§4 More on revisability

We now develop the general theory of revisable forcings further. In particular, we shall prove the fact mentioned after the proof of §3 Lemma 3.1 and examine its consequences. The reader who is more interested in applications can skip directly to the next chapter.

We recall that for any set of conditions $\mathbb{P}$ the canonical complete Boolean algebra $BA(\mathbb{P})$ is defined as the set of $X \subseteq \mathbb{P}$ s.t. $X = \neg\neg X$, where $\neg X =_{Df}$

$$=_{Df} \{p \mid \wedge q \in X \; p \perp q\} \text{ for } X \subseteq \mathbb{P}.$$

(Thus $\neg X = \{p \mid \wedge p' \leq p \; p' \notin X\}$ if X is closed under $\leq$. In particular $\neg X$ is closed under $\leq$, which easily gives:

$$\neg\neg X = \{p \mid \wedge p' \leq p \; \vee p'' \leq p' \; \vee q \in X \; p'' \leq q\} =$$
$$= \{p \mid \wedge p' \leq p \; \vee q \in X \; p' \| q\}. \; \mathbb{B} = BA(\mathbb{P}) \text{ is}$$

a complete BA with the complement operation $\neg$ and the operations:

$$\bigcap\nolimits^{\mathbb{B}} A = \bigwedge A, \quad \bigcup\nolimits^{\mathbb{B}} A = \neg\neg \bigcup A.)$$

For $q \in \mathbb{P}$ we set: $[q] = [q]^{\mathbb{B}} = \neg\neg\{q\}$.

(Hence $[q] = $ the smallest $b \in \mathbb{B}$ s.t.

$$\ni b = \{p \mid \wedge p' \leq p \; p' \| q\})$$

Lemma 1 Let $\mathbb{B} = BA(\mathbb{P})$ where $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ where $\mathcal{L}$ satisfies the basic axioms. Then $[p] \subset [q] \iff \mathcal{L}(p) \vdash \mathcal{L}(q)$.

prf.

$(\Longleftarrow)$ $\mathcal{L}(p) \vdash \mathcal{L}(q) \rightarrow \bigwedge_{p' \leq p} \mathcal{L}(p) \vdash \mathcal{L}(q)$

$\rightarrow \bigwedge_{p' \leq p} con(\mathcal{L}(p') \cup \mathcal{L}(q)) \iff$

$\iff \bigwedge_{p' \leq p} p' \| q \iff [p] < [q]$

$(\rightarrow)$ Let $\mathcal{L}(p) \not\vdash \mathcal{L}(q)$. We find $p' \leq p$ s.t. $\mathcal{L}(p') \cup \mathcal{L}(q)$ is inconsistent (hence $p' \perp q$). Let $\mathfrak{N}$ be a model s.t. $\mathfrak{N} \vDash \mathcal{L}(p)$, $\mathfrak{N} \not\vDash \mathcal{L}(q)$. Pick $\alpha \geq |p|, |q|$ s.t. for all $a \in R^p \cup R^q$ there is $a^* = \pi_{\alpha, \omega_1}^{\mathfrak{N}}{}^{-1} {}^{a} a$ with $\pi_{\alpha, \omega_1}^{\mathfrak{N}} : \langle \dot{M}_\alpha^{\mathfrak{N}}, a^* \rangle < \langle M, a \rangle$.

Define $p' \leq p$ by $: M^{p'} = \langle \dot{M}_i^{\mathfrak{N}} \mid i \leq \alpha \rangle$, $\pi^{p'} = \langle \dot{\pi}_{i,j}^{\mathfrak{N}} \mid i \leq j \leq \alpha \rangle$, $B^{p'} = \pi_{\alpha, \omega_1}^{\mathfrak{N}}{}^{-1} {}^{a} B^{\mathfrak{N}}$, and $F^{p'} = \{ \langle a, a^* \rangle \mid a \in R^p \cup R^q \}$.

Clearly $p' \in \mathbb{P}$, since $\mathfrak{N} \vDash \mathcal{L}(p')$. But then $p' \leq q$. We must show:

<u>Claim</u> $p' \perp q$

<u>Case 1</u> $M^q \neq \langle \dot{M}_i^{\mathfrak{N}} \mid i \leq |q| \rangle$ or $\pi^q \neq \langle \dot{\pi}_{i,j}^{\mathfrak{N}} \mid i \leq j \leq |q| \rangle$ or $b^q \neq b_{|q|}^{\mathfrak{N}} =_{df} \pi_{|q|, \omega_1}^{\mathfrak{N}}{}^{-1} {}^{a} B^{\mathfrak{N}}$.

This immediately gives: $\neg con(\mathcal{L}(q) \cup \mathcal{L}(p'))$.

__Case 2__ Case 1 fails.

Then there is $\langle a, \bar{a} \rangle \in F^q$ s.t.

$$\pi^{i \; \cup \eta}_{|q|, \omega_1} : \langle M^q_{|q|}, \bar{a} \rangle \not\prec \langle M, a \rangle,$$

since $\cup \eta \not\subseteq \mathcal{L}(q)$. Letting a^* be as above, we then have: $\langle a^*, a \rangle \in F^{p'}$ and

$$\pi^{p'}_{|q|, |p'|} : \langle M^q_{|q|}, \bar{a} \rangle \not\prec \langle M^{p'}_{|p'|}, a^* \rangle,$$

Hence $\neg con(\mathcal{L}(q) \cup \mathcal{L}(p'))$. QED (Lemma 1.1

Now let G be $\mathbb{P}$-generic over V, where $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$, $\mathbb{B} = BA(\mathbb{P})$. Define M^G, π^G, B^G as in Lemma 2.1. It is easily seen that M^G, π^G, B^G are, in fact, the G-interpretations of fixed canonical $\mathbb{P}$-names $M, \pi, B \in N^{*\mathbb{P}}$, where $N^* = \langle H_\delta, N, \langle, m \rangle$ for a regular $\delta > \beta^+$. This enables us to interpret the sentences of the language $\mathcal{L}$

in $N^* \mathbb{P}$ by interpreting $\check{M}, \check{\pi}, \mathring{B}$ as M, π, B
and $\underline{x}$ as $\check{x}$ for $x \in N$. (δ can be
any regular $\delta > \beta^+$.) Let $[\![\varphi]\!] \in B$
be their interpretation of φ. As usual
we write: $\Vdash \varphi \iff_{Df} [\![\varphi]\!] = 1$. We can
then define a new language $\mathcal{L}^*$
with the same formulae, but
taking $\{\varphi \mid \Vdash \varphi\}$ as the new set of
axioms. We can, of course, not
expect all theorems of $\mathcal{L}$ to become
theorems of $\mathcal{L}^*$, but many will,
and in fact we shall have:
$\mathbb{P} = \mathbb{P}_{\mathcal{L}^*}$. ($\mathcal{L}^* \subseteq N$ is, of course, not
necessarily N-definable, but it is
N'-definable, where $N' = \langle N, \mathcal{L}^* \rangle$ is
a ZFC^- model.) It is straight-
forward to see that:

Lemma 2.1 Assume that $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ adds
no new reals. Then all $\mathcal{L}$
basic axioms are in $\mathcal{L}^*$.

We recall that $p \in \mathbb{P} \iff con(\mathcal{L}(p))$,
where $\mathcal{L}(p) = \mathcal{L} + \varphi_p$. In order to
show: $\mathbb{P} = \mathbb{P}_{\mathcal{L}^*}$ we therefore need:

$con(\mathcal{L} + \varphi_p) <\longrightarrow con(\mathcal{L}^* + \varphi_p)$ for $p \in \widetilde{\mathbb{P}}$,

We first note:

Lemma 2.2 $con(\mathcal{L}^* + \varphi) <\longrightarrow [\![\varphi]\!] \neq \emptyset$

for all $\mathcal{L}$ - sentences φ,

Pf.

$con(\mathcal{L}^* + \varphi) <\longrightarrow \mathcal{L}^* \not\vdash \neg\varphi <\longrightarrow [\![\neg\varphi]\!] \neq 1 <\longrightarrow$

$<\longrightarrow [\![\varphi]\!] \neq \emptyset$,   QED (2.2)

Note At G is $\mathbb{P}$-generic over N^* we can
interpret $\mathcal{L}$-sentences in $N^*[G]$ by
letting $\underline{x}$ stand for x $(x \in N)$ and
$\dot{M}, \dot{\pi}, \dot{B}$ stand for M^G, π^G, B^G. Using this
interpretation, Lemma 2.2 tells us
that the following are equivalent:

- $\mathcal{L}^* + \varphi$ is consistent
- $\mathcal{L}^* + \varphi$ has a sound model in any collapse
  of 2^β to ω
- $\mathcal{L}^* + \varphi$ has a model of the form $N^*[G]$, G
  being $\mathbb{P}$-generic over N^*, in any collapse
  of 2^δ to ω
- $[\![\varphi]\!] \neq \emptyset$,

We now prove:

Lemma 2.3 Let $p \in \widetilde{\mathbb{P}}$, Then
$$p \in \mathbb{P} \Longleftrightarrow \mathcal{L}^*(p) \text{ is consistent,}$$

Pf
($\rightarrow$) Let $p \in \mathbb{P}$ & let $G \ni p$ be $\mathbb{P}$-generic
over N^*. Then $N^*[G]$ models $\mathcal{L}^* + \varphi_p$
by §3 Lemma 2.   QED ($\rightarrow$)

($\leftarrow$) Suppose not. Then $[\![\varphi_p]\!] \neq \emptyset$. Let $q \in [\![\varphi_p]\!]$ (hence $q \in \mathbb{P}$). We may also assume w.l.o.g. that $|q| \geq |p|$. Let $G \ni q$ be $\mathbb{P}$-generic. Then $N^*[G] \models \varphi_p$. Hence $M_i^P = M_i^q = M_i^G$, $\pi_{ij}^P = \pi_{ij}^q = \pi_{ij}^G$ for $i \leq j \leq |p|$ and $b^P = (\pi_{|q|,|p|}^q)^{-1} \text{``} b^q$. It follows that whenever $\mathcal{O}$ is a solid model of $\mathcal{L}(q)$, there is $\langle a, \bar{a} \rangle \in F^P$ s.t.

$\pi_{|p|, \omega_1}^{\mathcal{O}} : \langle M_{|p|}^P, \bar{a} \rangle \not\prec \langle M, a \rangle$, since otherwise we would have $\mathcal{O} \models \mathcal{L}(p)$ & hence $p \in \mathbb{P}$.

Set : $\Delta = _{Df}$ the set of $r \leq q$ s.t. for some $\langle \bar{a}, a \rangle \in F^P$, some $z \in \text{dom}(\pi_{|p|}^r)$ and some ψ we have:

$$\langle M_{|p|}^P, \bar{a} \rangle \models \psi(z) \not\leftrightarrow \langle M, a \rangle \models \psi(\pi_{|p|}^r(z)),$$

(where, as before, $\pi_i^r = F^r \circ \pi_{i, |r|}^r$).

<u>Claim</u> Δ is dense in $\{q' \mid q' \leq q\}$.

prf. Let $q' \leq q$. Let $\mathcal{O}$ be a solid model of $\mathcal{L}(q')$. Let $\langle a, \bar{a} \rangle \in F^P$, $z \in M_{|p|}^P = \dot{M}_{|p|}^{\mathcal{O}}$ s.t.

(1) $\langle M_{|p|}^P, \bar{a} \rangle \models \psi(z) \leftrightarrow \langle M, a \rangle \models \psi(\pi_{|p|, \omega_1}^{\mathcal{O}}(z))$.

Pick $\alpha > |q'|$ s.t. for all $a \in R = R^q \cup R^P \cup \{\pi_{|p|, \omega_1}^{\mathcal{O}}(z)\}$ there is a (necessarily unique) a^* s.t.

(2) $\ddot{\pi}^{\sigma}_{\alpha,\omega_1} : \langle \dot{M}^{\sigma}_{\alpha}, a^* \rangle \prec \langle M, a \rangle$.

(Hence, letting $\langle a_i \mid i < \omega \rangle$ enumerate $R^{\sigma} \cup R^p$ in V, we have $\langle a_i^* \mid i < \omega \rangle \in V$.)

Define $\tilde{\pi} \in \tilde{\mathbb{P}}$ by:

$M^{\pi} = \langle \dot{M}^{\sigma}_i \mid i \leq \alpha \rangle$, $\pi^{\pi} = \langle \dot{\pi}^{\sigma}_{i,j} \mid i \leq j \leq \alpha \rangle$,

$b^{\pi} = (\dot{\pi}^{\sigma}_{\alpha,\omega_1})^{-1} \, {}^{"} \dot{B}^{\sigma}$, $F^{\pi} = \{\langle a, a^* \rangle \mid a \in R\}$,

Then clearly, $\sigma \models \varphi_{\pi}$. Hence $\pi \in \mathbb{P}$. Hence $\pi \leq q'$.

<u>Claim</u> $\pi \in \Delta$.

Let $x = \dot{\pi}^{\sigma}_{|p|,\omega_1}(z)$, $x^* = (\dot{\pi}^{\sigma}_{|\pi|,\omega_1})^{-1}(x)$.

Then $x^* = \dot{\pi}^{\sigma}_{|p|,|\pi|}(z) = \pi^{\pi}_{|p|,|\pi|}(z)$. Hence

$x = F^{\pi} \pi^{\pi}_{|p|,|\pi|}(z) = \pi^{\pi}_{|p|}(z)$. Then

$\langle M^p_{|p|}, \bar{a} \rangle \models \varphi(z) \iff \langle M, a \rangle \models \varphi(\pi^{\pi}_{|p|}(z))$

QED (Claim)

Now let $G \ni q$ be $\mathbb{P}$-generic. Since $G \cap \Delta \neq \emptyset$ there is $\langle a, \bar{a} \rangle \in F^p$ s.t.

$\langle M^p_{|p|}, \bar{a} \rangle \models \varphi(z) \iff \langle M, a \rangle \models \varphi(\pi^G_{|p|,\omega_1}(z))$

for some $z \in M^p_{|p|}$, since $\pi^G_{|p|,\omega_1} \supseteq \pi^{\pi}_{|p|}$,

where $\pi \in G \cap \Delta$. Hence $N^*[G] \models \neg \varphi_p$.

But $N^*[G] \models \varphi_p$, since $q \in [\![\varphi_p]\!]$.

Contr! QED (Lemma 2.3)

<u>Note</u> Given the set of conditions $\mathbb{P}$, the definition of the partial order $\leq_{\mathbb{P}}$ was purely structural, so we have indeed shown that $\mathscr{L}$ and $\mathscr{L}^*$ yield the same partially ordered set $\mathbb{P} = \mathbb{P}_{\mathscr{L}} = \mathbb{P}_{\mathscr{L}^*}$.

We can then repeat our earlier proofs to get:

<u>Cor 2.4</u> $p \parallel q \longleftrightarrow \mathscr{L}^*(p) \cup \mathscr{L}^*(q)$ is consistent

<u>Cor 2.5</u> $[p] \subset [q] \longleftrightarrow \mathscr{L}(p^*) \vdash \mathscr{L}^*(q)$
$$\longleftrightarrow \mathscr{L}^* \vdash (\varphi_p \to \varphi_q).$$

But then:

<u>Cor 2.6</u> $[p] = \llbracket \varphi_p \rrbracket$ for $p \in \mathbb{P}$

proof.
($\subset$) since $p \Vdash \varphi_p$
($\supset$) If not, there is $q \in \llbracket \varphi_p \rrbracket$ s.t
$q \perp p$. Hence $\mathscr{L}^*(p) \cup \mathscr{L}^*(q)$ is inconsistent. But $q \Vdash \varphi_p \wedge \varphi_q$,
Hence $\mathscr{L}^*(p) \cup \mathscr{L}^*(q)$ is consistent
Contr! QED (2.6)

<u>Cor 2.7</u> $p \Vdash \psi \longleftrightarrow \mathscr{L}^*(p) \vdash \psi$
pf. $\mathscr{L}^*(p) \vdash \psi \longleftrightarrow \mathscr{L}^* \vdash (\varphi_p \to \psi) \longleftrightarrow$
$\longleftrightarrow \llbracket \varphi_p \rrbracket \subset \llbracket \psi \rrbracket$ (by 2.2) $\longleftrightarrow$
$\longleftrightarrow [p] \subset \llbracket \psi \rrbracket \longleftrightarrow p \Vdash \psi$.
QED (2.7)

__Cor 2.8__ Let G be $\mathbb{P}$-generic over V. Then G is definable from $\langle M^G, \pi^G, B^G \rangle$ since for all $p \in \mathbb{P}$ we have:

$$p \in G \iff (M^p = M^G \restriction (1+|p|) \wedge \pi^p = \pi^G \restriction (1+|p|)^2 \wedge$$
$$\wedge\; b^p = (\pi^G_{|p|,\omega_1})^{-1} \text{``} B^G \wedge$$
$$\wedge \bigwedge_{\langle a, \bar{a}\rangle \in F^p} \pi^G_{|p|,\omega_1} : \langle M^p_{|p|}, \bar{a}\rangle \prec \langle M, a\rangle)$$

pbf.
$$p \in G \iff [p] \cap G \neq \emptyset \iff [\![\varphi_p]\!] \cap G \neq \emptyset \iff$$
$$\iff N^*[G] \models \varphi_p. \qquad\qquad \text{QED } (2.8)$$

This theorem is frequently useful.

<u>Note</u> All of the previous theorems in this section hold if $\mathscr{L}$ satisfies the basic axioms and $\mathbb{P} = \mathbb{P}_{\mathscr{L}}$ adds no new reals. (Lemmas 1 and 2.2 – 2.6 in fact require only that $\mathscr{L}$ satisfy the basic axioms.)

<u>Lemma 3</u> Let $\mathbb{P} = \mathbb{P}_{\mathscr{L}}$ where $\mathscr{L}$ satisfies the basic axioms and $\mathbb{P}$ adds no new reals. Let G be $\mathbb{P}$-generic and let $cf(\beta) = \omega_1$ in $V[G]$. Then β^+ is a cardinal in $V[G]$

proof.

We can assume w.l.o.g. that $\Vdash cf(\check{\beta}) = \check{\omega_1}$. (Otherwise replace $\mathbb{P}, \mathscr{L}$ by $\mathbb{P}' = \mathbb{P}_{\mathscr{L}'}$, where $\mathscr{L}' = \mathscr{L}(p)$ and $p \in G$ with $p \Vdash cf(\check{\beta}) = \omega_1$. Then G is $\mathbb{P}'$-generic and $\Vdash_{\mathbb{P}'} cf(\check{\beta}) = \check{\omega_1}$.)

Thus it suffices to show:

<u>Lemma 3.1</u> Let $\mathbb{P}$ be as above where $\Vdash cf(\check{\beta}) = \check{\omega_1}$. Let $\mathbb{B} = \mathbb{B}A(\mathbb{P})$. Then $\mathbb{B} \setminus \{\emptyset\}$ has a dense subset of size β.

proof.

It suffices to show:

<u>Claim</u> Let $p \in \mathbb{P}$. There is an $\mathscr{L}$-statement $\psi \in M$ s.t. $\emptyset \neq [\![\psi]\!] \subseteq [p]$

It suffices to prove this for a dense set.
of $p \in \mathbb{P}$, so assume w.l.o.g. that p
conforms to $N^* = \langle H_{\beta^{++}}, N, < \rangle$. Let
G be $\mathbb{P}$-generic s.t. $p \in G$. Let $\alpha = |p|$.
Then $M^P = \langle M_i^G \mid i \leq \alpha \rangle$, $\pi^P = \langle \pi_{ij}^G \mid i \leq j \leq \alpha \rangle$,
and $b^P = (\pi_{\alpha, \omega_1}^G)^{-1} {}``B^G$. Let $\tilde{\beta} = \sup \pi_{\alpha, \omega_1}^G {}``B_\alpha^P$.
Then $\tilde{\beta} < \beta$. Set $\tilde{M} = J_{\tilde{\beta}}^A$ (where
$M = J_\beta^A$). For $a \in R^P$ set $\tilde{a} = a \cap \tilde{M}$.
Then $\pi_{\alpha \omega_1}^G : \langle \overline{M}, \overline{a} \rangle. \rightarrow \langle \tilde{M}, \tilde{a} \rangle$ is cofinal
and E_0-preserving. But then

$$(1) \quad \tilde{a} = \bigcup_{z \in M_\alpha^P} \pi_{\alpha \omega_1}^G (z \cap \overline{a}).$$

Let $\langle a_i \mid i < \omega \rangle$ enumerate R^P (in V).
Then $\langle \overline{a}_i \mid i < \omega \rangle \in H_{\omega_1}$. Moreover
$\langle \tilde{a}_i \mid i < \omega \rangle \in M$, since $\tilde{a}_i \in M = J_\beta^A$
for $i < \omega$ and $\operatorname{cf}(\beta)^V \geq \operatorname{cf}(\beta)^{V[G]} = \omega_1$.
Let ψ be the statement:
$$\dot{M} | \underline{\alpha + 1} = \underline{M^P} \wedge \dot{\pi} \restriction (\underline{\alpha + 1})^2 = \underline{\pi}^P \wedge$$
$$\wedge \underline{\tilde{\beta}} = \sup \dot{\pi}_{\underline{\alpha} \underline{\omega_1}} {}``\underline{\beta}^P \wedge$$
$$\wedge \bigwedge_{i < \omega} \underline{\tilde{a}_i} = \bigcup_{z \in M_\alpha^P} \dot{\pi}_{\underline{\alpha} \underline{\omega_1}} (z \cap \underline{\overline{a}_i}).$$
Then $\psi \in M$. But

(2) $N^*[G] \models \Psi$. Hence $[\![\Psi]\!] \neq \emptyset$.

Hence it suffices to show:

(3) $[\![\Psi]\!] \subset [\![\rho]\!]$.

Since $[\![\rho]\!] = [\![\varphi_\rho]\!]$, this is equivalent to

(4) $\Vdash_{I\!P}^{N^*} (\Psi \to \varphi_\rho)$

Let G be $I\!P$-generic over N^*. We must show:
$N^*[G] \models (\Psi \to \varphi_\rho)$. Let $N^*[G] \models \Psi$.
It suffices to show that:

<u>Claim</u> $\bar{\pi}_{\alpha,\omega}^G : \langle M_\alpha^\rho, \bar{a} \rangle \prec \langle M, a \rangle$ for $\langle a, \bar{a} \rangle \in F^\rho$.

Let $b = \{ \vec{z} \in M \mid \langle M, a \rangle \models \chi(\vec{z}) \}$. Then
$b \in R^\rho$ by the N^*-conformity of ρ.

Let $\langle b, \bar{b} \rangle \in F^\rho$. Then
$$\bar{b} = \{ \vec{z} \in M_\alpha^\rho \mid \langle M_\alpha^\rho, \bar{a} \rangle \models \chi(\vec{z}) \}$$
by the N^*-conformity of ρ. Hence:
$$\langle M_\alpha^\rho, \bar{a} \rangle \models \chi(\vec{z}) \quad \longleftrightarrow \quad \vec{z} \in \bar{b}$$
$$\longleftrightarrow \quad \pi_{\alpha \omega_\wedge}^G (\vec{z}) \in \tilde{b} = b \cap \tilde{M}$$
$$\longleftrightarrow \quad \langle M, a \rangle \models \chi(\pi_{\alpha \omega_\wedge}^G (\vec{z}))$$

since $\tilde{b} = \bigcup_{u \in M_\alpha^\rho} \pi_{\alpha, \omega_\wedge}^G (u \cap \bar{b})$

$\qquad\qquad\qquad$ QED (Lemma 3.1)

If, on the other hand, $cf(\beta) = \omega$ in $V[G]$, then 2^β is not a cardinal in $V[G]$. This follows from:

__Lemma 4.1__ Let β be a cardinal in an inner model W s.t. $2^\beta = \beta$ in W. Let $\gamma = 2^\beta$ in W. Suppose that in V we have:

$$\overline{\overline{\beta}} = \omega_1, \quad cf(\beta) = \omega, \quad 2^\omega = \omega_1.$$

Then $\overline{\overline{\gamma}} = \omega_1$ in V.

pf.

Let $\pi_{i,j} : M_i \prec M_j$ $(i \leq j \leq \omega_1)$ be a commutative continuous sequence of embeddings s.t. M_i is countable and transitive for $i < \omega_1$ and $M_{\omega_1} = H^W_\beta$. (This clearly exists in V by our assumptions.) For each $a \in P(\beta) \cap W$ there is a least i s.t.

$$\pi_{i,\omega_1} : \langle M_i, \overline{a} \rangle \prec \langle M, a \rangle \text{ for some } \overline{a}.$$

But then $\overline{a} \in H_{\omega_1}$ and $\langle M_i, \overline{a} \rangle$ is amenable. Thus $a = \bigcup_{x \in M_i} \pi_{i,\omega_1}(x \cap \overline{a}) = F(\langle i, \overline{a} \rangle)$, where F is a partial map on $\omega_1 \times H_{\omega_1}$. Hence $P(\beta) \cap W \subset F''(\omega_1 \times H_{\omega_1})$ and $\overline{\overline{\gamma}} = \overline{\overline{P(\beta) \cap W}} \leq \omega_1$. QED (4.1)

Cor 4.2   Let $\mathbb{P}$ be a set of conditions.

Let G be $\mathbb{P}$-generic and ~~let~~

$\quad 2^\omega = \omega_1 , \; \overline{\overline{\beta}} = \omega_1 , \; cf(\beta) = \omega \quad$ in $V[G]$,

where ω_1 is absolute in $V[G]$ and

$\quad 2^\omega = \omega_1 , \; 2^\beta = \beta \quad$ in V. Then $\overline{\overline{\mathbb{P}}} \geq 2^\beta$.

Moreover $cf(2^\beta) = \omega_1$ in $V[G]$ if $\overline{\overline{\mathbb{P}}} = 2^\beta$.

prf.

$card(2^\beta) = \omega_1$ in $V[G]$ by 4.1. But if

$\overline{\overline{\mathbb{P}}} < 2^\beta$, then 2^β would be a cardinal

in $V[G]$. Contr! At $\overline{\overline{\mathbb{P}}} = 2^\beta$ then

$cf(2^\beta) \neq \omega$ in $V[G]$, since other-

wise $card(2^{2^\beta}) = \omega_1$ in $V[G]$

by 4.1.   QED (Cor 4.2)

Def Q is reshapable iff for some β there is a revisable set of conditions $\mathbb{P} = \mathbb{P}_\beta$ s.t. $BA(Q) = BA(\mathbb{P})$.

It will turn out that many well known sets of forcing conditions are reshapable. The notion of "revisability" has something of the flavour of "properness", so it is not surprising that we get:

Lemma 5 Assume $2^\omega = \omega_1$ and $\beta = 2^\beta$, where $\beta > \omega_1$. Let Q be a proper set of conditions s.t. $\overline{\overline{Q}} \leq \beta$ and $\Vdash_Q (\overline{\overline{\beta}} = \omega_1 \wedge \mathcal{P}(\check{\omega}) = \mathcal{P}(\check{\omega})^\vee)$.

Then Q is reshapable.

Pf.

Assume w.l.o.g. that $Q \subseteq H_\beta$. Let $M_{\omega_1} = L_\beta^A = H_\beta$. Let $\Vdash_Q f : \check{\omega}_1 \xrightarrow{\text{onto}} \check{M}_{\omega_1}$. Let G be Q-generic. Set $f = f^G$ and $C = \{\alpha \leq \omega_1 \mid f''\alpha \prec M\}$. Let $\langle \alpha_i \mid i \leq \omega_1 \rangle$ be the monotone enumeration of C and set $\pi_{i,\omega_1} : M_i \xrightarrow{\sim} f''\alpha_i$, where M_i is transitive. Set $\pi_{ij} = \pi_{j,\omega_1}^{-1} \pi_{i,\omega_1}$ for $i \leq j \leq \omega_1$.

Set: $N = \langle H_{\beta^+}, M, <, f, \mathbb{Q}, ... \rangle$ in V.

Let $\mathcal{L}$ be our standard infinitary language on N (in particular with the constants $\underline{x}$ ($x \in N$), $\mathring{M}, \mathring{\pi}$). Set:
$M = \langle M_i \mid i \leq \omega_1 \rangle$, $\pi = \langle \pi_{ij} \mid i \leq j \leq \omega_1 \rangle$.

Let $\delta = \beta^{+V}$. (Hence $\delta = \omega_2^{V[G]}$.) Then $\langle H_\delta^{V[G]}, M, \pi, G \rangle$ models the basic axioms. Moreover, there are $\mathbb{Q}$-names $\ddot{M}, \ddot{\pi}, \ddot{G}$ which are N-definable s.t. $M = \ddot{M}^G, \pi = \ddot{\pi}^G, G = \ddot{G}^G$ whenever G is $\mathbb{Q}$-generic over N. Interpret $\mathcal{L}$ in $\ddot{N} = \langle N^{\mathbb{Q}}, \ddot{M}, \ddot{\pi}, \ddot{G} \rangle$, letting $\underline{x}$ be interpreted by $\check{x}$ for $x \in N$ and $\mathring{M}, \mathring{\pi}, \mathring{B}$ by $\ddot{M}, \ddot{\pi}, \ddot{G}$ respectively. This assigns to every $\mathcal{L}$-statement φ a truth value $[\![\varphi]\!] \in BA(\mathbb{Q})$. As usual, we write:

$\Vdash \varphi \longleftrightarrow_{nf} [\![\varphi]\!] = 1$, $p \Vdash \varphi \longleftrightarrow p \in [\![\varphi]\!]$

for $p \in \mathbb{Q}$ and $\mathcal{L}$-statements φ.

If G is $\mathbb{P}$-generic, set:
$$\ddot{N}[G] = \langle H_\delta^{V[G]}, \ddot{M}^G, \ddot{\pi}^G, G \rangle.$$

We write $\ddot{N}[G] \models \varphi$ if φ is true in $\ddot{N}[G]$ (i.e. interpreting $\dot{m}, \dot{\pi}, \dot{B}, \underline{x}$ by $\ddot{m}^G, \ddot{\pi}^G, \ddot{B}^G$ and $x = \check{x}^G$ resp.)

Add to $\mathcal{L}$ the axioms $\{\varphi \mid \Vdash \varphi\}$. Call the resulting theory $\mathcal{L}^*$. Then $\mathcal{L}^*$ is easily seen to contain the basic axioms. Set $\mathbb{P}^* = \mathbb{P}_{\mathcal{L}^*}$. It suffices to prove two propositions:

<u>Prop 1</u> $BA(\mathbb{Q}) = BA(\mathbb{P}^*)$

<u>Prop 2</u> $\mathbb{P}^*$ is revisable.

We start with Prop 1. Since $\mathbb{Q}$ is proper and adds no new reals, it in fact adds no new countable subsets of the ground model. Let $\mathbb{A} = BA(\mathbb{Q})$, $\mathbb{B} = BA(\mathbb{P}^*)$. By Lemma 1 we have

$$[p] \subset [q] \longleftrightarrow \mathcal{L}^* \Vdash (\varphi_p \to \varphi_q) \longleftrightarrow [\![\varphi_p]\!] \subset [\![\varphi_q]\!]$$

for $p, q \in \mathbb{P}^*$, where $[\![\varphi]\!]$ is the above mentioned interpretation in $\ddot{N}$. Hence it suffices to show:

<u>Claim</u> $\{[\![\varphi_p]\!] \mid p \in \mathbb{P}^*\}$ is dense in $\mathbb{A} \setminus \{\emptyset\}$,

since then the map $[p] \mapsto [\![\varphi_p]\!]$ extends to an $\sigma : \mathbb{A} \xrightarrow{\sim} \mathbb{B}$.

proof of <u>Claim</u> ;

Let $q \in Q$. We seek a $p \in \mathbb{P}^*$ s.t. $[\![\varphi_p]\!] \subset [q]$,

(where $[q] = [q]_{/A} = $ the smallest $a \in A$ with $q \in a$). Let $G \ni q$ be Q-generic. Set :

$\pi = \ddot{\pi}^G$, $M = \dot{M}^G$. (Of course $G = \ddot{B}^G$.)

Pick $\alpha < \omega_1$ s.t. $q \in rng (\pi_{\alpha \omega_1})$.

Note that $\pi_{\alpha \omega_1} \in V$, since Q adds no new countable sets. Define $p \in \mathbb{P}$ by: $M^p = M \restriction (\alpha + 1)$, $\pi^p = \pi \restriction (\alpha + 1)^2$,

$b^p = \pi_{\alpha \omega_1}^{-1} {}'' G$, $F^p = \pi_{\alpha, \omega_1}$. Clearly

$p \in \mathbb{P}^*$ since $\dot{N}[G] \models \varphi_p$.

<u>Claim</u> $[\![\varphi_p]\!] \subset [q]$.

It sufficies to show that for <u>any</u> Q-generic G we have:

$$G \cap [\![\varphi_p]\!] \neq \emptyset \rightarrow q \in G.$$

Let $G \cap [\![\varphi_p]\!] \neq \emptyset$. Hence $\dot{N}[G] \models \varphi_p$.

This means, however, that

$M^G \restriction (\alpha + 1) = M^p$, $\pi^G \restriction (\alpha + 1)^2 = \pi^p$,

$b^p = \pi_{\alpha, \omega_1}^G {}'' G$, where $\pi_{\alpha, \omega_1}^G = F^p = \pi_{\alpha, \omega_1}$.

But then $q = \pi_{\alpha, \omega_1}^G (\bar{q}) \in G$,

since $\bar{q} \in b^p$. QED (Prop 1)

We now prove Prop. 2

Let $N^* = \langle H_\mu, M, <, \mathring{f}, Q, \mathbb{P}^*, p, ... \rangle$ where $p \in \mathbb{P}^*$,
Let $p' \leq p$ conform to N^*. Let $\bar{N}^* = \bar{N}^*(p', N^*) =$
$= \langle \bar{H}, \bar{M}, <, \mathring{\bar{f}}, \bar{Q}, \bar{\mathbb{P}}^*, \bar{p}, ... \rangle$. Let $G \ni p'$ be
$\mathbb{P}^*$-generic. Let $\pi = \pi^G_{\alpha, \omega_1}$ where $\alpha = |p'|$. Then
π extends uniquely to $\sigma : \bar{N}^* \prec N^*$ s.t.
$\pi \cup F^{p'} \subseteq \sigma$. We choose $\mu > \delta$ big enough to
witness the properness of Q. Set $X = rng(\sigma)$.
Then $X \prec N^*$ is countable. Hence $X \in V$,
since $\mathbb{P}^*$ adds no countable set by
Prop 1. Since $[\![\varphi_p]\!] \neq \emptyset$ there is an N^*-
-definable $q \in [\![\varphi_p]\!]$. Hence $q \in X$.
Let $\pi(\bar{q}) = q$. By properness there is
a $\bar{q}' \leq \bar{q}$ s.t. $\tilde{G} = \{ r \in \bar{Q} \cap X \mid \bar{q}' \leq r \}$ is
$\bar{Q} \cap X$-generic over X. Set $\bar{G} = \pi^{-1} {}'' \tilde{G}$.
Then $\bar{G}$ is $\bar{Q}$-generic over $\bar{N}^*$ and $\bar{q} \in \bar{G}$.
Hence $\bar{G} \cap [\![\varphi_p]\!] \neq \emptyset$. Set:
$\bar{G}^* = \{ \rho \in \bar{\mathbb{P}}^* \mid [\![\varphi_\rho]\!] \cap \bar{G} \neq \emptyset \}$.
Then $\bar{G}^*$ is $\bar{\mathbb{P}}^*$-generic over $\bar{N}^*$ and $\bar{p} \in \bar{G}^*$.
Set: $\tilde{p} = \langle \langle M^{\bar{G}^*}, \pi^{\bar{G}^*}, \bar{G} \rangle, F^{p'} \rangle$.

$\underline{\text{Claim}}$ $\tilde{p} \in \mathbb{P}^*$.

Let $G \ni q'$ be Q-generic. Since

$$\bar{x} \left(\Vert_{\underset{Q}{\overline{N}^*}} \varphi(t_1 \ldots t_n) \right) \longleftrightarrow \sigma[\bar{x}] \left(\Vert_{\underset{Q}{N^*}} \varphi(\sigma(t_1) \ldots \sigma(t_n)) \right)$$

for $\bar{x} \in \overline{Q}$, $t_1, \ldots, t_n \in \overline{N}^* \overline{Q}$, and $\sigma'' \overline{G} = \tilde{G} \subset G$, it follows that σ extends to a $\sigma^* : \overline{N}^*[\overline{G}] \prec N^*[G]$ defined by:

$\sigma^*(t^{\overline{G}}) = \sigma(t)^G$. [Hence $\sigma^*(\overline{G}) = \sigma(\dot{G}^{\overline{G}}) = $

$= \dot{G}^G = G$, and $\sigma^*(x) = \sigma^*(\check{x}^{\overline{G}}) = \sigma(\check{x})^G =$

$= \sigma(x)$ for $x \in \overline{N}^*$.] But then

$$\sigma^* (M^{\overline{G}}, \pi^{\overline{G}}, \overline{G}) = M^G, \pi^G, G.$$

Since $\sigma^* \upharpoonright H_{\omega_1}^{\overline{N}^\sigma} = id$, we have:

$$M^{\overline{G}} \upharpoonright \alpha = M^G \upharpoonright \alpha , \quad \pi^{\overline{G}} \upharpoonright \alpha^2 = \pi^G \upharpoonright \alpha^2.$$

But $\langle M_\alpha^{\overline{G}}, \langle \pi_{i\alpha}^{\overline{G}} \mid i < \alpha \rangle \rangle =$

$= \lim (\langle M^{\overline{G}} \upharpoonright \alpha, \pi^{\overline{G}} \upharpoonright \alpha \rangle) = \langle M_\alpha^G, \langle \pi_{i\alpha}^G \mid i < \alpha \rangle \rangle.$

Hence:

(1) $M^{\overline{G}} = M^G \upharpoonright (\alpha+1), \quad \pi^{\overline{G}} = \pi^G \upharpoonright (\alpha+1)^2.$

But then

(2) $\pi_{\alpha \omega_1}^{\overline{G}} = \sigma \upharpoonright M_\alpha^{\overline{G}},$

since if $x \in M_\alpha^{\overline{G}}$, $x = \pi_{i\alpha}^{\overline{G}}(x')$, $i < \alpha$,

then $\sigma(x) = \sigma(\pi_{i\alpha}^{\overline{G}}(x')) = \sigma(\pi_{i\alpha}^{\overline{G}})(x') =$

$= \pi_{i\omega_1}^G(x') = \pi_{\alpha\omega_1}^G(x), \quad$ QED (2).

Since $FP' \subset \sigma$, we conclude that:

$\pi_{i\omega_1}^G : \langle M_\alpha^{\bar{G}}, \bar{a} \rangle \prec \langle M_\alpha^G, a \rangle$ for $\langle a, \bar{a} \rangle \in F_?^{p'}$

Hence $\check{N}[G] \models \varphi_{\tilde{p}}$. Hence $\mathcal{L}^*(\tilde{p})$ is

consistent and $\tilde{p} \in \mathbb{P}$. QED (Lemma 5)

. . . .

This proof shows more than we stated:

<u>Def</u> Let $2^\omega = \omega_2$, $2^\beta = \beta > \omega_1$. Let $S \subseteq [\beta]^\omega$

be stationary. $\mathbb{Q}$ is S-<u>proper</u> iff

whenever δ is big enough and

$X \prec \langle H_\delta, \mathbb{Q} \rangle$ is countable s.t. $X \cap \beta \in S$,

then for each $p \in X \cap \mathbb{Q}$ there is a $p' \leq p$

s.t. $p' \Vdash \check{G} \cap \check{X}$ is $\check{\mathbb{Q}} \cap \check{X}$-generic over $\check{X}$.

(<u>Note</u>: If $\mathbb{Q}$ adds no reals, this means

that there is $p' \leq p$ s.t. $G_{p'} = \{q \in \mathbb{Q} \cap X \mid p' \leq q\}$

is $\mathbb{Q} \cap X$-generic over $\check{X}$.)

<u>Def</u> Let β, S be as above. $\mathbb{Q}$ is S-

 — <u>superb</u> iff the following hold:

· $\mathbb{Q}$ is S-proper

· $\bar{\bar{\mathbb{Q}}} \leq \beta$

· $\mathbb{Q} \Vdash (\check{P}(\check{\omega}) = \check{P}(\omega) \wedge \mathrm{card}(\check{\beta}) = \omega_1)$

· $\mathbb{Q} \Vdash [\check{\beta}]^\omega \setminus \check{S}$ is not stationary

   in β.

Cor 5.1 Let Q be S-unpart. Then Q is reshapable.

prf (sketch)

Let $\Vdash_Q f: \check{\omega}_1 \xrightarrow{\text{onto}} \check{\beta}$ and $\Vdash_Q \dot{D} \subset \check{S}$ is club in $[\check{\beta}]^\omega$,

("club" means closed under countable ascending chains.)

Let G be Q-generic, $f = \dot{f}^G$, $D = \dot{D}^G$.

Set: $C = \{\alpha \leq \omega_1 \mid f''\alpha < M \wedge \beta \cap f''\alpha \in D\}$.

Then $C \cap \omega_1$ is club in ω_1 and $\omega_1 \in C$.

With this definition proceed as before, the details being virtually identical.

$$\text{QED (Cor 5.1)}$$

Without proof we mention the following theorem:

Lemma 6 Let Q be a set of conditions which adds no reals s.t. $\overline{\overline{Q}} < \beta$. Let $Q \Vdash (\dot{Q}'$ is $\check{\beta}$-reshapable). Then $\dot{Q} * \dot{Q}'$ is β-reshapable.

Hence:

Cor 6.1 Let $\mathbb{P}_0$ be β_0-reshapable and $\mathbb{P}_0 \Vdash \dot{\mathbb{P}}_1$ is $\check{\beta}_1$-reshapable. Then $\mathbb{P}_0 * \dot{\mathbb{P}}_1$ is β_1-reshapable.

(We use Lemmas 4.1 and 3.1 to show that $BA(\mathbb{P}_0) = BA(Q)$ for a Q s.t. $\overline{\overline{Q}} < \beta_1$.)

In §5 we note that there can be no similarly general theorem for iterations of length ω.