb{B}_\eta$ for $\eta \leq \nu$, $\bar{G}_\eta = \bar{G} \cap \bar{\mathbb{B}}_\eta$ for $\eta \leq \xi_k$.

(In particular we have fr. ins. that $\sigma_a(x_l) = \sigma_j(x_l)$ for $l \leq j$ where $k = j+1$.)

Whenever $\nu < \mu < \lambda$ and G is $\mathbb{B}_\nu$ - generic, we know that $\mathbb{B}_\mu / G$ is subcomplete in $V[G]$.

Thus, using (4) and repeating the construction of $c_{j+1}, \dot{\sigma}_{j+1}$ from c_j, σ_j in Case 1, we get:

(5) Let $a \in A_k$, $a = a^{i\nu\mu}$. There are $\tilde{a} \in \mathbb{B}_\mu$, $\dot{\sigma}_a' \in V^{\mathbb{B}_\mu}$ s.t. $h_\nu(\hat{a}) = a$ and whenever $G \ni \tilde{a}$ is $\mathbb{B}_\mu$-generic, $\sigma_a = \dot{\sigma}_a^G$, and $\sigma_a' = \dot{\sigma}_a'^G$, then:

(a) $\sigma_a' : \bar{W} \prec W$

(b) $\sigma_a'(\bar{\theta}, \bar{\mathbb{B}}, \bar{s}, \bar{\lambda}_i) = \theta, \mathbb{B}, s, \lambda_i$ $(i = 1, \ldots, m+1)$

(c) $\sup \sigma_a' {}'' \bar{\lambda}_i = \tilde{\lambda}_i$ $(i = 0, \ldots, m+1)$

(d) $\sigma_a' {}'' \bar{G}_{\bar{\xi}_{i+1}} \subset G$ (recall $\sigma_a(\bar{\xi}_{i+1}) = \mu$)

(e) Let r be least s.t. $\mu \leq \xi_r$. Then $\sigma_a'(x_i) = \sigma_a(x_i)$ for $i < r$.

(f) Let r be as above. Let $i = 0, \ldots, m+1$ and let $\sigma_a(\bar{\xi}_m^i) \leq \xi_r < \sigma_a(\bar{\xi}_{m+1}^i)$. Then $\sigma_a'(\bar{\xi}_l^i) = \sigma_a(\bar{\xi}_l^i)$ for $l \leq m+1$.

For each $a \in A_k$ we fix such a pair $\tilde{a}, \dot{\sigma}_a'$, which can be regarded as an instruction to be used later in forming b_r, where r is least s.t. $\mu \leq \xi_r$. If G is $\mathbb{B}_{\xi_r}$-generic and $a \cap b_r \in G$, we want:

$$\tilde{a} \in G \text{ and } \sigma_r = \dot{\sigma}_a^{\prime G}$$

(where $\sigma_r = \dot{\sigma}_r^{G}$). In particular, we need: $a \cap b_r = \tilde{a}$. But we shall also require $h_{\xi_k}(b_r) = c_k$. Hence we need $a \cap c_k = h_{\xi_k}(a \cap b_r) = h_\xi(\tilde{a})$. This is why b_k must be "shrunk" to c_k. Accordingly, we define c_k as follows:

<u>Def</u> Let b_k be given. Set $\bar{b} = b_k \setminus \bigcup A_k$

Then $c_k =_{Df} \bar{b} \cup \bigcup_{a \in A_k} h_{\xi_k}(\tilde{a})$.

We are working by induction on k. We assume I – VI to hold for all $j < k$ and III (a)–(c), IV to hold at k. We must now verify I, II, III (d) at k. III (d) is immediate; hence so is II. I (b) holds since $h_{\xi_i} h_{\xi_k}(\tilde{a}) = h_{\xi_i}(\tilde{a}) = h_{\xi_i} h_\nu(\tilde{a}) =$ $= h_{\xi_i}(a)$ for $a = a^{i\nu\mu} \in A_k$. Hence:

$h_{\xi_i}(c_k) = h_{\xi_i}(\bar{d}) \cup \bigcup_{a \in A} h_{\xi_i}(a) = h_{\xi_i}(b_k) = c_j$.

For I(a) note that $A_0 = \{a\}$ where $a = a^{0,0,\xi_1} = 1$, since $\sigma_0 = \sigma$ by III (c). Hence $c_0 = h_0(\tilde{a}) = 1$.

This completes the proof that I-IV hold at k, assuming III (a)-(c) and IV to hold at k and I-IV to hold below.

Now assume I-IV to hold below k. We must define b_k, $\mathring{\sigma}_k$ and verify III (a)-(c) and IV at k.

For $k = 0$ set: $b_k = 1$, $\dot{\sigma}_k = \check{\sigma}$. The verifications are trivial.

Now let $k = j+1$. Note that A_ℓ, $\langle \tilde{a} \mid a \in A_\ell \rangle$ have been defined for all $\ell \leq j$. Set:

<u>Def</u> $\hat{A}_j$ = the set of $a = a^{i\nu\mu} \in \bigcup_{\ell \leq j} A_\ell$ s.t. $\xi_j < \mu$.

Note that:

(6) Let $a \in a^{i\nu\mu} \in A_\ell$, $\ell \leq j$. Then $\sigma_j(\xi_h) = \sigma_\ell(\xi_h)$ for $h \leq i+1$.

proof. By induction on j we prove this for all j s.t. every $\ell \leq j$ satisfies I - IV.

At $\ell = j$ there is nothing to prove, so let

$l \leq n$ where $j' = n+1$. Then

$$\sigma_n(\bar{\xi}_i) = \nu \leq \xi_j < \mu = \sigma_n(\bar{\xi}_{i+1}),$$

The conclusion follows by II (f).

QED (6)

Using this we can repeat the proof of (1) to get:

(7) $a^{i\nu\mu} \wedge a^{i'\nu'\mu'} = 0$ if $\langle i,\nu,\mu\rangle \neq \langle i',\nu',\mu'\rangle$ and $a^{i\nu\mu}, a^{i'\nu'\mu'} \in \hat{A}_j$.

We now define:

<u>Def</u> $b_k = \bigcup \{h_{\xi_k}(\tilde{a}) \mid a \in \hat{A}_j\}$,

for $k = j+1$.

To define $\dot{\sigma}_k$ we set:

$\tilde{A}$ = the set of $a^{i\nu\mu} \in \hat{A}_j$ s.t. $\mu \leq \xi_k$.

$\dot{\sigma}_k \in V^{\mathbb{B}}$ is then a name s.t.

$[\![\dot{\sigma}_k = \dot{\sigma}_a']\!] = \tilde{a}$ if $a \in \tilde{A}$

$[\![\dot{\sigma}_k = \dot{\sigma}_j]\!] \cap b_k = b_k \setminus \bigcup \tilde{A}$.

It is straightforward to see that

(8) III (c) holds at k.

The proof is left to the reader.

III (a) holds vacuously at $k=j+1$. We prove:

(9) III (b) holds at k.

proof.

Clearly $h_{\xi_j}(b_k) = \bigcup_{a\in\hat{A}_j} h_{\xi_j}(\tilde{a})$. Hence we need:

Claim $c_j = \bigcup_{a\in\hat{A}_j} h_{\xi_j}(\tilde{a})$.

For $j=0$ this is trivial, so let $j=\ell+1$,

Recall that $c_j = \bar{b} \cup \bigcup_{a\in A_j} h_{\xi_j}(\tilde{a})$, where

$\bar{b} = b_j \setminus \bigcup_{a\in A_j} a$, so it suffices to

show:

Claim $\bar{b} = \bigcup_{a\in A'} h_{\xi}(\tilde{a})$ where $A' = \hat{A}_j \setminus A_j$.

($\supset$) Let $a' \in A'$. Then $a' \in \hat{A}_\ell$. Hence $h_{\xi_\ell}(\tilde{a}') \subset b_j$. Thus it suffices to note that $h_{\xi_\ell}(\tilde{a}') \cap h_{\xi_\ell}(\tilde{a}) = 0$ for all $a \in A_j$, since, in fact, $a' \cap a = 0$ by (7).

($\subset$) Suppose not. By the definition of b_j there is $a \in \hat{A}_\ell \setminus A'$ s.t. $h_{\xi_\ell}(\tilde{a}) \cap \bar{b} \neq 0$. Hence $a \cap \bar{b} \neq 0$.

We derive a contradiction. Let $G \ni a \cap \bar{b}$ be $\mathbb{B}_j$-generic. Note that

$a = a^{i\nu\mu}$ with $\nu \le \xi_\ell < \mu \le \xi_j$, since otherwise $a \in A'$. But then

$\sigma_l(\bar{\xi}_i) = \nu \leq \xi_l < \sigma_l(\bar{\xi}_{i+1}) = \mu$. Hence

$\sigma_1(\bar{\xi}_r) = \sigma_l(\bar{\xi}_r)$ for $r \leq i+1$. But then

$\sigma_1(\bar{\xi}_{i+1}) \leq \xi_1$ and there must be an

$m > i$ s.t. $\sigma_1(\bar{\xi}_m) \leq \xi_1 < \sigma(\bar{\xi}_{m+1})$.

Set $\nu^* = \sigma_1(\bar{\xi}_m)$, $\mu^* = \sigma_1(\bar{\xi}_{m+1})$.

Then $a^* = a^{m,\nu^*,\mu^*} \in G$. Hence

$a^* \in A_1'$. But $a^* \cap \bar{b} \neq 0$, since

$\bar{b} \in G$, contradicting the definition

of $\bar{b}$. QED (9)

It remains only to show:

(10) $\underline{\mathrm{IV}}$ holds at k.

As a preliminary we show:

(11) Let $a = a^{i\nu\mu} \in A_k$, $a' = a^{i'\nu'\mu'} \in \hat{A}_1$

s.t. $\mu' > \nu$. Then $a \cap a' = 0$.

Proof.

Suppose not. Let $a \cap a' \in G$ where

G is $\mathbb{B}_{\xi_k}$-generic. Then

$\sigma_1(\bar{\xi}_{i'}) = \nu \leq \xi_1 < \sigma_1(\bar{\xi}_{i'+1})$

Hence $\sigma_k(\bar{\xi}_l) = \sigma_j(\xi_l)$ for $l \le i'+1$. But then $\sigma_k(\bar{\xi}_i) = \nu < \mu' = \sigma_k(\bar{\xi}_{i'+1})$. Hence $i \le i'$. Thus $\nu = \sigma_k(\bar{\xi}_i) \le \sigma_k(\bar{\xi}_i) = \nu' \le \xi_i$. Contradiction!, since $a^{i\nu\mu} \in A_k$.

QED (11)

We can now prove (10). Let $a = a^{i\nu\mu} \in A_k$. Set A' = the set of $e = a^{i'\nu'\mu'} \in \hat{A}_j$ s.t. $\mu' \le \nu$. Since $b_k = \bigcup_{e \in \hat{A}_j} h_{\bar{\xi}_k}(\tilde{e})$ and $h_{\bar{\xi}_k}(\tilde{e}) = \tilde{e}$ for $e \in A'$, we conclude by (10) that: $a = a \cap b_k = \bigcup_{e \in A'} a \cap \tilde{e}$. But for $e = a^{i'\nu'\mu'} \in A'$ we have: $[\![\dot{\sigma}_k = \dot{\sigma}_e']\!] = \tilde{e} \in \mathbb{B}_{\mu'}$. Moreover, $[\![\dot{\sigma}_e'(\check{x}) = \check{y}]\!] \in \mathbb{B}_{\mu'}$, since $\dot{\sigma}_e', \check{x}, \check{y} \in V^{\mathbb{B}_{\mu'}}$ and "$\dot{\sigma}_e'(\check{x}) = \check{y}$" is Σ_0. (Cf. the note following (*) in Case 1.) Since $a = b_k \cap [\![\dot{\sigma}_k(\check{\bar{\xi}}_i) = \check{\nu}]\!] \cap [\![\dot{\sigma}_k(\bar{\xi}_{i+1}) = \check{\mu}]\!]$, we conclude: $a \cap [\![\dot{\sigma}_k(\check{x}) = \check{y}]\!] \cap \tilde{e} =$
$= \tilde{e} \cap [\![\dot{\sigma}_e'(\check{x}) = \check{y}]\!] \cap [\![\dot{\sigma}_e'(\check{\bar{\xi}}_i) = \check{\nu}]\!] \cap [\![\dot{\sigma}_e'(\bar{\xi}_{i+1}) = \mu]\!] \in \mathbb{B}_\mu \subset \mathbb{B}_\nu$.
Hence $a \cap [\![\dot{\sigma}_k(\check{x}) = \check{y}]\!] = \bigcup_{e \in A'} a \cap [\![\dot{\sigma}_k(\check{x}) = y]\!] \cap \tilde{e} \in \mathbb{B}_\nu$

QED (10)

QED (Theorem 1)