## Appendix to §5

In lemmas 1, 2 it appeared to be essential that $\beta > \omega_1$ be regular. We now show that we can modify this condition, letting $\beta > \omega_1$ be any cardinal s.t. $2^{\beta} = \beta^+$. In particular, if $cf(\beta) = \omega_1$, we shall construct a revisable $\mathbb{P}$ which gives every regular $\tau \in (\omega_1, \beta)$ cofinality $\omega$ while preserving $\beta^+$. Hence forcing with this $\mathbb{P}$ is certainly different from collapsing each regular $\tau \in (\omega_1, \beta)$ to $\omega_2$ and then applying Namba forcing, since $\beta^+$ would be $\leq$ the $\omega_3$ of the first extension + would therefore certainly be collapsed by the second (by §4 Lemma 4.1). We shall also see that this $\mathbb{P}$ is subcomplete in the sense of our later paper "Subproper and subcomplete forcing".

<u>Note</u> The question whether it is possible - for some $\beta$ with $cf(\beta) = \omega_1$ - to make every regular $\tau \in (\omega_1, \beta)$ $\omega$-cofinal while preserving $\omega_1$ and $\beta^+$. was first formulated by Moti Gitik.

Gitik also answered the question positively for a $\beta$ in a special model which he obtained by collapsing to make a supercompact cardinal become $\omega_1$. The more general result which we present here is implicit in §5, but we failed to notice it until we saw Gitik's work. (Gitik also used his special model to answer positively the harder question: Is it possible — for some strongly inaccessible $\beta$ — to make every regular $\tau \in (\omega_1, \beta)$ $\omega$-cofinal while preserving $\omega_1$ and $\beta^+$? A more general form of that result is as yet unknown.)

We first note that Facts 1–5 hold – in an appropriate reformulation – for a wider class of structures than the transitive $ZFC^-$ models. Call $M$ <u>smooth</u> iff $M$ is transitive and is either a model of $ZFC^-$ or a model of Zermelo set theory s.t. $M = \bigcup\{H_\tau^M \mid \tau \text{ is regular in } M\}$. As before, we call a map $\sigma : M \to M'$ <u>cofinal</u> iff $M' = \bigcup\{\sigma(u) \mid u \in M\}$. It is easily seen that if $M$ is smooth and $\sigma : M \to M'$ is a cofinal $\Sigma_0$-preserving map, then $M'$ is smooth. (Conversely, if $M'$ is smooth, then so is $M$.) Write $\sigma : M \to_{\Sigma_0} M'$ to mean that $\sigma$ is a $\Sigma_0$-preserving map.

We then have:

<u>Fact 1'</u> Let $\pi : M \to_{\Sigma_0} M'$, where $M$ is smooth. Set $\tilde{M} = M' | \bigcup_{u \in M} \pi(u)$. Then $\tilde{M} \prec_{\Sigma_0} M'$ and $\pi : M \to_{\Sigma_0} \tilde{M}$ cofinally.

If $\tau > \omega$ is regular in $M$, $M$ is smooth, and $\pi : M \to M'$, we define the notion: "$\pi$ is $\tau$-cofinal" exactly as before.

... that:

<u>Fact 2'</u> Let $\tau > \omega$ be regular in $M$, where $M$ is smooth. Let $\pi : M \xrightarrow[\Sigma_0]{} M'$. Set: $\bar{H} = H_\tau^M$, $\tilde{H} = \bigcup_{u \in \bar{H}} \pi(u)$, $\bar{\pi} = \pi \restriction \bar{H}$. Then $\bar{\pi} : \bar{H} \prec \tilde{H}$.

Now let $\tau > \omega$ be regular in $M$, where $M$ is smooth. Set $\bar{H} = H_\tau^M$ + let $\bar{\pi} : \bar{H} \prec \tilde{H}$ cofinally. By a <u>liftup</u> of $\langle M, \bar{\pi} \rangle$ we mean a pair $\langle M', \pi \rangle$ s.t. $M'$ is transitive, $\pi \restriction \bar{H} = \bar{\pi}$, and $\pi : M \xrightarrow[\Sigma_0]{} M'$ $\tau$-cofinally. (Hence $M'$ is smooth.) Exactly as before:

<u>Fact 3'</u> Let $\langle M, \bar{\pi} \rangle$ be as above. Then there is at most one liftup.

<u>Fact 4'</u> The liftup of $\langle M, \bar{\pi} \rangle$ exists iff $E$ is well founded, where $E \subset \mathbb{D}'$ is def. by:

$$\langle x, f \rangle E \langle z, g \rangle \longleftrightarrow \langle x, z \rangle \in \bar{\pi}(\{\langle u, v \rangle \mid f(u) \in g(v)\}),$$

Hence the <u>interpolation lemma</u>:

<u>Fact 5'</u> Let $\pi : M \xrightarrow[\Sigma_0]{} M'$. Let $\tau > \omega$ be regular in $M$, where $M$ is smooth. Let $\bar{H} = H_\tau^M$, $\bar{\pi} = \pi \restriction \bar{H}$, $\tilde{H} = \bigcup_{u \in \bar{H}} \pi(u)$. Then the liftup of $\langle M, \bar{\pi} \rangle$ exists. Moreover, there is a unique $\sigma : \tilde{M} \xrightarrow[\Sigma_0]{} M'$ s.t. $\sigma \tilde{\pi} = \pi$ and $\sigma \restriction \tilde{H} = \mathrm{id}$.

Note that if $\beta > \omega$ and $2^{<\beta} = \beta$, then $N = \langle H_\beta, \in, \ldots \rangle$ is always smooth, since either $\beta$ is regular, in which case $N$ is a $ZFC^-$ model, or else $2^\gamma < \beta$ for $\gamma < \beta$ (since otherwise, letting $\tau < \beta$ be regular s.t. $cf(\beta) \leq \tau$, we have $2^\tau = \beta$ + hence $cf(2^\tau) \leq \tau$. Contr.!).

Let $2^{\mathfrak{B}} = \beta$, $\beta > \omega_1$. Set $M = L^A_\beta =$
$= \langle L_\beta[A], A \rangle$, where $H_\beta = L_\beta[A]$.
Set $N = \langle H_{\beta^+}, M, <, \ldots \rangle$ where, as before, $<$ is a well ordering of $N$.
Let $\mathcal{L}$ be the language on $N$ containing the basic axioms of § 3 together with:

- $\mathbb{B} = \emptyset$
- Let $i < \underline{\omega}_1$, $\tilde{\beta}_i = \sup \dot{\pi}_{i\underline{\omega}_1}{}'' \dot{\beta}_i$, and $\tilde{M}_i = L^{\underline{A}}_{\tilde{\beta}_i}$. Then $\langle \tilde{M}_i, \dot{\pi}_{i,\underline{\omega}_1} \rangle$ is the liftup of $\dot{M}_i$ by $\dot{\pi}_{i,\omega_1} \upharpoonright H^{\dot{M}_i}_{\omega_2}$.
- $\underline{\tau} \in \operatorname{rng}(\dot{\pi}_{0\underline{\omega}_1})$ if $\tau = cf(\beta) < \beta$.
- $\tilde{\beta}_0 = \underline{\beta}$ if $\beta$ is regular.

By the last two clauses, $\mathcal{L} \vdash \tilde{\beta}_0 = \underline{\beta}$ whenever $cf(\beta) \neq \omega_1$. For $cf(\beta) = \omega_1$, however, we have: $\mathcal{L} \vdash \tilde{\beta}_i < \underline{\beta}$ for $i < \underline{\omega}_1$, since otherwise $\omega_1$ would be collapsed in a model of $\mathcal{L}$. For $\beta$ regular this $\mathcal{L}$ is like the $\mathcal{L}$ of § 5 except that we have omitted the axioms which say that the entire tower $\langle M_i \mid i \leq \omega_1 \rangle$, $\langle \pi_{ij} \mid i \leq j \leq \omega_1 \rangle$ is determined by $\pi_{0\omega_1}$. (In particular, ... but by the proof

We first show:

Lemma 1 $\mathcal{L}$ is consistent.

The proof is much like that of §5 Lemma 1.

Let $N^+ = \langle H_{(2^\beta)^+}, N, <, \ldots \rangle$.

Let $\sigma : \bar{N}^+ \prec N^+$, where $\bar{N}^+$ is countable and transitive. Let $\tilde{\sigma} : \bar{N}^+ \prec \tilde{N}^+$ be the liftup of $\bar{N}^+$ by $\sigma \upharpoonright H_{\omega_2}^{\bar{N}^+}$. Then there is $k : \tilde{N}^+ \prec N^+$ s.t. $k\tilde{\sigma} = \sigma$. It suffices to show:

Claim $\tilde{\mathcal{L}}$ is consistent, where $k(\tilde{\mathcal{L}}) = \mathcal{L}$.

We construct a solid model $\mathfrak{A}$ of $\tilde{\mathcal{L}}$ as follows. For $\alpha \leq \omega_1$ set:

$X_\alpha$ = the smallest $X \prec \tilde{N}^+$ s.t. $\alpha \cup \mathrm{rng}(\tilde{\sigma}) \subset X$.

For $i \leq \omega_1$ set:

$\alpha_i$ = the $i$-th $\alpha$ s.t. $\alpha = \omega_1 \cap X_\alpha$.

(Hence $\alpha_0 = \omega_1^{\bar{N}^+}$, $X_{\alpha_0} = \mathrm{rng}(\tilde{\sigma})$, $\alpha_{\omega_1} = \omega_1$, $X_{\omega_1} = \tilde{N}^+$.)

Set: $\tilde{\sigma}_i : \bar{N}_i^+ \leftrightarrow X_{\alpha_i}$, $\tilde{\sigma}_{il} = \tilde{\sigma}_l^{-1} \circ \tilde{\sigma}_i$ for $i \leq l \leq \omega_1$. Then $\tilde{\sigma}_{il} : \bar{N}_i^+ \prec \bar{N}_l^+$ is the liftup of $\bar{N}_i^+$ by $\tilde{\sigma}_{il} \upharpoonright H_{\omega_2}^{\bar{N}_i^+}$, since the embedding is $\omega_2^{\bar N_i^+}$ - cofinal.

Set : $M_i = \tilde{\sigma}_{i\omega}^{-1}(\tilde{M})$, where $k(\tilde{M}) = M$;

$\pi_{ij} = \tilde{\sigma}_{ij} \upharpoonright M_i$.

Then $\mathfrak{A} = \langle H_{(2^\beta)^{++}}, \langle M_i \mid i \le \omega_1 \rangle, \langle \pi_{ij} \mid i \le j \le \omega_1 \rangle \rangle$

models $\tilde{\mathcal{L}}$. QED (Lemma 1).

Now let $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$.

<u>Lemma 2</u> $\mathbb{P}$ is revisable

proof.

This follows mutatis mutandis by the proof of §5 Lemma 2. We omit the details.

<u>Lemma 3</u> Assume $cf(\beta) = \omega_1$. Let $G$ be $\mathbb{P}$-generic. Then $\beta^+$ is absolute in $V[G]$.

proof. By §4 Lemma 3.

<u>Note</u> In an earlier version of "$\mathcal{L}$-forcing" the proof of §4 Lemma 3.1 contained an error. Having set: $\tilde{\beta} = \sup \pi^G_{\alpha,\omega_1}{}''\beta^P_\alpha$ (p 11) and $\tilde{M} = J^A_{\tilde{\beta}}$, we claimed that $\pi^G_{\alpha,\omega_1} : M^P_\alpha \prec \tilde{M}$. This does not hold if $\beta$ is not regular.

However, we didn't use it in the proof.
For purposes of the proof it suffices to note that if for each $a \in R^P$ we set $\tilde{a} = a \cap \tilde{M}$, then $\pi^G_{\delta,\omega_1} : \langle M^P_\delta, \bar{a} \rangle \longrightarrow \langle \tilde{M}, \tilde{a} \rangle$ is a cofinal $\Sigma_0$-preserving map, where $\langle a, \bar{a} \rangle \in F^P$.

Finally, we note that - mutatis mutandis - the proof of §3.6 Lemma 3 in "Subproper and subcomplete forcing" shows:

<u>Lemma 4</u> $\mathbb{P}$ is subcomplete.

~~We have thus proven:~~

<u>Thm 1</u> Let $2^\omega = \omega_1$, $2^\beta = \beta$, where $\beta > \omega_1$. There is a subcomplete forcing $\mathbb{P}$ s.t. whenever $G$ is $\mathbb{P}$-generic, then in $V[G]$ we have:

(a) Let $\gamma \in (\omega_1, \beta]$ be regular. Then there is a countable $X \prec H_\gamma$ s.t. $H_\gamma$ = the smallest $Y \prec H_\gamma$ with $\omega_1 \cup X \subset Y$.

(b) If $\mathrm{cf}(\beta) = \omega_1$, then $\beta^+ = \omega_2^{V[G]}$.

<u>Note</u> (a) is equivalent to: There is a countable $\bar{H}$ and a $\sigma : \bar{H} \prec H$ which is $\omega_2^{\bar{H}}$-cofinal.

<u>Note</u> At $cf(\beta) \neq \omega_1$, it follows easily that (a) holds for $\gamma = \beta$. Thus $cf(\beta) = \omega$ in $V[G]$ and it follows by §4 Lemma 4.1 that $2^\beta$ has cardinality $\omega_1$ in $V[G]$. But then $2^\beta$ has cofinality $\omega_1$ in $V[G]$, since otherwise $(2^\beta)^+$ would be collapsed in $V[G]$ by §4 Lemma 4.1. This is impossible, since $\bar{\bar{\mathbb{P}}} < (2^\beta)^+$.

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For the case that $\beta$ is strongly inaccessible we can get another result:

<u>Thm 2</u> Let $2^\omega = \omega_1$. Let $\beta$ be strongly inaccessible. There is a subcomplete forcing $\mathbb{P}$ s.t. whenever $G$ is $\mathbb{P}$-generic, then in $V[G]$ we have:

(a) Let $\gamma \in (\omega_1, \beta)$ be regular. There is a countable $X \prec H_\gamma$ s.t. $H_\gamma =$ the smallest $Y \prec H_\gamma$ with $\omega_1 \cup X \subset Y$.

(b) $cf(\beta) = \omega_1$ and $\beta^+ = \omega_2^{V[G]}$.

The proof is a repetition of the proof of , except that in place of $\mathcal{L}$ we use the language $\mathcal{L}'$ obtained by replacing the axiom $\tilde{\beta}_0 = \underline{\beta}$ by: $\tilde{\beta}_i < \underline{\beta}$ for $i < \underline{\omega_1}$. We must first prove:

<u>Lemma 5</u> $\mathcal{L}'$ is consistent.

prf. Let $\mathfrak{A}$ be a solid model of the previous language $\mathcal{L}$. Since $\beta$ is strongly inaccessible there is $X \prec N = \langle H_{\beta^+}, M, <, m \rangle$ s.t. $V_{\tilde{\beta}} \subset X$ and $\overline{\overline{X}} = \beta'$ for a $\beta' < \beta$ s.t. $\beta' = 2^{\beta'}$ and $cf(\beta') = \omega_1$. Let $\sigma : N' \overset{\sim}{\longleftrightarrow} X$. Then $\sigma : N' \prec N$. Let $\sigma(\mathcal{L}'') = \mathcal{L}'$. It suffices to show:

<u>Claim</u> $\mathcal{L}''$ is consistent.

Note that $N' \in V_\beta \subset \text{wfcore}(\mathfrak{A})$; hence $M' = \sigma^{-1}(M) = L^{A'}_{\beta'} \in \mathfrak{A}$. Pick $i_0 < \omega_1$ s.t. $M' \in \text{rng}(\dot{\pi}^{\mathfrak{A}}_{i_0 \omega_1})$. Set:

$M'_i =_{\text{Df}} (\dot{\pi}^{\mathfrak{A}}_{i_0+i, \omega_1})^{-1}(M') \quad (i \le \omega_1),$

$$\tilde{\pi}'_{ij} =_{\mathrm{Df}} \pi^{\mathfrak{A}}_{i_0+i,\, i_0+j} \upharpoonright M'_i \quad (i \leq j \leq \omega_1).$$

Using the fact that $cf(\beta') = \omega_1$ in $V_\beta$ (hence in $\mathfrak{A}$), we easily get:

(1) $\tilde{\beta}_i < \beta'$ for $i < \omega_1$, where $\tilde{\beta}_i = \sup \pi''_{i\omega_1} \beta'_i$.

Note that $|M| = V_{\beta'}^{M}$, where $M = M^{\mathfrak{A}}_{\omega_1}$.

Since $\pi^{\mathfrak{A}}_{i_0+i,\,\omega_1} : M^{\mathfrak{A}}_{i_0+i} < M$ is $\tau'_i$-cofinal, where $\tau'_i = \omega_2^{M'_i} = \omega_2^{M_{i_0+i}}$, it follows that

$$\pi'_{i,\omega_1} : M'_i \xrightarrow[\Sigma_0]{} \tilde{M}'_i \text{ is } \tau\text{-cofinal},$$

where $\tilde{M}'_i = L^A_{\tilde{\beta}_i}$. (Hence $|M'| = V_{\tilde{\beta}_i}$.)

Thus:

(2) $\pi'_{i,\omega_1} : M'_i \to \tilde{M}'_i$ is the liftup of $M'_i$ by $\pi'_{i,\omega_1} \upharpoonright H^{M'_i}_{\tau'_i}$.

Let $\mathfrak{A} = \langle A, \in_{\mathfrak{A}}, M^{\mathfrak{A}}, \pi^{\mathfrak{A}} \rangle$. It follows that $\mathfrak{A}' = \langle A, \in_{\mathfrak{A}}, M', \pi' \rangle$ models $\mathcal{L}''$, where $M' = \langle M'_i \mid i \leq \omega_1 \rangle$, $\pi' = \langle \pi'_{ij} \mid i \leq j \leq \omega_1 \rangle$. QED (Lemma 5)

We then prove the analogues of Lemmas 2, 3, and 4 for $\mathbb{P}' = \mathbb{P}_{\mathcal{L}}$, exactly as before. (In particular, the analogue of Lemma 3 says that $\beta^+$ remains a cardinal in $V[G]$.) In $V[G]$ every cardinal $< \beta$ is collapsed to $\omega_1$ and $\beta = \sup_{i<\omega_1} \tilde{\beta}_i^G$. Hence $cf(\beta) = \bar{\bar{\beta}} = \omega_1$ in $V[G]$. Hence $\beta^+ = \omega_2^{V[G]}$.

QED (Thm 2)

<u>Note</u> Let $\mathbb{P}$ be the forcing of Thm 1, Let $G$ be $\mathbb{P}$-generic. Let $u$ be any countable set in $V[G]$ which lies cofinally in $H_{\omega_2}^V$. Then $u$, in fact, accomplishes all that $G$ was intended to accomplish — i.e. in $V[u]$ we have:

Fix Skolem functions for $H_{\omega_2}^V$ and let $X_\alpha$ be the Skolem closure of $\alpha \cup u$ for $\alpha \leq \omega_1$. Set: $\pi_\alpha : Q_\alpha \overset{\sim}{\longleftrightarrow} X_\alpha$.

Then for every $\tau \in (\omega_1, \beta]$ there are $\alpha < \omega_1$, $\bar{H} \in H_{\omega_1}$ s.t. $Q_\alpha = H_{\omega_2}^{\bar{H}}$ and $\pi_\alpha$ lifts to $\tilde{\pi} : \bar{H} \prec H$. Hence $cf(\tau) = \omega$ in $V[u]$, since $\tilde{\pi} \in V[u]$.

We also note that if $u'$ is also countable and cofinal in $H^V_{\omega_2}$, then $V[u] = V[u']$. (To see $(\supset)$ let $\alpha$ be big enough that $u' \subset \mathrm{rng}(\pi_\alpha)$. Then $u' = \pi_\alpha{}''\bar{u}$ for a $\bar{u} \in H_{\omega_1}$.)

However, we have now defined $\mathbb{P}$ in such a way that $V[G] \neq V[u]$ for any such $u$. In other words the canonical complete Boolean algebra $\mathbb{B} = BA(\mathbb{P})$ over $\mathbb{P}$ is not identical to the sub-algebra $\tilde{\mathbb{B}}$ generated by $[\![ \check{x} \in \mathring{u} ]\!]$ $(x \in H_{\omega_2})$, where $\mathring{u}$ is any term s.t.

$\Vdash$ $\mathring{u}$ is countable and cofinal in $\check{H}_{\omega_2}$.

We could rectify this in the case that $\mathrm{cf}(\beta) \neq \omega_1$ by adding to $\mathcal{L}$ the axiom:

$\mathrm{rng}(\dot{\pi}_{i+1, \underline{\omega}_1}) =$ the smallest $Y \prec \underline{M}$ s.t. $d_i \cup \mathrm{rng}(\dot{\pi}_{\underline{0}, \underline{\omega}_1}) \subset Y$.

(This is exactly what we did in §3.) The case $\mathrm{cf}(\beta) = \omega_1$ is somewhat more complex. Suffice it at this place to state without proof that $\tilde{\mathbb{B}}$ is, indeed, isomorphic to the complete BA over conditions $\tilde{\mathbb{P}} = \mathbb{P}_{\tilde{\mathcal{L}}}$ which are revisable.

(That $\tilde{\mathbb{B}}$ is subcomplete follows straightforwardly by the fact that $\tilde{\mathbb{B}} \subseteq \mathbb{B}$ and card$(\tilde{\mathbb{B}})$ = card$(\mathbb{B})$, where $\mathbb{B}$ is subcomplete.)

The same remarks apply mutatis mutandis to the forcing $\mathbb{P}'$ of Thm 2. (In this case the fact that $\bar{\bar{\beta}} = \omega_2$ in $V[u]$ is established as follows: Let $\Gamma \subset H_{\omega_1}$ be the set of $\langle \alpha, \bar{H}, \nu \rangle$ s.t. $\alpha < \omega_1$, $\bar{H} \models ZFC$, $\nu \in \bar{H}$, $Q_\alpha = H_{\omega_2}^{\bar{H}}$ and $\pi_\alpha$ lifts up to $\tilde{\pi} : \bar{H} \prec H_\tau$ for a regular $\tau < \beta$. Then Set: $F(\langle \alpha, \bar{H}, \nu \rangle) = \tilde{\pi}(\nu)$. Then $F \in V[u]$ maps $\Gamma$ onto $\beta$. Since $V[G] \supset V[u]$, it follows $cf(\beta) = \omega_1$ in $V[u]$.)

that $cf(\beta) = \omega_1$ in