$\mathcal{L}$ – Forcing

In this paper we define forcing conditions $\mathbb{P}$ with the help of an infinitary language $\mathcal{L}$ on a ZFC^- model of the form $N = \langle H_\delta, \ldots \rangle$, where δ is a regular cardinal. Such forcings can be used for many purposes. In §2 we develop a class of forcings defined to add a real number which collapses to turn a regular cardinal κ into ω_1. In typical applications it also makes κ^+ ω-cofinal. (κ could be ω_1 to start with.) We use this to get the following result: Let κ be measurable, $2^{\underset{\sim}{\kappa}} = \kappa$, $\beta > \kappa$ s.t. $2^{\underset{\sim}{\beta}} = \beta$, and U a normal measure on κ. There is a forcing extension in which $\kappa = \omega_1$ and there is a countable structure $\langle \bar{H}, \bar{U} \rangle$ which iterates up to $\langle H_\beta, U \rangle$ in ω_1 many steps. In §3 we then modify our method to get forcings which add no reals. We get: Let κ be measurable, $2^\omega = \omega_1$, $\beta > \kappa$ s.t. $2^{\underset{\sim}{\beta}} = \beta$ and U a normal measure on κ. There is a forcing extension adding no reals in which a structure $\langle \bar{H}, \bar{U} \rangle$ iterates up to $\langle H_\beta, U \rangle$ in exactly ω_1 steps. (We can also arrange that $\langle \bar{H}, \bar{U} \rangle$ iterate to $\langle H_\beta, U \rangle$ in α many steps for any given $\alpha \leq \varkappa$.) The forcing used here is a special case of a more class of $\mathcal{L}$-forcings which we call <u>revisable</u>. We introduce these forcings in §3 and discuss them in greater depth in §4. A great many forcings turn out to be equivalent to revisable forcings: We call a set of conditions $\mathbb{Q}$ <u>reshapable</u> iff the complete Boolean algebra $BA(\mathbb{Q})$ engendered by $\mathbb{Q}$ is isomorphic to $BA(\mathbb{P})$ where $\mathbb{P}$ is revisable. It turns out f.i., that whenever $2^\omega = \omega_1$, $\beta > \omega_1$, $2^\beta = \beta$, and $\mathbb{Q}$ is a proper set of conditions of size β which collapses β to ω_1 without adding new reals, then $\mathbb{Q}$ is reshapable. (This is proven in §4.) However, there are many improper revisable forcings, such as the forcing on a measurable described above. Assuming $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$, a particularly simple revisable forcing makes ω_2 cofinal with ω without adding new reals.

I originally considered this forcing, which I shall now call $\mathbb{P}$, a striking alternative to Namba forcing, since, indeed, the motivation of the construction and the combinatorics of the proof are very different. In §6, however, we show that $\mathbb{P}$ is equivalent to Namba forcing and hence that Namba forcing is reshapable. We also show that the variant of Namba forcing which Shelah calls Nm' is equivalent to a variant of $\mathbb{P}$ and is hence reshapable. (The forcing $\mathbb{P}$ is that described in "Example 1" in §5.)

We first discovered $\mathcal{L}$-forcing in the early 1990's. Initially we had only the forcing in §2, which adds a new real. We used it in our note "On some problems of Mitchell, Welch and Vickers", which is now appended to the present paper. Subsequently we worked out a form of $\mathcal{L}$-forcing which does not add reals. (Until recently, a very clumsy version of that could be found on my website.) We have now completely reworked the approach, arriving at the theory of "revisable forcing" developed in §3 – §6.