## § 3 Adding no reals

All of the forcings we have hitherto discussed add new reals. In the first application this was, of course, necessitated by the problem, since a countable $\bar{M}$ which iterates to $M$ in $\omega_1$ steps must be new. In the last application, however, the problem was simply to collapse $\omega_2$ to $\omega$ while giving it cofinality $\omega$. The emergence of new reals in the generic extension resulted only from the fact that our conditions say too little about the extension. In this chapter we define a class of analogous forcings which, however, do not add new reals. This will enable us fr. ins. to give a positive answer to the following question: Assume GCH. Let $\kappa$ be a measurable cardinal and $\beta > \kappa$ be a cardinal. Let $U$ be a normal measure on $\kappa$. Let $M = \langle L^A_\beta, U \rangle$, where $L^A_\beta = H_\beta$. Is there a forcing extension adding no reals in which an $\bar{M}$ iterates in $\omega_1$ many steps to $M$? ($\bar{M}$ will obviously have to be uncountable.)

In the following $\omega_1$ will play the role of $\kappa$ in § 1. We assume $2^\omega = \omega_1$. Let $\beta > \omega_1$ be a cardinal s.t. $2^\beta = \beta$.

Let $M = L^A_\beta = \langle L_\beta[A], A \rangle$, where $L_\beta[A] = H_\beta$.
Set $N = \langle H_{\beta^+}, M, <, m \rangle$, where $<$ well orders $N$.
Let $\mathcal{L}$ be any infinitary language on the ZFC$^-$-model $N$ with:

<u>Predicate</u> $\dot\in$

<u>Constants</u> $\underline{x}$ $(x \in N)$, $\dot{M}$, $\dot\pi$, $\dot{B}$

<u>Axioms</u>: ZFC$^-$, $\bigwedge v \, (v \in \underline{x} \longleftrightarrow \bigvee_{z \in x} v = \underline{z})$ (for all $x \in N$) and

- $\dot{M} = \langle \dot{M}_i \mid i \leq \underline{\omega_1} \rangle$, $\dot\pi = \langle \dot\pi_{ij} \mid i \leq j \leq \underline{\omega_1} \rangle$
- $\pi$ is a continuous commutative sequence of elementary embeddings $\dot\pi_{ij} : \dot{M}_i \prec \dot{M}_j$
- $\dot{M}_i$ is countable for $i < \underline{\omega_1}$
- $\dot{M}_{\underline{\omega_1}} = \underline{M}$ ; $\dot{M}_i$ is countable for $i < \underline{\omega_1}$
- $\dot\pi_{ij} \restriction \dot\alpha_i = id$, $\dot\pi_{ij}(\dot\alpha_i) = \dot\alpha_j$ where $\dot\alpha_i =_{\text{df}} \omega_1^{\dot{M}_i}$
- $\dot\beta_i < \dot\alpha_{i+1}$ for $i < \underline{\omega_1}$, where $\dot\beta_i =_{\text{df}} On \cap \dot{M}_i$
- $\dot{B} \subset \underline{M}$
- $H_{\omega_1} = \underline{H_{\omega_1}}$

We call these the *basic axioms*. Note that by the last axiom we have:

$\langle \dot{M}_i \mid i \leq \underline{\xi} \rangle, \langle \dot{\pi}_{ij} \mid i \leq j \leq \underline{\xi} \rangle \in \underline{H}_{\omega_1}$ for $\xi < \omega_1$.

Moreover, setting $\dot{M}_i^* =_{Df} J_{\beta_i}^{\dot{A}_i, \dot{B}_i}$ where

$\dot{B}_i = \dot{\pi}_{i\underline{\omega}_1}^{-1} \text{"} \mathring{B}$, $\dot{A}_i = \dot{\pi}_{i\underline{\omega}_1}^{-1} \text{"} \underline{A}$,

we have $\langle \dot{M}_i^* \mid i \leq \underline{\xi} \rangle \in \underline{H}_{\omega_1}$ for $\xi < \omega_1$.

We also set: $M^* = \dot{M}_{\underline{\omega}_1}^* = J_{\underline{\beta}}^{\underline{A}, B'}$.

<u>Note</u> The basic axioms do <u>not</u> say that $\mathring{\pi}_{ij}$ takes $\dot{M}_i^*$ to $M_j^*$, though $\mathring{\pi}_{ij}$ will, of course, be a structurally preserving embedding of $\langle \dot{M}_i, \dot{B}_i \rangle$ to $\langle \dot{M}_j, \dot{B}_j \rangle$

We now associate to each $\mathcal{L}$ satisfying the basic axioms a set of conditions $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$.

The conditions in § 2 gave - essentially - finite information about $\dot{M}, \mathring{\pi}$. The new conditions give countable information.

We first define a set $\tilde{\mathbb{P}}$ of preconditions:

Def $\tilde{\mathbb{P}}$ = the set of pairs $\langle p_0, p_1 \rangle$ s.t. for some $\gamma$:

(a) $p_0 = \langle M^p, \pi^p, b^p \rangle$, $p_0 \in H_{\omega_1}$

- $M^p = \langle M_i^p \mid i \leq \gamma \rangle$, $\pi^p = \langle \pi_{ij}^p \mid i \leq j \leq \gamma \rangle$
- $M_i^p = J_{\alpha_i}^{A_i}$ where $\alpha_i < \omega_1$
- $b^p \subset M_\gamma^p$
- $\pi^p$ is a continuous, commutative system of elementary embeddings $\pi_{ij}^p : M_i^p \prec M_j^p$ $(i \leq j)$

(b) $p_1$ is a countable set of pairs $\langle a, \bar{a} \rangle$ s.t. $a \subset M$, $\bar{a} \subset M_\gamma^p$.

We set: $|p| =_{\mathrm{Df}} \gamma$, where $\gamma$ is as above, for $p \in \tilde{\mathbb{P}}$.

Def Let $p \in \tilde{\mathbb{P}}$. $\varphi_p$ is the following $\mathcal{L}$-fmla:

$$\underline{M}^p = \langle \mathring{M}_i \mid i \leq \underline{|p|} \rangle \wedge \underline{\pi}^p = \langle \dot{\pi}_{ij} \mid i \leq j \leq \underline{|p|} \rangle \wedge$$

$$\wedge \bigwedge_{\langle a, \bar{a} \rangle \in p_1} \dot{\pi}_{\underline{|p|}, \underline{\omega_1}} : \underline{\langle M_{|p|}^p, \bar{a} \rangle} \prec \underline{\langle M, a \rangle} \wedge$$

$$\wedge \ \underline{b}^p = \dot{\pi}_{\underline{|p|}, \underline{\omega_1}}^{-1} {}'' \dot{B}.$$

Set $\mathcal{L}(p) = \mathcal{L} + \varphi_p$. We define $\mathbb{P} = \mathbb{P}_{\mathcal{L}} = \langle \mathbb{P}, \leq \rangle$ by:

Def $\mathbb{P} = \{p \in \tilde{\mathbb{P}} \mid \mathcal{L}(p)$ is consistent$\}$

$$p \leq q \longleftrightarrow (M^q = M^p \upharpoonright |q|+1 \wedge \pi^q = \pi^p \upharpoonright (|q|+1)^2 \wedge$$
$$\wedge \bigwedge \langle \bar{a}, a \rangle \in q_1 \; \bigvee a' (\langle a, a' \rangle \in p_1 \wedge$$
$$\wedge \; \pi^p_{|q|,|p|} : \langle M^q_{|q|}, \bar{a} \rangle \prec \langle M^p_{|p|}, a' \rangle \wedge$$
$$\wedge \; b^q = \pi^{p \; -1}_{|q|,|p|} {}'' b^p)$$

It is easily seen that $\leq$ is a partial ordering of $\mathbb{P}$.

<u>Lemma 0.1</u> $p_1^{-1}$ is a function

proof. Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$ (in some forcing collapse of $\beta^+$ to $\omega$).

Then $\langle a, \bar{a} \rangle \in p_1 \rightarrow \bar{a} = \dot{\pi}^{\mathfrak{A}\,-1}_{|p|,\omega_1} {}'' a$.

Def Let $p \in \mathbb{P}$. We set:

$F^p = p_1$, $R^p = \mathrm{rng}(p_1)$, $D^p = \mathrm{dom}(p_1)$.

For $a \in R^p$ set: $\bar{a}^p = p_1^{-1}(a)$.

For $i \leq |p|$ set:

$$F_i^p = \{\langle a, \bar{a} \rangle \mid a \in R^p \wedge \bar{a} = \pi^{p\;-1}_{i,|p|} {}'' \bar{a}^p \wedge$$
$$\wedge \; \pi^p_{i,|p|} : \langle M^p_i, \bar{a} \rangle \prec \langle M^p_{|p|}, \bar{a}^p \rangle\}$$

(Hence $F^p = F^p_{|p|}$)

$R_i^p = \mathrm{rng}(F_i^p)$, $D_i^p = \mathrm{dom}(F_i^p)$

<u>Lemma 0.2</u> Let $R^p$ be closed under set difference. Then $F^p$ is a bijection of $D^p$ onto $R^p$.

proof.

Let $\langle a, \bar{a}\rangle, \langle b, \bar{b}\rangle \in F^p$. It suffices to show:

<u>Claim</u> $\bar{a} \subset \bar{b} \to a \subset b$.

Set $\bar{c} = \bar{b} \setminus \bar{a}$, $c = b \setminus a$. Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$. Let $\pi = \pi^{\mathfrak{A}}_{|p|, \omega_1}$. Then

$F^{-1}(c) = \pi^{-1}{}''(b \setminus a) = \pi^{-1}{}''b \setminus \pi^{-1}{}''c = \bar{b} \setminus \bar{a} = \bar{c}$.

Hence $\bar{b} \subset \bar{a} \to \bar{c} = \emptyset \to c = \emptyset \to b \subset a$,

since $\pi : \langle M^p_{|p|}, \bar{c}\rangle \prec \langle M, c\rangle$. QED (0.2)

<u>Note</u> $p_1$ plays largely the role of $p_2$ in §2. The component $p_1 = \langle \pi^p_i \mid i \in D(p)\rangle$ of §2 is superfluous here, since the information it contains can be subsumed in our $p_1 = F^p$:

<u>Def</u> $\pi^p_i = \pi^p_{i, \omega_1} =_{Df} F^p \circ \pi^p_{i, |p|}$ for $i \leq |p|$

<u>Lemma 0.3</u> $\pi^p_i$ is a partial injection of $M^p_i$ into $M$

proof.

It suffices to prove this for $i = |p|$. Let $\langle x, \bar{x}\rangle \in F^p$ where $\bar{x} \in M^p_{|p|}$. ~~Let~~ $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$.

Then $\dot{\pi}^{\mathfrak{A}}_{|p|,\omega_1} : \langle M^p_{|p|}, \bar{x} \rangle \prec \langle M, x \rangle$. Hence $x \in M$ and $x = \dot{\pi}^{\mathfrak{A}}_{|p|,\omega_1}(\bar{x})$. QED (0.3)

Note It follows that if $\mathfrak{A}$ is a solid model of $\mathcal{L}(p)$, then $\pi^p_i \subset \dot{\pi}^{\mathfrak{A}}_{i,\omega_1}$ for all $i \leq |p|$.

Note It is easily seen that

$$p \leq q \rightarrow F^q_i \subset F^p_i \quad \text{for } i \leq |q|$$
$$\rightarrow \pi^q_i \subset \pi^p_i \quad \text{"} \quad .$$

We define:

Def Let $p, q \in \mathbb{P}$.

$p \| q \longleftrightarrow_{Df}$ $p, q$ are compatible in $\mathbb{P}$ (i.e. $\bigvee r\ r \leq p, q$)

$p \perp q \longleftrightarrow_{Df} \neg(p \| q)$.

Lemma 1.1 $p \| q \longleftrightarrow \mathcal{L}(p) \cup \mathcal{L}(q)$ is consistent.

prf.

($\rightarrow$) Let $r \leq p, q$. Then $\mathcal{L}(r) \vdash \mathcal{L}(p) \cup \mathcal{L}(q)$

($\leftarrow$) Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p) \cup \mathcal{L}(q)$. Since $\omega_1$ is regular in $\mathfrak{A}$ and $R^p \cup R^q$ is countable in $\mathfrak{A}$, there is an $\alpha < \omega_1$ s.t. $\alpha \geq |p|, |q|$ and for all $a \in R^p \cup R^q$ we have: $\dot{\pi}^{\mathfrak{A}}_{\alpha,\omega_1} : \langle M^{\mathfrak{A}}_\alpha, a^* \rangle \prec \langle M, a \rangle$, where $a^* = \dot{\pi}^{\mathfrak{A}\,-1}_{\alpha,\omega_1} {}'' a$.

Define $r \in \tilde{\mathbb{P}}$ by:

$M^r = \langle M_i^{\mathfrak{A}} \mid i \leq \alpha \rangle$, $\pi^r = \langle \pi_{ij}^{\mathfrak{A}} \mid i \leq j \leq \alpha \rangle$

$b^r = \pi_{\alpha,\omega_1}^{\mathfrak{A}\,-1}$ " $B^{\mathfrak{A}}$, $F^r = \{\langle a, a^* \rangle \mid a \in R^p \cup R^q\}$.

Then $r \in \mathbb{P}$, since $\mathfrak{A} \models \mathcal{L}(r)$. It follows easily that $r \leq p, q$. QED (1.1)

We also obtain the following lemmas on extending conditions:

<u>Lemma 1.2</u> Let $p \in \mathbb{P}$. Let $u \subset \mathcal{H}(M)$ be at most countable. There is $q \leq p$ s.t. $u \subset R^q$.

prf. Like 1.1

Hence:

<u>Cor 1.3</u> Let $p \in \mathbb{P}$, $u \subset M$, $\bar{\bar{u}} \leq \omega$. There is $q \leq p$ s.t. $u \subset \mathrm{rng}(\pi_{|q|}^q)$

<u>Lemma 1.4</u> Let $p \in \mathbb{P}$ + let $u \subset M_i^p$ be finite. There is $q \leq p$ s.t. $u \subset \mathrm{dom}(\pi_i^q)$.

prf.
Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$. Define $q$ by: $M^q = M^p$, $\pi^q = \pi^p$, $R^q = R^p \cup R$, where

$R = \{\langle \pi_{|p|\omega_1}^{\mathfrak{A}}(x), \pi_{i,|p|}^p(x) \rangle \mid x \in u\}$.

Then $\mathfrak{A} \models \mathcal{L}(q)$. Hence $q \leq p$.

QED (1.4)

<u>Lemma 1.5</u> Let $p \in \mathbb{P}$, $|p| \leq \alpha < \omega_1$. There is $q \leq p$ s.t. $|q| \geq \alpha$.

prf. Like 1.1

Using the extension lemma we easily get:

<u>Lemma 2</u> Let $G$ be $\mathbb{P}$-generic over $V$. Define $M^G = \langle M_i^G \mid i \leq \omega_1 \rangle$, $\pi^G = \langle \pi_{ij}^G \mid i \leq j \leq \omega_1 \rangle$, $B^G \subset M$ by:

$M^G \upharpoonright \omega_1 = \bigcup_{p \in G} M^p$, $\pi^G \upharpoonright \omega_1^2 = \bigcup_{p \in G} \pi^p$,

$M_{\omega_1}^G = M$, $\pi_{i,\omega_1}^G = \bigcup_{p \in G} \pi_i^p$,

$B^G = \bigcup_{p \in G} \pi_i^p {}'' b^p$.

Then the following hold:

(A) $M^G, \pi^G$ is a continuous commutative sequence of elementary embeddings.

(B) $\omega_1 \in \operatorname{rng}(\pi_{i,\omega_1}^G)$. Moreover, $\alpha_i^G = \operatorname{crit}(\pi_{i,\omega_1}^G)$, where $\alpha_i^G =_{\text{Df}} \pi_{i,\omega_1}^{G\,-1}(\omega_1)$.

(C) $\beta_i^G < \alpha_{i+1}^G$, where $\beta_i^G = \mathrm{On}_{M_i^G}$

(D) $b_i^p = \pi_{i\,\omega_1}^{G\,-1} {}'' B^G$ for all $p \in G$, $i \leq |p|$.

(E) If $p \in G$, $\alpha = \omega_1^V$, and $\langle a, \bar{a} \rangle \in F^p$, then $\pi_{|p|,\alpha} : \langle M_{|p|}^p, \bar{a} \rangle \prec \langle M, a \rangle$.

We now formulate a condition called <u>revisability</u> which will guarantee that $\mathbb{P}$ adds no new reals.

We first define:

<u>Def</u> Let $N^* = \langle H_\delta, M, <, \dots \rangle$ be a model of countable or finite type, where $\delta \geq \beta^+$ is a cardinal and $<$ well orders $H_\delta$. Let $p \in \mathbb{P}$. $p$ <u>conforms</u> to $N^*$ iff whenever $a_1, \dots, a_n \in \mathbb{R}^p$ and $b \subset M$ is $N^*$-definable in $a_1, \dots, a_n$ ($n \geq 0$), then $b \in \mathbb{R}^p$.

<u>Note</u> $\{p \mid p \text{ conforms to } N^*\}$ is dense in $\mathbb{P}$ by the extension lemmas.

<u>Note</u> If $p$ conforms to $N^*$, then $\mathbb{R}^p \neq \emptyset$ and $F^p : D^p \longleftrightarrow F^p$ by Lemma 0.2.

Before defining revisability we must prove a theorem:

<u>Lemma 3</u> Let $p$ conform to $N^*$. There is a unique $\bar{N}^* = \bar{N}^*(p, N^*)$ s.t.

(i) $\bar{N}^*$ is transitive and of the same type as $N^*$

(ii) If $a_1, \ldots, a_n \in \mathbb{R}^p$ $(n \geq 0)$ and $b \subset M$ is $N^*$-definable in $a_1, \ldots, a_n$, then $\bar{a}_1^p, \ldots, \bar{a}_n^p \in \bar{N}^*$ and $\bar{b}^p$ is $\bar{N}^*$-definable in $\bar{a}_1^p, \ldots, \bar{a}_n^p$ by the same definition.

(iii) Each $x \in \bar{N}^*$ is $\bar{N}^*$-definable from parameters in $M^p_{|p|} \cup D^p$

Moreover, If $\mathfrak{A}$ is a solid model of $\mathcal{L}(p)$, then $\dot{\pi}^{\mathfrak{A}}_{|p|, \omega_1} \cup F^p$ extends uniquely to a $\pi : \bar{N}^* \prec N^*$ s.t. $\pi(\bar{a}) = a$ whenever $\langle a, \bar{a} \rangle \in F^p$.

(Note that $M^p_{|p|} = \bar{M}^p \in \bar{N}^*$ by (b))

proof.

We first show the existence of $\bar{N}^*$ satisfying (i)–(iii). Exactly as in the proof of §2 Lemma 3 we get:

<u>Fact</u> For any $X \subset M$ the following are equivalent:

(a) $X \prec \langle M, a\rangle$ for all $a \in \mathbb{R}^p$

(b) Let $Y$ = the smallest $Y \prec N^*$ s.t. $X \cup \mathbb{R}^p \subset Y$. Then $Y \cap M = X$

((b) → (a) is trivial. (a) → (b) follows from the fact that each $z \in Y$ is $N^*$-definable from parameters in $X \cup \mathbb{R}^p$.)

Now collapse $\beta^{++}$ to $\omega$ and work in the resulting model $V[G]$. Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$. Then $\mathbb{R}^p \in N \subset \mathfrak{A}$, since $\mathbb{R}^p \subset N$ is countable. But $\dot{\pi}^{\mathfrak{A}}_{|p|, \omega_1} : \langle M^p_{|p|}, \bar{a}^p\rangle \prec \langle M, a\rangle$ for all $a \in \mathbb{R}^p$, or in other words, $X \prec \langle M, a\rangle$ for all $a \in \mathbb{R}^p$, where $X = \mathrm{rng}(\dot{\pi}^{\mathfrak{A}}_{|p|, \omega_1})$. Let $Y$ be the smallest $Y \prec N^*$ s.t. $X \cup \mathbb{R}^p \subset Y$. Let $\pi : \bar{N}^* \overset{\sim}{\longleftrightarrow} N^* | Y$ be the transitivization of $N^* | Y$.

<u>Claim 1</u> $\bar{N}^*$ satisfies (i) – (vii)

proof.

Since $X = Y \cap M$, we clearly have $\pi \upharpoonright M^p_{|p|} = \dot{\pi}^{\mathfrak{A}}_{|p|, \omega_1}$ and

$\pi^{-1}(a) = \pi^{-1}{}''(X \cap a) = \pi^{A\Omega}_{|p|,\omega_1}{}^{-1}{}''(X \cap a) =$
$= \bar{a}^p$ for $a \in \mathbb{R}^p$.
(ii), (iii) follow easily. QED(Claim 1)
But $\bar{N}^*$ was constructed in $V[G]$,
so we must show:

<u>Claim 2</u> $\bar{N}^* \in V$

proof:
Let $\tilde{Y}$ = the smallest $\tilde{Y} \prec N^*$ s.t. $M \cup \mathbb{R}^p \subset \tilde{Y}$. Then $\tilde{Y}$ has cardinality $\beta$ in $V$. Hence, letting $\tilde{\pi} : \tilde{N}^* \overset{\sim}{\longleftrightarrow} N^*|\tilde{Y}$ be the transitivization of $N^*|\tilde{Y}$, we have: $\tilde{N}^* \in N \subset \Omega$. Note that $\tilde{\pi} \upharpoonright M \cup \mathbb{R}^p = \mathrm{id}$. Hence, setting $\hat{Y}$ = the smallest $\hat{Y} \prec \tilde{N}^*$ s.t. $X \cup \mathbb{R}^p \subset \hat{Y}$, we have $\hat{Y} \in \Omega$. Hence $\hat{\pi}, \hat{N}^* \in \Omega$, where $\hat{\pi} : \hat{N}^* \overset{\sim}{\longleftrightarrow} \tilde{N}^*|\hat{Y}$ is the transitivization of $\tilde{N}^*|\hat{Y}$. But obviously $\hat{N}^* = \bar{N}^*$; hence $\bar{N}^* \in H^{\Omega}_{\omega_1} = H^{V}_{\omega_1}$. QED(Claim 2)

Note We have shown that if $\mathfrak{A}$ is a solid model of $\mathcal{L}(p)$, then there is an $\bar{N}^* \in V$ satisfying (i) – (iii) with the property that $\pi^{i\,\mathfrak{A}}_{|p|,\omega_1}$ extends uniquely to a $\pi : \bar{N}^* \prec N^*$.

The last part of the theorem is, therefore, established as soon as we show;

Claim 3 There is at most one $\bar{N}^*$ satisfying (i) – (iii)

prf.

Let $\bar{N}_0^*, \bar{N}_1^*$ satisfy (a) – (c). Then:

Let $x_1, \ldots, x_n \in M^p_{|p|}$, $b_1, \ldots, b_m \in D^p$.

(1) Then $\bar{N}_0^* \models \varphi(\vec{x}, \vec{b}) \longleftrightarrow \bar{N}_1^* \models \varphi(\vec{x}, \vec{b})$

for all $N^*$ – fmlae $\varphi$.

proof.

Let $b_i = \bar{a}_i^p$, where $a_i \in B^p$ $(i = 1, \ldots, m)$

Set: $c = \{\langle \vec{x} \rangle \in M \mid N^* \models \varphi(\vec{x}, \vec{a})\}$.

Then $\bar{c}^p = \{\langle \vec{x} \rangle \in M^p_{|p|} \mid \bar{N}_h^* \models \varphi(\vec{x}, \vec{b})\}$

for $h = 0, 1$. QED (1)

But it then follows easily that $\mathrm{id} \restriction (M^p_{|p|} \cup D^p)$ extends to a $\sigma : \bar{N}_0^* \overset{\sim}{\longleftrightarrow} \bar{N}_1^*$. Since $\bar{N}_0^*, \bar{N}_1^*$ are transitive, we conclude: $\bar{N}_0^* = \bar{N}_1^*$,

QED (Lemma 3)

As a corollary of the proof we get:

<u>Cor 3.1</u> Let $G$ be $\mathbb{P}$-generic over $V$. Let $p \in G$ conform to $N^*$. Then $\pi^G_{|p|,\omega_1} \cup F^p$ extends uniquely to a $\pi : \bar{N}^* \prec N^*$.

proof.

By a proof similar to (1) in Claim 3:

(1) Let $x_1, \ldots, x_n \in M^p_{|p|}$, $b_1, \ldots, b_n \in D^p$. Then $\bar{N}^* \models \varphi(\vec{x}, \vec{b}) \longleftrightarrow N^* \models \varphi(\pi^G_{|p|,\omega_1}(\vec{x}), F^p(\vec{b}))$.

The result follows easily, since each element of $\bar{N}^*$ is definable in parameters from $M^p_{|p|} \cup D^p$. QED (3.1)

<u>Note</u> If we assume that $\Vdash \mathcal{P}(\check{\omega}) \subset \check{V}$, we obtain the following fact, whose proof we shall bring later.

Set $\mathcal{L}^*$ = the set of $\varphi$ which are forced by $\mathbb{P}$ to hold in $N[\dot{G}]$, where $\dot{M}, \dot{\pi}, \dot{B}$ are interpreted by $M^{\dot{G}}, \pi^{\dot{G}}, B^{\dot{G}}$. Then

(a) $\mathbb{P} = \mathbb{P}_{\mathcal{L}^*}$

(b) $p \Vdash^N_{\mathbb{P}} \varphi \longleftrightarrow \mathcal{L}^*(p) \vdash \varphi$

If $G \ni p$ is $\mathbb{P}$-generic over $V$, then $\langle N[G], M^G, \pi^G, B^G \rangle$ is a solid model of $\mathcal{L}^*(p)$. Thus Cor 3.1 becomes a special case of Lemma 3.

The condition of revivability then says that certain conditions p can be revived, leaving $M^p_{|p|}$ and $F^p$ fixed, but changing $M^p \upharpoonright |p|$, $\pi^p \upharpoonright |p|^2$ and $b^p$.

Def $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ is revivable iff for all sufficiently large cardinals $\delta > 2^\beta$ we have:

(*) Let $p \leq r$ in $\mathbb{P}$, where $\mathbb{P}$ conforms to $N^* = \langle H_\delta, M, <, \mathbb{P}, r, m \rangle$. Let $\bar{N}^* = \bar{N}^*(p, N^*) = \langle \bar{H}, \bar{M}, \bar{\mathbb{P}}, \bar{r}, \bar{m} \rangle$ be as Lemma 3. (Hence $\bar{r} \in \bar{\mathbb{P}}$.) Then there is $G \ni \bar{r}$ s.t. $G$ is $\bar{\mathbb{P}}$-generic over $\bar{N}^*$ and, letting $M^G, \pi^G, B^G$ be defined over $\bar{N}^*[G]$ as in Lemma 2, we have: $q \in \mathbb{P}$, where $q = \langle \langle M^G, \pi^G, M^G \rangle, F^p \rangle$.

Note As defined here, revivability is a property of $\mathbb{P}$ rather than $\mathcal{L}$.

Note It follows easily that $q \leq r$.

Note In some applications we shall have a stronger form of revivability: For every $\bar{r} \in \bar{\mathbb{P}}$ there is $G \ni \bar{r}$ satisfying the above condition.

Def We say that $\mu$ witnesses the revivability of $\mathbb{P}$ iff (*) holds for all $\delta \geq \mu$.

We also define:

**Def** $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ is <u>weakly</u> <u>revisable</u> iff there exists $d$ s.t. for all sufficiently large cardinals $\delta > 2^{\beta}$ we have:

(**) Let $p \leq r$ in $\mathbb{P}$, where $\mathbb{P}$ conforms to $N^* = \langle H_\delta, M, <, d, \mathbb{P}, r, m \rangle$. Let $\bar{N}^* = \bar{N}^*(p, N^*) = \langle \bar{H}, \bar{M}, <, \bar{d}, \bar{\mathbb{P}}, \bar{r}, m \rangle$. Then there is $G \ni \bar{r}$ s.t. $G$ is $\bar{\mathbb{P}}$-generic over $\bar{N}^*$ and: $q \in \mathbb{P}$, where $q = \langle\langle M^G, \pi^G, M^G \rangle, FP \rangle$.

**Def** $\langle d, \mu \rangle$ <u>witnesses</u> the weak revisability of $\mathbb{P}$ iff (**) holds for all $\delta \geq \mu$.

<u>Lemma 3.2</u> $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ is revisable iff it is weakly revisable.

prf.

($\rightarrow$) is trivial. We prove ($\leftarrow$).

Say that $\mathbb{P}$ is $\langle d, \delta \rangle$-<u>revisable</u> iff (**) holds.

For $\eta \leq \infty$ set:

$\mu(\eta) \simeq$ the least $\mu$ s.t. there is a $d \in H_\mu$ with: $\mathbb{P}$ is $\langle d, \delta \rangle$ revisable for all cardinals $\delta \in [\mu, \eta)$.

$X_\eta \simeq$ the set of such $d \in H_{\mu(\eta)}$

Then $\mu(\eta) < \eta$ and $X_\eta$ are defined for sufficiently large cardinals $\eta$.

Moreover, if $\eta \leq \eta'$ and $\mu(\eta)$ is defined, then:

$\mu(\eta) \leq \mu(\eta') \leq \mu(\infty) \wedge X_\eta \supset X_{\eta'} \supset X_\infty$

Hence there are $\tilde{\mu}$, $\tilde{X}$ s.t. $\mu(\eta) = \tilde{\mu}$ and $X_\eta = \tilde{X}$ for sufficiently large $\eta < \infty$. It follows easily that $\tilde{\mu} = \mu(\infty)$, $\tilde{X} = X_\infty$. This means that if $\eta$ is a sufficiently large cardinal, then $\tilde{\mu}, \tilde{X} \in N^*$ are $N^*$-definable, where

$$N^* = \langle H_\eta, M, <, \mathbb{P}, \pi, \ldots \rangle.$$

Hence $d$ is $N^*$-definable, where $d$ is the $<$-least $d \in \tilde{X}$. Let $p \leq \pi$ be $N^*$-conforming. Let $\bar{N}^* = \bar{N}^*(p, N^*)$. Then $p$ is $\langle N^*, d \rangle$-conforming and $\langle \bar{N}^*, \bar{d} \rangle = \bar{N}^*(p, \langle N^*, d \rangle)$, where $\bar{d}$ is defined in $\bar{N}^*$ as $d$ was defined in $N^*$. But $\langle d, \tilde{\mu} \rangle$ witness the weak revisability of $\mathbb{P}$, where $\tilde{\mu} < \eta$. The conclusion follows easily.

QED (Lemma 3.2)

<u>Lemma 4</u> Let $\mathbb{P}$ be revisable. Then
$\Vdash \mathcal{R}(\check{\omega}) \subset \check{V}$.

prf.
Let $\Vdash \mathring{f} : \check{\omega} \to 2$. It suffices to show that $\Delta = \{p \mid \bigvee f \; p \Vdash \mathring{f} = \check{f}\}$ is predense in $\mathbb{P}$ - i.e. for each $q \in \mathbb{P}$ there is $p \in \Delta$ s.t. $p \| q$.

Let $N^* = \langle H_\delta, N, <, q, \mathring{f}, \mathcal{R}, \mathbb{P} \rangle$ with $\delta$ sufficiently large. Let $r \leq q$ conform to $N^*$. Let $\bar{N}^* = \bar{N}^*(r, N^*) = \langle \bar{H}, \bar{N}, <, \bar{q}, \bar{f}, \bar{\mathcal{L}}, \bar{\mathbb{P}} \rangle$ be as in Lemma 3. Then $\bar{q} \in \bar{\mathbb{P}}$. By revisability there is $\bar{G} \ni \bar{q}$ which is $\bar{\mathbb{P}}$-generic over $\bar{N}^*$ s.t. $p = \langle\langle \dot{M}^{\bar{G}}, \dot{\pi}^{\bar{G}}, \dot{B}^{\bar{G}} \rangle, F^r \rangle \in \mathbb{P}$.

<u>Claim</u> $p \| q$ and $p \in \Delta$.

Note that $p$ conforms to $N^*$.
Note that $\mathfrak{A}$ is a solid model of $\mathcal{L}(p)$.
(1) Let $\mathfrak{A}$ be a solid model of ... Let $\pi : \bar{N}^* \prec N^*$ be the canonical extension of $\dot{\pi}^{\mathfrak{A}}_{|p|, \omega_1}$. Let $\bar{s} \in \bar{G}$, $s = \pi(\bar{s})$. Then $\mathfrak{A} \models \mathcal{L}(s)$ (hence $s \| p$)

proof.
$\mathfrak{A} \models \mathcal{L}(s)$, since $M^s = M^{\bar{s}} = \dot{M}^{\mathfrak{A}} \restriction (|s|+1)$, $\pi^s = \pi^{\bar{s}} = \dot{\pi}^{\mathfrak{A}} \restriction (|s|+1)^2$, $b^s = b^{\bar{s}} = \dot{\pi}^{\mathfrak{A}-1}_{|s|, |r|} {}'' b^r = \dot{\pi}^{\mathfrak{A}-1}_{|s|, \omega_1} {}'' \dot{B}^{\mathfrak{A}}$, and

and $\dot{\pi}_{|s|,\omega_1} : \langle M^s_{|s|}, \bar{a}^s \rangle \prec \langle M, a \rangle$ for $a \in R^s$, since $\bar{a}^s = \bar{a}^{\bar{r}}$ and $a = \pi(a^{\bar{r}})$, where $\pi^p_{|s|,|p|} : \langle M^s_{|s|}, \bar{a}^s \rangle \prec \langle M^{\bar{r}}_{|\bar{r}|}, a^{\bar{r}} \rangle$ and $\dot{\pi}^{\mathfrak{A}}_{|p|,\omega_1} : \langle M^{\bar{r}}_{|\bar{r}|}, a^{\bar{r}} \rangle \prec \langle M, a \rangle$.

Thus $\mathfrak{A} \models \mathcal{L}(s) \cup \mathcal{L}(p)$. QED (1)

(2) $q \| p$, since $q = \pi(\bar{q})$.

Now set:

$f(i) =$ that $h$ s.t. $\forall s \in \bar{G}\ s \Vdash \bar{f}(\check{i}) = \check{h}$.

(3) $p \Vdash \dot{f} = \check{f}$

prf. Suppose not.

Then there is $p' \leq p$ s.t. $p' \Vdash \dot{f}(\check{i}) \neq \check{f(i)}$ for some $i$. Let $\bar{q}' \in \bar{G}$ s.t. $\bar{q}' \Vdash_{\bar{\mathbb{P}}} \bar{f}(\check{i}) = \check{f(i)}$. Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p')$. Then $\mathfrak{A}$ also models $\mathcal{L}(p)$ + we let $\pi : \bar{N}^* \prec N^*$ again be the canonical extension of $\dot{\pi}^{\mathfrak{A}}_{|p|,\omega_1}$. Then $q' = \pi(\bar{q}') \in \mathbb{P}$ and $q' \Vdash \dot{f}(\check{i}) = \check{f(i)}$. Hence $q' \perp p'$. But $q' \| p'$, since $\mathfrak{A} \models \mathcal{L}(q') \cup \mathcal{L}(p')$ by (1). Contr!

QED (Lemma 4)

The proof of (1) in Lemma 4 can be used to show:

<u>Cor 4.1</u> Let $p$ be as in Lemma 4 - i.e. $p \in \mathbb{P}$ is a revision of $r$, where $r$ is $N^*$-conforming and $N^* = \langle H_\delta, N, d, <, \mathbb{P}, r, m \rangle$. (Hence $p = \langle\langle M^{\bar{G}}, \pi^{\bar{G}}, B^{\bar{G}} \rangle, F^r \rangle$, where $\bar{G}$ is $\bar{\mathbb{P}}$-generic over $\bar{N}^* = \bar{N}^*(r, N^*)$.)
Let $G \ni p$ be $\mathbb{P}$-generic over $V$.
$\pi : \bar{N}^* < N^*$ be the canonical extension of $\pi^G_{|p|, \omega_1} \cup F^p$ given by Cor 3.1.

Then $\pi''\bar{G} \subset G$.

proof.
Recall that $\mathcal{L}(p) = \mathcal{L} + \varphi_p$. Repeating the proof of (1) in Lemma 4 with $V[G]$ in place of $\mathcal{U}$ gives us:
$V[G] \models \varphi_q$ for $\bar{q} \in \bar{G}$, $q = \pi(\bar{q})$.
Thus it suffices to show:
$G = \{ r \in \mathbb{P} \mid V[G] \models \varphi_r \}$.
$(\subset)$ is trivial. We prove $(\supset)$.
Let $V[G] \models \varphi_r$. Then
$r_0 = \langle M^G \restriction (|r|+1), \pi^G \restriction (|r|+1)^2, \pi^{G-1}{}''B^G \rangle$.
By the extension lemma,
$\{ p \mid R^r \subset R^p \}$ is dense in $\mathbb{P}$.
Hence there is $p \in G$ s.t. $R^r \subset R^p$.

We may assume w.l.o.g. that $|p| \geq |r|$. Since $\pi^{G}_{|r|,\omega_1} : \langle M^{r}_{|r|}, \bar{a}^{r} \rangle \prec \langle M, a \rangle$ and $\pi^{G}_{|p|,\omega_1} : \langle M^{p}_{|p|}, \bar{a}^{p} \rangle \prec \langle M, a \rangle$, and $\dot{\pi}^{G}_{|r|,|p|} = \pi^{p}_{|r|,|p|}$, it follows that $\pi^{p}_{|r|,|p|} : \langle M^{r}_{|r|}, \bar{a}^{r} \rangle \prec \langle M^{p}_{|p|}, \bar{a}^{p} \rangle$ for $a \in R^{r}$. Hence $r \leq p$. Hence $r \in G$. QED (4.1)

Since $\pi : \bar{N}^{*} \prec N^{*}$, we of course have:

$$q \Vdash^{\bar{N}^{*}}_{\bar{\mathbb{P}}} \varphi(\vec{x}) \longleftrightarrow \pi(q) \Vdash^{N^{*}}_{\mathbb{P}} \varphi(\pi(\vec{x}))$$

for $x_1, \ldots, x_n \in \bar{N}^{*}$, $q \in \bar{\mathbb{P}}$. Thus:

<u>Cor 4.2</u> Let $p, G$ be as above. Then $\pi^{G}_{|p|,\omega_1} \cup F^{p}$ extends uniquely to a $\pi^{*} : \bar{N}^{*}[\bar{G}] \prec N^{*}[G]$.