§1 Alternatives to Namba Forcing

Consider the following problem:
Let $M = L^u_\beta = \langle L_\beta[u], u\rangle$ be a countable, iterable structure, where u is normal on κ in M.
Letting $M_i = L^{u_i}_{\beta_i}$ be the i-th iterate, assume that $V\alpha$ u_α is a measure in L^{u_α}. Let $\alpha = \alpha(M)$ be the least such α. How large can α be?

We know:

(a) $\alpha \leq \omega_1$

(b) α can be any ordinal $< \omega_1$.

(A more precise version of (b) is:
con(ZFC + There is a measurable) $\rightarrow$
$\rightarrow$ con(ZFC + $\bigwedge \xi < \omega_1$ $\vee M$ $\alpha(M) = \xi$). We shall sketch the proof of this a bit later.)

In this note we show that $\alpha = \omega_1$ is possible. The strategy of the proof is to start with a model $L^{\tilde{u}}$, where $\tilde{u}$ is a measure on $\tilde{\kappa}$ in $L^{\tilde{u}}$.*) Letting $\tilde{\beta} > \tilde{\kappa}$ be a cardinal, we find a generic extension $L^{\tilde{u}}[G]$ in which:

(i) $\tilde{\kappa} = \omega_1$

(ii) There is a countable $M = L^u_\beta$ which iterates up to $L^{\tilde{u}}_{\tilde{\beta}}$.

*) In this paper we always write L^a_β to mean $\langle L_\beta[a], \in, a \cap L_\beta[a]\rangle$.

Note that for all $i < \kappa$, U_i is not a measure in L^{U_i}, since $\kappa_i < \tilde{\kappa}$ is not measurable in an inner model of $L^{\tilde{U}}$ and the property of being measurable in an inner model is absolute in generic extensions. Hence $\alpha(M) = \tilde{\kappa}$ in $L^{\tilde{U}}[G]$.

If we let e.g. $\tilde{\beta} = \tilde{\kappa}^+$, it follows that β is taken cofinally to $\tilde{\beta}$ by the iteration map from M to $\tilde{M} = L^{\tilde{U}}_{\tilde{\beta}}$. Thus $\tilde{\beta}$ becomes ω-cofinal, though it is not collapsed to ω. The standard method for accomplishing this is Namba forcing. Could one e.g. obtain the result by first collapsing $\tilde{\kappa}$ to ω_1 and then employing Namba forcing to make $\tilde{\kappa}^+$ ω-cofinal? In fact, no such stratagem will work, since in Namba's model any two cofinal subsets of $\tilde{\kappa}^+$ have the same degree of constructibility over the ground model, and no cofinal subset of $\tilde{\kappa}^+$ has the constructibility degree of a countable set over the ground model.

In this case, however, if $X \subset \tilde{\kappa}^+$ is the image of β under the iteration map, then X and M have the same constructibility degree over $L^{\tilde{u}}$, although M is countable. Thus we need an alternative to Namba forcing which not only collapses $\tilde{\kappa}^+$ but perforce adds new countable sets.

In this note we develop such a method and apply it to the above problem. Further applications to core model theory can be found in the paper "On some problems of Mitchell, Welch, and Vickers." We also describe a variant of the method which is applicable in other settings (e.g. generically ~~adding~~ a cofinal ω-sequence to ω_2 without collapsing ω_1. This forcing works over L, but is different from Namba forcing since the ω-sequence added has the constructibility degree of a real.

In §3 we develop a variant of our method which does not add reals. In §5 we use this to show that $\alpha(M) = \beta$ is possible for any $\beta \leq \omega_1$, even with M an uncountable structure with an uncountable critical point. (By the covering lemma we will of course have $\bar{\bar{M}} < \omega_2$.) This forcing is one of a general class of forcings which we call *revisable*. The simplest revisable forcing adds a cofinal ω-sequence to ω_2 without adding reals. This appeared to be a genuine alternative to Namba forcing, since the motivation and combinatorics of the proof are very different. In §6 we show, however, that it is equivalent to Namba forcing and that the variant of Namba forcing which Shelah calls Nm' is also equivalent to a revisable forcing.

All of our forcings have the property that they are definable from an infinitary language $\mathcal{L}$ on a transitive ZFC-model N. We call such forcings *$\mathcal{L}$-forcing*. A major tool is Barwise' completeness theorem, which we describe below.

§1.1 The Barwise Completeness Theorem

Let N be a transitive ZF^- model. Let $\mathcal{L}$ be a 1st order infinitary language on N, Let $Fml_{\mathcal{L}}$ be the set of $\mathcal{L}$-formulae. We suppose the syntactic operations to be defined in N. Thus $Fml_{\mathcal{L}} \subset N$ and $Fml_{\mathcal{L}}$ is closed under $\wedge, \vee, \rightarrow, \leftrightarrow, \neg$, $\bigwedge, \bigvee, \bigwedge\!\!\!\bigwedge, \bigvee\!\!\!\bigvee$, where $\bigwedge, \bigvee$ are the all + existence quantifiers and $\bigwedge\!\!\!\bigwedge, \bigvee\!\!\!\bigvee$ are the infinitary conjunction and disjunction operators defined on arbitrary sequences $\langle \varphi_i \mid i \in u \rangle \in N$ of $\mathcal{L}$-formulae. We write:

$$\bigwedge\!\!\!\bigwedge_{i \in u} \varphi_i = \bigwedge\!\!\!\bigwedge \langle \varphi_i \mid i \in u \rangle.$$

Similarly for $\bigvee\!\!\!\bigvee$. (One could f.i. take $(\varphi \wedge \psi) = \langle 0, \varphi, \psi \rangle$, $(\varphi \vee \psi) = \langle 1, \varphi, \psi \rangle$, $(\varphi \rightarrow \psi) = \langle 2, \varphi, \psi \rangle$, $(\varphi \leftrightarrow \psi) = \langle 3, \varphi, \psi \rangle$, $\neg\varphi = \langle 4, \varphi \rangle$, $\bigwedge x \varphi = \langle 5, x, \varphi \rangle$, $\bigvee x \varphi = \langle 6, x, \varphi \rangle$, $\bigwedge\!\!\!\bigwedge f = \langle 7, f \rangle$, $\bigvee\!\!\!\bigvee f = \langle 8, f \rangle$.) The language has $On \cap N$ many variables (e.g. $v_i = \langle 9, i \rangle$ for $i \in On \cap N$, where

v_i = the i-th variable). The predicates and function terms are also defined in N. If we pick our syntactic definitions judiciously, we get that $\mathrm{Fml}_{\mathcal{L}}$ is not only N-definable but also $\Delta_1(N)$. The provability rules are the same as those for ordinary (i.e. finitary) predicate logic except that we add the axiom schemes:

- $\varphi_i \to \bigvee_{j \in u} \varphi_j \qquad (i \in u)$
- $\bigwedge_{j \in u} \varphi_j \to \varphi_i \qquad (i \in u)$

and the rules:

- If $\varphi_i \to \psi$ is provable for $i \in u$, then so is $\bigvee_{i \in u} \varphi_i \to \psi$
- If $\psi \to \varphi_i$ is provable for $i \in u$, then so is $\psi \to \bigwedge_{i \in u} \varphi_i$.

The final step in specifying a language $\mathcal{L}$ is to specify the set of its <u>axioms</u> (which we shall always take to be sentences – i.e. without free variables).

If the set of axioms is N-definable, so is the set $\{\varphi \mid \vdash_{\mathcal{L}} \varphi\}$ of provable formulae. (In fact, if the axiom set is $\Sigma_1(N)$ in a parameter p, then so is $\{\varphi \mid \vdash_{\mathcal{L}} \varphi\}$.) The notion of a *model* $\mathfrak{A}$ of the $\mathcal{L}$-language and the set $\{\varphi \mid \mathfrak{A} \models \varphi\}$ of $\mathcal{L}$-sentences true in $\mathfrak{A}$ is defined in the usual way. We say that $\mathfrak{A}$ is a *model of* $\mathcal{L}$ iff all of the $\mathcal{L}$ axioms are true in $\mathfrak{A}$. *Barwise' completeness theorem* then says that if N is countable and $\mathcal{L}$ is consistent, then $\mathcal{L}$ has a model. Unfortunately, this does not hold for uncountable N. The *correctness theorem*, which says that a sentence provable in $\mathcal{L}$ is true in every model of $\mathcal{L}$, holds for arbitrary N. Together, these theorems give that if N is countable, then a sentence is provable in $\mathcal{L}$ iff it is true in every model of $\mathcal{L}$.

(We note in passing that the assumption "N is a transitive ZF^- model" can be replaced by the weaker assumption "N is admissible".)

We now restrict ourselves to languages $\mathcal{L}$ with the following features:

(*) • $\mathcal{L}$ has a designated predicate $\dot{\in}$

• For each $x \in N$ there is a designated constant $\underline{x}$ and $\langle \underline{x} \mid x \in N \rangle$ is N-definable (e.g. $\underline{x} = \langle 20, x \rangle$).

• All ZF^- axioms are axioms of $\mathcal{L}$

• Each of the following sentences is an axiom of $\mathcal{L}$:

$$\bigwedge v \, (v \in \underline{x} \longleftrightarrow \bigvee_{z \in x} v = \underline{z}) \quad (x \in N).$$

<u>Def</u> Let $\mathfrak{A} = \langle |\mathfrak{A}|, \in^{\mathfrak{A}}, \langle \underline{x}^{\mathfrak{A}} \mid x \in N \rangle, \ldots \rangle$ be a model of $\mathcal{L}$. By the <u>well founded core</u> of $\mathfrak{A}$ we mean the largest $X \subset \mathfrak{A}$ s.t. $\in^{\mathfrak{A}} \restriction X$ is well founded and X is closed under $\in^{\mathfrak{A}}$. (In other words, $z \in X$ iff $\in^{\mathfrak{A}} \restriction u(z)$ is well founded, where $u(z) =$ the closure of $\{z\}$ under $\in^{\mathfrak{A}}$.)

Denote this by $\mathrm{wfc}(\mathfrak{A})$.

Def $\mathfrak{A}$ is solid iff $wfc(\mathfrak{A})$ is a transitive set and $\in^{\mathfrak{A}} \cap wfc(\mathfrak{A})^2 = \in \cap wfc(\mathfrak{A})^2$.

It is clear that every model of $\mathcal{L}$ is isomorphic to a solid model, so we lose nothing by restricting ourselves to solid models. We then note the

Fact Let $\mathfrak{A}$ be a solid model of $\mathcal{L}$. Then $N \subset wfc(\mathfrak{A})$ and $\underline{x}^{\mathfrak{A}} = x$ for all $x \in N$.

. . . . .

A major advantage of Barwise' completeness theorem is that it enables us to construct well founded models using only "ordinary" model theory. However, these models are countable. We shall now combine this method with forcing so as to obtain uncountable models of certain languages.

Before turning to the forcing argument we use the Barwise completeness theorem to prove the result mentioned at the outset that, if ZFC + "There is a measurable" is consistent, then is in ZFC + "For every $\xi < \omega_1$ there is an M s.t. $\alpha(M) = \xi$".

We start with $L^{\tilde{U}}$, where $\tilde{U}$ is a normal measure on κ. Force to collapse κ^+ to ω, getting $W = L^{\tilde{U}}[G]$. We show that W has the desired property. Let $\tilde{M}_\xi = J^{\tilde{U}_\xi}_{\beta_\xi}$ be the ξ-th iterate of $\tilde{M} = J^{\tilde{U}}_{\kappa^{+L^{U}}}$. Then $\tilde{M}_\xi \in H_{\omega_1}$ for $\xi < \omega_1$. We show that for each $\xi < \omega_1$ there is an M s.t. $\tilde{M}_\xi$ is the ξ-th iterate of M and U_i is not a measure in L^{U_i} for $i < \xi$, where $M_i = L^{U_i}_{\beta_i}$ is the i-th iterate of M. Hence $\alpha(M) = \xi$. We work over $W' = L^{\tilde{U}_\xi}[G]$. Noting that G is generic over $L^{\tilde{U}_\xi}$, we see that no $\tau < \kappa_\xi$ is measurable in W'. But by Barwise' completeness theorem there will be an $M \in W'$ whose ξ-th iterate is $\tilde{M}_\xi$. To see this, let $N = L^{\tilde{U}_\xi}_\delta$, where $\delta =$ the least $\delta > \beta_\xi$ s.t. $L^{\tilde{U}_\xi}_\delta \models ZFC^-$.

Then N is countable in W' and $\tilde{M}_\xi = H^N_{\beta_\xi}$.
We define in N a language $\mathcal{L}$ satisfying (*) and, in addition, the following axioms involving two constants $\mathring{M}$ and $\mathring{\pi}$:

- $\mathring{M} = \langle \dot{M}_i \mid i \leq \underline{\xi} \rangle$, where $\dot{M}_i$ is countable for $i \leq \underline{\xi}$ and $\dot{M}_{\underline{\xi}} = \underline{\tilde{M}_\xi}$.
- $\mathring{\pi} = \langle \dot{\pi}_{ij} \mid i \leq j \leq \underline{\xi} \rangle$ is a commutative continuous system of elementary embeddings $\dot{\pi}_{ij} : \dot{M}_i \prec \dot{M}_j$.
- $\dot{\pi}_{i,i+1} : \dot{M}_i \longrightarrow_{\dot{u}_i} \dot{M}_{i+1}$ for $i < \underline{\xi}$, where $\dot{M}_i = L^{\dot{u}_i}_{\dot{\beta}_i}$

(Here $\pi : M \longrightarrow_u M'$ means: $M' = \mathrm{Ult}(M, u)$ and π is the canonical embedding.)

$\mathcal{L}$ is clearly consistent, since $\langle \tilde{M}_i \mid i \leq \xi \rangle$ gives a model of $\mathcal{L}$. By absoluteness, W' knows that $\mathcal{L}$ is consistent. But N is countable in W'. Hence $\mathcal{L}$ has a solid model $\mathfrak{A} \in W'$. Simple absoluteness considerations tell us that $\langle \dot{M}_i^{\mathfrak{A}} \mid i \leq \xi \rangle$, $\langle \dot{\pi}_{ij}^{\mathfrak{A}} \mid i \leq j \leq \xi \rangle$ is the desired iteration. QED