§5 Applications

In this section we give a number of concrete examples of $\mathcal{L}$-forcing. A major aim will be to answer the question posed at the outset in §1, except that we now suppose $M = L_\beta^u =$

$= \langle L_\beta[u], u \rangle$ to be uncountable. We again suppose M to be iterable and, letting $M_i = L_{\beta_i}^{u_i}$ to be the i-th iterate, we assume there is a r.t. U_α is a measure in L^{u_α}. What is the least such α? We again know that u_i is not ω-complete for $i < \alpha$. It follows that $\alpha \leq \omega_1$ and that $cf(\tau_i) = \omega$ for $\tau_i = \kappa_i^{+M_i}$, where κ_i is the critical point of u_i. But then $cf(\tau_\alpha) = cf(\tau_0) = \omega$ if $o < \alpha$. Hence $\tau_\alpha < \omega_2$ if $o < \alpha$ by the covering lemma for L^{u_α}.

We shall first show that α can be any countable ordinal, and then that it can be ω_1.

Thm1 Assume GCH. Let U be a normal measure on κ. Let $\beta > \kappa$ be regular. There is a generic extension of V with:

(a) $\mathcal{P}(\omega)$ is absolute

(b) For each $\alpha < \omega_1$ there is $\bar{M} = \langle \bar{H}, \bar{u} \rangle$ which iterates to $M = \langle H_\beta, u \rangle$ in exactly α many steps.

__Thm 2__ Assume GCH. Let u, $M = \langle H, u \rangle$ be as above. There is a generic extension of V with:

(a) $P(\omega)$ is absolute

(b) There is $\bar{M}$ which iterates to M in ω_1 steps.

Note that in the generic extension of Thm 1 u has countable cofinality, whereas it has cofinality ω in the extension of Thm 2.

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In order to prove these theorems we need the <u>interpolation</u> <u>lemma</u> for elementary embeddings. We define:

__Def__ Let $M = \langle |M|, \in, \cdots \rangle$ be a transitive ZFC^- model. Let $\pi : M \prec M'$ where M' is transitive. π is <u>cofinal</u> in M' iff $M' = \bigcup_{u \in M} \pi(u)$.

(__Note__ If π is cofinal in M', $\beta = On \cap M$, $\beta' = On \cap M'$, then $\beta' = \sup \pi''\beta$. Conversely, if $M = \langle L_\beta[A], \in, A, \cdots \rangle$ and $\beta' = \sup \pi''\beta$, then π is cofinal in M'.)

<u>Fact 1</u> Let $\pi : M \prec M'$ and set
$\widetilde{M} = M' \mid \bigcup_{u \in M} \pi(u)$. Then $\widetilde{M} \prec M'$ and
$\pi : M \prec \widetilde{M}$ cofinally.

(The proof uses the following fact,
which is a form of Łoś Thm and
is proven by induction on φ :
Let $x_1, ..., x_n \in \widetilde{M}$, $x_i \in \pi(u_i)$. Then
$\widetilde{M} \models \varphi(x_1, ..., x_n) \longleftrightarrow$
$\langle x_1, ..., x_n \rangle \in \pi(X)$
where $X = \{\langle \vec{z} \rangle \in u_1 \times ... \times u_m \mid M \models \varphi(\vec{z})\}$.)

An immediate corollary is :

<u>Fact 2</u> Let $\tau > \omega$ be regular in M,
where $\pi : M \prec M'$. Set :
$\bar{H} = H_\tau^M$, $\widetilde{H} = \bigcup_{u \in \bar{H}} \pi(u)$, $\bar{\pi} = \pi \upharpoonright \bar{H}$.
Then $\bar{\pi} : \bar{H} \prec \widetilde{H}$ cofinally.

<u>Def</u> Let $\pi : M \prec M'$. Let $\tau > \omega$ be
regular in M. π is τ-<u>cofinal</u> in M'
iff $M' = \bigcup_{u < \tau \text{ in } M} \pi(u)$.

(Hence τ-cofinality implies
cofinality.)

Def Let $\tau > \omega$ be regular in M, $\bar{H} = H_\tau^M$,
$\bar{\pi} : \bar{H} \prec H$ cofinally. By a __lift up__
of $\langle M, \bar{\pi} \rangle$ we mean a pair $\langle M', \pi \rangle$
s.t. M' is transitive, $\pi \upharpoonright \bar{H} = \bar{\pi}$,
and $\pi : M \prec M'$ τ - cofinally.
(Hence $H \subset M'$ and, in fact, $H = H_{\tau'}^{M'}$,
where $\tau' = \pi'(\tau) = On \cap H$.)

Fact 3 Let $\langle M, \bar{\pi} \rangle$ be as above. Then
there is at most one lift up.

proof.
Set $\mathbb{D} = \mathbb{D}_{\bar{\pi}}^M = $ the set of $\langle x, f \rangle$ s.t.
$f \in M$, $f : u \to M$ for a $u \in \bar{H}$, and $x \in \bar{\pi}(u)$.
For $\langle x, f \rangle \in \mathbb{D}$ set $k(\langle x, f \rangle) = \pi(f)(x)$.
Clearly k maps $\mathbb{D}$ onto M'. But
if $u_i = \langle x_i, f_i \rangle \in \mathbb{D}$ $(i = 1, \ldots, n)$, then
$M' \models \varphi(u_1, \ldots, u_n) \longleftrightarrow \langle x_1, \ldots, x_n \rangle \in \bar{\pi}(X_{\vec{u}})$,
where $X_{\vec{u}} = \{\langle z_1, \ldots, z_n \rangle \mid M \models \varphi(f_1(z_1), \ldots, f_n(z_n))\}$
(hence $X_{\vec{u}} \in \bar{H}$).
This means, in particular, that if
$\langle \pi', M'' \rangle$ is a second lift up, then
$$k(u) \in k(\sigma) \longleftrightarrow k'(u) \in k'(\sigma)$$
$$" \quad = \quad " \qquad \qquad " \quad = \quad " \quad ,$$
where $k'(\langle x, f \rangle) = \pi'(f)(x)$.
Hence we can define $\sigma : M' \overset{\sim}{\longleftrightarrow} M''$

by: $\sigma(k(u)) = k'(u)$. Hence $\sigma = id$, $M' = M''$, since M, M' are transitive. But then $k = k'$ and $\pi(x) = \pi'(x) = k(\langle \emptyset, f_x \rangle)$, where $f_x = \langle x, \emptyset \rangle$.   QED (Fact 3)

<u>Fact 4</u> The liftup of $\langle M, \bar{\pi} \rangle$ exists iff E is well founded, where $E \subset ID^2$ is defined by:

$$\langle x, f \rangle \, E \, \langle z, g \rangle \longleftrightarrow \langle x, z \rangle \in \bar{\bar{\pi}} \left(\{ \langle u, v \rangle \mid f(u) \in g(v) \} \right).$$

prf.

($\rightarrow$) trivial.

($\leftarrow$) Define $I \subset ID^2$ by

$$\langle x, f \rangle \, I \, \langle z, g \rangle \longleftrightarrow \langle x, z \rangle \in \bar{\bar{\pi}} \left(\{ \langle u, v \rangle \mid f(u) = g(v) \} \right).$$

Similarly set

$$\tilde{A} \langle x, f \rangle \longleftrightarrow x \in \bar{\pi} \left(\{ u \mid f(u) \in A^M \} \right)$$

where A is any predicate in the language of M. Interpreting $\dot{\in}, \dot{=}, A$ by $E, I, \tilde{A}$ we get <u>Łoś theorem</u> in the form:

$$ID \models \varphi(\vec{u}) \longleftrightarrow \langle x_1, \ldots, x_n \rangle \in \bar{\bar{\pi}}(X_{\vec{u}}),$$

where $u_i = \langle x_i, f_i \rangle \in ID$ $(i = 1, \ldots, n)$. The proof is by induction on φ and is just like the proof of Łoś theorem for ultrapowers. But then if we factor ID by I, we get a well founded model satisfying extensionality. Let $[u]$ be the equivalence class of

$u \in \mathbb{D}$ under I. There is then an
$h : \mathbb{D}/I \overset{\sim}{\hookrightarrow} M'$, where M' is transitive.
It follows easily that $\langle M', \bar{\pi} \rangle$ is
the liftup of $\langle M, \bar{\pi} \rangle$, where

$$\pi(x) = h([\langle \emptyset, f_x \rangle]), \quad f_x = \langle x, \emptyset \rangle.$$

$$\text{QED (Fact 4)}$$

By this we get the __interpolation__
__lemma__ for ZFC^- models:

__Fact 5__ Let $\pi : M \prec M'$. Let $\bar{\tau} > \omega$ be
regular in M, $\bar{H} = H^M_{\bar{\tau}}$, $\bar{\pi} = \pi \restriction \bar{H}$,
$\tilde{H} = \bigcup_{u \in \bar{H}} \pi(u)$. Then the liftup $\langle \tilde{M}, \tilde{\pi} \rangle$

of $\langle M, \bar{\pi} \rangle$ exists. Moreover, there is
a unique $\sigma : \tilde{M} \prec M$ s.t. $\sigma \tilde{\pi} = \pi$
and $\sigma \restriction \tilde{H} = id$.

prf
E is well founded, since

$$\langle x, f \rangle E \langle y, g \rangle \longleftrightarrow \pi(f)(x) \in \pi(g)(y).$$

Hence $\langle \tilde{M}, \tilde{\pi} \rangle$ exists. Define $\sigma : \tilde{M} \to M'$
by: $\sigma(\tilde{\pi}(f)(x)) = \pi(f)(x)$. Then σ is
elementary by Łoś theorem. It follows
easily that $\sigma \tilde{\pi} = \pi$ and $\sigma \restriction \tilde{H} = id$.
Conversely, if σ' has these
properties, then

$$\sigma'(\tilde{\pi}(f)(x)) = \pi(f)(x) = \sigma(\tilde{\pi}(f)(x))$$

$$\text{QED (Fact 5)}$$

Note Let $\tau \leq \kappa$ where κ is regular in M,
If π maps M τ-cofinally to M', then
$$H^{M'}_{\pi(\kappa)} = \bigcup_{\bar{u} < \tau \text{ in } H^M_\kappa} \pi(u) . \text{ Hence } \pi(u) = \sup \pi''\kappa$$
and $\pi \restriction H^M_\kappa$ maps τ-cofinally to $H^{M'}_{\pi(u)}$
if $\tau < \kappa$.

Note If $\pi : M \prec M'$, $\pi' : M' \prec M''$, $\pi(\bar{\tau}) = \bar{\tau}'$,
then $\pi'\pi$ is τ-cofinal to M'' iff
iff π is τ-cofinal to M' and
π' is τ'-cofinal to M''.

Note ZFC⁻ does <u>not</u> imply that H_τ
is a set for all regular τ. Thus in
the above definitions and results
we have not assumed that $H^M_\tau \in M$.
In our applications, however, we shall
normally have: $H^M_\tau = L_\tau[A]$ for an $A \in M$.
Hence $H^M_\tau \in M$ and $\pi''H^M_\tau$ lies cofinally
in $H^{M'}_{\pi(\tau)}$ iff $\pi''\tau$ lies cofinally in $\pi(\tau)$.

Example 1 We assume $\beta \geq \omega_2$ to be regular
s.t. $2^\beta = \beta$ and $2^\omega = \omega_1$. We devise an $\mathcal{L}$
s.t. $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ collapses each regular $\gamma \in (\omega_1, \beta]$,
making it ω-cofinal. We shall also ensure
that, if G is $\mathbb{P}$-generic, then $V[G] = V[x]$
where $x \subseteq \beta$ is a countable set. (We
can take $x = \text{rng}(\pi^G_{0,\omega_1})$)

In the following let $\tau = \omega_2$ and assume
that $\beta \geq \tau$ is regular, $2^\beta = \beta$, and $2^\omega = \omega_1$.
Set $M = L_\beta^A = \langle L_\beta[A], A \rangle$, where $L_\beta[A] = H_\beta$.
Set $N = \langle H_{\beta+}, M, <, \dots \rangle$, where $<$ well orders N.

Before we tackle the problem mentioned at
the outset of this section, we tackle a
simpler problem: Is there a forcing
extension of V which adds no reals
but in which every regular $\gamma \in (\omega_1, \beta]$
becomes ω-cofinal? (A positive answer
can, of course, be obtained by first
collapsing β onto ω_1 and then
applying Namba forcing. However,
we want to answer the problem with
our methods. The connection with
Namba forcing will be discussed later.)

Let $\mathcal{L}$ be the language on N which, in
addition to the basic axioms of §3, has
the following axioms:

- $\dot{\pi}_{i\underline{n}}$ is $\dot{\tau}_i$-cofinal in $\underline{M}$ ($i < n$), where
  $\dot{\tau}_i =_{\text{Df}} \omega_2^{\dot{M}_i}$

- $\text{rng}(\dot{\pi}_{i+1, \omega_1}) =$ the smallest $X \prec \underline{M}$ s.t.
  $\text{rng}(\dot{\pi}_{i, \omega_1}) \cup \{\dot{d}_i\} \subseteq X$ for $i <_{\underline{\omega_1}}$

- $\dot{B} = \emptyset$ (or better, just omit $\dot{B}$ and
  the axioms containing it)

<u>Note</u> The second axiom will guarantee that
$$V[G] = V[ring(\dot{\pi}^G_{0\,\omega_1})] \text{ for } \mathbb{P}_{\mathcal{L}}\text{-generic } G,$$

<u>Note</u> The first of these axioms says that $\langle \underline{M}, \dot{\pi}_{i\,\omega_1} \rangle$ is the liftup of $\langle \dot{M}_i, \dot{\pi}_{i,\omega_1} \restriction H^{M_i}_{\tau_i} \rangle$

The second axiom guarantees that the entire sequence $\dot{M}, \dot{\pi}$ is determined by $ring(\dot{\pi}_{0\,\omega_1})$.

We first show:

<u>Lemma 1</u> $\mathcal{L}$ is consistent.

proof.

Let $X \prec N$ be countable. Then $\alpha = \omega_1 \cap X$ is transitive and $\alpha = \omega_1^{\bar{M}}$, where $\sigma : \bar{N} \cong N|X$ and $\bar{N}$ is transitive.

Let $\sigma(\bar{M}) = M$. Let $\langle \tilde{N}, \tilde{\sigma} \rangle$ be the liftup of $\langle \bar{N}, \sigma \restriction H^{\bar{N}}_{\omega_2} \rangle$. Let $\tilde{M} = \tilde{\sigma}(\bar{M})$.

Since $\bar{M} = H^{\bar{N}}_{\bar{\beta}}$ (where $\sigma(\bar{\beta}) = \beta$), we see that $\tilde{M} = H^{\tilde{N}}_{\tilde{\beta}}$, where $\sigma(\bar{\beta}) = \tilde{\beta}$, and $\langle \tilde{M}, \tilde{\sigma} \restriction \bar{M} \rangle$ is the liftup of $\langle \bar{M}, \sigma \restriction H^{\bar{N}}_{\omega_2} \rangle$.

Note that $\tilde{N}$ is a ZFC^- model. More-over by the interpolation lemma there is $\sigma' : \tilde{N} \prec N$ s.t. $\sigma' \tilde{\sigma} = \sigma$. Clearly $\sigma'(\tilde{M}) = M$. Let $\tilde{\mathcal{L}}$ be defined from $\tilde{M}$ in $\tilde{N}$ as $\mathcal{L}$ was defined from M in N.

Since $\tilde{N}, N$ are ZFC^-- models, we have :

$\tilde{\mathcal{L}}$ is consistent $\longleftrightarrow \tilde{N} \models \tilde{\mathcal{L}}$ is consistent

$\qquad\qquad \longleftrightarrow N \models \mathcal{L}$ " "

$\qquad\qquad \longleftrightarrow \mathcal{L}$ is consistent.

Thus it suffices to show that $\tilde{\mathcal{L}}$ is consistent. We do this by displaying a model for $\tilde{\mathcal{L}}$. We define $M' = \langle M_i \mid i \leq \omega_1 \rangle$, $\overline{\pi} = \langle \pi_{ij} \mid i \leq j \leq \omega_1 \rangle$ as follows: Define $\langle X_i \mid i \leq \omega_1 \rangle$ by :

$X_0 = \text{rng}(\tilde{\sigma} \restriction \overline{M})$,

$X_{i+1} = $ the smallest $X \prec \tilde{M}$ s.t. $X_i \cup \{\alpha_i\} \subset X$, where $\alpha_i = \omega_1 \cap X_i$

$X_\lambda = \bigcup_{i < \lambda} X_i$ for $\lim(\lambda)$

Set : $\pi_{i\omega_1} : M_i' \overset{\sim}{\longrightarrow} \tilde{M} \mid X_i$, and

$\pi_{ij} = \pi_{j\omega_1}^{-1} \circ \pi_{i\omega_1}$ for $i \leq j$. Then

$\langle H_{\beta^{++}}, M', \overline{\pi} \rangle$ is a model of $\tilde{\mathcal{L}}$.

(The axiom $H_{\omega_1} = \underline{H}_{\omega_1}$ follows since

$H_{\omega_1} = \sigma(H_{\omega_1}^{\overline{N}}) = \tilde{\sigma}(H_{\omega_1}^{\overline{N}}) = H_{\omega_1}^{\tilde{N}}$,

since $H_{\omega_1}^{\overline{N}} \in H_{\omega_2}^N$.) QED (Lemma 1)

We then show :

<u>Lemma 2</u> $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ is revisable.

proof of Lemma 2.

Let p conform to $N^* = \langle H_\delta, N, <, \ldots \rangle$, where $\delta > \beta^+$ is regular. Let $\bar{N}^* = \bar{N}^*(p, N^*) = \langle \bar{H}, \bar{N}, <, \ldots \rangle$ be as in §3 Lemma 3. Let $\bar{\mathcal{L}}, \bar{\mathbb{P}} = \mathbb{P}_{\bar{\mathcal{L}}}$ be defined in $\bar{N}^*$ as $\mathcal{L}, \mathbb{P}$ were defined in N^*. Let $\bar{\mathcal{M}}$ be a solid model of $\bar{\mathcal{L}}(p)$. Let $\bar{G}$ be $\bar{\mathbb{P}}$-generic over $\bar{N}^*$ (where $\bar{G} \in V$). This gives us $M^{\bar{G}}, \pi^{\bar{G}}$. Define a new model $\tilde{\mathcal{M}} = \langle |\mathcal{M}|, \in^{\mathcal{M}}, \tilde{M}, \tilde{\pi} \rangle$ by:

$$\tilde{M}_i = \begin{cases} M_i^{\mathcal{M}} & \text{for } i \geq |p| \\ M_i^{\bar{G}} & \text{for } i \leq |p| \end{cases}$$

$$\tilde{\pi}_{i\,j} = \begin{cases} \pi_{i\,j}^{\mathcal{M}} & \text{for } |p| \leq i \leq j \\ \pi_{|p|\,j}^{\mathcal{M}} \, \pi_{i\,|p|}^{\bar{G}} & \text{for } i \leq |p| \leq j \\ \pi_{i\,j}^{\bar{G}} & \text{for } i \leq j \leq |p|, \end{cases}$$

$\tilde{\mathcal{M}}$ is easily seen to be a model of $\mathcal{L}(\tilde{p})$, where $\tilde{p} = \langle \langle M^{\bar{G}}, \pi^{\bar{G}}, \phi \rangle, F^p \rangle$.

Hence $\tilde{p} \in \mathbb{P}$.   QED (Lemma 2)

Now let G be $\mathbb{P}$-generic. It is easily seen that each $\pi_{i\,\omega_1}^G$ is τ_i-cofinal in M, since $\pi_{i\,j}^G$ is τ_i-cofinal in M_j^G for $i \leq j < \omega_1$ (where $\tau_i = \omega_1^{M_i^G}$).

But then if $\omega_2 \leq \gamma \in M$ is regular and $\pi^G_{i\omega_1}(\bar{\gamma}) = \gamma$ it follows easily that $\pi^G_{i\omega_1} \upharpoonright H^{M^G}_{\bar{\gamma}}$ is τ_i-cofinal in H^M_γ. Hence every regular $\gamma \in (\omega_1, \beta]$ is ω-cofinal in $V[G]$.

In the special case $\beta = \omega_2$, $\mathbb{P}$ simply makes ω_1 ω-cofinal. It is known, of course, that Namba forcing accomplishes the same, although the methods seem very different. To our great surprise, we discovered that forcing with $\mathbb{P}$ is the same as adding a Namba sequence. We defer the proof of that fact to the next chapter, however, and return to the problems stated at the outset. We are now in a position to prove Theorem 1.

We in fact show:

Lemma 3 Let κ be measurable. Let U be a normal ultrafilter on κ and let $\beta > \kappa$ be regular s.t. $2^\beta = \beta$. There is a generic extension $V[G]$ in which

- $\mathcal{P}(\omega)$ is absolute
- For each $\alpha < \omega_1$ there is $\langle H, U' \rangle$ which iterates to $\langle H_\beta, U \rangle$ in exactly α steps.

prf. of Lemma 3.

Set $M = L_\beta^A = \langle L_\beta[A], A \rangle$ s.t. $L_\beta[A] = H_\beta$ and u is M-definable. Set

$N = \langle H_{\beta^+}, M, <, \ldots \rangle$, where $<$ well orders N.

There is then a generic extension $V[G]$ in which H_{ω_1} is absolute and in which there is a countable transitive $\bar{M}$ and a $\sigma : \bar{M} \prec M$ which is $\bar{\tau}$-cofinal, where $\bar{\tau} = \omega_1^{\bar{M}}$, $\bar{\sigma} = \omega_1^{M}$, ($Thus $\langle M, \sigma \rangle$ is the liftup of $\langle \bar{M}, \sigma \restriction H_{\bar{\tau}}^{\bar{M}} \rangle$)

[This can be accomplished by the previously defined L-forcing. We can also do it by first generically collapsing β to ω_2 and then applying Namba forcing.]

We then have:

(1) for every $\alpha < \omega_1$ there is $\bar{M}, \sigma$ with the above properties s.t. $\omega_1^{\bar{M}} > \alpha$ and σ extends to a $\sigma' : \bar{N} \prec N$, where $\bar{N}$ is a transitive extension of $\bar{M}$ and $\sigma'(\bar{M}) = M$.

prf. Fix $\bar{M}', \sigma'$ with the above properties. Let $X \prec N$ be

the smallest $X \subseteq N$ s.t. range $(\sigma') \cup \{\alpha\} \subseteq X$.

Let $\sigma' : \bar{N} \xrightarrow{\sim} X$, where $\bar{N}$ is transitive. Let $\sigma'(\bar{M}) = M$ and let $\sigma = \sigma' \upharpoonright \bar{M}$. Then $\sigma'^{-1} \sigma : \bar{M}' \prec \bar{M}$ is $\omega_1^{\bar{M}'}$ cofinal and $\sigma' : \bar{M} \prec M$ is $\omega_1^{\bar{M}}$ cofinal. Clearly $\omega_1^{\bar{M}} = \omega_1 \cap X \geq \alpha$.

$$QED (1)$$

From now on let $\bar{M}, \sigma, \bar{N}, \sigma'$ be as in (1). As we showed at the end of §1, the infinitary language $\mathcal{L}$ on N which says that there exists $\langle M', u' \rangle$ which iterates to $\langle M, u \rangle$ in exactly α many steps is consistent. But then the corresponding language $\bar{\mathcal{L}}$ on $\bar{N}$ is consistent. Since $\bar{N}$ is countable, we can find a solid model of $\bar{\mathcal{L}}$. This gives us $\langle \bar{M}', \bar{u}' \rangle$ which iterates to $\langle \bar{M}, \bar{u} \rangle$ in exactly α-many steps (where $\sigma'(\bar{u}) = u$). We then use σ to "lift" $\langle \bar{M}', \bar{u}' \rangle$ to a $\langle M', u' \rangle$ which iterates to $\langle M, u \rangle$

Let $\langle \bar{M}_i, \bar{U}_i \rangle$ be the i-th iterate of $\langle \bar{M}, \bar{U} \rangle$ ($i \leq \alpha$) with iteration maps $\bar{\pi}_{ij}$ ($i \leq j \leq \alpha$).

(2) The liftup $\langle M_i, \sigma_i \rangle$ of $\langle \bar{M}_i, \sigma \restriction H_{\bar{\tau}}^{\bar{M}_i} \rangle$ exists (where $\bar{\tau} = \omega_2^{\bar{M}} = \omega_2^{\bar{M}_i}$).

proof.

We know that $\langle M_\alpha, \sigma_\alpha \rangle = \langle M, \sigma \rangle$ exists.
We must show that E_i is well founded

where

$$\langle x, f \rangle E_i \langle y, g \rangle \longleftrightarrow \langle x, y \rangle \in \sigma \left(\{ \langle u, v \rangle \mid f(u) \in g(v) \} \right)$$

where $\langle x, f \rangle, \langle g, q \rangle \in \mathbb{D}^{M_i}_{\sigma \restriction H_{\bar{\tau}}^{M_i}}$. But

Then
$$\langle x, f \rangle E_i \langle y, g \rangle \longleftrightarrow \langle x, \bar{\pi}_{i\alpha}(f) \rangle E_\alpha \langle y, \bar{\pi}_{i\alpha}(g) \rangle,$$
where E_α is well founded. QED (2)

By Łoś Theorem for liftups (as in the
proof of Fact 4) we can then define:

(3) $\pi_{ij} : M_i \prec M_j$ by:
$$\pi_{ij}(\sigma_i(f)(x)) = \sigma_j(\bar{\pi}_{ij}(f)(x))$$

for $\langle x, f \rangle \in \mathbb{D}^{M_i}_{\sigma \restriction H_{\bar{\tau}}^{M_i}}$.

But then there is U_i which is M_i –
definable by the same definition
as U from M. Hence U_i is a
normal measure in $\langle M_i, U_i \rangle$,

Since $\langle M_\alpha, U_\alpha \rangle = \langle M, U \rangle$, it suffices to show that $\langle \langle M_i, U_i \rangle \mid i \leq \alpha \rangle$ is the iteration of $\langle M_0, U_0 \rangle$ with iteration maps π_{ij} $(i \leq j \leq \alpha)$. We first show:

(4) $M_\lambda = \bigcup_{i < \lambda} rng(\pi_{i\lambda})$ for $Lim(\lambda)$, $\lambda \leq \alpha$

pf.

Let $x = \sigma_\lambda(f)(z) \in M_\lambda$, where $f \in \overline{M}_\lambda$,
$f: u \to \overline{M}_\lambda$, $z \in \sigma_\lambda(u)$. Then $f = \overline{\pi}_{i\lambda}(f')$
for an $i < \lambda$. Hence $x = \pi_{i\lambda}(\sigma_i(f')(z))$ QED (4)

(5) $\pi_{ij} \restriction H^{M_i}_{\kappa_i} = id$, where $\kappa_i = crit(U_i)$

pf

Let $x = \sigma_i(f)(z) \in H^{M_i}_{\kappa_i}$. Then
$f: u \to H^{\overline{M}_i}_{\overline{\kappa}_i}$, where $\overline{\kappa}_i = crit(\overline{U}_i)$.
Hence $f \in H^{\overline{M}_i}_{\overline{\kappa}_i}$ and $\overline{\overline{\pi}}_{ij}(f) = f$,
since $\overline{\overline{\pi}}_{ij} \restriction H^{M_i}_{\overline{\kappa}_i} = id$. But since
$H^{\overline{M}_i}_{\overline{\kappa}_i} = H^{\overline{M}_j}_{\overline{\kappa}_i}$, we have $\sigma_i \restriction H^{\overline{M}_i}_{\overline{\kappa}_i} =$
$= \sigma_j \restriction H^{\overline{M}_j}_{\overline{\kappa}_j}$. Hence $\pi_{ij}(x) =$
$= \pi_{ij}(\sigma_i(f)(z)) = \sigma_j \overline{\overline{\pi}}_{ij}(f)(z) = \sigma_i(f)(z) = x$.
QED (5)

(7) M_{i+1} = the Σ_0-closure of $\operatorname{rng}(\bar{\pi}_{i,i+1}) \cup \{u_i\}$ in M_{i+1}.

prf.

Let $x = \sigma_{i+1}(f)(z) \in M_{i+1}$, $f \in \overline{M}_{i+1}$,
$f: u \to \widetilde{M}_{i+1}$, $z \in \sigma(u)$ where $u \in H_{\bar{z}}^{\overline{M}}$.

Let $f = \bar{\pi}_{i,i+1}(g)(\bar{u}_i)$. Then $x =$

$$= \sigma_{i+1}(\bar{\pi}_{i,i+1}(g)(\bar{u}_i))(z) = (\overline{\pi_{i,i+1} \sigma_i}(g)(u_i))(z),$$

where $z \in \operatorname{rng}(\bar{\pi}_{i,i+1})$ by (5), QED (7)

(8) Let $x \in \mathcal{R}(u_i) \cap M_i$. Then

$$x \in u_i \iff u_i \in \overline{\pi}_{i,i+1}(x)$$

prf.

Let $x = \sigma_i(f)(z)$ as before. Then
$$x \in u_i \iff z \in \sigma_i(\{v \mid f(v) \in \bar{u}_i\})$$
$$\iff z \in \sigma_{i+1}(\{v \mid \bar{u}_i \in \overline{\pi}_{i,i+1}(f)(v)\})$$
$$\iff u_i \in \overline{\pi}_{i,i+1}(f)(z) = \overline{\pi}_{i,i+1}(x)$$

since $\sigma_{i+1} \restriction H_{\bar{z}}^{\overline{M}} = \sigma_i \restriction H_{\bar{z}}^{\overline{M}} = \sigma \restriction H_{\bar{z}}^{\overline{M}}$,

and $f(v) \in \bar{u}_i \iff \bar{u}_i \in \overline{\pi}_{i,i+1}(f(v)) = \overline{\pi}_{i,i+1}(f)(v)$

for $v \in \operatorname{dom}(f) \in H_{\bar{z}}^{\overline{M}}$, QED (8)

(1)–(8) establish the desired conclusion.

$$\text{QED (Lemma 3)}$$

We note that this result holds not only
for normal measures but also for "sufficiently
closed" _extenders_. We adopt the following
notion of extenders:

Def Let M be a transitive ZFC^- model.
E is an _extender_ at κ of length γ on M
iff $\kappa \in M$ and:

(a) $\gamma > \kappa$ is closed under Gödel tuple function $\langle \ \rangle$.

(b) $\kappa \in M$ and $E = \langle E_\xi \mid \xi < \gamma \rangle$, where
each E_ξ is an ultrafilter on $\mathcal{P}(\kappa)^M$

(c) The transitive ultrapower $M' = Ult(M, E)$
exists - i.e. there is $\pi : M \prec M'$ s.t.

→ $M' = $ the Σ_0 - closure of $\gamma \cup \text{ran}(\pi)$ in M'

• $X \in E_\xi \iff \xi \in \pi(X)$ for $X \in \mathcal{P}(\kappa)^M$, $\xi < \gamma$

• $\mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)^{M'}$.

Note π, M' are uniquely determined by
these conditions + π is called the
canonical embedding. We also write
$\pi : M \xrightarrow{E} M'$ to mean that $M' = Ult(M, E)$
and π is the canonical embedding.

Los Thm for ultrapower says:

$$M' \models \varphi(\pi(f_1)(\xi_1), ..., \pi_n(f_m)(\xi_m)) \iff$$

$$\iff \{\langle \eta_1, ..., \eta_m \rangle \mid M \models \varphi(f_1(\eta_1), ..., f_m(\eta_m))\} \in E_{\langle \xi_1, ..., \xi_m \rangle}$$

where $\xi_1, ..., \xi_m < \gamma$ and $f_i \in M$, $f_i : \kappa \to M$
for $i = 1, ..., m$. (Here $\langle \ \rangle$ is Gödel tuple
function on On.)

Def Let E be an extender of length i
on M. E is _nicely closed_ iff
iff $(\aleph^{\omega_1})^M = (\aleph^{\omega_1})^{Ult(M,E)}$.

We then get:

Lemma 4 Let E be a nicely closed extender
of length γ on V. Let $\beta > \gamma$ be regular
s.t. $\beth_\beta = \beta$. There is a generic extension
V[G] in which

. $\sharp(\omega)$ is absolute

. For each $\alpha < \omega_1$ there is $\langle H, E'\rangle$ which
iterates to $\langle H_\beta, E\rangle$ in exactly α steps.

prf:
Set $M = L_\beta^A$, where $L_\beta[A] = H_\beta$ and E is M-definable.
Set $N = \langle H_{\beta+}, M, <, \ldots\rangle$ where $<$ wellorder W,
Work in the same generic extension as before.
Let $\alpha < \omega_1$. (1) holds just as before. Let
$\bar{M}, \sigma, \bar{N}, \sigma'$ be as in (1). We see as before
that the infinitary language $\bar{\mathcal{L}}$ on $\bar{N}$
which says that some $\langle \bar{M}, \bar{E}'\rangle$ iterates
to $\langle \bar{M}, \bar{E}\rangle$ in exactly α many steps
is consistent. (Here $\bar{E}$ is the extender on
$\bar{N}$ (hence on $\bar{M}$) which is defined in $\bar{N}$ as
E was defined on N.) We then use
σ to "lift" $\langle \bar{M}, \bar{E}'\rangle$ to $\langle M, E'\rangle$ as before.
Let $\langle \bar{M}_i, \bar{E}_i\rangle$ be the i-th iterate of $\langle \bar{M}, \bar{E}\rangle$
$(i \leq \alpha)$ with iteration maps $\bar{\pi}_{ij}$ $(i \leq j \leq \alpha)$,

(2) and (3) follow exactly as before. But then there is E_i which is M_i-definable by the same definition as E in M. It follows that E_i is an extender on $\langle M_i, E_i \rangle$. Since $\langle M_\alpha, E_\alpha \rangle = \langle M, E \rangle$, it suffices to show that $\langle \langle M_i, E_i \rangle \mid i \leq \alpha \rangle$ is the iteration of $\langle M_0, E_0 \rangle$ with iteration maps $\bar\pi_{ij}$ $(i \leq j \leq \alpha)$.

(4), (5) follow exactly as before.

We note that the fact that $(\aleph^{\omega_1})^M = (\aleph^{\omega_1})^{Ult(M,E)}$ if $Ult(M,E)$ exists is expressible by $\langle M, E \rangle \models \varphi$ for a certain statement φ. (This is left to the reader. Then each $\langle \bar M_i, \bar E_i \rangle$ has the same property. By this we get:

(6) Let $f \in \bar M_i$, $f : \omega_i \xrightarrow{\bar M_0} \bar\gamma_i$ (where $\bar\gamma_i$ is the length of $\bar E_i$). Then $\bar\sigma_i(f) = \bar\sigma_{i+1}(f)$. Moreover $\sigma_i \upharpoonright (\bar\gamma_i + 1) = \sigma_{i+1} \upharpoonright (\bar\gamma_i + 1)$.

proof:

Since σ_i, σ_{i+1} are liftups of $\langle \bar M_i, \sigma_i \upharpoonright \bar E \rangle$, $\langle \bar M_{i+1}, \sigma_i \upharpoonright \bar{\bar E} \rangle$ resp., where $\bar E = \omega_2^{\bar M} = \omega_2^{\bar M_i} = \omega_2^{\bar M_{i+1}}$, we know that each element of $\bar\gamma_i$ has the form $\sigma_i(f)(\xi)$

where $f_i : \omega_1^{\bar{m}} \to \bar{\delta}_i$ in $\bar{M}_i$ and $\mathfrak{z} < \omega_1$. But
Then $\bar{\delta}_i$ is also the set of $\sigma_{i+1}(f)(\mathfrak{z})$
for the same collection of f, since
$(\bar{\delta}_i \omega_1)\bar{M}_i = (\bar{\delta}_i \omega_1)\bar{M}_{i+1}$. But then

$$\sigma_i(f)(\mathfrak{z}) < \sigma_i(g)(\mathfrak{s}) \iff$$

$$\iff \langle \mathfrak{z}, \mathfrak{s} \rangle \in \sigma(\{\langle \mu, \tau \rangle \mid f(\mu) < g(\tau)\})$$

$$\iff \sigma_{i+1}(f)(\mathfrak{z}) < \sigma_{i+1}(g)(\mathfrak{s})$$

for $f, g : \omega_1^{\bar{m}} \to \bar{\delta}_i$ in $\bar{M}_i$, $\mathfrak{z}, \mathfrak{s} < \omega_1$.

But then there is an isomorphism
$h : \langle \bar{\delta}_i, < \rangle \xrightarrow{\sim} \langle \bar{\delta}_i, < \rangle$ defined by
$h(\sigma_i(f)(\mathfrak{z})) \iff \sigma_{i+1}(f)(\mathfrak{z})$. Hence $h = \mathrm{id}$
and $\sigma_i(f)(\mathfrak{z}) = \sigma_{i+1}(f)(\mathfrak{z})$. It follows
easily that $\sigma_i \restriction (\bar{\delta}_i + 1) = \sigma_{i+1} \restriction (\bar{\delta}_i + 1)$.

$$\text{QED}(6).$$

The proof of (7), (8) are virtually
as before, using (6). To prove (7)
we let $x = \sigma_{i+1}(f)(\mathfrak{z}) \in M_{i+1}$, where
$f \in \bar{M}_{i+1}$, $f : \omega_1^{\bar{m}} \to \bar{M}_{i+1}$, $\mathfrak{z} < \omega_1$.
Let $f = \bar{\pi}_{i, i+1}(g)(\mu)$, where $g : \bar{\pi}_i \to \bar{M}_i$,
$g \in \bar{M}_i$, $\mu < \bar{\delta}_i$. Then
$x = \sigma_{i+1}(\bar{\pi}_{i, i+1}(g)(\mu))(\mathfrak{z}) =$
$= (\bar{\pi}_{i, i+1} \sigma_i(g)(\sigma_i(\mu))(\mathfrak{z})$, where

$\sigma_i(\mu) < \bar{\delta}_i$ and $\xi < \omega_1$ (hence $\bar{\pi}_{i,i+1}(\xi) = \xi$).

This proves (7). We reformulate (8) as

(8') Let $X \in \mathcal{P}(u_i) \cap M_i$, $\mu < \delta_i$. Then

$$X \in E_\mu \quad \longleftrightarrow \quad \mu \in \overline{\pi}_{i,i+1}(X),$$

proof

Let $X = \sigma_i(f)(\xi)$, $f \in \bar{M}_i$, $f: \omega_1^{\bar{M}} \to \mathcal{P}(\bar{u}_i)^{M_i}$,

and $\mu = \sigma_i(h)(\varsigma)$, $h \in \bar{M}_i$, $h: \omega_1^{\bar{M}} \to \bar{\delta}_i$,

and $\xi, \varsigma < \omega_1$. Then

$$X \in E_\mu \quad \longleftrightarrow \quad \langle \xi, \varsigma \rangle \in \sigma_i(\{\langle \xi', \varsigma' \rangle \mid f(\xi') \in \bar{E}_{h(\varsigma')}\})$$

$$\longleftrightarrow \quad \langle \xi, \varsigma \rangle \in \sigma_{i+1}(\{\langle \xi', \varsigma' \rangle \mid h(\varsigma') \in \overline{\pi}_{i,i+1}(f)(\xi')\})$$

$$\longleftrightarrow \quad \sigma_{i+1}(h)(\varsigma) \in \overline{\pi}_{i,i+1}\,\sigma_i(f)(\xi)$$

$$\overset{\shortparallel}{\mu} \qquad\qquad \overset{\shortparallel}{\in \overline{\pi}_{i,i+1}(X)},$$

$$\text{QED (Lemma 4)}$$

We now turn to the proof of Thm 2. We note that the proof of Thm 1 gives:

__Fact 6__ Let $\bar{U} \subset \bar{M}$ be a normal measure on $\bar{\kappa}$ which is $\bar{M}$-definable. Let $\langle \bar{M}_i, \bar{U}_i \rangle$ be the i-th iterate of $\langle \bar{M}, \bar{U} \rangle$ with iteration maps $\bar{\pi}_{ij}$ $(i' \leq i \leq \alpha)$. Let $\bar{\tau} \leq \bar{\kappa}$ be regular in $\bar{M}$. Set $\bar{H} = H_{\bar{\tau}}^{\bar{M}}$. (Hence $\bar{H} = H_{\bar{\tau}}^{\bar{M}_i}$ and $\bar{\pi}_{ij} \restriction \bar{H} = id.$) Let $\sigma : \bar{H} \prec H$ cofinally and let $\langle M_i, \sigma_i \rangle$ be the liftup of $\langle \bar{M}_i, \sigma \rangle$. Let U_i be definable over M_i as $\bar{U}$ was defined over $\bar{M}$. Then $\langle M_i, U_i \rangle$ is the i-th iterate of $\langle M_0, U_0 \rangle$. Moreover, if π_{ij} $(i \leq i \leq \alpha)$ are the iteration maps, then $\pi_{ij} \sigma_i = \sigma_j \bar{\pi}_{ij}$ $(i \leq i \leq \alpha)$.

__Note__ By a straightforward modification of the proof of Lemma 4, the corresponding statement holds for $\langle \bar{M}, \bar{E} \rangle$, where $\bar{E} \subset \bar{M}$ is an extender of length δ on $\bar{M}$ which is $\bar{M}$ definable and $\bar{E}$ is nicely closed wrt. $\bar{\tau}$ — i.e. $(\gamma^{\bar{\tau}})^{\bar{M}} = (\gamma^{\bar{E}}) \cap \mathrm{ult}(\bar{M}, \bar{E})$,

__Note__ If $\bar{M}, \bar{\tau}, \bar{\kappa}, \bar{M}_i, \ldots, M, M_i, \sigma_i$ are as in Fact 6, and $\bar{\kappa}_i = \mathrm{crit}(\bar{U}_i)$, then $\sigma_i \restriction (\bar{\kappa}_i + 1) = \sigma_j \restriction (\bar{\kappa}_i + 1)$ for $i \leq i$. This is because $\sigma_i \restriction \bar{H} = \sigma_j \restriction \bar{H}$, where $H = H_{\bar{\kappa}_i^+}^{\bar{M}_i} = H_{\bar{\kappa}_i^+}^{\bar{M}_j}$, since $\langle H, \sigma_i \restriction \bar{H} \rangle = \langle H, \sigma_j \restriction \bar{H} \rangle$ is the liftup of $\langle \bar{H}, \sigma \restriction H_{\bar{\tau}}^{\bar{M}} \rangle$, where $H = H_{\kappa_i^+}^{M_i} = H_{\kappa_i^+}^{M_j}$.

<u>Note</u> By the preceding note Fact 6 has a converse. Let U be an M-definable measure on κ in M and suppose that $\langle M, U \rangle$ has iterates $\langle M_i, U_i \rangle$ $(i \leq \alpha)$ with iteration maps π_{ij} $(i \leq j \leq \alpha)$. Let $\sigma : \bar{M} \prec M$ and let $\bar{U}$ be defined over $\bar{M}$ like U over M. As is well known that $\langle \bar{M}, \bar{U} \rangle$ then has iterates $\langle \bar{M}_i, \bar{U}_i \rangle$ $(i \leq \alpha)$ and that there are <u>unique</u> maps $\sigma_i : \bar{M}_i \prec M_i$ with the properties: $\sigma_0 = \sigma$,

$$\sigma_j \cdot \bar{\pi}_{ij} = \pi_{ij} \cdot \sigma_i, \quad \text{and} \quad \sigma_i \restriction (\bar{\kappa}_i + 1) = \sigma_j \restriction (\bar{\kappa}_i + 1)$$

for $i \leq j \leq \alpha$, where $\bar{\pi}_{ij} : \bar{M}_i \prec \bar{M}_j$ are the iteration maps and $\bar{\kappa}_i = \mathrm{crit}(\bar{U}_i)$.

If we then assume σ to be $\bar{\tau}$-cofinal in M for some $\bar{\tau} \leq \bar{\kappa}_0$, it follows by the previous note that σ_i is $\bar{\tau}$-cofinal in M_i for $i \leq \alpha$.

(A similar converse can be formulated for the case that E is an M-definable extender which is nicely closed w.r.t. τ, where $\tau = \sigma(\bar{\tau}) \leq \kappa = \mathrm{crit}(E)$.)

<u>Example 2</u> We now construct an $\tilde{\mathcal{L}}$ s.t. $\mathbb{P} = \mathbb{P}_{\tilde{\mathcal{L}}}$ proves Theorem 2. We shall in fact show:

Lemma 5 Let u, U, β be as in Lemma 3,
There is a generic extension $V[G]$ in
which

- $\#(\omega)$ is absolute
- There is $\langle H, U' \rangle$ which iterates to $\langle H_\beta, U \rangle$
  in exactly ω_1 steps.

proof.
Define $M = L_\beta^A = \langle H_\beta, A \rangle$, $N = \langle H_{\beta^+}, N, <, in \rangle$
as before. Let $\tilde{\mathscr{L}}$ be the language on N
which, in addition to the basic axioms,
contains the axioms:

- $\dot{B} = \emptyset$
- $\underline{\omega_2} = \sup \dot{\pi}_{i \underline{\omega_1}} " \dot{\tau}_i$, where $\dot{\tau}_i =_{pf} \omega_2^{\dot{M}_i}$ for $i \leq \underline{\omega_1}$
- Let $\langle M', \sigma \rangle$ be the lift up of
  Let u' be defined over
  M' as $\underline{u}$ over $\underline{M}$. The $\langle M', u' \rangle$ iterates
  to $\dot{M}_j$ in $j - i$ steps. Moreover, if
  $h : M' \to \dot{M}_j$ is the iteration map,
  then $\dot{\pi}_{ij} = h\sigma$ $(i \leq j \leq \underline{\omega_1})$.

We first show:

Lemma 5.1 $\tilde{\mathscr{L}}$ is consistent.

pf.
Let $\langle N', u' \rangle$ be the ω_1-th iterate of
$\langle N, u \rangle$ with iteration map k,

Let $M' = k(M)$. Then $\langle M', u' \rangle$ is the a_1-th iterate of $\langle M', u' \rangle$ with iteration map $k \upharpoonright M$. Let $\tilde{\mathcal{L}}'$ be N'-definable as $\mathcal{L}$ was defined in N. Then

$$k : \langle N, M, \mathcal{L} \rangle \prec \langle N', M', \mathcal{L}' \rangle$$

and it suffices to show:

Claim $\tilde{\mathcal{L}}'$ is consistent.

Pf.

Let $\mathcal{L}$ be the language on N constructed in Example 1 and let G be $\mathbb{P}_{\mathcal{L}}$-generic over U. We construct a model of $\tilde{\mathcal{L}}'$ in $U[G]$. Let $M^G = \langle M_i \mid i \leq \omega_1 \rangle$, $\pi^G = \langle \pi_{ij} \mid i \leq j \leq \omega_1 \rangle$ be defined as before. Then $\pi_{ij} : M_i \prec M_j$ is $\tau_i = \omega_1^{M_i}$-cofinal for $i \leq j \leq \omega_1$ and $M = M_{\omega_1}$. Let u_i be defined over M_i as u over M.

Since $\pi_{i\omega_1} : \langle M_i, u_i \rangle \prec \langle M_{\omega_1}, u_{\omega_1} \rangle$, $\langle M_i, u_i \rangle$ is iterable. Let $\langle M_{i,\ell}, u_{i,\ell} \rangle$ be the ℓ-th iterate of $\langle M_i, u_i \rangle$ for $\ell \leq i \leq \omega_1$. Let $k_{\ell j} : M_{i\ell} \prec M_{ij}$ be the iteration map ($\ell \leq j \leq i \leq \omega_1$).

Set $H_i =_{Df} H_{\tau_i}^{M_i}$, where $\tau_i = \omega_1^{M_i}$.

Since $\pi_{ij} : \langle M_i, U_i \rangle \prec \langle M_j, U_j \rangle$, there are maps
$\pi_{ij}^{\ell} : \langle M_{i\ell}, U_{i\ell} \rangle \prec \langle M_{j\ell}, U_{j\ell} \rangle$ $(\ell \leq i \leq j \leq \omega_1)$
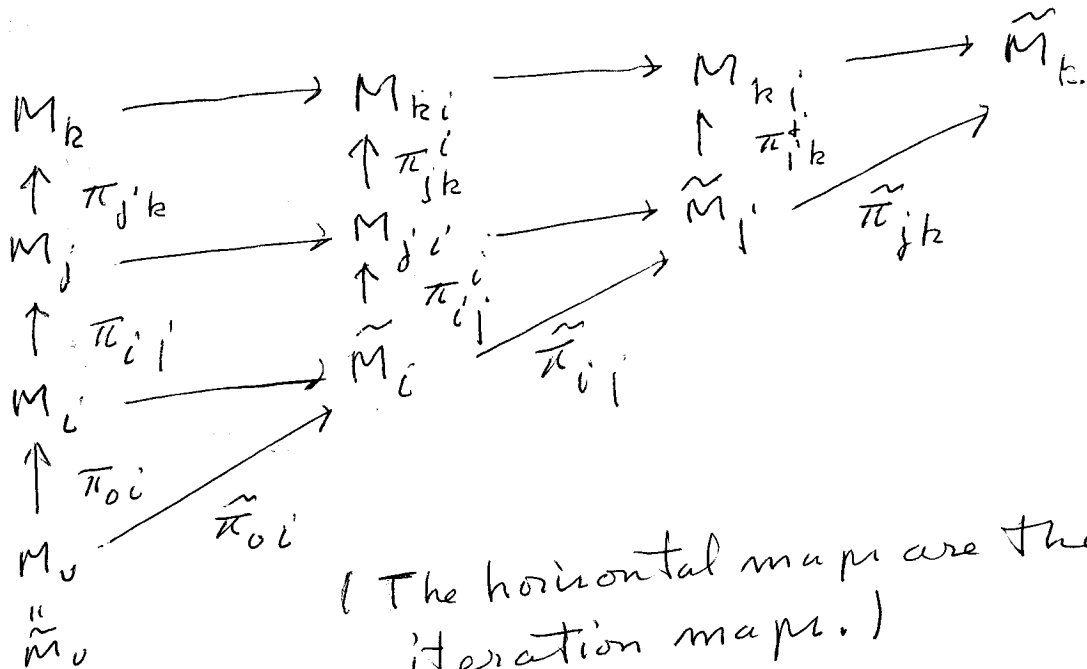uniquely defined by the conditions:

$\pi_{ij}^{0} = \pi_{ij}$, $\pi_{ij}^{\ell} k_{h\ell}^{i} = k_{h\ell}^{j} \pi_{ij}^{h}$, and

$\pi_{ij}^{h} \upharpoonright (\kappa_{i,h} + 1) = \pi_{ij}^{\ell} \upharpoonright (\kappa_{i,h} + 1)$, where

$\kappa_{i,h} = \mathrm{crit}(U_{i,h})$ $(h \leq \ell \leq i \leq j \leq \omega_1)$.

Set: $\widetilde{M}_i = M_{ii}$ + define $\widetilde{\pi}_{ij} : \widetilde{M}_i \prec \widetilde{M}_j$

by $\widetilde{\pi}_{ij} = k_{ij}^{j} \pi_{ij}^{i}$. Then:

(The horizontal maps are the iteration maps.)

Since π_{ij} is τ_i-cofinal, it follows by the remarks following Fact 6 that each π_{ij}^{ℓ} is τ_i-cofinal. But then
$\langle H_{\beta^+}^{V[G]}, \langle \widetilde{M}_i \mid i \leq \omega_1 \rangle, \langle \widetilde{\pi}_{ij} \mid i \leq j \leq \omega_1 \rangle \rangle$
is a model of $\widetilde{\mathcal{L}}$. QED (Lemma 5.1)

Now let G be $\mathbb{P} = \mathbb{P}_{\tilde{\mathcal{L}}}$ - generic. Let

$M^G = \langle \tilde{M}_i \mid i \le \omega_1 \rangle$, $\tilde{\pi}^G = \langle \tilde{\pi}_{i,j} \mid i \le j \le \omega_1 \rangle$.

Set $H_i = H_{\tau_i}^{\tilde{M}_i}$, where $\tau_i = \omega_2^{\tilde{M}_i}$.

Let $\langle M', \sigma' \rangle =$ the liftup of $\langle M_0, \tilde{\pi}_{0,\omega_1} \restriction H_0 \rangle$.

Then $\langle M', u' \rangle$ iterates to $\langle M, u \rangle$ in exactly

ω_1 many steps.

It remains only to show that $\mathcal{L}(\omega)$ is

absolute in $V[G]$. This, of course, follows

from:

__Lemma 5.2__ $\mathbb{P} = \mathbb{P}_{\tilde{\mathcal{L}}}$ is revisable.

proof.

Let $p \in \mathbb{P}$ conform to $N^* = \langle H_\delta, N, <, \dots \rangle$,

where $\delta \ge \beta^+$ is a regular cardinal.

Let $\bar{N}^*$ be defined as usual. Let $\bar{\mathcal{L}}$

be defined in $\bar{N}^*$ as $\tilde{\mathcal{L}}$ was defined

in N^*. Let $\bar{\mathbb{P}} = \mathbb{P}_{\bar{\mathcal{L}}}$ and let $\bar{G}$

be $\bar{\mathbb{P}}$ - generic over $\bar{N}^*$. Define q

by: $q = \langle \langle M^{\bar{G}}, \pi^{\bar{G}}, \emptyset \rangle, F^p \rangle$.

__Claim__ $q \in \mathbb{P}$.

We construct a solid model of $\tilde{\mathcal{L}}(q)$.

Let $\mho$ be a solid model of $\tilde{\mathcal{L}}(p)$.

Define a new model $\tilde{\mathcal{M}}$ by:

$\tilde{\mathcal{M}} = \langle |p|, \tilde{M}, \tilde{\pi} \rangle$, where:

$$\tilde{M}_i = \begin{cases} \dot{M}_i^{\mathcal{M}} & \text{for } |p| \leq i \leq \omega_1 \\ M_i^{\bar{G}} & \text{for } i \leq |p| \end{cases}$$

$$\tilde{\pi}_{ij} = \begin{cases} \pi_{ij}^{\mathcal{M}} & \text{for } |p| \leq i \leq j \leq \omega_1 \\ \pi_{hj}^{\mathcal{M}} \, \pi_{ih}^{\bar{G}} & \text{for } i \leq |p| \leq j \leq \omega_1 \\ \pi_{ij}^{\bar{G}} & \text{for } i \leq j \leq |p| \end{cases}$$

Then $\tilde{\mathcal{M}} \vDash \varphi$, where $\tilde{\mathcal{L}}(q) = \mathcal{L} + \varphi_q$.

It suffices to show:

<u>Claim</u> $\tilde{\mathcal{M}} \vDash \tilde{\mathcal{L}}$.

Let $i < j$ and let $\langle M, \sigma \rangle$ be the liftup of $\langle \tilde{M}_i, \tilde{\pi}_{ij} \restriction H_i \rangle$, where $H_i = H_{\tau_i}^{\tilde{M}_i}$, $\tau_i = \omega_2^{\tilde{M}_i}$. We must show that M' iterates to $\tilde{M}_j$ in exactly $j - i$ steps and that $\tilde{\pi}_{ij} = k\sigma$, where k is the iteration map. If $j \leq |p|$ or $|p| < i$ this is clear, so let $i < |p| < j$. Let $\langle M_0, \sigma_0 \rangle$ be the liftup of $\langle \tilde{M}_i, \tilde{\pi}_{i,|p|} \restriction H_i \rangle$. Then M_0 iterates to $\tilde{M}_{|p|}$ in $|p| - 1$ steps with iteration map k_0. Moreover

$\tilde{\tilde{\pi}}_{i,|p|} = k_0 \sigma_0$. Similarly if $\langle M_1, \sigma_1 \rangle$ is the lift up of $\langle \tilde{m}_{|p|}, \tilde{\tilde{\pi}}_{|p|j}, \upharpoonright H_{|p|} \rangle$, then M_1 iterates to $\tilde{m}_j$ in $j - |p|$ steps with iteration map k_1, and $\tilde{\tilde{\pi}}_{|p|j} = k_1 \sigma_1$. By Fact 6 it follows, however, that if $\langle M', \sigma' \rangle$ is the liftup of $\langle M_0, \tilde{\tilde{\pi}}_{|p|j}, \upharpoonright H_{|p|} \rangle$, then M' iterates to M_1 in $|p| - j$ steps, if k' is the iteration map, then $k' \sigma' = \sigma_1 k_0$. Hence M' iterates to $\tilde{m}_j$ in $(|p| - i) + (j - |p|) = j - i$ steps with iteration map $k_1 \circ k'$. But then $\tilde{\tilde{\pi}}_{ij} = \tilde{\tilde{\pi}}_{|p|j} \circ \tilde{\tilde{\pi}}_{|p|i} = k_1 \sigma_1 \circ k_0 \sigma_0 = k_1 k' \sigma' \sigma_0$, where $\langle M', \sigma' \sigma_0 \rangle$ is the liftup of $\langle \tilde{m}_i, \tilde{\tilde{\pi}}_{ij}, \upharpoonright H_i \rangle$.

$$\text{QED (Lemma 5)}$$

A modification of the proof again yields:

Lemma 6 Let E be a nicely closed extender of length γ on V. Let $\beta > \gamma$ be regular r.t. $2^{\bar{\beta}} = \beta$. There is a generic extension $V[G]$ in which:

- $\#(\omega)$ is absolute
- There is $\langle H, E' \rangle$ which iterates to $\langle H_\beta, E \rangle$ in exactly ω_1 steps.

The proof is left to the reader.

Lemmas 5, 6 and Cor 6.1 of §4 give us
further examples of revisable conditions.
Recall that in §4 we defined a set of con-
ditions Q to be β-__reshapable__ iff there
is a __revisable__ $IP = IP_{\mathcal{L}}$ s.t $BA(Q) \simeq BA(IP)$
and $\mathcal{L} \vdash \dot{M}_{\omega_1} = \underline{M}$, where $M = J_\beta^A$.

By §4 Lemma 6 we have:

__Example__ 3 Let $IP =$ the standard countable
conditions for collapsing β to ω_1,
where $2^\omega = \omega_1$ and $\omega_1 < \beta = 2^\beta$.
Then IP is reshapable.
This is because IP satisfies the
assumptions of §4 Lemma 5. The
verification is left to the reader.

__Example__ 4 Let IP be as in Example 3.
Let $S \subseteq \omega_1$ be stationary and let
Q be the set of countable conditions
for shooting a club through S
(- i.e. $p \in Q$ iff $p: \beta + 1 \to S$ is a
normal fcn for some $\beta < \omega_1$).
Then $Q \times IP$ is reshapable.
prf (sketch)

This, in fact, follows directly from §4 Lemma 5.1, since $\mathbb{Q} * \mathbb{P}$ is easily shown to be S^*-superb, where $S^* =$ the set of countable $a \subset \beta$ s.t. $a \cap \omega_1 \in S$.

Example 5 Let S be a stationary set of points of cofinality ω in β. Let $\mathbb{P}$ be the countable condition for shooting a club of length ω_1 through S (i.e. $p \in \mathbb{P}$ iff $p : \beta + 1 \to S$ is normal for some $\beta < \omega_1$). Then $\mathbb{P}$ is reshapable.

This is because $\mathbb{P}$ is S^*-superb, where $S^* =$ the set of countable $a \subset \beta$ s.t. $\sup a \in S$.

Note: It is possible to prove a 2-step iteration theorem for reshapable conditions: If $\mathbb{P}$ is β-reshapable and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is γ-reshapable, then $\mathbb{P} * \dot{\mathbb{Q}}$ is γ-reshapable. Example 4 can be used to show that a corresponding result for iterations of length ω cannot be proven.