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# $K$ WITHOUT THE MEASURABLE

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**Abstract.** We show in ZFC that if there is no proper class inner model with a Woodin cardinal, then there is an absolutely definable *core model* that is close to $V$ in various ways.

**§1. The main theorem.** If the universe $V$ of sets does not have within it very complicated canonical inner models for large cardinal hypotheses, then it has a canonical inner model $K$ that in some sense is as large as possible. $K$ is absolutely definable, its internal structure can be analyzed in fine-structural detail, and yet it is close to the full universe $V$ in various ways.

If $0^\sharp$ does not exist,[1] then $K = L$. Set forcing cannot add $0^\sharp$ or change $L$, so $K^V = K^{V[G]} = L$ whenever $G$ is set-generic over $V$. The fine-structure theory of [7] produces a detailed picture of the first order theory of $L$. Jensen's Covering Theorem [1] describes one of the most important ways $L$ is close to $V$: any uncountable $X \subseteq L$ has a superset $Y$ of the same cardinality such that $Y \in L$.

If $0^\sharp$ does exist, then $L$ is quite far from $V$, and so $K$ must be larger than $L$. Dodd and Jensen developed a theory of $K$ under the weaker hypothesis that there is no proper class inner model with a measurable cardinal in [2], [3], and [4]. This hypothesis is compatible with the existence of $0^\sharp$, and if $0^\sharp$ exists, then $0^\sharp$ in $K$, and hence $K$ is properly larger than $L$. Under this weaker anti-large-cardinal hypothesis, $K$ is again absolutely definable, admits a fine structure theory like that of $L$, and is close to $V$, in that every uncountable $X \subseteq K$ has a superset $Y$ of the same cardinality such that $Y \in K$.

Several authors have extended the Dodd–Jensen work over the years. We shall recount some of the most relevant history in the next section. In this paper, we shall prove a theorem which represents its ultimate extension in one direction. Our discussion of the history will be clearer if we state that theorem now.

THEOREM 1.1. *There are $\Sigma_2$ formulae $\psi_K(v)$ and $\psi_\Sigma(v)$ such that, if there is no transitive proper class model satisfying* ZFC *plus "there is a Woodin cardinal", then*

(1) $K = \{v \mid \psi_K(v)\}$ *is a transitive proper class premouse satisfying* ZFC,
(2) $\{v \mid \psi_\Sigma(v)\}$ *is an iteration strategy for $K$ for set-sized iteration trees, and moreover the unique such strategy,*

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[1] One can think of $0^\sharp$ as a weak approximation to a canonical inner model with a measurable cardinal.


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(3) (*Generic absoluteness*) $\psi_K^V = \psi_K^{V[g]}$, *and* $\psi_\Sigma^V = \psi_\Sigma^{V[g]} \cap V$, *whenever* $g$ *is* $V$*-generic over a poset of set size,*
(4) (*Inductive definition*) $K \mid (\omega_1^V)$ *is* $\Sigma_1$ *definable over* $J_{\omega_1}(\mathbb{R})$,
(5) (*Weak covering*) *For any* $\lambda \geq \omega_2^V$ *such that* $\lambda$ *is a successor cardinal of* $K$, $\mathrm{cof}(\lambda) \geq |\lambda|$; *thus* $\alpha^{+K} = \alpha^+$, *whenever* $\alpha$ *is a singular cardinal of* $V$.

It is easy to formulate this theorem without referring to proper classes, and so formulated, the theorem can be proved in ZFC. The theorem as stated can be proved in GB.

For definiteness, we use here the notion of premouse from [25], although the theorem is almost certainly also true if we interpret *premouse* in the sense of [9]. See the footnotes to section 3.5 below.[2] A proper class premouse is sometimes called an extender model. Such models have the form $(L[\vec{E}], \in, \vec{E})$, where $\vec{E}$ is a coherent sequence of extenders, and what (1) says is that the distinguished extender sequence of $K$ is definable over $V$ by $\psi_K$. One can show that $K$ satisfies $V = K$.[3]

The hierarchy of an iterable premouse has condensation properties like those of the hierarchy for $L$, and this enables one to develop their first order theories in fine-structural detail. For example, since $K$ is an iterable extender model, it satisfies $\square$ at all its cardinals. (See [18] and [19].)

Items (1)–(4) say that $K$ is absolutely definable. Notice that by items (3) and (4), for any uncountable cardinal $\mu$, $K \mid \mu$ is $\Sigma_1$ definable over $L(H_\mu)$, uniformly in $\mu$. This is the best one can do if $\mu = \omega_1$ (see [22, §6]), but for $\mu \geq \omega_2$ there is a much simpler definition of $K \mid \mu$ due to Schindler (see [6]).

The weak covering property (5) is due to Mitchell and Schimmerling [13], building on [14]. The strong covering property can fail once $K$ can be complicated enough to have measurable cardinals. Weak covering says that $K$ is close to $V$ in a certain sense. There are other senses in which $K$ can be shown close to $V$; for example, every extender which coheres with its sequence is on its sequence [17], and if there is a measurable cardinal, then $K$ is $\Sigma^1_3$-correct [22, §7].

The hypothesis that there is no proper class model with a Woodin cardinal in Theorem 1.1 cannot be weakened, unless one simultaneously strengthens the remainder of the hypothesis, i.e., ZFC. It is in this respect that Theorem 1.1 is the ultimate result in one direction. For suppose $\delta$ is Woodin, that is, $V$ is our proper class model with a Woodin. Suppose toward contradiction we had a formula $\psi_K(v)$ defining a class $K$, and that (3), (4), and (5) held. Let $g$ be $V$-generic for the full stationary tower below $\delta$.[4] Let

$$j \colon V \to M \subseteq V[g],$$

where $M^{<\delta} \subseteq M$ holds in $V[g]$. We can choose $g$ so that $\mathrm{crit}(j) = \aleph_{\omega+1}^V$. Let $\mu = \aleph_\omega^V$. Then

$$(\mu^+)^K = (\mu^+)^V < (\mu^+)^M = (\mu^+)^{j(K)} = (\mu^+)^K,$$

[2] The authors are quite sure that there is at most *one* core model, but the project of translating between the two types of premouse is not complete. See [5].

[3] This follows easily from [22, 8.10], for example.

[4] The reader who is not familiar with stationary tower forcing needn't worry, as we shall not use it in this paper.

a contradiction. The first relation holds by (5), the second by the choice of $j$, the third by (5) applied in $M$, and the last by (3) and (4), and the agreement between $M$ and $V[g]$.[5]

As a corollary to Theorem 1.1, we get

COROLLARY 1.2. *If* ZFC + *"there is a pre-saturated ideal on* $\omega_1$*" is consistent, then* ZFC + *"there is a Woodin cardinal" is consistent.*

The corollary follows from the theorem via a straightforward transcription of the argument in section 7 of [22]. Shelah has proved the converse relative consistency result. The proof of the corollary illustrates one of the main ways core model theory is applied: if there is a pre-saturated ideal on $\omega_1$, then there cannot be a $K$ as in the conclusion to 1.1, and therefore there is a proper class model with a Woodin cardinal.

Core model theory can be used to produce inner models with more than one Woodin cardinal. In this respect, 1.1 is not the end of the line. But so far, what takes its place are relativizations of 1.1 that are proved by the same method. See [23] for one example of such an argument.

## §2. Some history.

Our theorem grows out of, and in some sense completes, a long line of research in core model theory. In order to set the stage properly, we review some of this prior work.

Core model theory began in the mid-1970's with the work of Dodd and Jensen, [2], [3], [4], who proved Theorem 1.1 with its anti-large-cardinal hypothesis strengthened to "$0^\dagger$ does not exist", and indeed reached much stronger conclusions regarding the covering properties of $K$ under that assumption.

The theory was further developed under the weaker anti-large-cardinal hypothesis that there is no sharp for a proper class model of ZFC with a measurable $\kappa$ of order $\kappa^{++}$ by Mitchell [12], [11]. Mitchell's work introduced ideas which have played a prominent role since then. Of particular importance for us is the technique of constructing a preliminary model $K^c$ which is close enough to $V$ to have weak covering properties, and yet is constructed from extenders which have "background certificates", so that one can prove the model constructed is iterable. The weak covering properties of $K^c$ are then used to obtain the true, generically absolute $K$ as a certain Skolem hull of $K^c$.

In 1990, Steel extended Mitchell's work so that it could be carried out under the weaker anti-large-cardinal hypothesis that there is no proper class model with a Woodin cardinal. He needed, however, to assume that there is a measurable cardinal $\Omega$. Under that hypothesis, he could develop the basic theory of $K^{V_\Omega}$, including proofs of (1)–(4) of Theorem 1.1. (See [22].) At this level, the iterability of $K^c$ required a stronger background condition than the one Mitchell had used, which had just been countable completeness. Steel introduced such a condition, and used it to prove iterability, but he was not able to prove that his preliminary $K^c$ computed any successor cardinals correctly without resorting to the ad hoc assumption that there is a measurable cardinal in $V$. As a result, one could not obtain sharp relative consistency results at the level of one Woodin cardinal, such as

[5]The same proof shows there is no formula $\psi$ such that (3) holds, and (5) holds in all set generic extensions of $V$.

Corollary 1.2, using the theory Steel developed. Our work here removes the ad hoc assumption that there is a measurable cardinal, and thereby remedies this defect.

In 1991–94 Mitchell, Schimmerling, and Steel proved weak covering for the one-Woodin $K$ Steel had constructed in [22]. See [14] and [13]. The techniques of [14] will be important for us here, as we shall use them in a measurable-cardinal-free proof of weak covering for one of our preliminary versions of $K$. Thus by 1994 all parts of our main theorem had been proved, but in the theory Kelley–Morse augmented by a predicate $\mu$, with axioms stating that $\mu$ is a normal, non-principal ultrafilter on the class $\Omega$ of all ordinals.

The first step toward eliminating the measurable cardinal from the theory of [22] was to find a background condition weaker than Steel's which would suffice to prove iterability. This was first done in early 2001 by Mitchell and Schindler. They showed that if $\Omega \geq \omega_2$ is regular, and $2^{<\Omega} = \Omega$ then (provided all mice are tame), there is an iterable mouse $W$ of height $\Omega$ which is *universal*, in the sense that no premouse of height $\Omega$ iterates past $W$. The existence of such $\Omega$ follows from GCH, but it is not provable in ZFC alone. Subsequently, in 2003, Jensen [8] found a probably weaker[6] background condition, showed it suffices for iterability, and showed without any GCH assumptions that it allows enough extenders on the sequence of $K^c$ that $K^c$ is universal. The reader should see [10] for further discussion of these background certificate conditions, their relationships, and the resulting universal models.

The construction of a "local $K^c$" of height some regular $\Omega$, and universal among all mice of ordinal height $\Omega$, was an important advance. Previously, the universality of $K^c$ and $K$ had been generally understood, so far as their basic theory is concerned, in terms of proper-class sized comparisons with proper-class sized competitors.[7] However, once one gets close to Woodin cardinals, it becomes possible that there are definable, proper-class sized iteration trees on $K^c$ (whatever $K^c$ may be) which have no definable, cofinal branches. This makes class-sized comparisons of class-sized premice pretty much useless, once one gets near Woodin cardinals. In contrast, lemmas 2.3 and 2.4(b) of [22] easily imply

THEOREM 2.1. *Suppose there is no proper class model with a Woodin cardinal, and let $M$ be a countably iterable premouse of height $\Omega$, where $\Omega$ is regular; then for any cardinal $\kappa$, there is a unique $(\kappa, \kappa)$-iteration strategy for $M$.*

If there is no proper class model with a Woodin cardinal, then $K^c$ constructions of [15] and [8] produce countably iterable premice, and hence by 2.1, they produce fully iterable premice. Thus the fact that they produce mice which are universal at regular $\Omega$ is potentially quite useful. In the context of ZFC, universality at a regular cardinal is much more useful than universality at OR.

Nevertheless, universality at regular cardinals is not enough to implement Mitchell's method for obtaining true $K$ as a Skolem hull of $K^c$. For that, one

[6]The precise relationship between the two conditions is not known. There is a common weakening of the two which seems to suffice for iterability, but this has not been checked carefully.

[7]It is shown in [17] that $K \mid \Omega$ is universal vis-a-vis "stable" competitors of height $\Omega$, whenever $\Omega$ is a regular cardinal $\geq \omega_2$. (Stability is defined in section 3.1 below. The need to restrict attention to stable competitors was overlooked in [17].) However, this is "after the fact", so to speak, in that one needs the basic theory of $K$ from [22] in order to prove it. In similar fashion, [14] and [13] imply that $K \mid \Omega$ is universal vis-a-vis stable competitors of height $\Omega$, whenever $\Omega$ is the successor of a singular cardinal, but the proof uses the theory of [22].

needs some form of weak covering, and a corresponding notion of "thick hull". Jensen took the key step forward here in 2006, with his theory of stacking mice. Jensen's results are described in section 3 of [10], and we shall make heavy use of those results here. Jensen described this work to Steel in early May of 2006, and after some ups and downs, in summer 2007 the two of them finished the proof of Theorem 1.1.

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§3. **Plan of the proof.** Our main goal will be to construct mice which are universal at some regular cardinal because they satisfy weak covering. Having done that, it will be a routine matter to adapt Mitchell's notion of thick sets to define local $K$'s, and show they fit together into a single $K$ using the local inductive definition of $K$ from section 6 of [22].

We reach our main goal by proving:

THEOREM 3.1. *Assume there is no proper class model with a Woodin cardinal. Let $\kappa$ be a singular strong limit cardinal; then there is an iterable mouse $M$ such that $(\kappa^+)^M = \kappa^+$.*

Ordinarily, one would expect that the mouse $M$ witnessing 3.1 would be $K$ itself, and the proof of 3.1 would involve the basic theory of $K$, as it does in [14]. Thus we would have no way to get started. But we shall show that one need not go all the way to $K$ to get the desired $M$. Instead, the mouse $M$ witnessing 3.1 will be a *psuedo-$K$*, constructed using versions of thick sets and the hull and definablitity properties in which the measurable cardinal $\Omega$ of [22] is replaced by a large regular cardinal. All of the new work lies in carrying over enough of the [22] theory of $K$ to psuedo-$K$; having done that properly, it will be completely routine to adapt the proof of weak covering in [14].

The construction of psuedo-$K$ goes roughly as follows. Let $\kappa$ be as in 3.1, and let $\kappa < \tau < \Omega$, where $\tau$ and $\Omega$ are regular, $2^{<\tau} < \Omega$, and $\forall\alpha < \Omega(\alpha^\omega < \Omega)$. Let

$W =$ output of the robust-background-extender $K^c$-construction
up to $\Omega$, with background extenders having
critical point of $V$-cofinality $\tau$ forbidden.

Jensen [8] shows that $W$ is countably iterable. As there is no proper class with a Woodin cardinal, $W$ is fully iterable.

There are three cases:

CASE 1. $W$ has no largest cardinal.

In this case, Jensen [8] shows that $W$ is universal, in that no mouse of height $\leq \Omega$ iterates past $W$. By the bicephalus argument, any robust extender that coheres with the sequence of $W$ is on the sequence of $W$. Let $S(W)$ be the stack over $W$ as defined in section 3 of [10]. By the proof[8] of theorem 3.4 of [10], we have

$$\operatorname{cof}(o(S(W)) \geq \Omega,$$

[8]This is very nearly the *statement* of 3.4 of [10], but unfortunately, a superfluous instance of GCH crept into the definition of "certified $K^c$" given there.

where we use the notation $o(H)$ for $H \cap \mathrm{OR}$. This enables us to define thick sets as $\tau$-clubs in $o(S(W))$. Mitchell's arguments carry over, and one can then define our psuedo-$K$, call it $\tilde{K}(\tau, \Omega)$, as the intersection of all thick hulls of $S(W)$. Sharpening some arguments which in [22] brought in the measurable cardinal again, we show that

$$\tau \subseteq \tilde{K}(\tau, \Omega).$$

This is done in 4.31 below. It is not hard then to show that $\tilde{K}(\tau, \Omega) \mid \tau$ satisfies the inductive definition of $K$ in section 6 of [22]. So in this case, our psuedo-$K$, up to $\tau$, is the real $K$. In particular, the proof of weak covering in [14] easily shows that $M = \tilde{K}(\tau, \Omega)$ witnesses the truth of Theorem 3.1.[9]

CASE 2. $W$ has a largest cardinal $\gamma$, and $W \models \mathrm{cof}(\gamma)$ is not measurable.

This case is much easier. It is easy to see that $W$ is universal. We now just take thick sets to be $\tau$-clubs in $\Omega$, and define $\tilde{K}(\tau, \Omega)$ to be the intersection of all thick hulls of $W$. Again, $\tilde{K}(\tau, \Omega) \mid \tau$ is true $K$ in the sense of the local inductive definition, and witnesses the truth of 3.1.

CASE 3. $W$ has a largest cardinal $\gamma$, and $W \models \mathrm{cof}(\gamma)$ is measurable.

The trouble here is that if $\mu$ is a measure of $W$ on $\mathrm{cof}(\gamma)^W$, then $\mathrm{Ult}(W, \mu)$ has ordinal height $> \Omega$. So $W$ is "unstable", making the notion of universality for it problematic. So what we do is replace $W$ by

$$W^* = \mathrm{Ult}(W, \mu) \mid \Omega,$$

where $\mu$ is the order zero measure of $W$ on $\mathrm{cof}(\gamma)^W$. It is not hard to see $\gamma$ is also the largest cardinal of $W^*$, and not of measurable cofinality in $W^*$. So $W^*$ is stable, and universal vis-a-vis other stable mice of height $\leq \Omega$. We can then use the procedure of case 2 to derive $\tilde{K}(\tau, \Omega)$ from $W^*$. We won't have that $\tilde{K}(\tau, \Omega) \mid \tau$ is true $K$ in this case, however, because replacing $W$ by $W^*$ may have gotten rid of some measures at ordinals of $V$-cofinality $\eta$, where $\eta$ is the $V$-cofinality of $\gamma$, which are in true $K$. Nevertheless, the proof of weak covering for $K$ in [14] goes through for $\tilde{K}(\tau, \Omega)$ with only minor changes, so that again, $\tilde{K}(\tau, \Omega)$ witnesses the truth of 3.1.

We now turn to the details. Section 4 is devoted to constructing $\tilde{K}(\tau, \Omega)$. Section 4.5 shows that $\tau \subseteq \tilde{K}(\tau, \Omega)$. Section 5 contains the routine adaptation of [14] needed in case 3, and there by completes the proof of Theorem 3.1. Finally, in section 6 we prove Theorem 1.1.

## §4. Psuedo-$K$.

We assume for the rest of this paper that there is no proper class model with a Woodin cardinal.

We fix throughout this section a regular cardinal $\tau \geq \omega_3$, and a regular cardinal $\Omega$ such that $2^{<\tau} < \Omega$, $\tau^{++} < \Omega$, and $\forall \alpha < \Omega(\alpha^\omega < \Omega)$. We shall construct a psuedo-$K$ of ordinal height $\tau$. Psuedo-$K$ will depend on $\tau$ and $\Omega$, but there will be no other arbitrary choices involved in its definition. We shall call it $\tilde{K}(\tau, \Omega)$.

[9]In this case, we have already produced true $K$ up to $\tau$, so we don't really need to prove 3.1, and produce $K \mid \tau$ again by the procedure we outlined after the statement of 3.1.

**4.1. Stably universal weasels.**

DEFINITION 4.1. *A* weasel *is an iterable premouse of height* $\Omega$.

DEFINITION 4.2. *Let* $W$ *be a weasel; then*

(a) $W$ *is a* mini-universe *iff* $W \models$ *"there are unboundedly many cardinals".*
(b) $W$ *is a* collapsing weasel *iff* $W \models$ *"there is a largest cardinal". In this case, we let* $\gamma^W$ *be the largest cardinal of* $W$*, and* $\eta^W$ *be the* $W$*-cofinality of* $\gamma^W$*.*
(c) $W$ *is* stable *iff* $W$ *is a mini-universe, or* $W$ *is collapsing and* $\eta^W$ *is not the critical point of a total-on-*$W$ *extender from the* $W$*-sequence.*
(d) $W$ *is* stably universal *iff* $W$ *is stable, and whenever* $R$ *is a mouse such that* $o(R) < \Omega$*, or* $R$ *is a stable weasel, then* $R$ *does not iterate past* $W$*.*

Farmer Schlutzenberg [21] has shown that for iterable 1-small mice $M$ satisfy ing enough of ZFC, $M \models \eta$ is measurable iff $\eta$ is the critical point of a total-on-$M$ extender from the $M$-sequence. So clause (c) above could be re-phrased as: $\eta^W$ is not measurable in $W$. We shall not use this fact, however.

DEFINITION 4.3. *A mouse* $M$ *is* stable *iff* $o(M) < \Omega$*, or* $M$ *is a stable weasel.*

With this definition, we can say $W$ is stably universal iff $W$ is a stable weasel, and no stable mouse iterates past $W$. Moreover, if $\mathscr{T}$ is an iteration tree of length $< \Omega$ on a stable mouse, then all models of $\mathscr{T}$ are stable.

PROPOSITION 4.4. (1) *If* $W$ *is an unstable collapsing weasel, then* $\mathrm{Ult}(W, U) \mid \Omega$ *is a stable collapsing weasel, where* $U$ *is the order zero measure of* $W$ *on* $\eta^W$*.*
(2) *Any stable collapsing weasel is stably universal.*
(3) *If there is a collapsing weasel, then there is no universal mini-universe.*
(4) *If* $W$ *and* $R$ *are collapsing weasels, then* $\gamma^W$ *and* $\gamma^R$ *have the same* $V$*-cofinality.*

PROOF. This is all straightforward. ⊣

We shall adopt the terminology of CMIP concerning phalanxes, and iteration trees on phalanxes. See [22, 9.6, 9.7, 6.6]. Here is definition 9.6 of [22], slightly revised.[10]

DEFINITION 4.5. *A phalanx is a pair of sequences*

$$\Phi = (\langle (M_\beta, k_\beta) \mid \beta \leq \gamma \rangle, \langle (\nu_\beta, \lambda_\beta) \mid \beta < \gamma \rangle),$$

*such that for all* $\beta \leq \gamma$

(1) $M_\beta$ *is a protomouse (possibly a premouse),*
(2) *if* $\beta < \alpha < \gamma$*, then* $\nu_\beta < \nu_\alpha$ *and* $\lambda_\beta \leq \nu_\alpha$*,*
(3) *if* $\beta < \alpha \leq \gamma$*, then* $\lambda_\beta$ *is the least* $\eta \geq \nu_\beta$ *such that* $M_\alpha \models \eta$ *is a cardinal, and moreover,* $\rho_{k_\alpha}(M_\alpha) > \lambda_\beta$*,*
(4) $\lambda_\beta \leq o(M_\beta)$*, and*
(5) *if* $\beta < \alpha \leq \gamma$*, then* $M_\beta$ *agrees with* $M_\alpha$ *(strictly) below* $\lambda_\beta$*.*

We say $\Phi$ has length $\gamma + 1$, and call $M_\gamma$ the *last model* of $\Phi$. Roughly speaking, the $\lambda_\beta$ measure the agreement of $M_\beta$ with later models, while the $\nu_\beta$ tell you which model to go back to when forming normal trees on $\Phi$.[11] We demand that $\lambda_\beta$ be a cardinal in $M_\alpha$, whenever $\beta < \alpha$. The $k_\beta$ bound the degrees of ultrapowers taken of models lying above $M_\beta$ in a tree on $\Phi$, in the case one has not dropped reaching that model.

[10] Clause 2 is now a bit stronger.

[11] $\lambda_\beta$ is determined by $\nu_\beta$ and $M_{\beta+1}$, as the least cardinal of $M_{\beta+1}$ which is $\geq \nu_\beta$.

If $\mathcal{T}$ is a normal iteration tree of length $\gamma+1$, then $\Phi(\mathcal{T})$ is the phalanx of length $\gamma+1$ with $M_\beta = M_\beta^{\mathcal{T}}$, $k_\beta = \deg^{\mathcal{T}}(\beta)$, $\nu_\beta = \nu(E_\beta^{\mathcal{T}})$, and $\lambda_\beta = \mathrm{lh}(E_\beta^{\mathcal{T}})$ if $E_\beta^{\mathcal{T}}$ is of type II, while $\lambda_\beta = \nu(E_\beta^{\mathcal{T}}$ otherwise.

If $\Phi$ is a phalanx, and $\langle M, k, \nu, \lambda\rangle$ is a 4-tuple such that lengthening each sequence in $\Phi$ by the corresponding entry of $\langle M, k, \nu, \lambda\rangle$ yields a phalanx, then we write

$$\Phi^\frown\langle M, k, \nu, \lambda\rangle$$

for this new phalanx.

The phalanxes with which we deal are mostly of the form $\Phi(\mathcal{T})^\frown\langle M, k, \nu, \lambda\rangle$ for $\mathcal{T}$ some normal iteration tree on a mouse, or generated from such a phalanx by lifting it up via a family of extender ultrapowers.

Normal (i.e., $\omega$-maximal) iteration trees on phalanxes are defined in [22, 6.6]. One thing to notice is that we require $\mathrm{lh}(E_\xi^{\mathcal{T}}) \geq \lambda_\beta^\Phi$ whenever these are defined. Thus $M_\beta^\Phi$ agrees with all non-root models of $\mathcal{T}$ up to $\lambda_\beta$.

Remark 4.6. Suppose $\Phi^\frown\langle M, k, \nu, \lambda\rangle$ is a phalanx, and $\nu$ is the sup of all the $\nu_\beta^\Phi$. Then no normal iteration tree on $\Phi^\frown\langle M, k, \nu, \lambda\rangle$ ever visits the last model of $\Phi$, so for iteration purposes, one can think of $M$ as having replaced the last model of $\Phi$.

We need a notion of stability for phalanxes as well.

Definition 4.7. *A phalanx* $(\langle (P_\xi, k_\xi) \mid \xi \leq \alpha\rangle, \langle(\nu_\xi, \lambda_\xi) \mid \xi < \alpha)$ *is* stable *iff*

(1) *each $P_\xi$ is stable, and*

(2) *if $\xi < \alpha$ and $P_\xi$ is a collapsing weasel such that for $\eta = \eta^{P_\xi}$, we have $(\eta^+)^{P_\xi} \leq \lambda_\xi$, then for all $\gamma \geq \xi$, $\eta^{P_\xi}$ is not a measurable cardinal of $P_\gamma$.*

Lemma 4.8. *Let $\Phi$ be a stable phalanx, and $\mathcal{T}$ an iteration tree on $\Phi$ such that $\mathrm{lh}(\mathcal{T})$ is a successor ordinal $< \Omega$; then $\Phi(\mathcal{T})$ is stable. In particular, all models of $\mathcal{T}$ have ordinal height $\leq \Omega$.*

Proof. Let $\Phi(\mathcal{T}) = (\langle (P_\xi, k_\xi) \mid \xi \leq \gamma\rangle, \langle(\nu_\xi, \lambda_\xi) \mid \xi < \gamma)$. Clause (2) of stability is an easy consequence of the agreement of models in an iteration tree. For let $\xi < \gamma$ and $P_\xi$ be a collapsing weasel such that for $\eta = \eta^{P_\xi}$, we have $(\eta^+)^{P_\xi} \leq \lambda_\xi$. Suppose that $\eta$ is measurable in $P_\gamma$, say via the normal measure $U$. Let $\alpha + 1 = \mathrm{lh}(\Phi)$. If $\alpha \leq \xi$, the agreement of models in an iteration tree gives $U \in P_\xi$, contrary to the stability of $P_\xi$. If $\xi < \gamma \leq \alpha$, we have a contradiction to our assumption about $\Phi$. Finally, if $\xi < \alpha < \gamma$, then $U \in P_\alpha$ by the agreement properties of $\mathcal{T}$, noting that its first extender has length at least $\lambda_\xi$. But this then contradicts our assumption on $\Phi$.

Clause 1 of stability now reduces to: $o(P_\xi) \leq \Omega$ for all $\xi \leq \gamma$. We prove this by induction on $\gamma$. The base case of the induction is $\Phi(\mathcal{T}) = \Phi$, and is given by hypothesis.

Assume first that $\gamma$ is a limit ordinal. We must see that $o(P_\gamma) \leq \Omega$. But suppose not, and let $\Omega = i_{\eta,\gamma}^{\mathcal{T}}(\mu)$, where $\eta < \gamma$. By induction, $i_{\eta,\nu}(\mu) < \Omega$ for all $\nu <_T \gamma$. But letting $X_\nu = i_{\nu,\gamma}``(i_{\eta,\nu}(\mu))$, we have $\Omega = \bigcup_{\eta <_T \nu <_T \gamma} X_\nu$. Thus $\Omega$ is a union of $\gamma$ sets of size $< \Omega$, contrary to $\Omega$ being regular.

Now let $\gamma = \alpha + 1$. Let $\xi = \mathrm{pred}_T(\gamma)$, and $P_\gamma = \mathrm{Ult}_k(Q, E)$, where $Q \trianglelefteq P_\xi$ and $E = E_\alpha^{\mathcal{T}}$. If $o(P_\gamma) > \Omega$, then $P_\xi$ is a collapsing weasel and $\eta^{P_\xi} = \mathrm{crit}(E)$. Since $P_\alpha$ is stable, we must have $\xi < \alpha$ and $\mathrm{crit}(E) < \nu(E_\xi^{\mathcal{T}}) \leq \lambda_\xi$. Moreover, $\alpha + 1 \notin D^{\mathcal{T}}$, so $(\eta^{P_\xi})^{+,P_\xi} \leq \lambda_\xi$. So $\eta^{P_\xi}$ is measurable in $P_\alpha$, contrary to the fact that $\Phi(\mathcal{T} \restriction \gamma)$ satisfies clause 2. ⊣

**4.2. Thick sets and $K^c$.** The efficient $K^c$ constructions give stably universal weasels, with universality insured by thick sets. To see this in the case that our $K^c$ is a mini-universe, we need some results on stacking mice from [10]. We now briefly recall them.

LEMMA 4.9. *Let $W$ be a countably iterable mini-universe, and let $W \trianglelefteq M$, where $M$ is a countably iterable $k$-sound mouse, with $k <$ omega such that $\rho_k(M) = \Omega$; then*

(a) *$\rho_\omega(M) = \Omega$, and*
(b) *if also $W \trianglelefteq N$, where $N$ is countably iterable, $i$-sound, and $\rho_i(N) = \Omega$, then either $M \trianglelefteq N$ or $N \trianglelefteq M$.*

PROOF. For (a), suppose $A$ is a bounded, $M$-definable subset of $\Omega$ such that $A \notin W$. Let $\pi\colon H \to V_\theta$ with $\theta$ large, and $\operatorname{crit}(\pi) = \alpha < \Omega$, and $\pi(\alpha) = \Omega$, and $\pi(\bar{M}) = M$. By condensation (see [16, §8]), we have $\bar{M} \trianglelefteq W$. But $A$ is definable over $\bar{M}$ by the elementarity of $\pi$, so $A \in W$, a contradiction.

The proof of (b) is similar: we reflect the incomparability of $M$ and $N$ to the incomparability of some $\bar{M}$ and $\bar{N}$, where $\bar{M}$ and $\bar{N}$ are both initial segments of $W$. This is a contradiction. ⊣

So if $W$ is a mini-universe, we can stack all mice extending $W$ and projecting exactly to $\Omega$ into a single mouse $S(W)$ extending $W$.

DEFINITION 4.10. *Let $W$ be a mini-universe; then $S(W)$ is the stack of all sound mice $M$ extending $W$ such that for some $k$, $\rho_k(M) = \Omega$. If $W$ is a collapsing weasel, then we set $S(W) = W$.*

The following observation is useful:

PROPOSITION 4.11. *Let $W$ be a mini-universe, and $M$ a premouse such that $W \trianglelefteq M$, and $\rho_k(M) = \Omega$ where $k < \omega$. The following are equivalent:*

(1) *$M \trianglelefteq S(W)$,*
(2) *for club many $\alpha < \Omega$, $H^M(\alpha \cup p_k(M)) \trianglelefteq W$,*
(3) *for stationary many $\alpha < \Omega$, $H^M(\alpha \cup p_k(M)) \trianglelefteq W$.*

PROOF. (1) implies (2) by condensation. To see (3) implies (1), we must show $M$ is countably iterable. But this follows from (3) and the fact that $W$ is countably iterable. ⊣

We call $S(W)$ the *completion of $W$*. If $W$ is a mini-universe, we also call $S(W)$ the *stack over $W$*. Notice that in either case, $S(W)$ has a largest cardinal.

LEMMA 4.12. *Let $W$ be stably universal, and $M$ be a countably iterable premouse such that $S(W)$ is a cutpoint initial segment of $M$; then $\rho_\omega(M) \geq o(S(W))$.*

PROOF. This is easy if $W$ is a stable collapsing weasel, so assume $W$ is a mini-universe. Let $M$ be a minimal counterexample. If $\rho_\omega(M) = \Omega$, then $M$ is one of the mice stacked in $S(W)$, contradiction. So let $\rho = \rho_\omega(M) < \Omega$. Let $\bar{M}$ be the transitive collapse of $X$, where $X \prec M$ with $X \cap \Omega = \bar{\Omega}$ with $\rho < \bar{\Omega} < \Omega$. Thus $\bar{M}$ agrees with $W$ up to $\bar{\Omega}$. Using condensation applied to the proper initial segments of $S(W)$ which are in $X$, we get $\bar{M}$ agrees with $W$ up to the collapse of $o(S(W))$. But $\bar{M}$ has the collapse of $o(S(W))$ as a cutpoint, so using the universality of $W$, we get that $\bar{M}$ is an initial segment of $W$. This implies the new subset of $\rho$ defined over $M$ is actually in $M$, a contradiction. ⊣

REMARK 4.13. So far as we can see, there could be a mouse $M$ such that $M \mid \Omega$ is a universal mini-universe, but $\rho_\omega(M) < \Omega$. One could not have $\Omega = \rho_k(M)$ for some $k$, however, by 4.9.

COROLLARY 4.14. *Let $W$ be stably universal; then $L[S(W)] \models o(S(W))$ is a cardinal.*

DEFINITION 4.15. *Let $W$ be a weasel, and let $C \subseteq o(S(W))$; then we say $C$ is* strongly $W$-thick *iff*

(a) $\operatorname{cof}(o(S(W))) \geq \Omega$, *and $C$ is $\tau$-club in $o(S(W))$, and*
(b) *for all $\eta \in C$, $\operatorname{cof}(\eta)^{S(W)}$ is not the critical point of a total-over-$W$ extender from the $W$-sequence.*

*We say a set $\Gamma \subseteq S(W)$ is $W$*-thick *iff $\Gamma$ has a strongly $W$-thick subset.*

It might be more natural to say that $C$ is strongly $(\tau, W)$-thick, but we have fixed $\tau$ for this section.

PROPOSITION 4.16. *Let $W$ be a weasel.*

(a) *The intersection of $< \Omega$ strongly $W$-thick sets is strongly $W$-thick.*
(b) *If $S(W)$ is $W$-thick, then $W$ is universal, and the collection of all $W$-thick sets constitutes an $\Omega$-complete filter.*

Let us say a $K^c$ construction *forbids critical points of cofinality in $X$* iff whenever $F$ is the last extender of some level $\mathcal{N}_\gamma$ of the construction, then $\operatorname{crit}(F)$ does not have $V$-cofinality in $X$. We say a construction is *$X$-maximal* iff it puts on extenders whenever possible, subject to this restriction, and to whatever background condition the construction employs.

We shall use *robustness* as our background condition on the extenders added in a $K^c$-construction. See [8], or [10, 2.5] for the definition. Robustness follows from being *hull-certified* in the following sense.

DEFINITION 4.17. *Let $M$ be an active premouse with last extender $F$, where $\kappa = \operatorname{crit}(F)$ and $\nu = \nu(F)$. We say $M$ (or sometimes, $F$) is* hull-certified by $\pi$ *iff*

(1) *$\pi \colon H \to H_\xi$ is elementary, $H$ is transitive, $H$ and $H_\xi$ are closed under $\omega$-sequences, and $M \mid (\kappa^+)^M \in H$, and*
(2) *$F \restriction \nu = (E_\pi \restriction \nu) \cap M$; that is, $F$ is the (trivial completion of) the $(\kappa, \nu)$-extender over $M$ induced by $\pi$.*

This is close to the notion of being certified by a collapse in [10, 2.2], but unfortunately that definition required $\xi$ be regular and $2^{<\xi} = \xi$, which is too much GCH. One still has, by a straightforward proof:

LEMMA 4.18. *Let $M$ be hull-certified; then $M$ is robust.*

PROOF. See [10, Lemma 2.6]. ⊣

The following is a preliminary weak covering theorem for the robust $K^c$. It is essentially Theorem 3.4 of [10], although unfortunately that theorem had the superfluous hypothesis that $2^{<\Omega} = \Omega$.

THEOREM 4.19. *Let $R$ be the output of the $\{\tau\}$-maximal $K^c$-construction of length $\Omega$ all of whose levels are robust. Suppose $R$ is a mini-universe; then $S(R)$ is $R$-thick.*

PROOF SKETCH. Let $C = \{\alpha < o(S(R)) \mid \operatorname{cof}(\alpha) = \tau\}$. We claim that $C$ is strongly $R$-thick. Clause (b) in the definition of strong $R$-thickness follows easily from the fact that critical points of cofinality $\tau$ were forbidden in the construction of $R$. For clause (a), we need to see that

$$\operatorname{cof}(o(S(R))) \geq \Omega.$$

This is proved exactly as in the proof of 3.4 of [10], using "hull-certified" in place of "certified by a collapse" everywhere.[12] ⊣

Preliminary weak covering in the collapsing weasel case is easier:

THEOREM 4.20. *Suppose there is a collapsing weasel $W$, and let $\eta$ be the $V$-cofinality of $\eta^W$. Let $R$ be the output of the $\{\tau, \eta\}$-maximal $K^c$-constructions of length $\Omega$, all of whose levels are robust. Then $R$ is a stable collapsing weasel, and $\Omega$ is $R$-thick.*

PROOF. If $W \models \eta^W$ is measurable, let $W^* = \mathrm{Ult}(W, \mu) \mid \Omega$. where $\mu$ is the order zero measure on $\eta^W$. Otherwise, let $W^* = W$. By part (1) of proposition 4.4, $W^*$ is a stable collapsing weasel, whose largest cardinal has $V$-cofinality $\eta$.

By [8], if $R$ is a mini-universe, it must be universal. (This also follows from 4.19 and 4.24 below.) But that contradicts proposition 4.4, part (3). Thus $R$ is a collapsing weasel. We claim that $R$ is stable. If not, letting $\gamma = \operatorname{cof}(\eta^R)^V$, we have $\gamma = \eta$ by part (4) of proposition 4.4. But critical points of $V$-cofinality $\eta$ were not allowed in the construction of $R$, contradiction. Thus $R$ is stable. Letting $C = \{\alpha < o(S(R)) \mid \operatorname{cof}(\alpha) = \tau\}$, it is clear that $C$ is strongly $R$-thick. ⊣

Combining 4.19 and 4.20 we have

COROLLARY 4.21. *There is a stably universal weasel $W$ such that $S(W)$ is $W$-thick.*

**4.3. Preservation of thickness under hulls and iterations.** For iterations, we have:

LEMMA 4.22. *Let $\Phi$ be a stable phalanx, let $W$ be a weasel on $\Phi$, and suppose $i\colon W \to R$ is an iteration map coming from a normal iteration tree $\mathcal{U}$ of length $\leq \Omega + 1$ on $\Phi$, and that $i``\Omega \subseteq \Omega$. Let $E$ be the long extender of of length $\Omega$ over $W$ derived from $i$; then* $\mathrm{Ult}(S(W), E) = S(R)$.

PROOF. This is trivial if $W$ is a collapsing weasel, so assume $W$ is a mini-universe, and that 4.22 fails for $W$. Let $\pi\colon H \to V_\theta$ be elementary, where $H$ is transitive, $\operatorname{crit}(\pi) = \alpha < \Omega$, $\pi(\alpha) = \Omega$, and everything relevant is in $\operatorname{ran}(\pi)$. Let

$$\pi(\bar{\mathcal{U}}) = \mathcal{U}, \pi(\bar{N}) = N, \pi(\overline{S(W)}) = S(W),$$

where $\mathcal{U}$ is the tree giving rise to $i$, and $N$ is the first collapsing level of $S(R)$ above $\mathrm{Ult}(S(W), E)$.

Now $\bar{N}$ is a level of $R$ projecting to $\alpha$ by condensation. Thus $\bar{N}$ is an initial segment of $\mathcal{M}_\alpha^{\mathcal{U}}$. Also $\mathrm{Ult}(\overline{S(W)}, \bar{E})$ is a proper initial segment of $\bar{N}$. It follows that there is a first level $W \mid \gamma$ of $W$ such that $W \mid \gamma$ projects to $\alpha$, and $\overline{S(W)}$ is an initial segment of $W \mid \gamma$.

But then

$$\bar{N} = \mathrm{Ult}(W \mid \gamma, \bar{E}),$$

so we get $W \mid \gamma$ in $H$ as $\bar{E}$ and $\bar{N}$ are there. (Note that $W \mid \gamma$ is the transitive collapse of $H_n^{\bar{N}}(i``\alpha \cup p_n(\bar{N})$, and $\bar{E}$ determines $i \restriction \alpha$.) But $\alpha$ is a cardinal of $W$,

[12] It is at this point that we use $\tau^{++} < \Omega$, which gives us two regular cardinals that *are* allowed as cofinalities of critical points in the construction of $R$.

so $\rho_\omega(W \mid \gamma) = \alpha$. Thus $W \mid \gamma$ witnesses that $\overline{S(W)}$ is not the maximal stack over $W \mid \alpha$ in $H$. This contradicts the elementarity of $\pi$. $\dashv$

THEOREM 4.23. *Let $\Phi$ be a stable phalanx, let $W$ be a model of $\Phi$ such that $S(W)$ is $W$-thick, and suppose $i \colon W \to R$ is an iteration map coming from a normal iteration tree $\mathscr{U}$ of length $\leq \Omega + 1$ on $\Phi$, and that $i``\Omega \subseteq \Omega$. Let $E$ be the long extender of length $\Omega$ over $W$ derived from $i$, and let $i^* \colon S(W) \to \mathrm{Ult}(S(W), E)$ be the canonical extension of $i$; then*

(1) $\mathrm{Ult}(S(W), E) = S(R)$,
(2) $\{\alpha \mid i^*$ *is continuous at* $\alpha\}$ *is $W$-thick, and*
(3) $\mathrm{ran}(i^*)$ *is $R$-thick.*

We show now that the universality of a mini-universe is determined by the cofinality of the stack over it.

THEOREM 4.24. *Let $W$ be a mini-universe; then $W$ is universal iff*

$$\mathrm{cof}(o(S(W))) \geq \Omega.$$

PROOF. Suppose first that $W$ is a universal mini-universe. Let $R$ be the robust $K^c$ of Theorem 4.19. Then $R$ is also universal, and by 4.23 and 4.9, the comparison of $W$ with $R$ is in fact a comparison of $S(W)$ with $S(R)$, and yields iteration maps

$$i \colon S(W) \to S(Q) \text{ and } j \colon S(R) \to S(Q).$$

It follows from the continuity of $i$ and $j$ at $o(S(W)$ and $o(S(R))$ that

$$\Omega \leq \mathrm{cof}(o(S(R))) = \mathrm{cof}(o(S(Q))) = \mathrm{cof}(o(S(W))),$$

as desired.

Conversely, suppose $W$ is not universal, and let $M$ be a mouse of height $\leq \Omega$ that iterates past $W$. Let $\mathscr{T}$ and $\mathscr{U}$ be the comparison trees on the $W$ and $M$ respectively. Let $R$ be the last model of $\mathscr{T}$, and $N = \mathscr{M}^{\mathscr{U}}_\Omega$ be the last model of $\mathscr{U}$, so that $W$-to-$R$ does not drop, and $R \trianglelefteq N$. Let

$$j \colon S(W) \to S(R)$$

be the iteration map, extended to $S(W)$ via 4.23.

CLAIM 1. $S(R) \trianglelefteq N$.

PROOF. If not, we have $P$ such that $P \trianglelefteq S(R)$, $\rho_\omega(P) = \Omega$, and $P \ntrianglelefteq N$. Let

$$\pi \colon H \to V_\theta$$

be elementary, with everything relevant in $\mathrm{ran}(\pi)$, and

$$\pi(\alpha) = \Omega, \text{ for } \alpha = \mathrm{crit}(\pi).$$

For notational simplicity, let us assume $\mathscr{T}$ has been padded so as to keep pace with $\mathscr{U}$, which has length $\Omega$ because $M$ is iterating past $W$. We then have

$$\alpha = \mathrm{crit}(i^{\mathscr{U}}_{\alpha,\Omega}) \leq \mathrm{crit}(i^{\mathscr{U}}_{\alpha,\Omega}),$$

and

$$\mathscr{M}^{\mathscr{U}}_\alpha \mid (\alpha^+)^{\mathscr{M}^{\mathscr{U}}_\alpha} = \mathscr{M}^{\mathscr{T}}_\alpha \mid (\alpha^+)^{\mathscr{M}^{\mathscr{T}}_\alpha} = R \mid (\alpha^+)^R,$$

by standard arguments. But let $\pi(\bar{P}) = P$; then by condensation, $\bar{P} \trianglelefteq R$, and hence $\bar{P} \trianglelefteq \mathscr{M}^{\mathscr{U}}_\alpha$. But $\pi(\mathscr{M}^{\mathscr{U}}_\alpha) = \mathscr{M}^{\mathscr{U}}_\Omega = N$, so $P \trianglelefteq N$, contradiction. $\dashv_{\text{CLAIM 1}}$

Claim 2. $S(R) = N \mid (\Omega^+)^N$.

Proof. Otherwise, noting that $\Omega$ is a cardinal of $N$, we get that $S(R)$ is not the full stack over $R$, a contradiction. $\dashv_{\text{Claim 2}}$

Now let $\alpha < \Omega$ be large enough that $i = i^{\mathcal{U}}_{\alpha,\Omega}\colon \mathcal{M}^{\mathcal{U}}_\alpha \to N$ is defined. Let $i(\kappa) = \Omega$. Then $i$ maps $(\kappa^+)^{\mathcal{M}^{\mathcal{U}}_\alpha}$ cofinally into $(\Omega^+)^N = o(S(R))$. Thus $o(S(R))$ has cofinality $< \Omega$. But $j$ maps $o(S(W))$ cofinally into $o(S(R))$, contradiction. This completes the proof of 4.24. $\dashv$

For hulls we have the following. Let $\Gamma \subseteq S(W)$; then we put

$$H^{S(W)}(\Gamma) = \{x \mid x \text{ is definable over } S(W) \text{ from parameters in } \Gamma\}.$$

Then

Lemma 4.25. *Let $\Gamma$ be $W$-thick, and let $\pi\colon N \cong H^{S(W)}(\Gamma) \prec S(W)$, where $N$ is transitive; then*

(a) *$H^{S(W)}(\Gamma)$ is cofinal in $\Omega$,*
(b) *$N = S(N \mid \Omega)$,*
(c) *$\{\alpha < o(N) \mid \pi(\alpha) = \sup \pi``\alpha\}$ is $N$-thick*
(d) *$N \mid \Omega$ is universal.*

Proof. (a) is clear if $W$ is a collapsing weasel. Suppose $W$ is a mini-universe, but $H^{S(W)}(\Gamma)$ is bounded in $\Omega$. It is clear then that $N$ is a collapsing weasel. This contradicts part (3) of proposition 4.4.

For (b), it is clear that $N \trianglelefteq S(N \mid \Omega)$. Suppose that $P$ is least such that $P \trianglelefteq S(N \mid \Omega)$ and $P \ntrianglelefteq N$ and $\rho_\omega(P) = \Omega$). We can form

$$Q = \mathrm{Ult}_k(P, E_\pi \mid \Omega),$$

and we have that $\rho_k(Q) = \Omega$, and $Q$ properly extends $S(W)$ because $\Gamma$ is cofinal in $o(S(W))$. But for club many $\alpha < \Omega$, $\mathrm{Hull}^Q_k(\alpha \cup p_k(Q)) \trianglelefteq W$, so $Q \trianglelefteq S(W)$ by proposition 4.11, a contradiction.

Part (c) is clear, and (d) follows from (c) and Theorem 4.24. $\dashv$

**4.4. The hull property.** The proof from [22] that $K^c$ has the hull property at club many $\alpha < \Omega$ does not generalize to our current situation. However, there is in fact a much simpler proof.

Definition 4.26. *Let $S(W)$ be $W$-thick, and suppose $\alpha < \Omega$; then we say $W$ has the* hull property *at $\alpha$ iff whenever $\Gamma$ is $W$-thick, then $P(\alpha)^W$ is contained in the transitive collapse of $H^{S(W)}(\Gamma \cup \alpha)$.*

Lemma 4.27. *Suppose $S(W)$ is $W$-thick; then there are club many $\alpha < \Omega$ such that $W$ has the hull property at $\alpha$.*

Proof. Since $L[S(W)] \models \Omega$ is not Woodin, we can pick $A \in S(W)$ least such that no $\kappa < \Omega$ is $A$-reflecting in $\Omega$ in $L[S(W)]$. Thus there are club many $\alpha < \Omega$ such that whenever $\kappa < \alpha$ and $E$ is on the $W$-sequence and $\mathrm{crit}(E) = \kappa$, then

$$i_E(A) \cap \alpha \neq A \cap \alpha.$$

We claim that $W$ has the hull property at any such $\alpha$.

To see this, let $\Gamma$ be $W$-thick, and let $\pi\colon S(H) \cong H^{S(W)}(\Gamma \cup \alpha) \prec S(W)$, where $H$ is transitive. Note $A \in \mathrm{ran}(\pi)$. We now compare $(W, H, \alpha)$ with $W$. By Dodd–Jensen, the comparison ends up above $H$ on the phalanx side, and yields iteration maps

$$i\colon H \to P,$$

and

$$j\colon W \to P,$$

such that $\operatorname{crit}(i) \geq \alpha$. We can extend $i$ and $j$ so they act on $S(H)$ and $S(W)$, and since $A$ is definable over $L[S(W)]$, we have that

$$i(\pi^{-1}(A)) = j(A).$$

But then if $\operatorname{crit}(j) = \kappa < \alpha$, we would have that the first extender used in $j$ witnesses that $\kappa$ is $A$-reflecting up to $\alpha$ in $W$. So $\operatorname{crit}(j) \geq \alpha$. But then

$$P(\alpha) \cap W = P(\alpha) \cap P = P(\alpha) \cap H,$$

which is what we need to show. ⊣

Remark. For the duration of this remark, we drop our assumption that there is no proper class model with a Woodin cardinal. Indeed, suppose instead that $\Omega$ is Woodin in $V$, that $V_\Omega$ is fully iterable. Let $N$ be the output of the *full* background extender $K^c$ construction of $V_\Omega$. Our iterability assumption implies that this construction does not halt before $\Omega$, so that $N$ is a mini-universe, and that $N$ is fully iterable. Lemma 11.1 of [24] shows that $N$ is universal. In fact, the proof of 4.24 goes over to this situation, and one has that $\operatorname{cof}(o(S(N)) \geq \Omega$. We can thus define $\tau$-thick sets, for example with $\tau = \Omega$. *If* we could show that $N$ has the hull property at club many $\alpha < \Omega$, we could go on to define true $K$ up to $\Omega$ as the intersection of all thick hulls of $S(N)$. This could be very useful, for example, in proving the Mouse Set Conjecture. (See [24].)

Unfortunately, our proof of Lemma 4.27 used very heavily that $\Omega$ was not Woodin in $L[S(W)]$! This certainly fails for $L[S(N)]$. It is open whether $N$ has the hull property at club many $\alpha < \Omega$.

**4.5. $\tilde{K}(\tau, \Omega)$ contains $\tau$.** We now define our psuedo-$K$, and show it contains $\tau$.

Definition 4.28. *Suppose $S(W)$ is $W$-thick; then we set*

$$\operatorname{Def}^W = \bigcap \{H^{S(W)}(\Gamma) \mid \Gamma \textit{ is } W\textit{-thick}\}.$$

Lemma 4.29. *Suppose $S(W)$ is $W$-thick and $S(R)$ is $R$-thick; then* $(\operatorname{Def}^W, \in) \cong (\operatorname{Def}^R, \in)$.

Proof. Comparing $W$ with $R$, we get $i\colon W \to Q$ and $j\colon R \to Q$, iteration maps to a common weasel. By 4.23, these give rise to $i^*\colon S(W) \to S(Q)$ and $j^*\colon S(R) \to S(Q)$. It is easy then to use 4.23 to see $(i^*)``\operatorname{Def}^W = \operatorname{Def}^Q = (j^*)``\operatorname{Def}^R$. ⊣

Our psuedo-$K$ is

Definition 4.30. *$\tilde{K}(\tau, \Omega)$ is the common transitive collapse of all* $\operatorname{Def}^W$, *for $W$ such that $S(W)$ is $W$-thick.*

The proof in [22] of the counterpart to the following lemma used the measurable cardinal a second time.

Lemma 4.31. *$\tilde{K}(\tau, \Omega)$ has ordinal height at least $\tau$.*

Proof. The collapsing weasel case is easy: let $W$ be any stable collapsing weasel, and $\gamma$ its largest cardinal. For each $\xi < \gamma$, let $\Gamma_\xi$ be strongly $W$ thick and such that

$$\xi \notin H^W(\Gamma_\xi),$$

with $\Gamma_\xi = \Omega$ if there is no thick hull omitting $\xi$. Let

$$\Gamma = \bigcap_{\xi<\gamma} \Gamma_\xi.$$

So $\Gamma$ is strongly $W$-thick, and

$$H^W(\Gamma) \cap \gamma = \mathrm{Def}^W \cap \gamma.$$

But then

$$H^W(\Gamma) = \mathrm{Def}^W,$$

because if $\Lambda \subseteq \Gamma$ is strongly $W$-thick, and $\xi \in H^W(\Gamma)$, we can find a function $f \in H^W(\Lambda)$ with domain $\gamma$ such that $\xi \in \mathrm{ran}(f)$. But then $\xi = f(\mu)$ for $\mu \in H^W(\Gamma)$, so $\xi = f(\mu)$ for $\mu \in \mathrm{Def}^W$, so $\xi \in H^W(\Lambda)$. Since $H^W(\Gamma) = \mathrm{Def}^W$, we have $\Omega \subseteq \tilde{K}(\tau, \Omega)$, which is more than we claimed.

Now let $W = K^c_\tau$ be the output of the robust $K^c$-construction of length $\Omega$, and suppose $W$ is a mini-universe. Suppose toward contradiction that $\mathrm{Def}^W \cap \Omega$ has order type $\beta < \tau$. As above, we can find a strongly $W$-thick set $\Gamma_0$ such that $H^W(\Gamma_0) \cap \Omega$ has $\mathrm{Def}^W$ for its first $\beta$ elements. Let $b_0$ be least in $H^W(\Gamma_0) \cap \Omega \setminus \mathrm{Def}^W$. Now pick a decreasing sequence $\langle \Gamma_\nu \mid \nu < \Omega \rangle$ such that letting

$$b_\nu = \text{least ordinal in } H^W(\Gamma_\nu) \setminus \mathrm{Def}^W,$$

we have that $\nu < \xi \Rightarrow b_\nu < b_\xi$, for all $\nu, \xi < \Omega$.

The proof of the following claim is due to Mitchell [11].

Claim. There is no $\nu < \Omega$ such that $\forall \xi < \nu (b_\xi < \nu)$ and $\nu \in H^W(\nu \cup \Gamma_{\nu+1})$.

Proof. Fix such a $\nu$. We can then find $c < \nu$ and $d \in (\Gamma_{\nu+1})^{<\omega}$, and a Skolem term $\tau$, such that $\nu = \tau^W[c, d]$. But then we have $\xi < \nu$ such that $c < b_\xi$, so

$$H^W(\Gamma_\xi) \models \exists c < b_\xi (b_\xi < \tau[c, d] < b_{\nu+1}).$$

But the witness $e$ to the existential quantifier here is in $H^W(\Gamma_\xi) \cap b_\xi$, and hence in $\mathrm{Def}^W$. It follows that

$$b_\xi < \tau^W[e, d] < b_{\nu+1},$$

and $\tau^W[e, d] \in H^W(\Gamma_{\nu+1})$, a contradiction. ⊣

Because the lemma fails, we have an $\tau^+$-club $C \subseteq \Omega$ such that for all $\nu \in C$, $\mathrm{cof}(\nu) = \tau^+$, and

(1) $\xi < \nu \Rightarrow b_\xi < \nu$,
(2) $\nu \notin H^W(\nu \cup \Gamma_\nu)$,
(3) $W$ has the hull property at $\nu$.

For $\nu \in C$, let

$$\sigma_\nu : N_\nu \cong H^W(\nu \cup \Gamma_{\nu+1}) \prec W,$$

where $N_\nu$ is transitive, and let $F_\nu$ be the $(\nu, \sigma_\nu(\nu))$ extender of $\sigma_\nu$. Note $F_\nu$ measures all sets in $W$, by the hull property at $\nu$. $F_\nu$ coheres with $W$, and not all of its initial segments are of type $Z$, on the $W$-sequence, or an ultrapower away. (Otherwise $W$ has reached a Shelah cardinal.) So we have some $\beta$ such that $(W \mid \beta, F_\nu \restriction \beta)$ is a non-type-$Z$ premouse, but is not robust. (Note here that $\nu$ is not forbidden as a critical point.) Let $\beta(\nu)$ be the least such $\beta$.[13 14]

So for each $\nu \in C$, we pick a witness $U_\nu$ that $F_\nu \restriction \beta_\nu$ is not robust with respect to $W \mid \beta_\nu$. This means the following: for any $\beta$, let $C_{\beta,\xi}$ be the $\xi^{\text{th}}$ level of the Chang model built over $W \mid \beta$. (See [8] or [10].) Let $\mathscr{L}_0$ be the common language of the $C_{\beta,\xi}$. If $U \subseteq W \mid \beta$ and $\sup(U \cap \Omega) = \beta$, put

$$\begin{aligned}\mathrm{Sat}(U) = \{(\varphi, x) \mid x \in U^\omega \text{ and}\\ \varphi \text{ is a } \Sigma_1 \text{ formula of } \mathscr{L}_0 \text{ and } C_{\beta,\Omega} \models \varphi[x]\}.\end{aligned}$$

If $U \subseteq W \mid \beta$ and $\psi\colon U \to W \mid \gamma$, with $\sup(\mathrm{ran}(\psi) \cap \Omega = \gamma)$, we set

$$\begin{aligned}\mathrm{Sat}(U, \psi) = \{(\varphi, x) \mid x \in U^\omega \text{ and}\\ \varphi \text{ is a } \Sigma_1 \text{ formula of } \mathscr{L}_0 \text{ and } C_{\gamma,\Omega} \models \varphi[\pi \circ x]\}.\end{aligned}$$

Then our counterexample $U_\nu$ to robustness has the following properties:

(1) $U_\nu$ is a countable subset of $W \mid \beta(\nu)$,
(2) there is no map $\psi\colon \mathscr{U}_\nu \to W \mid \nu$ with the properties that, setting $\beta = \sup(U_\nu \cap \beta(\nu))$ and $\bar{\beta} = \sup(\psi``\beta)$, we have
  (i) $\psi \restriction U_\nu \cap \nu$ is the identity,
  (ii) $\mathrm{Sat}(U) = \mathrm{Sat}(U, \psi)$, and
  (iii) for all $a \in [U_\nu \cap \beta(\nu)]^{<\omega}$ and all $X \subseteq [\nu]^{|a|}$ such that $X \in U_\nu$, we have $a \in \sigma_\nu(X) \Leftrightarrow \psi(a) \in X$.

Since $\forall\alpha < \Omega(\alpha^\omega < \Omega)$, we can simultaneously fix $\omega$-many regressive ordinal valued functions on a $\tau^+$-stationary set. In particular, we can fix $U_\nu \cap \nu$ on an $\tau^+$-stationary set. Let $S_0 \subseteq C$ be $\tau^+$-stationary, and $y_n$ for $n < \omega$ such that

$$U_\nu \cap \nu = \{y_n \mid n < \omega\}$$

for all $\nu \in S_0$.

Let us pick enumerations

- $\langle z_n^\nu \mid n < \omega\rangle$ of $U_\nu$,
- $\langle a_n^\nu \mid n < \omega\rangle$ of $[U_\nu \cap \beta(\nu)]^{<\omega}$,
- $\langle X_n^\nu \mid n < \omega\rangle$ of $U_\nu \cap \bigcup_{n<\omega} P([\nu]^n)$.

Let $\gamma_\nu = \sup(U_\nu \cap \beta_\nu)$. Let $\mathscr{L}_1$ be the expansion of $\mathscr{L}_0$ with constant symbols $\dot{z}_n$, $\dot{a}_n$, $\dot{X}_n$, $\dot{y}_n$, for all $n < \omega$, as well as constant symbols $\dot{f}$ for all $f \in \omega^\omega$. Let $\mathscr{U}_\nu^*$ be the obvious expansion of $C_{\beta_\nu,\Omega}$ to a structure for $\mathscr{L}_1$, where we interpret $\dot{f}$ by

[13] In fact, $\beta$ is unique by the initial segment condition. At this moment, in order to be accurate with the details, one must choose between using $\lambda$-indexing as in [9], and using ms-indexing, as in [16] and [25]. No doubt either would do, but we shall be following the weak covering proof of [14], which uses ms-indexing, so we have chosen it. This means that the iterability and universality arguments using robustness of [8] have to be translated to ms-indexing, so as to prove that theorems 4.19 and 4.20 do indeed hold with the ms-indexing. We see no difficulty in doing this, and it may be less work than re-doing the weak covering proof of [14] in $\lambda$-indexing. Schindler [20] proves weak covering in $\lambda$-indexing for a $K^c$ with many strong cardinals, but no one has written up a full analog of [14].

[14] In any case, one could avoid re-doing the robustness work in ms-indexing by forcing $2^{<\Omega} = \Omega$, using [15] (which is done in ms-indexing) wherever we are using [8] in this paper, thereby obtaining $\tilde{K}(\tau, \Omega)$ in the generic extension, and then arguing that $\tilde{K}(\tau, \Omega)$ is in $V$ by homogeneity.

the function $h(n) = z^{\nu}_{f(n)}$. (So $\mathscr{U}^*$ is a structure for a language of size $2^{\omega}$.) Then, let $S_1 \subseteq S_0$ be $\tau^+$ stationary, and such that the first order theory of $\mathscr{U}^*_\nu$ is constant on $S_1$.

Now let $\xi, \nu \in S_1$ be such that $\beta_\xi < \nu$. There is a bijection $\psi$ between $\mathscr{U}_\nu$ and $\mathscr{U}_\xi$ given by

$$\psi(z^{\nu}_n) = z^{\xi}_n.$$

Since $\mathscr{U}^*_\xi$ and $\mathscr{U}^*_\nu$ are elementarily equivalent, we have that $\mathrm{Sat}(U_\nu) = \mathrm{Sat}(U_\nu, \psi)$. Also, $\psi \restriction U_\nu \cap \nu$ is the identity. So we just have to see that for a proper choice of $\xi$ and $\nu$, $\psi$ satisfies the "typical object" condition (iii) above.

For each $\nu \in S_1$, and $n < \omega$, we can write

$$\sigma_\nu(X^{\nu}_n) = \tau^{\nu}_n[\alpha^{\nu}_n, d^{\nu}_n],$$

where $\tau^{\nu}_n$ is a Skolem term, $\alpha^{\nu}_n < \nu$, and $d^{\nu}_n \in \Gamma^{<\omega}_{\nu+1}$. By Fodor again, we can thin $S_1$ to a $\tau^+$ stationary set $S_2$ such that we have $\tau_n$ and $\alpha_n$ for $n < \omega$ with

$$\tau^{\nu}_n = \tau_n \text{ and } \alpha^{\nu}_n = \alpha_n$$

for all $\nu \in S_2$.

For $\nu \in S_2$, let

$$f(\nu) = \{\langle n, k\rangle \mid a^{\nu}_n \in \sigma_\nu(X^{\nu}_n)\}.$$

We thin $S_2$ to a $\tau^+$-stationary $S_3$ such that $f$ is constant on $S_3$.

Finally, for $\nu \in S_3$, put

$$R^{\nu}(n, \theta, \mu) \Leftrightarrow \theta \in \tau_n[\mu, d^{\nu}_n]^W \text{ and } \theta, \mu \in \mathrm{Def}^W.$$

We thin $S_3$ to an $\tau^+$-stationary $S_4$ such that $R^\nu$ is constant for $\nu \in S_4$. This is where we use $2^{<\tau} < \Omega$.

Now let $\xi, \nu \in S_4$ be such that $\sigma_\xi(\xi) < \nu$. Let $\psi(z^{\nu}_n) = z^{\xi}_n$. It will be enough to show that $\psi$ satisfies the typical object condition (iii) above. This amounts to showing that for all $n, k$

$$a^{\nu}_n \in \sigma_\nu(X^{\nu}_k) \Leftrightarrow a^{\xi}_n \in X^{\nu}_k.$$

But because we are in $S_3$, we have $a^{\nu}_n \in \sigma_\nu(X^{\nu}_k) \Leftrightarrow a^{\xi}_n \in \sigma_\xi(X^{\xi}_k)$. Thus it is enough to show

$$\sigma_\xi(X^{\xi}_k) = X^{\nu}_k \cap [\sigma_\xi(\xi)]^{<\omega},$$

for all $k$. Suppose this fails for $k$. Notice now that $\sigma_\xi(\xi) \leq b_{\xi+1}$, since the latter is above $\xi$ and in $H^W(\Gamma_{\xi+1})$. Then we get

$$W \models \exists\theta < b_{\xi+1} \exists\mu < b_{\xi+1}(\theta \in \tau_k[\mu, d^{\xi}_k] \Leftrightarrow \theta \notin \tau_k[\mu, d^{\nu}_k]).$$

The displayed formula is a fact about elements of $H^W(\Gamma_{\xi+1})$, so there are witnesses $\theta, \mu$ to it in $H^W(\Gamma_{\xi+1})$. Since $\theta, \mu < b_{\xi+1}$, we must have $\theta, \mu \in \mathrm{Def}^W$. But this implies $R^\xi \neq R^\nu$, a contradiction which completes the proof of lemma 4.31. $\dashv$

**§5. The weak covering proof.** In this section, we prove Theorem 3.1. From this point on, the proof is so much like that in [14] that there is no point in writing it all down again. We shall describe here the relatively minor changes needed, assuming that the reader has [14] in hand. We begin at the beginning of §3 of [14].

Let $\kappa$ be singular strong limit cardinal. Let $\Omega$ be a regular cardinal large enough that $\tilde{K}(\kappa^+, \Omega)$ has height at least $\kappa^+$. We now adopt the terminology regarding stable weasels, thick sets, and so forth given above, associated to $\tau = \kappa^+$ and our choice of $\Omega$. We shall show that $\tilde{K}(\kappa^+, \Omega)$ computes $\kappa^+$ correctly.

Fix a very soundness witness $W_0$ for $\tilde{K}(\kappa^+, \Omega) \mid \kappa^+$; that is, let $W_0$ be such that $S(W_0)$ is $W_0$-thick, and $\kappa^+ \subseteq \mathrm{Def}^{W_0}$. Suppose toward contradiction that $(\kappa^+)^{W_0} < \kappa^+$. Set $\lambda = (\kappa^+)^{W_0}$.

REMARK 5.1. The proof in [14, §3] delays the moment where one enters a proof by contradiction, and so could we, but we won't.

If $W_0$ is a mini-universe, or $W_0$ is a collapsing weasel with $\mathrm{cof}(\gamma^{W_0})^V > \kappa$, set $W = W_0$.

If $W_0$ is a collapsing weasel, and $\mathrm{cof}(\eta^{W_0})^V < \kappa$, we say $W_0$ is *phalanx-unstable*. In this case, $W_0$ is not suitable for the role of the weasel $W$ in [14]. The reason is that although $\eta^{W_0}$ is not measurable in $W_0$, nevertheless the phalanxes which show up in the weak covering proof may not be stable. Let us set

$$\nu = \mathrm{cof}(\eta^{W_0})^V.$$

What we need for $W$ is a weasel none of whose measurable cardinals below $\kappa^+$ have $V$-cofinality $\nu$. We obtain $W$ by linearly iterating $W_0$ via normal measures: letting $W_\alpha$ be the $\alpha^{\text{th}}$ model of this iteration, we set

$$W_{\alpha+1} = \mathrm{Ult}(W_\alpha, U),$$

where $U$ is the order zero measure of $W_\alpha$ on the least measurable cardinal $\mu$ of $W_\alpha$ such that $\mu < \kappa^+$ and $\mathrm{cof}(\mu)^V = \nu$. If there is no such $\mu$, the iteration is over. The critical points in the iteration are increasing, so it is normal, and ends in $\leq \kappa^+$ steps. Let $W$ be its final model. Note that in this phalanx-unstable case

(1) $W$ is a stable collapsing weasel, and $\mathrm{cof}(\gamma^W)^V = \nu$,
(2) $W$ has the hull property at all $\mu < \kappa^+$,
(3) for $\mu < \kappa^+$, $W$ has the definability property at $\mu$[15] iff $\mu$ was not a critical point in the iteration; in particular, $W$ has the definability property at all $\mu$ such that $\mathrm{cof}(\mu)^V \neq \nu$, and
(4) if $\mu < \kappa^+$ and $\mathrm{cof}(\mu)^V = \nu$, then $W \models \mu$ is not measurable.

This completes our definition of the weasel $W$. It is easy to see that in either case, $(\kappa^+)^W = (\kappa^+)^{W_0}$. So we have $(\kappa^+)^W < \kappa^+$.

Now let

$$\pi \colon N \to V_{\Omega+\omega}$$

be such that $N$ is transitive, $|N| < \kappa$, $\mathrm{ran}(\pi)$ is cofinal in $\lambda$, everything of interest is in $\mathrm{ran}(\pi)$, and $N$ is closed under $\omega$-sequences. We further demand that if $W$ is collapsing, and $\nu = \mathrm{cof}(\eta^W)^V < \kappa$, then $N$ is closed under $\nu$ sequences.

[15] This means that $\mu \in H^{S(W)}(\mu \cup \Gamma)$, for all $W$-thick sets $\Gamma$.

We have now reached the top of page 233 in our transcription of [14]. As there, we now compare $W$ with $\bar{W}$, and the main thing we have to show is that $\bar{W}$ does not move in this comparison. Here, we find it convenient to depart a bit from the way [14] is organized: we shall organize our argument as an induction on the cardinals of $\bar{W}$, rather than as an induction on the cardinals of the last model above $\bar{W}$ in its comparison with $W$. Of course, if $\bar{W}$ does not move, these two models are the same.

So let

$$\kappa_\alpha = \alpha^{\text{th}} \text{ infinite cardinal of } \bar{W}$$

enumerate the cardinals of $\bar{W}$. Let

$$\pi(\kappa_\theta) = \kappa^+.$$

Thus $\kappa_{\bar{\kappa}} = \bar{\kappa}$, $\kappa_{\bar{\kappa}+1} = \bar{\lambda}$, and $\bar{\kappa} + 2 \leq \theta$. We shall prove by induction on $\alpha$:

$\mathbf{I}_\alpha$ There is a normal iteration tree on $W$ whose last model agrees with $\bar{W}$ below $\kappa_\alpha + 1$.

Main Lemma 5.2. *$\mathbf{I}_\alpha$ holds, for all $\alpha \leq \theta + 1$.*

Remark 5.3. $\mathbf{I}_{\bar{\kappa}+1}$ is all we really need.

Proof of 5.2.

Claim 5.4. *$\mathbf{I}_0$ holds, and if $\alpha \leq \theta$ is a limit ordinal such that $\mathbf{I}_\beta$ holds for all $\beta < \alpha$, then $\mathbf{I}_\alpha$ holds.*

Proof. Let $\mathcal{T}_\beta$ be a normal tree of minimal length witnessing $\mathbf{I}_\beta$. Then $\beta < \gamma$ implies $\mathcal{T}_\gamma$ extends $\mathcal{T}_\beta$. Let $\mathcal{T}$ be the union of the $\mathcal{T}_\beta$ for $\beta < \gamma$, extended by adding the direct limit along its unique cofinal wellfounded branch if this union has limit length. Let $P$ be the last model of $\mathcal{T}$, so that $P \mid \kappa_\alpha = \bar{W} \mid \kappa_\alpha$. The tree witnessing $\mathbf{I}_\alpha$ is $\mathcal{T}$ if $P \mid \kappa_\alpha$ is passive, and otherwise it is $\mathcal{T}$ extended by using the extender from $P$ with index $\kappa_\alpha$. $\dashv_{\text{Claim 5.4}}$

We now work towards showing $\mathbf{I}_\alpha \Rightarrow \mathbf{I}_{\alpha+1}$.

Claim 5.5. *Assume $\alpha \leq \theta$, $\mathbf{I}_\alpha$ holds, and let $\mathcal{T}$ be the normal tree of minimal length on $W$ which witnesses $\mathbf{I}_\alpha$; then $\Phi(\mathcal{T})^\frown\langle \bar{W}, \omega, \kappa_\alpha, \kappa_\alpha\rangle$ is a stable phalanx.*

Proof. We first check that $\Phi(\mathcal{T})^\frown\langle \bar{W}, \omega, \kappa_\alpha, \kappa_\alpha\rangle$ is a phalanx. By assumption, $\bar{W}$ agrees with the last model of $\mathcal{T}$ below $\kappa_\alpha + 1$, and $\kappa_\alpha$ is a cardinal in $\bar{W}$. If $\xi < \text{lh}(\mathcal{T}) - 1$, then $\text{lh}(E_\xi^{\mathcal{T}}) \leq \kappa_\alpha$, and $\text{lh}(E_\xi^{\mathcal{T}})$ is a cardinal in the last model of $\mathcal{T}$, so $\text{lh}(E_\xi^{\mathcal{T}})$ is a cardinal of $\bar{W}$, and $\bar{W}$ agrees with $M_\xi^{\mathcal{T}}$ below $\text{lh}(E_\xi^{\mathcal{T}})$.

By 4.8, $\Phi(\mathcal{T})$ is stable. Thus if $\Psi = \Phi(\mathcal{T})^\frown\langle \bar{W}, \omega, \kappa_\alpha, \kappa_\alpha\rangle$ is unstable, we must have some $\xi \leq lh(\mathcal{T}) - 1$ such that $M_\xi^\Psi$ is a collapsing weasel, and for $\eta = \eta^{M_\xi^\Psi}$, $(\eta^+)^{M_\xi^{\mathcal{T}}} \leq \lambda_\xi^\Psi$ and $\bar{W} \models \eta$ is measurable.[16] But then $\eta^W \leq \eta < o(\bar{W}) < \kappa$ (our singular cardinal), so we are in the phalanx-unstable case in the definition of $W$. Letting $\nu = \text{cof}(\eta^W)^V$, we have that $i_{0,\xi}^{\mathcal{T}}$ is continuous at points of cofinality $\nu$, so that $\nu = \text{cof}(\eta)$. But also, $\pi$ is continuous at points of cofinality $\nu$, so $\nu = \text{cof}(\pi(\eta))$. But then $\pi(\eta)$ is not measurable in $W$, while $\eta$ is measurable in $\bar{W}$, contradiction. $\dashv_{\text{Claim 5.5}}$

[16]At this point we are using that $\kappa_\alpha$ is a cardinal in the full $\bar{W}$.

CLAIM 5.6. *Assume $\alpha \leq \theta$, $\mathbf{I}_\alpha$ holds, and let $\mathcal{T}$ be the normal tree of minimal length on $W$ which witnesses $\mathbf{I}_\alpha$. Suppose also that the phalanx $\Phi(\mathcal{T})^\frown\langle \bar{W}, \omega, \kappa_\alpha, \kappa_\alpha\rangle$ is iterable; then $\mathbf{I}_{\alpha+1}$ holds.*

PROOF. We compare $\Phi(\mathcal{T})$ with $\Phi(\mathcal{T})^\frown\langle \bar{W}, \omega, \kappa_\alpha, \kappa_\alpha\rangle$. Note the two last models agree below $\kappa_\alpha+1$, so all extenders used are at least that long. We think of the tree on $\Phi(\mathcal{T})$ as a tree $\mathcal{U}$ on $W$ extending $\mathcal{T}$. Let $\mathcal{V}$ be the tree on $\Phi(\mathcal{T})^\frown\langle \bar{W}, \omega, \kappa_\alpha, \kappa_\alpha\rangle$. Let $N$ be the last model of $\mathcal{U}$ and $P$ the last model of $\mathcal{V}$; then $P \trianglelefteq N$ because $W$ stably universal, and $\Phi(\mathcal{T})^\frown\langle \bar{W}, \omega, \kappa_\alpha, \kappa_\alpha\rangle$ is stable.

CLAIM. $P$ is above $\bar{W}$ in $\mathcal{V}$, and the branch $\bar{W}$-to-$P$ does not drop.

PROOF. If $W_0$ is not phalanx unstable, the proof is completely standard. But we must take a little care with the phalanx unstable case.

Suppose $P$ is above $M = M_\xi^{\mathcal{T}}$ instead. By stable universality of $W$, we get that the branches $W$-to-$M$ and $M$-to-$P$ do not drop, and that $P = N$. Let $E$ be the first extender used in $M$-to-$P$, and $\mu = \text{crit}(E)$, so that $\mu < \kappa_\alpha$. Using the fact that $W$ has the hull property everywhere, we see that $P$ has the hull property at $\mu$, and fails to have the hull property at all cardinals $\rho$ in the interval $(\mu, \kappa_\alpha)$.

Suppose first $(\mu^+)^{\bar{W}} < \kappa_\alpha$. Then looking at the pattern of the hull property in $N$ determined by the branch $W$-to-$N$ of $\mathcal{U}$, we see there is an extender $F$ with critical point $\mu$ used in this branch. $F$ is also applied to $M$ in this branch of $\mathcal{U}$. Letting $i\colon M \to P$ and $j\colon M \to N$ be the embeddings given by $\mathcal{V}$ and $\mathcal{U}$, we have $\text{ran}(i) \cap \text{ran}(j)$ is thick in $N = P$. Standard arguments with the hull property at $\mu$ then show $E$ is compatible with $F$, contradiction.

Suppose then that $(\mu^+)^{\bar{W}} = \kappa_\alpha$. We claim that $M$ has the definability property at $\mu$. This is well-known in the case that $W_0$ is not phalanx unstable. Suppose instead that $\nu = \text{cof}(\gamma^W) < \kappa$. In this case, $W$ has the the hull property everywhere. This implies by a well-known induction that $M$ has the definability property at all points $\rho \geq \sup(\{\nu(E_\delta^{\mathcal{T}}) \mid \delta + 1 <_T \xi\})$ except those of the form $i_{0,\xi}(\delta)$, where the definability property fails in $W$ at $\delta$. Each such $\delta$ has cofinality $\nu$, and since $i_{0,\xi}$ is then continuous at $\delta$, $i_{0,\xi}(\delta)$ has cofinality $\nu$. Now $\mu \geq \sup(\{\nu(E_\delta^{\mathcal{T}}) \mid \delta + 1 <_T \xi\})$ because $E$ was applied to $M$ and $\mathcal{T}$ is normal. Thus if the definability property fails at $\mu$ in $M$, then $\text{cof}(\mu)^M = \nu$. However, $\bar{W} \models \mu$ is measurable, so $\text{cof}(\mu) \neq \nu$.

Since $\text{crit}(E) = \mu$, $P = N$ does not have the definability property at $\mu$. This implies that the first extender $F$ used in the branch $M$-to-$N$ of $\mathcal{U}$ has critical point $\mu$. Again, the hull property at $\mu$ in $M$ yields $E$ is compatible with $F$, contradiction. $\dashv_{\text{CLAIM}}$

Thus $P$ is above $\bar{W}$ in $\mathcal{V}$. By the universality of $W$, we get that $\bar{W}$-to-$P$ does not drop, and $P \trianglelefteq N$. All critical points in $\bar{W}$-to-$P$ are $\geq \kappa_\alpha$, so

$$\bar{W} \mid \kappa_{\alpha+1} \trianglelefteq P \trianglelefteq N.$$

Letting $\mathcal{M}_\gamma^{\mathcal{T}}$ be the last model of $\mathcal{T}$, $\text{lh}(E_\xi^{\mathcal{T}}) > \kappa_\alpha$ for all $\xi \geq \gamma$, and thus

$$\bar{W} \mid \kappa_{\alpha+1} \trianglelefteq \mathcal{M}_\gamma^{\mathcal{T}}.$$

The tree which witnesses $\mathbf{I}_{\alpha+1}$ is then $\mathcal{T}$ if $\mathcal{M}_\gamma^{\mathcal{T}} \mid \kappa_{\alpha+1}$ is passive, and the normal extension of $\mathcal{T}$ via the extender of $\mathcal{M}_\gamma^{\mathcal{T}}$ with index $\kappa_{\alpha+1}$ otherwise. $\dashv_{\text{CLAIM 5.6}}$

The claim in the proof of 5.6 gives:

Corollary 5.7. *Assume $\alpha \leq \theta$, $\mathbf{I}_\alpha$ holds, and let $\mathcal{T}$ be the normal tree of minimal length on $W$ which witnesses $\mathbf{I}_\alpha$. Suppose also that the phalanx $\Phi(\mathcal{T})^\frown\langle \bar{W}, \omega, \kappa_\alpha, \kappa_\alpha\rangle$ is iterable; then there is a normal iteration tree $\mathcal{U}$ extending $\mathcal{T}$, and an initial segment $P$ of the last model of $U$, and an embedding $j : \bar{W} \to P$ such that* $\operatorname{crit}(j) \geq \kappa_\alpha$.

So our proof of the Main Lemma 5.2 is done when we show:

Claim 5.8. *Let $\mathcal{T}$ be the normal tree of minimal length on $W$ which witnesses $\mathbf{I}_\alpha$; then the phalanx $\Phi(\mathcal{T})^\frown\langle \bar{W}, \omega, \kappa_\alpha, \kappa_\alpha\rangle$ is iterable.*

To prove this, it helps to re-organize $\Phi(\mathcal{T})$, and in doing this, we shall rejoin the notation established on page 233 of [14]. For $\beta < \alpha$, set

$$\begin{aligned} \eta(\beta) &= \text{least } \xi < \operatorname{lh}(\mathcal{T}) - 1 \text{ such that } \nu(E_\xi^{\mathcal{T}}) > \kappa_\beta, \\ &= \operatorname{lh}(\mathcal{T}) - 1, \text{ if there is no such } \xi, \end{aligned}$$

$$\lambda_\beta = \kappa_{\beta+1},$$

$$\begin{aligned} P_\beta &= \mathcal{M}_{\eta(\beta)}^{\mathcal{T}} \mid \gamma, \text{ where } \gamma \text{ is least s.t. } \rho_\omega(\mathcal{M}_{\eta(\beta)}^{\mathcal{T}} \mid \gamma) < \lambda_\beta, \\ &= \mathcal{M}_{\eta(\beta)}^{\mathcal{T}}, \text{ if there is no such } \gamma, \end{aligned}$$

and

$$k_\beta = \text{largest } k \leq \omega \text{ such that } \lambda_\beta < \rho_k(P_\beta).$$

Notice here that for $\beta < \alpha$, $\mathcal{M}_{\eta(\beta)}^{\mathcal{T}}$ agrees with the last model of $\mathcal{T}$, and hence with $\bar{W}$, below $\lambda_\beta$. (If $\eta(\beta) < \operatorname{lh}(\mathcal{T}) - 1$, then $\operatorname{lh}(E_{\eta(\beta)})$ is a cardinal of the last model of $\mathcal{T}$ and $\operatorname{lh}(E_{\eta(\beta)}) \leq \kappa_\alpha$, so $\lambda_\beta = \kappa_{\beta+1} \leq E_{\eta(\beta)})$, and we have the desired agreement. If $\eta(\beta) = \operatorname{lh}(\mathcal{T}) - 1$, then the fact that $\lambda_\beta \leq \kappa_\alpha$ gives the desired agreement.) Thus our definition of $P_\beta$ makes sense, and $P_\beta$ agrees with $\bar{W}$ below $\lambda_\beta$. It is possible that $\lambda_\beta$ is active in $P_\beta$, in which case $P_\beta$ disagrees with $\bar{W}$ at $\lambda_\beta$, and $\lambda_\beta = o(P_\beta)$.

Note also that for $\xi + 1 \in [0, \eta(\beta)]_T$, we have $\nu(E_\xi^{\mathcal{T}}) \leq \kappa_\beta$. This easily yields

Claim 5.9. *Let $\beta < \alpha$; then either*

(1) *$k_\beta < \omega$, $\rho_{k_\beta+1}(P_\beta) \leq \kappa_\beta < \rho_{k_\beta}(P_\beta)$, and $P_\beta$ is $\kappa_\beta$-sound, or*
(2) *$k_\beta = \omega$, $P_\beta$ is a weasel such that $S(P_\beta)$ is $P_\beta$-thick, $P_\beta$ has the hull property at all $\mu \geq \kappa_\beta$, and for all $\mu \geq \kappa_\beta$, either $P_\beta$ has the definability property at $\mu$, or $W_0$ was phalanx unstable, and* $\operatorname{cof}(\mu) = \operatorname{cof}(\eta^W)$.

Note that in a normal tree $\mathcal{U}$ on $\Phi(\mathcal{T})^\frown\langle \bar{W}, \omega, \kappa_\alpha, \kappa_\alpha\rangle$, if an extender $E = E_\eta^{\mathcal{U}}$ with $\operatorname{crit}(E) < \kappa_\alpha$ is used, then we have a $\beta < \alpha$ such that $\operatorname{crit}(E) = \kappa_\beta$, and $\eta(\beta)$ is the $U$-predecessor of $\eta + 1$, and $\mathcal{M}_{\eta+1}^{\mathcal{U}} = \operatorname{Ult}_{k_\beta}(P_\beta, E)$. Thus normally iterating $\Phi(\mathcal{T})^\frown\langle \bar{W}, \omega, \kappa_\alpha, \kappa_\alpha\rangle$ is equivalent to normally iterating the phalanx $\Phi_\alpha$, where

Definition 5.10. *For any $\xi \leq \alpha$, $\Phi_\xi$ is the phalanx*

$$(\langle (P_\beta, k_\beta) \mid \beta \leq \xi\rangle^\frown\langle \bar{W}, \omega\rangle, \langle (\lambda_\beta, \lambda_\beta) \mid \beta < \xi\rangle).$$

Our proof of 5.5 shows

Claim 5.11. *For all $\xi \leq \alpha$, $\Phi_\xi$ is stable.*

We shall show by induction on $\xi \leq \alpha$:

$(4)_\xi$ $\Phi_\xi$ is iterable.

(The numbering here corresponds to the numbering of inductive hypotheses in [14, p. 236].)

Simultaneously, we show the iterability of some phalanxes associated to $\Phi_\xi$. First, set for $\beta < \alpha$:

$$R_\beta = \mathrm{Ult}_{k_\beta}(P_\beta, E_\pi \restriction \pi(\kappa_\beta)),$$

and let

$$\pi_\beta \colon P_\beta \to R_\beta$$

be the ultrapower map, and

$$\Lambda_\beta = \pi_\beta(\lambda_\beta) = \sup(\pi``\lambda_\beta).$$

Note that $R_\beta$ agrees with $W$ below $\Lambda_\beta$, since $P_\beta$ agrees with $\bar{W}$ below $\lambda_\beta$. It is possible that some of the $R_\beta$ are protomice, but not premice.

We need

Claim 5.12. *For all $\beta < \alpha$, $o(R_\beta) \leq \Omega$.*

Proof. In the case that $o(P_\beta) < \Omega$, or $P_\beta$ is a mini-universe, this is easy. So suppose that $P_\beta$ is a collapsing weasel. Let $\gamma = \gamma^{P_\beta}$ be its largest cardinal. We are done if we show the ultrapower map from $P_\beta$ to $R_\beta$ is continuous at $\gamma$. Assume not; then we have a finite set $a \subseteq \pi(\kappa_\beta)$ and a function $f : [\mu]^{|a|} \to \gamma$, where $[\mu]^{|a|}$ is the space of $(E_\pi)_a$, such that $f$ is not bounded on any set of $(E_\pi)_a$ measure one. But then

$$\mathrm{cof}(\gamma) \leq \mu \leq \bar{\kappa} < \kappa.$$

Set $\nu = \mathrm{cof}(\gamma)$, and note

$$\mathrm{cof}(\gamma^W)^V = \mathrm{cof}(\gamma^{P_\beta})^V = \nu,$$

by 4.23. We are then in the phalanx-unstable case, so that $N$ is closed under $\nu$-sequences.

Because of this, $(E_\pi)_a$ is $\nu$-complete: if $X_\xi \in (E_\pi)_a$ for all $\xi < \nu$, then $a \in \bigcap_{\xi<\nu} \pi(X_\xi) = \pi(\bigcap_{\xi<\nu} X_\xi)$, so $V \models \pi(\bigcap_{\xi<\nu} X_\xi) \neq \emptyset$, so $N \models \bigcap_{\xi<\nu} X_\xi \neq \emptyset$.

But pick $\langle \mu_\xi \mid \xi < \nu \rangle$ cofinal in $\gamma$, and for $\xi < \nu$, let

$$X_\xi = \{u \mid f(u) > \mu_\xi\}.$$

Clearly, each $X_\xi$ is in $(E_\pi)_a$, but the intersection is empty. $\dashv_{\text{Claim 5.12}}$

Definition 5.13. *For $\xi \leq \alpha$, let $\Psi_\xi$ be the phalanx*

$$(\langle (R_\beta, k_\beta) \mid \beta \leq \xi \rangle^\frown \langle W, \omega \rangle, \langle (\Lambda_\beta, \Lambda_\beta) \mid \beta < \xi \rangle).$$

We need to modify the definitions to do with special phalanxes, definitions 2.4.5, 2.4.6, and 2.4.7 of [14]. The reason is that in the phalanx-unstable case, the class parameter and class projectum defined on p. 226 of [14] do not behave properly. What we have is

Claim 5.14. *For any $\xi \leq \alpha$, $\Psi_\xi$ satisfies all clauses in the definition of* very special phalanx of protomice *except those to do with the class parameter and projectum* (*i.e.*, (iv) *of* 2.4.5 *and the first item in* 2.4.6) *from* [14]. *Moreover, $\Psi_\xi$ is stable.*

Proof. Stability is proved just as it was for $\Phi_\xi$. The rest is easy. $\dashv_{\text{Claim 5.14}}$

The soundness properties of $\Psi_\xi$ replacing those to do with the class parameter and projectum are just:

CLAIM 5.15. *Let $\beta < \alpha$; then either*

(1) $k_\beta < \omega$, $\rho_{k_\beta+1}(R_\beta) \leq \pi(\kappa_\beta) < \rho_{k_\beta}(R_\beta)$, *and $R_\beta$ is $\pi(\kappa_\beta)$-sound, or*

(2) $k_\beta = \omega$, *$R_\beta$ is a weasel such that $S(R_\beta)$ is $R_\beta$-thick, $R_\beta$ has the hull property at all $\mu \geq \pi(\kappa_\beta)$, and for all $\mu \geq \pi(\kappa_\beta)$, either $R_\beta$ has the definability property at $\mu$, or $W_0$ was phalanx unstable, and* $\operatorname{cof}(\mu) = \operatorname{cof}(\gamma^W)$.

This is easy to prove.

Along with $(4)_\xi$, we show by induction

$(5)_\xi$ $\Psi_\xi$ is iterable, with respect to special iteration trees.

See [14, 2.4.6] for the definition of "special". It demands one consequence of normality, and it demands that when an extender is applied to $R_\beta$, its critical point should be $\pi(\kappa_\beta)$.

LEMMA 5.16. *For any $\xi \leq \alpha$, $(5)_\xi \Rightarrow (4)_\xi$.*

See [14, 3.17] for a proof.

The fact that the $R_\beta$ may not be premice complicates our argument. We persevere by introducing premice $S_\beta$ which in some sense replace them, along with premice $Q_\beta$ downstairs replacing $P_\beta$ in parallel fashion. These are defined on page 234 of [14]. The construction insures that $S_\beta$ agrees with $R_\beta$, and hence with $W$, below $\Lambda_\beta$. Let $k_\beta = n(P_\beta, \pi(\kappa_\beta))$.

DEFINITION 5.17. *For $\xi \leq \alpha$, let*

$$\Psi^*_\xi = (\langle (S_\beta, k_\beta) \mid \beta \leq \xi \rangle^\frown \langle W, \omega \rangle, \langle (\Lambda_\beta, \Lambda_\beta) \mid \beta < \xi \rangle).$$

Again, we have

CLAIM 5.18. *For any $\xi \leq \alpha$, $\Psi^*_\xi$ satisfies all clauses in the definition of* very special phalanx of premice *except those to do with the class parameter and projectum* (*i.e.*, (iv) *of* 2.4.5 *and the first item in* 2.4.6) *from* [14]. *Moreover, $\Psi^*_\xi$ is stable.*

The soundness properties of the models in $\Psi^*_\xi$ are given by

CLAIM 5.19. *Let $\beta < \alpha$; then either*

(1) $k_\beta < \omega$, $\rho_{k_\beta+1}(S_\beta) \leq \pi(\kappa_\beta) < \rho_{k_\beta}(R_\beta)$, *and $S_\beta$ is $\pi(\kappa_\beta)$-sound, or*

(2) $k_\beta = \omega$, *$S_\beta$ is a weasel such that $S(S_\beta)$ is $S_\beta$-thick, and there is a finite set $t$ of ordinals such that*

  (a) *$S_\beta$ has the $t$ hull property at all $\mu \geq \pi(\kappa_\beta)$, and*

  (b) *for all $\mu \geq \pi(\kappa_\beta)$, either $S_\beta$ has the $t$ definability property at $\mu$, or $W_0$ was phalanx unstable, and* $\operatorname{cof}(\mu) = \operatorname{cof}(\gamma^W)$.

See [14, 3.5, 3.6] for a proof.[17] In that paper, the parameter $t$ in part (2) is identified using the definability property over $S_\beta$. In the phalanx-unstable case, we are not able to characterize $t$ this way. However, this does not matter for our argument, as we will never actually compare $\Psi^*_\xi$, or any other phalanx having $S_\beta$ as a backup model, with another phalanx.

It will be enough to prove

$(6)_\xi$ The phalanx $\Psi^*_\xi$ is iterable.

[17] The parameter $t$ witnessing (2) of 5.19 is the accumulation of the Dodd parameters of extenders used in getting from $P_\beta$ to $Q_\beta$, lifted up by $\pi_\beta$.

CLAIM 5.20. *For any $\xi \leq \alpha$, if $\Psi^*_\xi$ is iterable, then $\Psi_\xi$ is iterable.*

This is lemma 3.18 in [14]. No changes in that proof are needed here.

To prove $(6)_\xi$, we use

$(2)_\beta$ $(\langle (W,\omega),(S_\beta,k_\beta)\rangle,(\pi(\kappa_\beta),\pi(\kappa_\beta)))$ is iterable.

CLAIM 5.21. *Let $\xi \leq \alpha$, and suppose that $(2)_\beta$ holds, for all $\beta < \xi$; then $(6)_\xi$ holds.*

This is lemma 3.19 of [14], and its proof does not change. It represents the fundamental step in the inductive definition of $K$ from [22].[18]

To prove $(2)_\beta$ for $\beta < \xi$ we use

$(3)_\beta$ The phalanx $(\langle (\bar{W},\omega),(Q_\beta,k_\beta)\rangle,(\kappa_\beta,\kappa_\beta))$ is iterable.

CLAIM 5.22. *For any $\beta < \alpha$, $(3)_\beta$ implies $(2)_\beta$.*

This is lemma 3.13 of [14], and again, the proof does not change. The proof uses the countable completeness of $E_\pi$ to realize countable elementary submodels of $(\langle (W,\omega),(S_\beta,k_\beta)\rangle,(\pi(\kappa_\beta),\pi(\kappa_\beta)))$ back in $(\langle (\bar{W},\omega),(Q_\beta,k_\beta)\rangle,(\kappa_\beta,\kappa_\beta))$.

Finally, we close the circle with

CLAIM 5.23. *Assume $(4)_\gamma$ holds for all $\gamma < \beta$; then $(3)_\beta$ holds.*

This is lemma 3.16 of [14]. No changes are needed. It is the fundamental step in the inductive definition of $K$ once more, but this time downstairs.

This completes our proof of the Main Lemma 5.2. ⊣

We now complete the proof of 3.1. Fix $\alpha = \bar{\kappa}+1$, and for $\beta < \alpha$, let $P_\beta, R_\beta, Q_\beta, S_\beta$, etc., be defined as above in our proof that $\Phi_\alpha$ is iterable. We have by $(2)_{\bar{\kappa}}$ that (simplifying our phalanx notation for readability) $(W,S_{\bar{\kappa}},\kappa)$ is iterable. We also have that $S_{\bar{\kappa}}$ agrees with $W$ below $\Lambda_{\bar{\kappa}}$. Let us compare $(W,S_{\bar{\kappa}},\kappa)$ with $W$. As in the proof of 5.21, we get an iteration tree $\mathcal{U}$ on $W$, and and embedding $j\colon S_{\bar{\kappa}} \to H$, where $H$ is an initial segment of the last model of $\mathcal{U}$, and $\mathrm{crit}(j) \geq \kappa$.

Note that $S_{\bar{\kappa}}$ agrees with $W$ below $\Lambda_{\bar{\kappa}} < (\kappa^+)^W$. Thus

$$P(\kappa)^{S_{\bar{\kappa}}} = P(\kappa)^W = P(\kappa)^H.$$

CASE 1. $S_{\bar{\kappa}}$ is not a weasel.

Then by 5.19, $S_{\bar{\kappa}}$ is $\kappa$-sound and projects to $\kappa$. This implies by standard arguments that $S_{\bar{\kappa}} \in W$, contrary to $P(\kappa)^W \subseteq S_{\bar{\kappa}}$.

CASE 2. $S_{\bar{\kappa}}$ is a weasel.

Then $Q_{\bar{\kappa}}$ is a weasel, and from its construction, we have an iteration map

$$i\colon W \to Q_{\bar{\kappa}}.$$

Moreover, if $\mathrm{crit}(i) < \mathrm{crit}(\pi)$ and $W_0$ was phalanx-unstable, then $\mathrm{cof}(\mathrm{crit}(i)) \neq \mathrm{cof}(\eta^W)$. But also

$$S_{\bar{\kappa}} = \mathrm{Ult}(Q_{\bar{\kappa}}, E_\pi \restriction \kappa),$$

and letting $k\colon Q_{\bar{\kappa}} \to S_{\bar{\kappa}}$ be the canonical embedding, $\mathrm{crit}(k) = \mathrm{crit}(\pi)$. Letting

$$\mu = \mathrm{crit}(k \circ i),$$

it follows that $\mu \leq \mathrm{crit}(\pi)$, and in the phalanx-unstable case, $\mathrm{cof}(\mu) \neq \mathrm{cof}(\eta^W)$.

[18] Comparing $(\langle (W,\omega),(S_\beta,k_\beta)\rangle,(\pi(\kappa_\beta),\pi(\kappa_\beta)))$ with $W$, we get $j_\beta\colon S_\beta \to N_\beta$ with $\mathrm{crit}(j_\beta) \geq \pi(\kappa_\beta)$ and $N_\beta$ and initial segment of the last model of an iteration tree on $W$. One can then use the $j_\beta$ to lift a tree on $\Psi^*_\xi$ to a tree on a $W$-generated phalanx.

Since $S_{\bar\kappa}$ is a weasel, and $S(S_{\bar\kappa})$ is $S_{\bar\kappa}$-thick, we see that $H$ is the last model of $\mathscr{U}$, and there was no dropping in $\mathscr{U}$ from $W$ to $H$. Further, $\mathrm{ran}(j \circ k \circ i)$ is $H$-thick, so $H$ does not have the definability property at $\mu$. Letting $H = \mathscr{M}_\gamma^{\mathscr{U}}$, and using that $W$ has the definability property at $\mu$, we get

$$\mathrm{crit}(i_{0,\gamma}^{\mathscr{U}}) < \kappa.$$

By 5.19, there is a finite set $t$ of ordinals such that $S_{\bar\kappa}$ has the $t$ hull property at $\kappa$. Since $\mathrm{crit}(j) \geq \kappa$, this implies that $H$ has the $j(t)$ hull property at $\kappa$. We can now pull this back to the first model after $W$ on the branch $[0, \gamma]_U$: letting $\eta + 1$ be least in $[0, \gamma]_U$, we have a finite set $s$ of ordinals such that $\mathscr{M}_\eta^{\mathscr{U}}$ has the $s$ hull property at $\kappa$. (See [14, p. 239], claim 1.)

Let $E = E_\eta^{\mathscr{U}}$, so that $\mathrm{crit}(E) < \kappa$. Let $a \subseteq \nu(E)$ be such that $s = [a, f]_E$ for some $f \in W$. Now let

$$\sigma \colon \mathrm{Ult}(W, E \restriction (\kappa + 1) \cup \{a\}) \to \mathrm{Ult}(W, E)$$

be the factor map, so that $\mathrm{crit}(\sigma) > \kappa$ and $s \in \mathrm{ran}(\sigma)$. We have $P(\kappa)^W \subseteq P(\kappa)^H \subseteq P(\kappa)^{\mathscr{M}_\eta^{\mathscr{U}}}$, and since $\mathrm{ran}(\sigma)$ is $\mathscr{M}_\eta^{\mathscr{U}}$-thick, we get that

$$P(\kappa) \cap W = P(\kappa) \cap \mathrm{Ult}(W, E \restriction (\kappa + 1) \cup \{a\}).$$

But now notice that $E \restriction (\kappa + 1) \cup \{a\}$ is coded by some $C \subseteq \kappa$ in $\mathscr{M}_\eta^{\mathscr{U}}$. By the agreement properties of iteration trees, $C \in W$. This implies $P(\kappa) \cap W$ has cardinality $\kappa$ in $W$, a contradiction.

This finishes the proof of the weak covering theorem. $\dashv$

**§6. Proof of the main theorem.** We can now prove Theorem 1.1. Suppose for the rest of this section that there is no proper class model with a Woodin cardinal. We obtain the class $K$ witnessing the truth of this theorem by piecing together the approriate $\tilde{K}(\tau, \Omega)$. To do that, we use

Lemma 6.1. *Let $\mu$ be a singular strong limit cardinal, $\tau = \mathrm{cof}(\mu)$, and $\Omega = \mu^+$; then $\tilde{K}(\tau, \Omega) \mid \tau$ satisfies the local inductive definition of $K$ given in* [22, §6].

Proof. By Theorem 3.1 and the proof of Proposition 4.4, there is a stable collapsing weasel $W$ such that $\mu$ is the largest cardinal of $W$. By lemma 4.31, we can choose $W$ so that also $\tau \subseteq \mathrm{Def}^W$, and hence $W \mid \tau = \tilde{K}(\tau, \Omega) \mid \tau$. So we must see that $W \mid \tau$ satisfies the local inductive definition. It is easy to see that the proof in [22, §6] that $\mathrm{Def}^{K^c}$ satisfies this inductive definition works in our situation, provided we can show:

Claim. Let $\alpha$ be a cardinal of $W$ such that $\alpha \leq \tau$, and suppose that the phalanx $(W, M, \alpha)$ is iterable, where $|M| < \Omega$ and $\rho_k(M) \geq \alpha$. Then there is an iteration tree $\mathscr{T}$ on $W$ with last model $\mathscr{P}$, and such that all extenders used in $\mathscr{T}$ have length at least $\alpha$, and a fully elementary

$$\pi \colon M \to \mathscr{P},$$

such that $\pi \restriction \alpha = \text{identity}$.

Proof. To reconcile our notation with that of definition 4.5: the phalanx we refer to here is $\langle (W, \omega), (M, k) \rangle$ paired with $\langle (\alpha, \alpha) \rangle$.

We prove the claim as usual, by comparing $\Phi = (W, M, \alpha)$ with $W$. The key point is that both are stable! In the case of $W$, this is simply by construction. In the case of $\Phi$, we need to check clause (2) of definition 4.7. But $\eta^W \geq \tau$, as $\mu$ is the largest cardinal of $W$, and its $V$-cofinality $\tau$ is $\leq$ its cofinality inside $W$. Thus clause (2) is vacuously true.

Since $\tau \subseteq \mathrm{Def}^W$, standard arguments show the comparison ends above $M$ on the $\Phi$-side, and that this gives us the desired $\pi$. $\dashv_{\text{CLAIM}}$

This proves 6.1. $\dashv$

COROLLARY 6.2. *Let $\mu$ and $v$ be singular strong limit cardinals, with $V$-cofinalities $\tau$ and $\sigma$, where $\tau \leq \sigma$; then $\tilde{K}(\tau, \mu^+) \mid \tau = \tilde{K}(\sigma, v^+) \mid \tau$.*

PROOF. This follows from 6.1, noting that the inductive definition in question is independent of $\tau$ and $\mu$. $\dashv$

This leads to

DEFINITION 6.3. *$K$ is the unique proper class premouse $W$ such that for any singular strong limit cardinal $\mu$, $W \mid \mathrm{cof}(\mu) = \tilde{K}(\mathrm{cof}(\mu), \mu^+) \mid \mathrm{cof}(\mu)$.*

What is left in the proof of Theorem 1.1 is already present in the literature.

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