§1 Normal iterability implies iterability

In our previous notes [ANFS] (Addenda to A New Fine Structure for Higher Core Models) we showed that if θ is an inaccessible cardinal satisfying three conditions A1–A3, then any premouse $M \in V_\theta$ which is normally iterable in V_θ is, in fact, a mouse in V_θ. We also show that any countable premouse which is countably normally iterable is a mouse. It follows from this that if M is weakly normally iterable, then M is a mouse. (The assumption for all of these results is that M is 1-small!) Our three assumptions read:

A1 Either no $\tau < \theta$ is Woodin in an inner model or else V_θ is closed under #

A2 Let $M \in V_\theta$ be a 1-small premouse and $\mathcal{I}$ a normal iteration of M of length θ. Then $\mathcal{I}$ has a cofinal branch.

A3 θ is Mahlo.

The main result of this section is that A3 is superfluous. We recall that A3 was needed to ensure that a particularly strong background condition for placing extenders on our construction of K^c could be met. With the help of A3, we then showed that K^c is a universal weasel in V_Θ. Without A3 it was possible to construct K^c with a weaker background condition (§1 Def 1.2 in the published version of [S] (The Core Model Iterability Problem).) A1-A2 then implied that K^c is iterable in the Mitchell-Steel sense (hence normally iterable) and our proof showed that K^c is universal (hence that every normally iterable premouse is Mitchell-Steel iterable). Our present result shows that this K^c is fully iterable. Hence all notions of iterability for 1-small mice coincide.

Consider the assumption:

(+) Let M be a countable, countably iterable 1-small premouse. Let $\mathcal{I}$ be an iteration of length ω_1. Then $\mathcal{I}$ has a cofinal branch.

(<u>Note</u> If (+) holds, then the results of [NFS] §7-8 hold for all weak mice. (+) follows from either of the assumptions:

(a) ω_1 is not Woodin in an inner model.

(b) $a^\#$ exists for all $a \subset \omega_1$.)

Thus A1 → (+), Our main theorem reads:

<u>Theorem 1</u> Assume (+). Let $M \in V_\theta$ be a 1-small premouse which is $\theta + 1$-normally iterable. Then M is a weak mouse.

("δ-normally iterable" means that every normal iteration of length $< \delta$ can be continued.)

(We recall that M is a <u>weak mouse</u> iff whenever $\sigma : \bar{M} \to_{\Sigma^*} M$ and $\bar{M}$ is countable, then $\bar{M}$ is countably iterable. M is <u>weakly normally iterable</u> iff each such $\bar{M}$ is countably normally iterable.)

Now assume A1+A2 and form Steel's K^c in V_θ, using the abovementioned weaker background condition.

<u>Corollary 2</u> Assume A1+A2. Then K^c is defined and θ-iterable. Moreover K^c is universal in V_θ.

proof. (sketch)

Steel's methods show that K^c is defined and MS-iterable (hence normally iterable) below θ. By A2, each proper segment of K^c is a weak mouse. Now suppose e.g. that no inner model of V_θ has a Woodin cardinal. Then there are arbitrarily large initial segments $K^c_\alpha = \langle J^E_\alpha, E_{\omega\alpha}\rangle$ satisfying "There is no $\nu \leq \alpha$ s.t. $J^E_\beta \models$ there is a Woodin cardinal, and $E_\nu \neq \emptyset$ or $\nu = \omega\alpha$." The smooth iterations of such segments have unique cofinal branches by [NFS] §6. Since K^c_α is a weak mouse, it follows by a method of Woodin that K^c_α is iterable in V_θ. Hence K^c is iterable in V_θ. If V_θ is closed under # but there is an inner model with a Woodin cardinal, we use the fact that each K^c_α is a segment of a premouse M which is

countably normally iterable and whose normal iterations have unique cofinal branches. (The core of M is Woodin's Y_0, the first mouse which is not 1-small
By closure under #, it follows that M is normally iterable in V_θ. Hence by A2 M is $\theta+1$-iterable, hence a weak mouse by Thm 1. But the smooth iterations of M have unique branches. Hence M is iterable in V_θ. Hence so is K^c_α. Hence so is K^c. Using A2 we then prove exactly as before that K^c is universal in V_θ.

QED (Corollary 2)

We can then repeat the proof of [ANFS] §3 Thm 7 to get:

<u>Corollary 3</u> Assume A1 + A2. Let Q be a countable one-small premouse which is countably normally iterable. Then Q is iterable in V_θ.

Hence

<u>Corollary 3.1</u> Assume A1 + A2. Let $Q \in V_\theta$ be a weakly normally iterable 1-small premouse. Then Q is a weak mouse.

Thus, assuming A1 + A2, the notions "iterable" and "normally iterable" coincide, as do the notions "weakly iterable" and "weakly normally iterable".

<u>Remark</u> Steel's proofs involve machinery which is in some ways different from that of [NFS] (e.g. n-ultrapowers, degree functions etc.). It is, in fact, possible to prove Corollary 2 using only the methods of [NFS]. To this end we must directly modify the proof of Thm 1 the way the proof of [ANFS] §3 Corollary 6.1 was modified to give [ANFS] §3 Thm 7, in order to get Corollary 3 above <u>before</u> we prove Corollary 2. We then know that every weakly normally iterable 1-small premouse is a weak mouse, + hence satisfies the theorems in [NFS] §7, §8. Thus we can form the sequence of premice $\langle N_i \rangle$ which converges to K^c and show inductively that each N_i is weakly normally iterable, hence a weak mouse. This enables us to continue the construction up to Θ.

It remains to prove Thm 1. Let Q be a premouse which is $\theta+1$ - normally iterable. Our strategy will be to provide Q with a "background array" $\langle N_i \mid i \leq \xi \rangle$ similar to the array which converges to K^c. Each element of this array will be a weak mouse. It will turn out, however, that the array stops at a $\xi < \theta$ and that N_ξ is a simple iterate of Q, thus showing that Q is a weak mouse. In fact, each N_i will be a segment of a simple normal iterate Q_i of Q (in the sense that either $N_i = Q_i \| \alpha$ or $N_i = \langle J_\alpha^E, \emptyset \rangle$ and $Q_i \| \alpha = \langle J_\alpha^E, E_{\omega\alpha} \rangle$ for some α). We stop when we reach a point ξ s.t. $N_\xi = Q_\xi$.

[However, the Q_i will not in general be the components of a single normal iteration. We may from time to time revise the iteration which gave Q_i by cutting back to an initial segment and proceeding differently from then on.

<u>Def</u> By an <u>array</u> we mean a sequence $\langle N_i \mid i < \Gamma \rangle$ s.t.

(a) N_i is a 1-small weak mouse for $i+1<$

(b) $N_0 = \langle \emptyset, \emptyset \rangle$

(c) Let $i+1 < \Gamma$, where $\langle J^E_\nu, E_{\omega\nu} \rangle = \mathrm{core}(N_i)$. Then either $N_{i+1} = \langle J^E_{\nu+1}, \emptyset \rangle$ or else $E_{\omega\nu} = \emptyset$ and $N_{i+1} = \langle J^E_\nu, F \rangle$ where $F \neq \emptyset$.

(d) Let $\mathrm{Lim}(\lambda)$, $\lambda < \Gamma$. For $\xi < \lambda$ set:

$$\kappa_\xi = \kappa_{\xi,\lambda} = \inf \{ \omega\rho^\omega_{N_i} \mid \xi \leq i < \lambda \}$$

$$\mu_\xi = \mu_{\xi,\lambda} = \kappa_\xi^{+N_\xi} \quad */.$$

If

$$J^{E^{N_\xi}}_{\mu_\xi} = J^{E^{N_i}}_{\mu_\xi} \text{ for } \xi \leq i < \lambda, \text{ then}$$

$$N_\lambda = \Big\langle \bigcup_{\xi < \lambda} J^{E^{N_\xi}}_{\mu_\xi}, \emptyset \Big\rangle.$$

$*/$ $\kappa^{+M} =_{df} \begin{cases} ht(M) \text{ if } \kappa = \omega \cdot ht(M) \\ \tau \text{ if not, where } \tau \leq ht(M) \text{ is maximal s.t. } \kappa = \text{the largest cardinal in } J^{E^M}_\tau \end{cases}$

We shall employ the notation $\kappa_{\xi,\lambda}, \mu_{\xi,\lambda}$ in the above sense even when $\lambda > \xi$ is not a limit ordinal for $\lambda \leq \Gamma$. Set: $M_i =_{df} \mathrm{core}(N_i)$.

The proofs of the following theorems are essentially contained in [NFS] §10, 11.

<u>Fact 1</u> Let $\kappa = \omega\rho^{\omega}_{M_\xi} = \omega\rho^{\omega}_{M_{\xi+1}}$. Then

$$\kappa^{+M_{\xi+1}} = \kappa^{+N_{\xi+1}} = ht(M_\xi)+1 > \kappa^{+M_\xi}$$

<u>Fact 2</u> Let $\kappa = \kappa_{\ell i} = \omega\rho^{\omega}_{N_\ell}$. Then

$$\ell < h < j < i \longrightarrow \kappa < \mu_{hi} < \mu_{ji}$$

<u>Fact 3</u> If $\mathrm{Lim}(\lambda)$, then $\bigwedge i<\lambda \bigvee j<\lambda\, (i<j \wedge \mu_{i\lambda}<)$

<u>Fact 4</u> $J^{E^{W_\xi}}_{\mu_{\xi,i}} = J^{E^{N_h}}_{\mu_{\xi,i}}$ for $\xi \leq h < i \leq \Gamma$

(Thus if the length Γ of the array is ∞ or a limit ordinal, we may define $N_\Gamma = \langle \bigcup_{\nu<\Gamma} J^{E^{N_\nu}}_{\mu_{\nu,\Gamma}}, \emptyset \rangle$.)

(Fact 1 is (1) following the definition of the array $\langle N_i \mid i < \theta \rangle$ in [NFS] §10. Fact 2 follows by induction on i, using Fact 1. Fact 3 is proven (though not exactly so stated) in (2) following (1) in [NFS] §10. This is followed by a sketch of the proof of Fact 4 in [NFS] §10.)

The following facts were proven in [NFS] §11. Since their proofs were rather garbled, we reprove them here.

Let $\gamma \leq \Gamma \leq \infty$ s.t. $Lim(\gamma)$. Let $ht(N_\gamma) = \mu$. Let $\omega < \lambda < \mu$ s.t. λ is a limit ordinal (or $\lambda = \infty$) and is cardinally absolute in N_γ (i.e. if $\tau < \lambda$ is a cardinal in $J_\lambda^{E^{N_\gamma}}$, then τ is a cardinal in N_γ). Set:

$$\delta = \delta(\lambda) = \delta(\lambda, \gamma) = \sup\{\xi < \gamma \mid \mu_{\xi,\gamma} < \lambda\}$$

<u>Fact 5</u> δ is a limit ordinal.

prf. Suppose not. Let $\delta = \alpha + 1$.

(1) $ht(M_\alpha) \geq \lambda$

since otherwise $ht(N_{\alpha+1}) \leq ht(M_\alpha) + 1 < \lambda$, where $\mu_{\alpha\gamma} \leq ht(N_{\alpha+1})$. Contr!

Let $\kappa = \kappa_{\alpha\gamma}$.

(2) $\omega\rho^\omega_{N_\alpha} = \kappa$,

since otherwise $\omega\rho^\omega_{N_\alpha} > \kappa$ is a cardinal in $N_{\alpha+1}$. Hence $\mu_{\alpha+1,\gamma} = \kappa^{+N_{\alpha+1}} = \kappa^{+N_\alpha}$ $= \mu_{\alpha,\gamma} < \lambda$. Contr!

But then the case $N_\alpha = \langle J^E_\beta, \emptyset\rangle$, where $E_\beta \neq \emptyset$ is excluded, since then $\mu_{\alpha,\gamma} \geq \kappa \geq \omega\rho^\omega_{N_\alpha} = ht(N_\alpha) \geq \lambda$. Hence

(3) $N_{\alpha+1} = \langle J^{\bar{E}}_{\beta+1}, \emptyset\rangle$, where

$$M_\alpha = \langle J^{\bar{E}}_\beta, \bar{E}_{\omega\beta}\rangle,$$

(4) $\mu_{\alpha,\gamma}$ is not a cardinal in $N_{\alpha+1}$, since $\mu_{\alpha,\gamma} > \kappa = \kappa_{\alpha,\gamma}$ by (1) and κ is the largest cardinal in $N_{\alpha+1}$.

(5) $\mu_{\alpha\gamma} = \kappa^{+M_\alpha}$ by [NSF] §8 Lemma 5.

Let β be as in (3). Then $\beta+1 \geq \lambda$. Hence $\beta \geq \lambda$, since Lim(λ). Hence $\mu_{\alpha\gamma} < \lambda$ is a cardinal in $M_\alpha = \langle J_\beta^{\bar{E}}, \bar{E}_{\omega\beta}$ hence in $J_\lambda^{\bar{E}}$. But $J_{\beta+1}^{\bar{E}} =$

$= J_{\mu_{\alpha+1,\gamma}}^{E^{M_{\alpha+1}}} = J_{\mu_{\alpha+1,\gamma}}^{E^{N_\gamma}}$. Hence $\mu_{\alpha,\gamma}$ is a cardinal in $J_\lambda^{E^{N_\gamma}}$, hence in N_γ, hence in $N_{\alpha+1}$. Contr!

QED (Fact 5)

<u>Fact 6</u> $\mu_{i\gamma} = \mu_{i\delta}$ for $i < \delta$.

proof. Suppose not.
Let $\alpha \geq \delta$ be least s.t. $\omega\rho^\omega_{N_\alpha} = \kappa_{\alpha,\gamma} = \kappa$.

Set $\kappa_0 = \sup_{i<\delta} \kappa_{i\delta}$. Then $\kappa_0 > \kappa$ and either $\kappa_0 = \lambda$ is a limit of cardinals in N_δ or else $\kappa_0 > \kappa$ is a cardinal in N_δ. Either $\delta = \alpha$ or $\kappa_{\delta\alpha} > \kappa$ is

a cardinal in N_α, where $J^{E^{N_\alpha}}_{\kappa_{\delta\alpha}} = J^{E^{N_\delta}}_{\kappa_{\delta\alpha}}$.
It follows easily that if $\kappa' =$
$= \kappa_0$ if $\delta = \alpha$; $\kappa' = \min(\kappa_0, \kappa_{\delta\alpha})$ if not, then $\kappa < \kappa' < \lambda$ and κ' is a cardinal in N_α. Hence $\mu_{\alpha,\gamma} = \kappa^{+N_\alpha} \leq$
$\leq \kappa' < \lambda$. Contr! QED (Fact 6)

<u>Fact 7</u> $N_\delta = \langle \bigcup_{i<\delta} J^{E^{N_i}}_{\mu_{i\gamma}}, \emptyset \rangle = \langle J^{E^{N_\gamma}}_\lambda, \emptyset \rangle$

prf.
The first equation is immediate by Fact 6.
For the second we need $ht(N_\delta) = \lambda$.
If not, then $\mu_{\delta\gamma} \leq ht(N_\delta) < \lambda$. Contr.

<u>Fact 8</u> $M_\delta = N_\delta$ and $\mu_{\delta\gamma} = \lambda$.

prf.
<u>Case 1</u> λ is a cardinal in N_γ. Then $\mu_{\delta\gamma} = \kappa_{\delta\gamma} = \lambda$. In particular $wp^\omega_{N_\delta} = \lambda$ + hence $M_\delta = N_\delta$.

<u>Case 2</u> Case 1 fails. Then λ is not a limit of cardinals in N_δ, since it would then be a limit cardinal in N_γ, by cardinal absoluteness. Let $\kappa =$
$=$ the largest cardinal in N_δ. Then κ is a cardinal in N_γ, but λ is not,

It follows easily that $\kappa_{\delta\gamma} = \kappa$. Hence $\mu_{\delta\gamma} = \kappa^{+N_\delta} = \lambda$. QED (Fact 8).

An immediate corollary of this proof is:

<u>Fact 9</u> If λ is a cardinal in N_γ, then $\kappa_{\delta\gamma} = \mu_{\delta\gamma} = \lambda$.

(This holds, in particular, when λ is a limit of cardinals in N_δ.)

. . . .

<u>Def</u> Let $\mathcal{J} = \langle\langle P_i\rangle, \langle \nu_i \mid i \in D\rangle, \langle \eta_i\rangle, \langle \pi_{i_1 i}\rangle, T\rangle$ be a normal iteration of a 1-small premouse P. Let $\lim(\lambda)$, $\lambda \leq lh(\mathcal{J})$. We say that a branch b cofinal in λ is <u>economical</u> iff P_b is well founded and $\lambda = \sup D \cap \lambda$ and $E_\nu^{P_b} = \emptyset$ whenever $\nu \geq \sup\{\nu_i \mid i \in D \cap \lambda\}$. We know by [NFS] §6 that there is at most one noneconomical well founded cofinal branch for λ. We say that $\mathcal{J}$ obeys the <u>economical strategy</u> iff $b_\lambda = \{i \mid i T \lambda\}$ is chosen to be economical wherever possible. Note that if b_λ is economical, then $\mathcal{J}|(\lambda+1)$ can be only trivially extended - i.e. $\nu \notin D$, $P_\nu = P_\lambda$ for all $\nu \geq \lambda$. Moreover, for

$\mu < \lambda$, $\lim(\mu)$, b_μ is the unique well founded cofinal branch. Thus unique branches are chosen at all points except

. . . .

We now assume $Q \in V_\theta$ where θ is inaccessible and Q is a $\theta+1$-normally iterable 1-small premouse. We construct an array $\langle N_i \mid i < \Gamma \rangle$ $(\Gamma \leq \theta)$ Simultaneously we construct sets $D_i \subset i$ and normal iterations

$\mathcal{J}^i = \langle \langle Q^i_h \rangle, \langle \nu_h \mid h \in D_i \rangle, \langle \eta^i_h \rangle, \langle \pi^i_{hj} \rangle, T^i \rangle$

of Q of length $i+1$ s.t.

(a) $h \leq i \longrightarrow h \cap D_i \subset D_h$

(b) Let $D^* =$ the closure of D in i. Then

$h \in D^* \longrightarrow \mathcal{J}^h = \mathcal{J}^i | (h+1)$

(Hence $D_h = h \cap D_i$)

(c) $Q_i = Q^i_i$

(d) $\mathcal{J}^i$ obeys the economical strategy

(e) Let $\alpha < i$ s.t. $\kappa_{\beta i} < \kappa_{\alpha i}$ whenever $\beta < \alpha$. Then $\mathcal{J}^\alpha = \mathcal{J}^i | (\alpha+1)$,

(Note that ν_h is defined for $h \in \bigcup_i D_i$ independently of i.)

(f) If $h \in D_i$, then $\nu_h = \mathrm{ht}(N_h)$.

We again define:

<u>Def</u> Let $M = \langle J_\alpha^E, F\rangle$ be a premouse s.t. $F \neq \emptyset$. Let $\kappa = \mathrm{crit}(F)$. $\langle N, G\rangle$ is a <u>background</u> <u>certificate</u> for M iff

(i) N is a transitive ZFC$^-$ model s.t. $V_\kappa \in N$

(ii) G is an extender of length $> \lambda = lh(F)$ at κ on N,

(iii) Let $\pi : N \longrightarrow_G N'$. Then $V_{\lambda+2} \subset N'$

(iv) $F(X) = \lambda \cap G(X)$ for $X \in \mathbb{P}(\kappa) \cap N \cap M$.

<u>Note</u> This is the same as the definition given in [NFS] §11 except that we now require $V_{\lambda+2} \subset N'$ instead of $V_{\lambda+1} \subset N'$. Unfortunately we were unable to carry out the present proof or, in fact, the proofs of the results stated in [NFS] §11 and [ANFS] without this change.

We construct N_i, Q_i, $\mathcal{T}^i$ by induction on i, verifying (a)—(h) at each stage.

<u>Case 1</u> $i = 0$. Set $Q_0 = Q$, $N_0 = \langle\emptyset, \emptyset\rangle$. $\mathcal{T}^0$ is then the iteration of Q of length 1.

<u>Case 2</u> $i = j+1$.
If N_j is not a weak mouse, then N_{j+1}, $\mathcal{T}^i$ are undefined. Otherwise:

<u>Case 2.1</u> $N_j = Q_j$ and π_{0j}^j is total. Then N_{j+1}, $\mathcal{T}^{j+1}$ are undefined.

<u>Case 2.2</u> $N_j = Q_j$ and Case 2.1 fails. Then there is a truncation on the branch to j. Let $h+1 \leq_T j$ be maximal s.t. $\eta_h^j < ht(Q_\xi^j)$ where $h \in D^j$, $\xi = T^j(h+1)$.

Then $M_j = \mathrm{core}(N_j) = Q_\xi^j \| \eta_h^j$. We set:
$N_{j+1} = Q_\xi \| (\eta_h^j + 1)$, $\mathcal{T}^{j+1} | (\xi+1) = \mathcal{T}^\xi$, $D_{j+1} = D_\xi$. (Hence $Q_\ell^{j+1} = Q_\xi$ for $\xi \leq \ell \leq j+1$ and $Q_{j+1} = Q_\xi$.) The verification of (a)—(d), (f) are trivial. We verify (e).

We first note that if $\kappa_{\beta i} < \kappa_{\alpha i}$ for all $\beta < \alpha$, then $\kappa_{\beta i} = \kappa_{\beta l} < \kappa_{\alpha, l} \le \kappa_{\alpha i}$ for all l s.t. $\alpha \le l \le i$. For $\alpha \le \xi$ the conclusion is then trivial, since $\mathcal{T}^{i+1} | (\alpha+1) = \mathcal{T}^{\xi} | (\alpha + 1)$. We now show that the case $\xi < \alpha \le j$ is excluded. It suffices to show: *)

<u>Claim</u> $\omega\rho^{\omega}_{N_\alpha} \ge \nu_\xi$ for $\xi < \alpha < j$.

Suppose not. Let α be the least counterexample. Since $ht(N_\xi) = \nu_\xi$ and $\omega\rho^{\omega}_{N_l} \ge \nu_\xi$ for $\xi < l < \alpha$, it follows that $ht(N_\alpha) > \nu_\xi$. But $\xi \in D_j \cap \alpha \subset D_\alpha$. Hence ν_ξ is a cardinal in Q_α. Hence $N_\alpha = Q_\alpha$ since otherwise $\nu_\xi < ht(N_\alpha) \le ht(Q_\alpha)$ and N_α is a "proper segment" of Q_α, which implies $\omega\rho^{\omega}_{N_\alpha} \ge \nu_\xi$. But then Case 2.2 applied at α and there are $k+1 \le_{T^\alpha} \alpha$, $\mu = T^\alpha(k+1)$ s.t. $M_\alpha = \mathrm{core}(N_\alpha) = Q_\mu \| \eta^\alpha_k$ where $\eta^\alpha_k < ht(Q_\mu)$ and $D_{\alpha+1} = D_\mu$. But

*) Since $\omega\rho^{\omega}_{\bar{Z}_i} \le \mathrm{crit}(E^{Q_i}_{\nu_i}) < \nu_\xi$

$\xi \in D_i \cap (\alpha+1) \subset D_{\alpha+1} = D_\mu$. Hence $\xi < \mu$ and ν_ξ is a cardinal in Q_μ. Since M_α is a proper segment of Q_μ, we conclude $\omega\rho^\omega_{N_\alpha} = \omega\rho^\omega_{M_\alpha} \geq \nu_\xi$. Contr! QED (Case 2.2)

<u>Note</u> This shows generally that if $\xi \in D_i$, then $\omega\rho^\omega_{N_\alpha} \geq \nu_\xi$ for all $\alpha \in (\xi, i)$.

<u>Case 2.3</u> $N_i \neq Q_i$

Then N_i is a "proper segment" of Q_i (i.e. either $N_i = Q_i \| \alpha$ for an $\alpha < ht(Q_i)$ or $N_i = \langle J^E_\alpha, \emptyset \rangle$, where $\alpha \leq ht(Q_i)$ and $Q_i \| \alpha = \langle J^E_\alpha, F \rangle$, $F \neq \emptyset$.)

Thus N_i is sound. Hence $M_i = N_i$. We set: $\mathcal{J}^{i+1} | (i+1) = \mathcal{J}^i$ (hence $D_{i+1} \cap i = D_i$). Let $\nu = ht(N_i)$

<u>Case 2.3.1</u> $N_i = Q_i \| \nu$

Define $\mathcal{J}^{i+1}$ by setting: $i \notin D_{i+1}$ (hence $Q_{i+1} = Q_i$). Set: $N_{i+1} = Q_i \| \nu + 1$.

<u>Case 2.3.2</u> Case 2.3.1 fails.

Then $N_i = \langle J^E_\nu, \emptyset \rangle$, where $Q_i \| \nu = \langle J^E_\nu, F \rangle$ and $F \neq \emptyset$. We first ask whether:

(*) $\kappa > ht(Q_0)$ is a cardinal, where $\kappa = crit$

If not, set $i \in D_{i+1}$, $\nu_i = \nu$;

This defines $\mathcal{J}^{i+1}$ with $E_\nu^{Q_{i+1}} = \emptyset$. We then set:

$$N_{i+1} = Q_{i+1} \| (\nu+1) = \langle J_{\nu+1}^{E^{Q_i} \cap J_\nu^{Q_i}}, \emptyset \rangle.$$

However, in order that $\mathcal{J}^{i+1}$ be normal we must still verify:

Claim $\nu > \nu_\xi$ for $\xi \in D_i$

Suppose not. Let ξ be a counterexampl. Since ν is a limit ordinal and $N_i = \langle J_\nu^E, \emptyset \rangle$, it follows that i is a limit ordinal and $\nu = \sup_{\alpha < i} \mu_{\alpha i}$, where $\mu_{\alpha i} < \nu \leq \nu_\xi$ for all $\alpha < i$. But then there is $\alpha \in (\xi, i)$ s.t. $\omega\rho_{N_\alpha}^\omega < \nu_\xi$. Contr!

QED (Claim)

Now suppose that (*) holds. Since $\mathcal{J}^\kappa$ is a normal iteration of Q of length κ, we have $\mathcal{J}^\kappa \subset H_\kappa$. Set: $\mathfrak{U} = \mathfrak{U}_\kappa = L_\alpha[\mathcal{J}^\kappa]$, where α is least s.t. $L_\alpha[\mathcal{J}^\kappa]$ is admissible. (Hence $\alpha < \kappa^+$

Note that if $\kappa \leq \kappa_{\kappa, i}$, then $\kappa_{\beta, i} < \kappa_i$ for $\beta < \kappa$, in which case $\mathcal{J}^\kappa = \mathcal{J}^i(\kappa+1)$.

We ask whether:

(**) $\kappa = ht(N_\kappa) \leq \kappa_{\kappa j}$ and F has a background certificate $\langle N, F^* \rangle$ s.t. $\mathcal{M} \in N$.

If not, proceed exactly as before. Now let (**) hold. We define $\mathcal{J}^{j+1}$ by: $j \notin D_{j+1}$ (hence $Q^{j+1} = Q^j$) and set: $N_{j+1} = Q_{j+1} \| \nu = \langle J_\nu^E, F \rangle$.

This completes the construction in Case 2.3. We verify (a)—(f). (a) — (d) and (f) are immediate. (e) follows by $\mathcal{J}^{j+1} | j+1 = \mathcal{J}^j$ and the fact that if $\alpha \leq j$ and $\kappa_{\beta, j+1} < \kappa_{\alpha, j+1}$ for $\beta < \alpha$, then $\kappa_{\beta j} = \kappa_{\beta j+1} < \kappa_{\alpha j}$ for $\beta < \alpha$. QED (Case 2.3)

<u>Case 3</u> $i = \lambda$, $\lim(\lambda)$

We are constrained to set:

$$N_\lambda = \langle J_\mu^E, \emptyset \rangle = \langle \bigcup_{i<\lambda} J_{\mu_{i\lambda}}^{E^{N_i}}, \emptyset \rangle,$$

where $J_{\mu_{i\lambda}}^{E^{N_i}} = J_{\mu_{i\lambda}}^{E^{Q_i}}$ and $\mu_{i\lambda} < \mu$ for $i < \lambda$.

Case 3.1 There is $\kappa < \mu$ s.t. $\kappa_{i\lambda} = \kappa$ for sufficiently large $i < \lambda$. Then $J_\kappa^{E^i} = J_\kappa^E$ for sufficiently large $i < \lambda$ and κ is a cardinal in N_λ. Set:

$$\delta = \delta(\kappa, \lambda) = \begin{cases} 1 \text{ if } \kappa = \omega, \\ \sup\{\xi < \lambda \mid \mu_{\xi\lambda} < \kappa\} \text{ if not} \end{cases}$$

Then δ is a limit cardinal if $\kappa > \omega$. We also have: $N_\delta = \langle J_\tau^E, \emptyset \rangle$ where $\kappa = \omega\tau$; $\kappa = \kappa_{\delta\lambda} = \mu_{\delta\lambda}$. and $\mu_{i\lambda} = \kappa^{+N_i} > \kappa$ for $\delta < i < \lambda$. (At $\kappa = \omega$, these facts are obvious. Otherwise the follow from Facts 1 - 9 stated earlier.)

Claim Let $\delta < i < \lambda$. Then $\kappa^{+Q_i} \leq \kappa^{+Q_{\delta+1}}$ and $(J_{\kappa^+}^E)^{Q_i} \subset (J_{\kappa^+}^E)^{Q_{\delta+1}}$.

prf. Suppose not.
Let i = the least counterexample. Then $i > \delta+1$.
Let $h = \text{lub}\, D_i$. Then $h = i$, since otherwise $Q_i = Q_j$ for a $j < i$ + i is not minimal. Pick $j \in D_i$ s.t. $j \geq \delta + 1$. Then $\nu_j = \text{ht}(N_j) > \kappa$, and $J_{\nu_j}^{EQ_j} = J_{\nu_j}^{EQ_i}$, where ν_j is a cardinal in Q_i. Hence $(J_{\kappa^+}^{EQ_i}) = (J_{\kappa^+}^{EQ_j}) \subset (J_{\kappa^+}^{EQ}$

QED (Claim)

But then for $\delta < i < \lambda$ we have:
$\mu_{i\lambda} = \kappa^{+N_i} \leq \kappa^{+Q_i} \leq \kappa^{+Q_{\delta+1}}$ and $J_{\mu_{i\lambda}}^{EQ_i} = J_{\mu_{i\lambda}}^{EQ_{\delta+1}}$. But $N_\lambda = \langle J_\mu^E, \emptyset$

where $J^E_\mu = \bigcup_{\delta < i < \lambda} J^{E^{Q_i}}_{\mu_{i\lambda}} \subset (J^E_{\mu^+})^{Q_{\delta+1}}$,

We set: $\mathcal{T}^\lambda | (\delta+2) = \mathcal{T}^{\delta+1}$, $D_\lambda = D_{\delta+1}$;

(Hence $Q_\lambda = Q_i^\lambda = Q_{\delta+1}$ for $\delta < i \leq \lambda$).

We verify (a)–(f). (a)–(d), (f) are immediate. To verify (e), we note that if $\alpha < \lambda$ s.t. $\kappa_{\beta\lambda} < \kappa_{\alpha\lambda}$ for all $\beta < \alpha$, then $\alpha \leq \delta$ + hence $\mathcal{T}^\lambda | (\alpha+1) = \mathcal{T}^\delta | (\alpha+1) = \mathcal{T}^\alpha$.

<u>Case 3.2</u> Case 3.1 fails.

Let S = the set of $\alpha < \lambda$ s.t. $\kappa_{\beta\lambda} < \kappa_{\alpha\lambda}$ for all $\beta < \lambda$. Then $\sup S = \lambda$. Thus $\mathcal{T}^\gamma = \mathcal{T}^\alpha | \gamma+1$ for $\gamma < \alpha$; $\gamma, \alpha \in S$. Hence $\mathcal{T} = \bigcup_{\alpha \in S} \mathcal{T}^\alpha$ is an iteration of length λ. Pick a cofinal well founded branch b in $\mathcal{T}$ obeying the economical strategy. (<u>Note</u> If there is a truncation in b, then b was, in fact, the unique choice.) This gives us $\mathcal{T}^\lambda$ of length $\lambda+1$ with $\mathcal{T}^\lambda | \lambda = \mathcal{T}$ and $b = \{i \mid i \leq_T \lambda \text{ in } \mathcal{T}^\lambda\}$. We recall that $N_\lambda = \langle J^E_\mu, \emptyset \rangle$, where $J^E_\mu = \bigcup_{i<\lambda} J^{E^{N_i}}_{\mu_{i\lambda}}$.

We claim:

<u>Claim</u> $J_\mu^E = J_\mu^{EQ_\lambda}$.

prf. Set $k = \text{lub } D_\lambda$. If $k < \lambda$, then $Q_\lambda = Q_i$ for sufficiently large $i < \lambda$. But then $J_{\mu_i}^E = J_{\mu_i}^{EN_i} = J_{\mu_i}^{EQ_i} = J_{\mu_i}^{EQ_\lambda}$. Now let $k = \lambda$. For $i \in D_\lambda$ we have:

$\mu_i = \nu_i = \text{ht}(N_i)$ (since $N_i = \langle J_{\nu_i}^{EQ_i}, \emptyset \rangle$ is a ZFC model). Hence $J_{\nu_i}^{EN_i} = J_{\nu_i}^{EQ_i} = J_{\nu_i}^{EQ_\lambda}$. QED (Claim)

This completes the construction. The verification of (a)–(f) is straightforward.

. . . . .

We recall that $\vec{N}$ is constructed in V_Θ where Θ is inaccessible. If N_i is undefined for some $i < \Theta$, then $\vec{N} = \langle N_i \mid i \leq \xi \rangle$ where $\xi < \Theta$. In this case one of the following holds:

(a) $N_\xi = Q_\xi$ is a simple iterate of Q

(b) N_ξ is not a weak mouse.

We shall show that (b) cannot occur.

First, however, we prove:

<u>Lemma 4</u> N_i is undefined for some $i < \theta$.

prf.

Suppose not. Define $N = N_\theta$, $Q' = Q_\theta$, $\mathcal{Y} = \mathcal{Y}^\theta$, $D = D_\theta$ exactly as in the other limit cases. Set $T = T^\theta$, $\pi_{ij} = \pi^\theta_{ij}$. Then $ht(N) = \theta$. Hence $Q' \notin V_\theta$, since N is a segment of Q'. Hence $\sup D = \theta$. Note that $N, Q', \mathcal{Y}, D$ are uniquely definable from Q, since all branches chosen are unique. (The case $Q_i = N_i$ without truncation on the branch to i has not occurred. The branch to θ is unique because θ is regular.) Let $H_{\theta^+} = L^A_{\theta^+}$. Let $H = L^A_\gamma$ = the small $H \prec L^A_{\theta^+}$ s.t. $\theta \subset H$. Let $X \prec H$ s.t. $Q \in X$, $\bar{\bar{X}} < \theta$, and $\bar{\theta} = X \cap \theta$ is transitive. Let $\sigma : \bar{H} \longleftrightarrow X$, where $\bar{H} = L^{\bar{A}}_{\bar{\gamma}}$. Then $\bar{\theta} = crit(\sigma)$ and $L^{\bar{A}}_{\bar{\theta}} = L^A_{\bar{\theta}}$. Let $(\bar{Q}', \bar{\mathcal{Y}}, \bar{D}, \bar{N}) = Q', \mathcal{Y}, D, N$.

It is easily seen that $D \cap \bar{\theta} = \bar{D}$ and $\mathcal{Y} | \bar{\theta} = \bar{\mathcal{Y}} | \bar{\theta}$.

Since branches are unique, we conclude that $\bar{\mathcal{T}} = \mathcal{T} | \bar{\theta}+1$. But then $\sigma \upharpoonright Q_{\bar{\theta}} = \pi_{\bar{\theta}\theta}$, since for $x \in Q_{\bar{\theta}}$, $x = \pi_{i\bar{\theta}}(\bar{x})$, $i < \theta$, we have:

$$\sigma(x) = \sigma(\pi_{i\bar{\theta}}(\bar{x})) = \pi_{i\theta}(\bar{x}) = \pi_{\bar{\theta}\theta}(x).$$

Let j be least s.t. $j+1 \leq_T \theta$ and $j \in D$. (This must exist, since otherwise $Q' = Q_{\bar{\theta}} \in V_\theta$.) Let $\xi = T(j+1)$. Then $\xi \geq_T \bar{\theta}$ and $\pi_{\bar{\theta}\xi} = \mathrm{id}$. Hence

$$\pi_{\bar{\theta}, j+1} : Q_{\bar{\theta}} \xrightarrow[F]{*} Q_{j+1}, \text{ where}$$

$F = E^{Q_j}_{\nu_j}$. Since $\mathrm{crit}(\pi_{j+1,\theta}) \geq \lambda_j$, we have: $F(X) = \lambda_j \cap \pi_{\bar{\theta},\theta}(X) = \lambda_j \cap \sigma(X)$ for $X \in \mathcal{P}(\kappa_j) \cap Q_j$.

Clearly $\mathrm{crit}(F) = \bar{\theta}$. Set $\bar{\mathfrak{M}} = \mathfrak{M}_{\bar{\theta}} = L_{\bar{\alpha}}[\mathcal{T}^{\bar{\theta}}]$, where $\bar{\alpha}$ is the least ordinal admissible in $\mathcal{T}^{\bar{\theta}}$.

It $\mathfrak{A} = \mathfrak{A}_\theta$ has the same definition in Y, then Y is H-definable in Q and hence $\bar{\mathfrak{A}} = \sigma^{-1}(\mathfrak{A}) \in \bar{H}$. Set:
$F^{**} = \sigma \upharpoonright (\mathbb{P}(\bar{\theta}) \cap \bar{H})$. It follows easily that $\sigma : \bar{H} \longrightarrow_{F^{**}} H$.
But then there is $\alpha < \theta$ s.t., setting $F^{*}(X) = \alpha \cap F^{**}(X)$, $\sigma' : \bar{H} \longrightarrow_{F^{*}} H'$, we have:
$V_{\lambda_i + 2} \subset H'$. Thus $\langle \bar{H}, F^{*} \rangle$ is a sufficient background certificate for F, meaning that, according to our definition, $i \notin D^{i+1} \supset D$ and $Q_{i+1} = Q_i$. Contradiction!

QED (Lemma 4)

Thus $\vec{N} = \langle N_i \mid i \leq \xi \rangle$ for some $\xi < \theta$, ξ is thus a definable point (in the parameter Q) in V_θ. It remains only to prove:

<u>Lemma 5</u> N_ξ is a weak mouse.

The proof will stretch over several sublemmas.

We recall our definition from [ANFS] §:

<u>Def</u> Let F be an extender at κ, λ on M. Let $W \subset \lambda \cup (\mathbb{P}(\kappa) \cap M)$. F <u>fixes</u> W as witnessed by $k : \lambda \cap W \longrightarrow \kappa$ iff whenever $\alpha_1, \ldots, \alpha_n \in \lambda \cap W$, $X \in \mathbb{P}(\kappa) \cap W$, then $\langle k(\alpha_1), \ldots, k(\alpha_n) \rangle \in X \longleftrightarrow \langle \alpha_1, \ldots, \alpha_n \rangle \in F(X)$.

We first note that Lemma 3 of [ANFS] §2 can be improved to:

<u>Lemma 5.1</u> Let $\sigma : \bar{M} \longrightarrow_{\Sigma^*} M \ \min(\vec{p})$. Let F be an extender on M at κ, λ. Let $\langle \sigma, g \rangle : \langle \bar{M}, \bar{F} \rangle \xrightarrow{**} \langle M|\rho_0, F \rangle$ where $\bar{F}$ is weakly amenable (hence close to $\bar{M}$). Let $W_1 \subset \mathcal{P}(\kappa) \cap M$ s.t. $\mathrm{rng}(f) \subset W_1$ whenever f is a partial map from a $\xi < \kappa$ to $\mathcal{P}(\kappa) \cap M$, which is $\Sigma^*(M)$ in parameters from $\mathrm{rng}(\sigma) \cup \{\rho_i \mid i < \omega\}$. Let $W_0 \subset \lambda$ s.t. $\mathrm{rng}(g) \subset W_0$. Suppose, moreover, that $\bar{M}$ has cardinality $< \mathrm{cf}(\kappa)$. Let F fix $W = W_0 \cup W_1$ as witnessed by $k : W_0 \longrightarrow \kappa$. Then:

(a) There is $\pi : \bar{M} \xrightarrow[\bar{F}]{*} \bar{M}'$

(b) There is $\sigma' : \bar{M}' \longrightarrow_{\Sigma^*} M \ \min(\vec{p})$ defined by $\sigma'(\pi(f)(\alpha)) = \sigma(f)(kg(\alpha))$ whenever $f \in \Gamma^*(\bar{\kappa}, \bar{M})$, $\bar{\alpha} < \bar{\lambda}$ (where $\bar{F}$ is at $\bar{\kappa}, \bar{\lambda}$). (Hence $\sigma'\pi = \sigma$ and $\sigma \upharpoonright \bar{\lambda} = kg$.)

proof.

The proof is virtually the same as that of [ANFS] §2 Lemma 3. We content ourselves with describing the points of difference. Just as before we

verify the exitence of $\bar{\pi}: \bar{M} \longrightarrow_{\bar{F}}^{*} \bar{M}'$ and $\sigma': \bar{M}' \longrightarrow M$, getting for Than in the form: $M \models \varphi(\sigma(f)(kg(\vec{\alpha}))) \bmod (\vec{p}') \longleftrightarrow$
$\longleftrightarrow \{\xi \mid \bar{M} \models \varphi(f(\xi))\} \in \bar{F}_{\alpha} \longleftrightarrow$
$\longleftrightarrow \bar{M}' \models \varphi(\bar{\pi}(f)(\vec{\alpha}))$, for φ which is $\Sigma_0^{(n)}$ for an n s.t. $\bar{\kappa} < \omega\rho^n_{\bar{M}}$. Hence $\sigma: \bar{M}' \longrightarrow_{\Sigma_0^{(n)}} M$ if $\bar{\kappa} < \omega\rho^n_{\bar{M}}$. If $\kappa < \omega\rho^u_{\bar{M}}$ we are done. If not, let $\omega\rho^{n+1} \leq \bar{\kappa} < \omega\rho^n$ in $\bar{M}$. We need a new proof of:

<u>Claim</u> $\omega\rho_{n+1} < \kappa < \omega\rho_n$.

proof.

$\kappa < \omega\rho_n$ is trivial. We show by ind. on $i < \omega$ that $\omega\rho_m(i) < \kappa$ for $m > n$.

For $i = 0$ this is trivial, since $\mathrm{card}(\bar{M}) < \mathrm{cf}(\kappa)$, where $\omega\rho_m(0) = \sup \sigma''\omega\rho^m_{\bar{M}}$ and $\sigma''\omega\rho^m_{\bar{M}} \subset \kappa$. Now let it hold for i. Let h be a partial map from $\omega\rho_{m+1}(i)$ to $\omega\rho_m$ which is $\Sigma_1^{(m)}(M, \vec{p})$ in parameters from $\mathrm{rng}(\sigma)$. Note that $\omega\rho_m \leq \kappa$, since $\bar{M} \models \bigwedge v^m (v^m \in On \to v^m < \bar{\kappa})$

and hence $M \models \bigwedge v^m (v^m \in On \to v^m < \kappa)$.

<u>Subclaim</u> $\sup \mathrm{rng}(h) < \kappa$.

prf. Set $X(\xi) \simeq \kappa \setminus h(\xi)$ for $\xi < \omega\rho^{(i)}_{m+1}$.
Then $X(\xi) \in F_\kappa$ and hence $k(\kappa) \in X(\xi)$.
Hence $h(\xi) < k(\kappa)$ whenever $h(\xi)$ is defined.

QED (Subclaim)

But $\omega\rho_m(i+1)$ is the supremum of all such $\sup \mathrm{rng}(h)$. Since there are only $\mathrm{card}(\overline{M})$ many such h, we conclude $\omega\rho_m(i+1) < \kappa$. Finally we note that

$\omega\rho_m = \sup_{i<\omega} \omega\rho_m(i) < \kappa$, since $\omega < cf(\kappa)$.

QED (Claim)

The rest of the proof is a literal repetition of [ANFS] §2 Lemma 3.

QED (Lemma 5.1)

We must now prove a lemma which verifies the efficacy of our background criterion for placing an extender on N_{i+1} in the inductive definition of N_i.

<u>Lemma 5.2</u> Let $\kappa > \overline{\overline{Q}}_0$ s.t. κ is a cardinal and $ht(N_\kappa) = \kappa \leq \kappa_{\kappa_j}$, where $j > \kappa$. Let $\mathfrak{A} = \mathfrak{A}_\kappa = L_\alpha[\mathcal{J}^\kappa]$, where α is the least ordinal admissible in $\mathcal{J}^\kappa$. Then

$$\mathcal{P}(\kappa) \cap \underline{\Sigma}^*(Q_j) \subset \mathfrak{A}.$$

In order to prove Lemma 5.2 we must first develop some machinery. Let

$$\mathcal{I} = \langle\langle Q_i\rangle, \langle \nu_i \mid i \in D\rangle, \langle \eta_i\rangle, \langle \pi_{ij}\rangle, T\rangle$$

be a normal iteration of length Γ. Let $l : \bar{\Gamma} \to \Gamma$ be order preserving and let $\sigma : \bar{Q} \xrightarrow[\Sigma^*]{} Q$, where $\mathcal{I}$ is an iteration of Q. We say that $\langle \sigma, k\rangle$ is good for $\mathcal{I}$ iff it induces a normal iteration

$$\bar{\mathcal{I}} = \langle\langle \bar{Q}_i\rangle, \langle \bar{\nu}_i \mid i \in \bar{D}\rangle, \langle \bar{\eta}_i\rangle, \langle \bar{\pi}_{ij}\rangle, \bar{T}\rangle$$

of length $\bar{\Gamma}$ and a sequence $\langle \sigma_i \mid i < \bar{\Gamma}\rangle$ s.t.

(a) $\sigma_i : \bar{Q}_i \xrightarrow[\Sigma^*]{} Q_{l(i)}$ and $\sigma_0 = \pi_{0,l(0)}\sigma$.

(b) $i \bar{T} j \longleftrightarrow l(i) T l(j)$

(c) Set $\tilde{l}(i) = \text{lub } l''i$. Then $\tilde{l}(i) \leq_T l(i)$ and $\pi_{\tilde{l}(i), l(i)}$ is a total function on $Q_{\tilde{l}(i)}$.

(d) $\sigma_i \bar{\pi}_{hi} = \pi_{l(h) l(i)} \sigma_h$ for $h \leq i$ in $\bar{T}$

(e) $\sigma_i \restriction \bar{\lambda}_h = \sigma_h \restriction \bar{\lambda}_h$ if $h \in \bar{D} \cap i$

(f) Let $\bar{\xi} = \bar{T}(i+1)$, $\xi = T(l(i)+1)$. Then $l(\bar{\xi}) = \xi$. If $\bar{\xi} \notin \bar{D}$, then $\xi \notin D$. If $\bar{\xi} \in \bar{D}$, then $\xi \in D$, $\sigma_i(\bar{\nu}_i) = \nu_{l(i)}$, and $\sigma_{\bar{\xi}}(\bar{\eta}_i) = \eta_{l(i)}$ (where $\sigma_{\bar{\xi}}(On \cap \bar{Q}_i) =_{\text{df}} On \cap Q_{l(i)}$).

<u>Lemma 5.2.1</u> Let $\langle \sigma, \ell \rangle$ be good for $\mathfrak{I}$ inducing $\bar{\mathfrak{I}}$. Then $\mathrm{rng}(\sigma_i) \subset \mathrm{rng}(\pi_{\tilde{\ell}(i), \ell(i)})$. (Hence $\sigma_i = \pi_{\tilde{\ell}(i), \ell(i)} \tilde{\sigma}_i$, where $\tilde{\sigma}_i = \pi^{-1}_{\tilde{\ell}(i), \ell(i)} \sigma_i$).

prf.

We prove this by induction on i. In the process we shall give a precise description of the structure of $\tilde{\sigma}_i$.

<u>Case 1</u> $i = 0$. Trivial by (a) with $\tilde{\sigma}_0 = \sigma$

<u>Case 2</u> $i = j+1$

<u>Case 2.1</u> $j \notin \bar{D}$. Then $\sigma_i = \pi_{\ell(j) \ell(i)} \sigma_j$ by (d), since $\bar{\pi}_{j i} = \mathrm{id}$. But $\tilde{\ell}(i) = \ell(j) + 1$; hence $\sigma_i = \pi_{\tilde{\ell}(i) \ell(i)} \tilde{\sigma}_i$, where

$\tilde{\sigma}_i = \pi_{\ell(j), \ell(j)+1} \sigma_j = \sigma_j$. QED (Case 2.1)

<u>Case 2.2</u> $j \in \bar{D}$. Let $\bar{\xi} = T(j+1)$, $\xi = \ell(\bar{\xi}) = T(\ell(j)+1)$, $\bar{Q}^* = \bar{Q}^*_j = \bar{Q}_{\bar{\xi}} \| \bar{\eta}_j$, $Q^* = Q^*_{\ell(j)} = Q_{\xi} \| \eta_{\ell(j)}$, $\sigma^* = \sigma_{\xi} \upharpoonright \bar{Q}^*_i$. Let $\bar{F} = E^{\bar{Q}_j}_{\bar{\nu}_j}$, $F = E^{Q_{\ell(j)}}_{\nu_{\ell(j)}}$, $\bar{\kappa} = \mathrm{crit}(\bar{F})$, $\kappa = \mathrm{crit}(F)$. Then

with $f \in \Gamma^*(\bar{\kappa}, \bar{Q}^*)$, $\alpha < \bar{\lambda}_j$

for $x = \bar{\pi}_{\bar{\xi}, j+1}(f)(\alpha) \in \bar{Q}_{j+1}$, we have:

$\sigma_{j+1}(x) = \pi_{\xi, \ell(j+1)}(\sigma^*(f))(\sigma_j(\alpha))$ by (d), (e).

Thus $\sigma_{j+1}(x) = \pi_{\ell(j)+1,\, \ell(j+1)}\, \tilde{\sigma}_{j+1}(x)$,

where $\tilde{\sigma}_{j+1}(x) = \pi_{\xi,\, \ell(j)+1}(\sigma^*(f))(\sigma_j(\alpha))$,

by (c), since crit$(\pi_{\ell(j)+1,\ell(j)+1}) \geq$

$\geq \lambda_j > \sigma_j(\alpha)$. QED (Case 2)

<u>Case 3</u> $i = \lambda$, $\lim(\lambda)$,

Let $x \in Q_\lambda$, $x = \bar{\pi}_{h\lambda}(\bar{x})$, $h \mathrel{T} \lambda$. Then

$\ell(h) \leq_T \tilde{\ell}(\lambda) \leq_T \ell(\lambda)$ and $\sigma_\lambda(x) =$

$= \sigma_\lambda \bar{\pi}_{h\lambda}(\bar{x}) = \pi_{\ell(h)\ell(\lambda)}\, \sigma_h(\bar{x}) =$

$= \pi_{\tilde{\ell}(\lambda)\ell(\lambda)}\, \tilde{\sigma}(x)$, where

$\tilde{\sigma}(x) = \pi_{\ell(h),\tilde{\ell}(\lambda)}\, \sigma_h(\bar{x})$. QED (Lemma 5.2.1)

As a corollary of the proof:

<u>Lemma 5.2.2</u> If $\bar{\pi}_{hi}$ is a total fcn on $\bar{Q}_h$, then $\pi_{\ell(h)\ell(i)}$ is total on $Q_{\ell(h)}$ (hence $\pi_{\tilde{\ell}(h)\ell(i)} = \pi_{\ell(h)\ell(i)}\pi_{\tilde{\ell}(h)\ell(h)}$ is total on $Q_{\tilde{\ell}(h)}$)

prf. Ind. on i

$i = 0$ is trivial. If $i = j+1$, it suffices to show that $\pi_{\ell(\xi)\ell(j+1)}$ is total, where $\xi = T(j+1)$. But this is apparent from the above. The limit case is straightforward. QED (Lemma 5.2.2)

We note that in Case 2.2 of the proof of Lemma 5.2.1 the map $\tilde{\sigma} = \tilde{\sigma}_{j+1}$ is defined by:

$$\tilde{\sigma}(\pi_{\bar{\xi}, j+1}(f)(\alpha)) = \pi_{\xi, l(j)+1}\, \sigma^*(f)(\sigma_j(\alpha))$$

where $f \in \Gamma^*(\bar{\kappa}, \bar{Q}^*)$, $\alpha < \bar{\lambda}_j$. The existence of such a $\tilde{\sigma} : \bar{Q}_{j+1} \to Q_{l(j)+1}$ and the fact that it is Σ^*-preserving will follow directly from:

$$(*)\ \langle \sigma^*, \sigma_j \restriction \bar{\lambda}_j \rangle : \langle \bar{Q}^*, \bar{F} \rangle \xrightarrow{*} \langle Q^*, F \rangle.$$

<u>Lemma 5.2.3</u> Let $\langle \sigma, l \rangle$ be good for $\mathcal{J}$ inducing $\bar{\mathcal{J}}$, $\langle \sigma_i \mid i < \bar{\mu} \rangle$. Let $j \in \bar{D}$ and let $\bar{\xi}, \xi, \bar{Q}^*, Q^*, \bar{F}, F, \sigma^*$ be as in Case 2.2 in the proof of Lemma 5.2.1. Then $(*)$ holds.

This, in turn, follows from:

<u>Lemma 5.2.4</u> Let $\mathcal{I}$ be a normal iteration of length $\mu+1$. Let $\langle \sigma, \ell \rangle$ be good for $\mathcal{I}$ inducing $\bar{\mathcal{I}}$ of length $\bar{\mu}+1$ and $\langle \sigma_i \mid i \leq \bar{\mu} \rangle$, where $\ell(\bar{\mu}) = \mu$. Let $\bar{\nu} \leq ht(\bar{Q}_{\bar{\mu}})$ s.t. $\bar{F} = E_{\bar{\nu}}^{\bar{Q}_{\bar{\mu}}} \neq \emptyset$ and $\bar{\nu} > \bar{\nu}_i$ for $i \in \bar{D}$. Set $\nu = \sigma_{\bar{\mu}}(\bar{\nu})$,

$F = E_{\nu}^{Q_{\mu}}$, $\bar{\kappa} = \mathrm{crit}(\bar{F})$, $\kappa = \mathrm{crit}(F)$,

$\bar{\delta} =$ the least $\delta \leq \bar{\mu}$ s.t. $\delta = \bar{\mu}$ or $\bar{\kappa} < \bar{\lambda}_\delta$

$\delta =$ " $\delta \leq \mu$ s.t. $\delta = \mu$ or $\kappa < \lambda_\delta$

Set: $\bar{\tau} = \bar{\kappa}^{+J_{\bar{\nu}}^{E^{\bar{Q}_{\bar{\mu}}}}}$, $\tau = \kappa^{+J_{\nu}^{E^{Q_{\mu}}}}$;

$\bar{\eta} =$ the max $\bar{\eta} \geq \bar{\lambda}_{\bar{\delta}}$ s.t. $\bar{\tau}$ is a cardinal in $J_{\bar{\eta}}^{E'}$

$\eta =$ " $\eta \geq \lambda_\delta$ " " J_{η}^{E}

$\bar{Q}^* = \bar{Q}_{\bar{\delta}} \| \bar{\eta}$, $Q^* = Q_\delta \| \eta$. Suppose that $\ell(\bar{\delta}) = \delta$ and $\sigma(\bar{\eta}) = \eta$. Set: $\sigma^* = \sigma \upharpoonright \bar{Q}^*$. Then:

$$\langle \sigma^*, \sigma_{\bar{\mu}} \upharpoonright \bar{\lambda} \rangle : \langle \bar{Q}^*, \bar{F} \rangle \xrightarrow{*} \langle Q^*, F \rangle$$

where $\bar{\lambda} = \bar{F}(\bar{\kappa})$, $\lambda = F(\kappa)$.

The proof of Lemma 5.2.4 will be quite similar to those of corresponding lemmas in [NFS].

If $\bar{\nu} \in \bar{Q}_{\bar{\mu}}$ (hence $\nu \in Q_\mu$), the conclusion is trivial, since for $\bar{\alpha} < \bar{\lambda}$, $\alpha = \sigma_{\bar{\mu}}(\bar{\alpha})$ we have: $\bar{F}_{\bar{\alpha}} \in J_{\bar{\nu}_{\bar{\delta}}}^{E^{\bar{Q}_{\bar{\mu}}}} = J_{\bar{\nu}_{\bar{\delta}}}^{E^{\bar{Q}^*}} \subset \bar{Q}^*$ and $F_\alpha = \sigma_{\bar{\mu}}(\bar{F}_{\bar{\alpha}}) = \sigma^*(\bar{F}_{\bar{\alpha}}) \in Q^*$. Thus only the case: $\bar{\nu} = ht(\bar{Q}_{\bar{\mu}})$ (hence $\nu = ht(Q_\mu)$) remains. The conclusion follows from:

<u>Lemma 5.2.5</u> Let $\mathcal{I}$, $\langle \sigma, l \rangle$, $\bar{\mathcal{I}}$, $\bar{\delta}$, δ, $\bar{Q}^*$, Q^*, σ^* etc. be as in Lemma 5.2.4, where $\bar{\nu} = ht(Q_{\bar{\mu}})$, $\nu = ht(Q_\mu)$. Set: $\bar{\tau} = \bar{\kappa}^{+\bar{Q}_{\bar{\mu}}}$, $\tau = \kappa^{+Q_\mu}$. Let $\bar{A} \subset \bar{\tau}$ be $\Sigma_1(Q_{\bar{\mu}})$ in a parameter $\bar{p}$ and $A \subset \tau$ be $\Sigma_1(Q_\mu)$ in $p = \sigma_{\bar{\mu}}(\bar{p})$ by the same def. Then $\bar{A}$ is $\Sigma_1(\bar{Q}^*)$ in a parameter $\bar{q}$ and A is $\Sigma_1(Q^*)$ in $q = \sigma^*(\bar{q})$ by the same definition.

(We note that $\bar{\tau} < \bar{\nu} \leq ht(\bar{Q}^*)$. To obtain the desired result, we let $\bar{A}, A$ be appropriate codes of $\bar{F}_{\bar{\alpha}}, F_\alpha$ respectively, where $\bar{\alpha} < \bar{\lambda}$, $\alpha = \sigma_{\bar{\mu}}(\bar{\alpha})$.)

proof of Lemma 5.2.5

Suppose not. Let μ be the least counter-example. With μ fixed, let $\bar{\mu}$ be the least counterexample. Let $\langle\sigma, l\rangle$ be a counterexample inducing $\bar{\mathfrak{I}}, \langle\sigma_i \mid i \leq \bar{\mu}\rangle$

(1) $\bar{\mu} = i+1$ for some i.

pf. Suppose not.

Let $\bar{A} \subset \bar{\tau}$ be $\Sigma_1(\bar{Q}_{\bar{\mu}})$ in $\bar{p}$ and $A \subset \tau$ be $\Sigma_1(Q_\mu)$ in $p = \sigma_{\bar{\mu}}(\bar{p})$ by the same def.

Pick $\mu' \,\bar{\mathsf{T}}\, \mu$ s.t. $\mu' > \delta$, $\bar{\pi}_{\mu',\bar{\mu}}$ is total and $\bar{\pi}_{\mu',\bar{\mu}}(\bar{p}') = \bar{p}$. Define

$l' : \mu'+1 \to \mu+1$ by: $l' \restriction \mu' = \mathrm{id}$; $l'(\mu') = \mu$.

Then l' induces $\bar{\mathfrak{I}}' = \bar{\mathfrak{I}} \mid \mu'+1$, $\langle\sigma_i' \mid i \leq \mu'\rangle$ where $\sigma_h' = \sigma_h$ for $h < \mu'$ and $\sigma_{\mu'}' =$

$= \sigma_{\bar{\mu}} \bar{\pi}_{\mu',\mu} = \pi_{l(\mu'),\mu} \sigma_{\mu'}$. Since $\bar{\pi}_{\mu',\mu} \restriction \bar{\tau}+1 = \mathrm{id}$

it follows easily that $\bar{A}$ is $\Sigma_1(\bar{Q}_{\mu'})$ in $\bar{p}'$ by the same def. Clearly, $\bar{Q}^*, \sigma^*, Q^*$ have the same definition wrt. $\langle\sigma, l'\rangle$. But by the minimality of $\bar{\mu}$, we conclude: $\bar{A}$ is $\Sigma_1(\bar{Q}^*)$ in a $\bar{q}$ and A is $\Sigma_1(Q^*)$ in $q = \sigma^*(\bar{q})$ by the same def. Contr! QED(1)

(2) $\mu = l(i)+1$, where $\bar\mu = i+1$.

prf. Suppose not.

Then $l(i)+1 < \mu$ and $\sigma_{\bar\mu} = \pi_{l(i)+1\,\mu}\tilde\sigma_{\bar\mu}$.

Define $l' : \bar\mu+1 \to \mu'+1$ $(\mu' = l(i)+1)$ by $l' \upharpoonright \bar\mu = \mathrm{id}$, $l'(\bar\mu) = \mu'$. Then l' induces $\bar{J}$, $\langle \sigma'_h \mid h \le \bar\mu \rangle$ wrt $J|\mu'+1$, where $\sigma'_h = \sigma_h$ for $h < \bar\mu$, $\sigma'_{\bar\mu} = \tilde\sigma_{\bar\mu}$. Then, letting $\bar{A}, A, \bar{p}, p$ be as above, $\bar{Q}^*, Q^*$, σ^* have the same definitions wrt $\langle \sigma, l' \rangle$, $J' = J|\mu'+1$ and A is $\Sigma_1(Q_{\mu'})$ in $p' = \pi^{-1}_{\mu'\mu}(p)$ by the same definition. Hence the minimality of μ gives us a contradiction just as before. QED(2)

We have: $\bar\mu = i+1$, $\mu = l(i)+1$. Since $\langle \sigma, l \rangle$ is a counterexample, we obviously have:

(3) $\bar\delta < \bar\mu$ (hence $\delta < \mu$).

Set $\bar{\xi}_i = \bar{T}(i+1)$, $\xi = l(\bar{\xi}_i) = T(l(i)+1)$,

Set: $\bar{Q}_i^* = \bar{Q}_{\bar{\xi}_i} \| \bar{\eta}_i$, $Q_i^* = Q_{\xi_i} \| \eta_{l(i)}$,

$\sigma_i^* = \sigma_{\bar{\xi}_i} \restriction \bar{Q}_i^*$. Then:

(4) $\bar{\kappa} < \bar{\kappa}_i$ (hence $\kappa < \kappa_{l(i)}$ and

$\bar{\pi}_{\bar{\xi}_i, \bar{\mu}} \restriction \bar{\tau} + \bar{Q}_i^* = \mathrm{id}$)

prf. Suppose not. Let $\bar{\kappa}' = \bar{\pi}^{-1}_{\bar{\xi}_i, \bar{\mu}}(\bar{\kappa}) =$ $\mathrm{crit}(E^{\bar{Q}_i^*}_{\mathrm{On} \cap \bar{Q}_i^*})$. Then $\bar{\kappa}' < \bar{\kappa}_i$, since otherwise $\bar{\kappa} = \bar{\pi}_{\bar{\xi}_i, \bar{\mu}}(\bar{\kappa}') \geq \bar{\pi}_{\bar{\xi}_i, \bar{\mu}}(\bar{\kappa}_i) = \bar{\lambda}_i$, hence $\bar{\delta} = \bar{\mu}$ Contr! Hence $\bar{\kappa} = \pi_{\bar{\xi}_i \bar{\mu}}(\bar{\kappa}') = \bar{\kappa}' < \bar{\kappa}_i$. QED (4)

(5) $\bar{\delta} \leq \bar{\xi}_i$ (hence $\delta \leq \xi_i$), since $\bar{\kappa} < \bar{\kappa}_i < \bar{\lambda}_{\bar{\xi}}$

(6) $\omega\rho^1_{\bar{Q}_{\bar{\mu}}} \leq \bar{\tau}$ (hence $\omega\rho^1_{Q_\mu} \leq \tau$).

prf. Suppose not. Let $\bar{A} \subset \bar{\tau}$ be $\Sigma_1(\bar{Q}_{\bar{\mu}})$ in $\bar{p}$ and $A \subset \tau$ be $\Sigma_1(Q_\mu)$ in $p = \sigma_{\bar{\mu}}(p)$ by the same definition. Then $\bar{A} \in \bar{Q}_{\bar{\mu}}$ and $A = \sigma_{\bar{\mu}}(\bar{A}) \in Q_\mu$, since $x = A \cap \tau$ is $\Sigma_0^{(1)}$ in p. Hence $\bar{A} \in J^{E^{\bar{Q}_{\bar{\mu}}}}_{\lambda_{\bar{\delta}}} = J^{E^{\bar{Q}_{\bar{\delta}}}}_{\lambda_{\bar{\delta}}} \subset \bar{Q}^*$ is $\Sigma_1(\bar{Q}^*)$ in $\bar{A}$ and $A = \sigma^*(\bar{A})$ is $\Sigma_1(Q^*)$ in A by the same def. Contr!

QED (6)

(7) $\omega\rho^1_{\bar{Q}^*_i} \le \bar{\tau}$ (hence $\omega\rho^1_{Q^*_{\ell(i)}} \le \tau$).

pf. $\pi_{\bar{\xi}\bar{\mu}}$ is Σ^*-preserving and $\mathrm{crit}(\pi_{\bar{\xi}\bar{\mu}})$
$> \bar{\tau} + 1$. QED(7)

(8) Let $\bar{A}$ be $\Sigma_1(\bar{Q}_{\bar{\mu}})$ in $\bar{p}$ and A be $\Sigma_1(Q_\mu)$ in $p = \sigma_{\bar{\mu}}(\bar{p})$ by the same def., where $\bar{A} \subset \bar{\tau}$. Then $\bar{A}$ is $\Sigma_1(\bar{Q}^*_i)$ in some $\bar{q}$ and A is $\Sigma_1(Q^*_{\ell(i)})$ in $q = \sigma^*_i(\bar{q})$ by the same def.

pf.
We have $\pi_{\bar{\xi},\bar{\mu}} : \bar{Q}^*_i \longrightarrow_{\bar{F}} \bar{Q}_{\bar{\mu}}$, $\pi_{\xi,\mu} : Q^* \longrightarrow_F Q_\mu$, where $\bar{F} = E^{\bar{Q}_i}_{\bar{\nu}_i}$, $F = E^{Q_{\ell(i)}}_{\nu_{\ell(i)}}$ by (7). But $\sigma_{\bar{\mu}}$ is then defined by:
$\sigma_{\bar{\mu}}(\pi_{\bar{\xi}\bar{\mu}}(f)(\alpha)) = \pi_{\xi\mu}\sigma^*_i(f)(\sigma_i(\alpha))$, where $\langle \sigma^*_i, \sigma_i \upharpoonright \bar{\lambda}_i \rangle : \langle \bar{Q}^*_i, \bar{F} \rangle \longrightarrow^* \langle Q^*_{\ell(i)}, F \rangle$. The result follows by [NFS] §1 Lemma 9. QED(8)

(9) $\bar{\xi} > \bar{\delta}$ (hence $\xi > \delta$)
Suppose not. Then $\bar{\xi} = \bar{\delta}$ by (5). Hence $\bar{\eta}_i \le \bar{\eta}$ since $\bar{\tau} < \bar{\kappa}_i < \bar{\tau}_i$. Hence $\bar{Q}^*_i \subset \bar{Q}^*$ and $\sigma^*(\bar{Q}^*_i) = Q^*_{\ell(i)}$ if $\bar{Q}^*_i \ne \bar{Q}^*$. Thus if $\bar{A}, A$ are as in (8), then $\bar{A}$ is $\Sigma_1(\bar{Q}^*)$ in some $\bar{q}$ and A is $\Sigma_1(Q^*)$ in $q = \sigma^*(\bar{q})$ by the same def. Contr! QED(9)

(10) $\bar{Q}^*_i = \bar{Q}_{\bar{\xi}}$ (hence $Q^*_{l(i)} = Q_\xi$).

Prf. If not, then $\bar{\tau}^{+\bar{Q}_{\bar{\xi}}} > \omega\bar{\eta}_i = \mathrm{On} \cap \bar{Q}^*_i$ by (7). But $\bar{\tau} < \bar{\lambda}_{\bar{\delta}}$, where $\bar{\lambda}_{\bar{\delta}}$ is a limit cardinal in $\bar{Q}_{\bar{\xi}}$, by (9).

Hence $\bar{\tau}^{+\bar{Q}_{\bar{\xi}}} = \bar{\tau}^{+J^{\bar{Q}_{\bar{\xi}}}_{\lambda_{\bar{\delta}}}} \subset \bar{Q}^*_i$, since $\bar{\eta}_i \geq \bar{\lambda}_{\bar{\xi}} > \bar{\lambda}_{\bar{\delta}}$. Contr! QED (10)

We now note that $\langle \sigma, l \upharpoonright (\bar{\xi}+1) \rangle$ induces $\bar{\mathfrak{I}} \upharpoonright (\bar{\xi}+1), \langle \sigma_h \mid h \leq \bar{\xi} \rangle$ wrt $\mathfrak{I} \upharpoonright (\xi+1)$.

By (10) we have $\bar{\pi}_{\bar{\xi},\bar{\mu}} : \bar{Q}_{\bar{\xi}} \longrightarrow_{\Sigma^*} \bar{Q}_{\bar{\mu}}$, $\pi_{\xi,\mu} : Q_\xi \longrightarrow_{\Sigma^*} Q_\mu$, where $\sigma_{\bar{\mu}} \bar{\pi}_{\bar{\xi}\bar{\mu}} = \pi_{\xi\mu} \sigma_{\bar{\xi}}$. Let $\bar{\nu}' = \mathrm{ht}(\bar{Q}_{\bar{\xi}})$, $\nu' = \mathrm{ht}(Q_\xi)$, $\bar{F}' = E^{\bar{Q}_{\bar{\xi}}}_{\bar{\nu}'}$, $F' = E^{Q_\xi}_{\nu'}$.

Then $\bar{\kappa} = \mathrm{crit}(\bar{F}')$, $\kappa = \mathrm{crit}(F')$, $\bar{\tau} = \bar{\kappa}^{+\bar{Q}_{\bar{\xi}}}$, $\tau = \kappa^{+Q_\xi}$ by (4). Thus $\bar{Q}^*, Q^*, \sigma^*$ have the same definition wrt. $\bar{F}', F'$ as wrt $\bar{F}, F$. But then if $\bar{A}, A$ are as in (8), it follows by (8) and the minimality of $\bar{\mu} > \bar{\xi}$ that $\bar{A}$ is $\Sigma_1(\bar{Q}^*)$ in some $\bar{r}$ and A is $\Sigma_1(Q^*)$ in $r = \sigma^*(\bar{r})$ by the same definition. Contr!

QED (Lemma 5.2.5.

Thus Lemma 5.2.4 is proven. We make use of it in proving:

<u>Lemma 5.2.6</u> Let $\mathcal{I}$ be an iteration of length Γ. Let $\mu+1<\Gamma$. Let $u\subset\Gamma$ be finite. Let $a_i\subset Q_i$ be finite for $i\in u$. Then there exist $n<\omega$, $l:(\mu+n+1)\to\Gamma$ inducing $\bar{\mathcal{I}}$, $\langle\sigma_i \mid i\leq\mu+n\rangle$ s.t. $l\upharpoonright\mu+1=\mathrm{id}$, $u\subset\mathrm{rng}(l)$ and $a_{l(i)}\subset\mathrm{rng}(\sigma_i)$ for $i\in u$.

proof.

Suppose not. Let Γ be a minimal counterexample. It follows easily that $\Gamma=i+2$ where $i+1=\max(u)>\mu$. Set $a_j=\emptyset$ for $j\notin u$. Let $a_{i+1}=\pi_{\xi,i+1}(f)(\alpha)$ where $\xi=T(i+1)$, $f\in\Gamma^*(\kappa_i,Q_i^*)$, $\alpha<\lambda_i$. Let $p\in Q_i^*$ s.t. $p=f$ or f is $\Sigma_1^{(n)}(Q_i^*)$ in p, where $\kappa_i<\omega\rho^n_Q$. (Here $Q_i^*=Q_\xi\|\eta_i$.) Set: $u'=(u\cap(i+1))\cup\{i,\xi\}$. For $j\in u'$ set: $a'_j=a_j$ if $j\neq i,\xi$;

$$a'_i=\begin{cases}a_i\cup\{\alpha\} & \text{if } \nu_i=ht(Q_i)\\ a_i\cup\{\alpha,\nu_i\} & \text{if not}\end{cases}$$

$$a'_\xi=\begin{cases}a_i\cup\{p\} & \text{if } \eta_i=ht(Q_\xi)\\ a_i\cup\{p,\eta_i\} & \text{if not.}\end{cases}$$

By the minimality of $\Gamma=i+2$ we know there is $l:(\mu+m+1)\to(i+1)$ s.t. $u'\subset\mathrm{rng}(l)$, $l(\mu+m)=i$, and l

induces $\bar{\mathcal{J}}$, $\langle \sigma_i \mid i \leq \mu+m \rangle$ s.t. $a'_{l(i)} \in \mathrm{rng}(\sigma_i)$ for $l(i) \in U'$, where $l \upharpoonright \mu+1 = \mathrm{id}$. Let $\sigma'_{\mu+m}(\bar{\nu}, \bar{\alpha}) = \nu_i, \alpha$ and $\sigma'_{\bar{\xi}}(\bar{p}, \bar{\eta}) = p, \eta_i$, where $l(\bar{\xi}) = \xi$. Then $\bar{\xi}$ = the least $\bar{\xi}$ s.t. $\bar{\kappa} = \mathrm{crit}(E_{\bar{\nu}}^{\bar{Q}_{\mu+m}}) < \bar{\lambda}_{\bar{\xi}}$; $\bar{\eta}$ = the maximal $\bar{\eta} \leq \mathrm{ht}(Q_{\bar{\xi}})$ s.t. $\bar{\tau} = \bar{\kappa}^{+ J_{\bar{\nu}}^{E^{\bar{Q}_{\mu+m}}}}$ is a cardinal in $\bar{Q}^* = \bar{Q}_{\bar{\xi}} \| \bar{\eta}$. By Lemma 5.2.4 we have:

$$\langle \sigma^*, \sigma_{\mu+m} \upharpoonright \bar{\lambda} \rangle : \langle \bar{Q}^*, \bar{F} \rangle \xrightarrow{*} \langle Q^*, F \rangle,$$

where $\sigma^* = \sigma_{\bar{\xi}} \upharpoonright \bar{Q}^*$, $\eta = \sigma_{\bar{\xi}}(\bar{\eta})$, $Q^* =$

$$= Q^*_i = Q_{\xi} \| \eta, \quad \bar{F} = E_{\bar{\nu}}^{\bar{Q}_{\mu+m}}, \quad F = E_{\nu}^{Q_i}.$$

Thus we can extend $\bar{\mathcal{J}}$ to $\bar{\mathcal{J}}'$ of length $\mu+m+2$ and define $\sigma' =$

$$= \sigma'_{\mu+m+1} : \bar{Q}'_{\mu+m+1} \xrightarrow{}_{\Sigma^*} Q_{i+1}$$ by setting:

$$\bar{\pi} = \bar{\pi}'_{\bar{\xi}, \mu+m+1} : \bar{Q}^* \xrightarrow{*}_{\bar{F}} \bar{Q}'$$ and defining:

$$\sigma'(\bar{\pi}(f)(\alpha)) = \pi_{\xi, i+1} \sigma^*(f)(\sigma_{\mu+m}(\alpha)).$$

for $f \in \Gamma^*(\bar{\kappa}, \bar{Q}^*)$, $\bar{\kappa} = \mathrm{crit}(\bar{F})$, $\alpha < \bar{\lambda} = \bar{F}(\bar{\kappa})$

Extend l to $l' : \mu+m+2 \longrightarrow \Gamma$ by $l'(\mu+m+1) = i+1$. Then l' induces $\bar{\mathcal{J}}'$, $\langle \sigma_h \mid h \leq \mu+m+1 \rangle$ and $a_i \in \mathrm{rng}(\sigma_{\mu+m})$

QED (Lemma 5.2.6)

We are now ready to prove Lemma 5.2

Clearly $\kappa_{\kappa i} = \kappa$, since $\kappa_{\kappa i} \leq \omega\rho^{\omega}_{N_\kappa} \leq$ $\leq ht(N_\kappa) = \kappa$. Moreover, if $\beta < \kappa$ then $\kappa_{\beta i} \leq ht(N_\beta) < \kappa$. Hence $\mathfrak{J}^i|(\kappa+1) = \mathfrak{J}^\kappa$.

Thus $D_\kappa = D_i \cap \kappa$. We know that $\kappa = \sup D_\kappa$, since otherwise $Q_\kappa = Q_h$ for an $h < \kappa$; hence $ht(N_\kappa) \leq ht(Q_h) < \kappa$. Contr! But then $\kappa = \sup\{\lambda \mid \in D_\kappa\}$. (To see this, note that $\kappa = \sup\{\kappa_{h\kappa} \mid h \in D_\kappa\}$. But $\kappa_{h\kappa} \leq ht(N_h) = \nu_h < \lambda_i$, where $h < i$ and $h, i \in D_\kappa$.)

Now let $A \subset \kappa$ be $\Sigma^*(Q_i)$ in the parameter p. Let $l : (\kappa+m+1) \longrightarrow i+1$ s.t. $l \upharpoonright (\kappa+1) = id$, $l(\kappa+m) = i$ induce $\bar{\mathfrak{J}}, \langle \sigma_h \mid h \leq \kappa+m \rangle$ wrt $\mathfrak{J}^i$ s.t. $p \in rng(\sigma_{\kappa+m})$. Then $\sigma_\kappa = id \upharpoonright Q_\kappa$, where $Q_\kappa = \bar{Q}_\kappa$ and $\sigma_{\kappa+m} \upharpoonright \lambda_h = \sigma_\kappa \upharpoonright \lambda_h = id$ for $h < \kappa$. Hence $\sigma_{\kappa+m} \upharpoonright \kappa = id$.

Let $\sigma_{\kappa+m}(\bar{p}) = p$. Since $\sigma_{\kappa+m}$ is Σ^* preserving, it follows that A is $\Sigma^*(\bar{Q}_{\kappa+m})$ in $\bar{p}$ by the same definition. But $\bar{\mathcal{I}}$ is a finite extension of $\mathcal{I}^u$, where $\mathcal{I}^u \in \mathfrak{A}$ and $\mathfrak{A}$ is admissible. Hence $\bar{\mathcal{I}} \in \mathfrak{A}$. Hence $A \in \mathfrak{A}$. QED (Lemma 5.2)

We now turn to the proof of Lemma 5. We have constructed $\langle N_i \mid i \leq \xi \rangle$, $\langle Q_i \mid i \leq \xi \rangle$ where $Q_\xi = N_\xi$ is an iterate of Q. Thus it suffices to show that N_ξ is a weak mouse – i.e. we must show that if $\sigma : P \xrightarrow[\Sigma^*]{} N_\xi$ and P is countable, then P is countably iterable. We note that if $\sigma : P \xrightarrow[\Sigma^*]{} N_\xi$, then $\sigma : P \xrightarrow[\Sigma^*]{} N_\xi \min(\vec{\rho}$ where $\vec{\rho} = \min(\sigma, N_\xi, \langle \rho^i_{N_\xi} \mid i < \omega \rangle)$. Hence the result is easily seen to follow from the next lemma. Call $\mathcal{I}$ a <u>putative iteration</u> iff $\mathcal{I}$ is like a normal iteration in all respects except that the last element (if there is one) may not be well founded.

Then:

<u>Lemma 5.3</u> Let $\sigma : P \underset{\Sigma^*}{\longrightarrow} N_\eta$ min$(\vec{\rho})$ $(\eta \leq \xi)$. Let $\mathcal{J} = \langle\langle P_i \rangle, \ldots, T\rangle$ be a direct putative iteration of P of countable length. Then one of the following holds:

(A) $\mathcal{J}$ has a last element P_γ and there is $\sigma' : P_\gamma \underset{\Sigma^*}{\longrightarrow} N_{\eta'}$, min$(\vec{\rho}')$ s.t.

(i) If $\pi_{0\gamma}$ is not total, then $\eta' < \eta$

(ii) If $\pi_{0\gamma}$ is total, then $\eta' = \eta$, $\vec{\rho}' = \vec{\rho}$ and $\sigma' \pi_{0\gamma} = \sigma$

(B) $\mathcal{J}$ has a maximal branch b of limit length s.t. there is $\sigma' : P_b \underset{\Sigma^*}{\longrightarrow} N_{\eta'}$, min$(\vec{\rho}')$ with

(i) If π_b is not total, then $\eta' < \eta$

(ii) If π_b is total, then $\eta' = \eta$, $\vec{\rho}' = \vec{\rho}$ and $\sigma' \pi_b = \sigma$.

We note that if $\mathcal{J}$ employs the economical strategy and is of limit length, then the branch in (B) must be cofinal in $lh(\mathcal{J})$. We obtain a smooth countable iteration strategy for P by always choosing branches as in (B). Thus P is countably iterable.

Following tradition, we sketch the proof of Lemma 5.3 for the special case:

$lh(\mathcal{T}) = \omega$ and $\mathcal{T}$ has no truncation.

Hence we must find a branch b and a $\delta' : P_b \xrightarrow[\Sigma^*]{} N_\gamma \min(\vec{\rho})$ as in (B)(iii).

Assume that there are no such and that $\gamma = \eta$ is the least counterexample. Thus γ is definable from Q. We define a tree $\mathcal{U}$ of attempts to construct such b, δ'.

<u>Def</u> $\mathcal{U} = \mathcal{U}(\delta, N_\gamma, \vec{\rho})$ is the set of pairs $\langle i, \delta' \rangle$ s.t. $\delta' : P_i \xrightarrow[\Sigma^*]{} N_\gamma \min(\vec{\rho})$ and $\delta' \pi_{0i} = \delta$. $\mathcal{U}$ is ordered by the relation:

$$\langle i, \delta' \rangle < \langle j, \delta'' \rangle \text{ iff } (i > j \text{ and } \delta' \pi_{ji} = \delta'')$$

By our assumptions, $\mathcal{U}$ is a well founded poset. Let $|\mathcal{U}|$ denote the order type of $\mathcal{U}$ and $|p|_{\mathcal{U}}$ the type of $\{q \mid q \leq_{\mathcal{U}} p\}$ for $p \in \mathcal{U}$.

We define a sequence $\langle \delta_i, S_i, R_i, \vec{\rho}^{\,i} \rangle$ with the following properties:

(1) $\mathbb{R}_i = \langle R_i, \in, \theta_i \rangle$ is a coarse premouse s.t. ${}^{\omega}\mathbb{R}_i \subset \mathbb{R}_i$

(2) $\sigma_i : P_i \xrightarrow[\Sigma^*]{} S_i$ $\min(\vec{\beta}^i)$

(3) $Q \in V_{\theta_i}^{\mathbb{R}_i}$ and there exist sequences $\vec{N}^{\mathbb{R}_i} = \langle N_j^{\mathbb{R}_i} \mid j \le \xi_i \rangle$, $\vec{Q}^{\mathbb{R}_i} = \langle Q_j^{\mathbb{R}_i} \mid j \le \xi_i \rangle$, $\vec{\mathcal{J}}^{\mathbb{R}_i} = \langle \mathcal{J}_j^{\mathbb{R}_i} \mid j \le \xi_i \rangle$ with $\langle \nu_h^{\mathbb{R}_i} \mid h \in \bigcup_{j \le \xi_i} D_j \rangle$ defined in $V_{\theta_i}^{\mathbb{R}_i}$ from Q exactly as $\vec{N}, \vec{Q}, \vec{\mathcal{J}}$ were defined from Q in V_θ.

(4) $S_i = N_{\gamma_i}^{\mathbb{R}_i}$ where γ_i is defined in $V_{\theta_i}^{\mathbb{R}_i}$ as γ was defined in V_θ.

<u>Remark</u> $\vec{N}, \vec{Q}, \vec{\mathcal{J}}$ are "defined" only up to the choice of a single economical branch in $\mathcal{J}_\xi$; all other choices are unique. The same holds for $\vec{N}^{\mathbb{R}_i}, \vec{Q}^{\mathbb{R}_i}, \vec{\mathcal{J}}^{\mathbb{R}_i}$.

In stating the further conditions on $\langle \mathbb{R}_i, \sigma_i, S_i, \vec{\beta}^i \rangle$, we shall need to use the "resurrection sequence" defined in [NSF] §10. This gives us for any $\nu \le N_\gamma$ s.t. $E_\nu^{N_\gamma} \ne \emptyset$ an $\eta^* = \eta^*[\gamma, \nu]^{\vec{N}}$ and a $\sigma^* = \sigma^*[\gamma, \nu]^{\vec{N}}$ s.t.

$\sigma^* : N_\gamma \| \nu \xrightarrow[\Sigma^*]{} N_\eta$, where $\eta \le \gamma$.

If $\nu = ht(N_\gamma)$, then $\eta^* = \gamma$ and $\sigma^* = id$. Otherwise σ^* is defined as a composition of core maps and we have: $\sigma^* \upharpoonright \rho = id$, where $\rho =$ the least $\omega\rho^\omega_{N_\gamma \| \beta}$ s.t. $\nu \leq \beta < ht(N$

We set:

$$\sigma_i = \sigma^*[\gamma_i, \delta_i(\nu_i)]^{\vec{N}^{R_i}}$$

$$\eta_i = \eta^*[\gamma_i, \delta_i(\nu_i)]^{\vec{N}^{R_i}}$$

$$\nu_i^* = \sigma_i \delta_i(\nu_i) = ht(N_{\eta_i}) \; ; \; \lambda_i^* = \sigma_i \delta_i(\lambda_i).$$

We require:

(5) Let $h < i$. Then $\lambda_h^* < \lambda_i^*$,

$V^{R_h}_{\lambda_h^*+2} = V^{R_i}_{\lambda_h^*+2}$, and $\delta_i \upharpoonright \lambda_h = \sigma_h \delta_h \upharpoonright \lambda_h$.

<u>Note</u> It follows that $j < h < i \rightarrow \delta_h \upharpoonright \lambda_j = \delta_i \upharpoonright \lambda_j$ since $\delta_h(\lambda_j)$ is a cardinal in S_h; hence $\omega\rho^\omega_{S_h \| \beta} \geq \delta_h(\lambda_j)$ for $\delta_h(\nu_h) \leq \beta < ht(S_h)$. Hence $\sigma_h \upharpoonright \delta_h(\lambda_j) = id$.

Set: $U_i = U(\delta_i \pi_{0i}, S_i, \vec{\rho}^{\,i})^{\vec{N}^{R_i}}$. Then $\langle i, \delta_i \rangle \in U_i$. We assume:

(6) U_i is well founded and R_i has $|\langle i, \delta_i \rangle|_{U_i}$ many cutoff points.

Finally we want:

(7) $\mathbb{R}_{i-1} \in \mathbb{R}_i$ for $i > 0$.

This gives the desired contradiction.

. . . . . .

<u>Remark</u> Let $h = T(i+1)$, $\kappa = \kappa_i^* = \sigma_i \delta_i(\kappa_i)$, $\tau = \tau_i^* = \sigma_i \delta_i(\tau_i)$. <u>We claim</u>

(8) $\delta_i \upharpoonright (\tau_i + 1) = \delta_h \upharpoonright (\tau_i + 1)$

(9) $\sigma_i \upharpoonright (\delta_i(\tau_i) + 1) = \sigma_h \upharpoonright (\delta_h(\tau_i) + 1) = id$

Hence:

(9.1) $\tau < \lambda_h^* = \sigma_h \delta_h(\lambda_h)$

prf.

τ_i is a successor cardinal in P_h, since otherwise there would be a truncation. Hence $\delta_h(\tau_i)$ is a successor cardinal in S_h. Hence $\delta_h(\tau_i) \leq \rho$, where

$\rho = \sup\{ \omega\rho^{\omega}_{N_{\gamma_h} \| \beta} \mid \delta_h(\nu_h) \leq \beta < ht(N_{\gamma_h}) \}$

in $\mathbb{R}_h$. Hence $\sigma_h \upharpoonright \rho = id$ and ρ is a cardinal in N_{γ_h} if $\rho < ht(N_{\gamma_h})$.

Hence $\sigma_h \upharpoonright (\delta_h(\tau_i) + 1) = id$.

Hence $\delta_i \upharpoonright (\tau_i + 1) = \sigma_h \delta_h \upharpoonright (\tau_i + 1) = \delta_h \upharpoonright (\tau_i + \cdot$

proving (8). (9) is immediate if $i = h$. Otherwise $\delta_i(\lambda_h)$ is a cardinal in S_i and hence $\pi_i \upharpoonright \delta_i(\gamma_h) = id$. Hence (8) holds. QED

<u>Remark</u> Let h, i, κ be as above. Argue in R_i. Since $E^{N_{\eta_i}}_{\nu_i^*} \neq \emptyset$ and $\kappa = crit(E^{N_{\eta_i}}_{\nu_i^*})$, we know that $\kappa > \bar{Q}$ is a cardinal and that $\kappa = ht(N_\kappa) = \kappa_{\kappa, \eta_i}$ where $\kappa < \eta_i$. But then all branches in $\mathcal{T}^\kappa$ are unique, since otherwise an economical branch without truncation would occur $\leq \kappa$. But then $Q_\ell = Q_\kappa$ for all $\ell \geq \kappa$ and $E^{Q_\kappa}_\nu = \emptyset$ for $\nu \geq \kappa$. But $E^{Q_{\eta_i}}_{\nu_i^*} = E^{N_{\eta_i}}_{\nu_i^*} \neq \emptyset$. Contr!

But then the same branches exist and are unique in R_h, since $V^{R_h}_{\lambda_h^*+1} = V^{R_i}_{\lambda_h^*+1}$. <u>Hence</u>:

(9) $(\mathcal{Y}^\kappa)^{R_h} = (\mathcal{Y}^\kappa)^{R_i}$; hence

(10) $\mathfrak{A}_\kappa^{R_h} = \mathfrak{A}_\kappa^{R_i}$, where

$\mathfrak{A}_\kappa = L_\alpha[\mathcal{Y}^\kappa]$ is defined as above.

<u>Remark</u> Let h, i, κ be as above. Set $F = E^{P_i}_{\nu_i}$, $F' = E^{N\gamma_i}_{\nu_i^*}$. We shall need:

(11) $\langle \sigma_h, \delta_i^* \upharpoonright \lambda_i \rangle : \langle P_h, F \rangle \xrightarrow{**} \langle S_h | \mathfrak{f}_0^h, F' \rangle$

where $\delta_i^* = \sigma_i \delta_i$. This is proven inductively during the construction of $\langle R_i, \delta_i, S_i, \vec{\rho}^i \rangle$, using the methods of [NFS]. We shall skip this, however, and verify only (1) – (6). (In fact, we do not know how to verify (11) without enhancing the following construction of $\langle R_i, \delta_i, S_i, \vec{\rho}^i \rangle$ in the successor case. The details are in §3.)

We now construct $\langle R_j, \delta_j, S_j, \vec{\rho}^j \rangle$ by induction on j and verify (1) – (6) at each stage.

<u>Case 1</u> $j = 0$. Set $R_0 = \langle V_{\theta+\mu}, \in, \theta \rangle$, where μ is so chosen that there are $|U_0|$ many cutpoints $< \mu$, where $U_0 = U(\delta, N_\eta, \vec{\rho})$, $\delta_0 = \delta$, $\vec{\rho}^0 = \vec{\rho}$ (hence $\eta_0 = \eta$).

<u>Case 2</u> $j = i+1$. Let $h = T(i+1)$, Let $F = F^{R_i}_{\nu_i}$, $F' = E^{N_{\eta_i}}_{\nu_i^*}$. Then in R_i $N_{\eta_i} = \langle J^E_{\nu_i^*}, F' \rangle$ where $F' \neq \emptyset$. Hence N_{η_i} is derived from $N_{\eta_i - 1}$ by Case 2.3, where (∗), (∗∗) hold. Hence F' has a background certificate $\langle N, F^* \rangle$ s.t. $\mathfrak{A}_\kappa \in N$, where $\kappa = \mathrm{crit}(F') = \kappa_i^*$. But $\mathfrak{A}_\kappa^{R_i} = \mathfrak{A}_\kappa^{R_h}$. (In particular, $f \in \mathfrak{A}_\kappa$ whenever f is a partial map of some $\xi < \kappa$ to $H(\kappa) \cap S_h$ which is $\underline{\Sigma}^*(Q_h)$.) Set $\delta_i^* = \sigma_i \delta_i$.

Let $\pi : N \xrightarrow[F^*]{} N^*$. Then λ^*_i, $\delta^*_i \restriction \lambda_i \in N'$

Pick a $\beta < lh(F^*)$ s.t. $\delta^*_i \restriction \lambda_i = \pi(\tilde{\delta})(\beta)$ and $\lambda^*_i = \pi(\tilde{\lambda})(\beta)$, where $\tilde{\delta}, \tilde{\lambda} \in N$ map κ into $V_\kappa^{P_i}$ (hence $\tilde{\delta}, \tilde{\lambda} \in V_{\lambda^*_h}^{P_i} = V_{\lambda^*_h}^{P_h}$).

<u>Claim</u> For F^*_β many ξ there exist in V_κ^N a coarse premouse R s.t. ${}^\omega R \subset R$, a model S and a map $\delta : P_{i+1} \xrightarrow[\Sigma^*]{} S$ min($\tilde{\rho}$) s.t.

(i) $\vec{N}^R, \vec{Q}^R, \vec{\eta}^R$ are defined in R by the usual definition and $S = N^R_{\gamma R}$.

(ii) $V^R_{\tilde{\lambda}(\xi)+2} = V^N_{\tilde{\lambda}(\xi)+2}$

(iii) $\delta \restriction \lambda_i = \tilde{\delta}(\xi)$; $\delta(\lambda_i) > \tilde{\lambda}(\xi)$

(iv) Let $U = U(\delta\pi_{0,i+1}, S, \vec{\rho})$. Then U is well founded and there is order type at least $|\langle i+1, \delta\rangle|_U$ many cutoff points of R.

proof.
Suppose not. Let X = the set of ξ for which the statement holds.
Then $X \notin F_\beta$.

Set: $W_0 = \{\beta\} \cup \sigma_i^{*}{}''\lambda_i$;

W_1 = the union of all $\mathrm{rng}(f)$ s.t. f is a partial map of a $\xi < \kappa$ to $\mathbb{P}(\kappa) \cap S_h$ which is $\Sigma^*(S_h)$ in parameters from $\mathrm{rng}(\delta_h) \cup \{p^h_m \mid m < \omega\}$.

(Hence $W_1 \subset \mathfrak{A}_\kappa$, since each such f is $\underline{\Sigma}^*(Q_h)$. Since ${}^{\omega}N \subset N$ and there are just countably many such f, we have $W_1 \in N$ and $\overline{\overline{W_1}} < \kappa$ in N.) Set:

W_2 = the set of $Y \subset \kappa$ which are N-definable from $\tilde{\sigma}, \tilde{\lambda}$ and parameters from $TC(\mathcal{J})$.

(Hence $W_2 \subset N$ is countable; hence $W_2 \in N$)

Set: $W = W_0 \cup W_1 \cup W_2$. Then

(12) F^* fixes W.

proof.

Let $f \in N$, $f : \xi \leftrightarrow W_1 \cup W_2$, $\xi < \kappa$.

Let $g : \omega \leftrightarrow W_0$. For $s \in \omega^{<\omega}$, $s = \langle s_1, ..., s_n \rangle$ set:

$$\tilde{f}_s = \{ \zeta < \xi \mid f(\zeta) \in F^*_{\langle g(s_1), ..., g(s_n) \rangle} \}$$

Then $\tilde{f} = \langle \tilde{f}_s \mid s \in \omega^{<\omega} \rangle \in N$, since ${}^{\omega}N \subset N$.
It suffices to show:

<u>Claim</u> There is $k : \omega \xrightarrow{1-1} \kappa$ s.t. whenever $\langle i_1, \ldots, i_n \rangle \in \omega^{<\omega}$, then:

$$\langle k(\vec{i}) \rangle \in f(\xi) \longleftrightarrow \xi \in \tilde{f}_{\langle \vec{i} \rangle}$$

for all $\xi < \zeta$.

It suffices to note that the same claim holds of $\pi(\kappa), \pi(f), \pi(\tilde{f})$ in N', where $\pi : N \longrightarrow_{F^*} N'$. This is instanced by $k = f^{\cdot}$. QED (12)

Now let $\sigma : W_0 \longrightarrow \kappa$ fix W wrt. F^*;
(i.e. $\langle \sigma(\vec{\alpha}) \rangle \in X \longleftrightarrow X \in F_{\langle \vec{\alpha} \rangle}$ for $\alpha_1, \ldots, \alpha_n \in W_0$, $X \in W_1 \cup W_2$.)

Then σ fixes $W_0 \cup W_1$ wrt. F'.

By (11) and Lemma 5.1 we can define a map $\delta' : P_{i+1} \longrightarrow_{\Sigma^*} S_h$ min$(\vec{p}^h)$ by: $\delta'(\pi_{h,i+1}(f)(\alpha)) = \delta_h(f)(\sigma \delta_i^*(\alpha))$, where $f \in \Gamma^*(\kappa_i, P_h)$, $\alpha < \lambda_i$.

Note that $X \in W_2$. Let $\bar{\beta} = \sigma(\beta)$. Let $U_h = U(\delta_h \pi_{0h}, S_h, \vec{p}^h)$. Then P_h has at least $|\langle h, \delta_h \rangle|_{U_h}$ many cutoff points. But $|\langle i+1, \delta' \rangle|_{U_h} < |\langle h, \delta_h \rangle|_{U_h}$, since $\delta' \pi_{h,i+1} = \delta_h$. Hence there is $\gamma \in R_h$ s.t. $\gamma =$ the $|\langle i+1, \delta' \rangle|_{U_h}$'th cutoff of R_h. Pick $Y \prec V_\gamma^{R_h}$ s.t. $V^{R_h}_{\tilde{\lambda}(\bar{\beta})+1} \cup \{S_h\} \subset Y$, $\bar{\bar{Y}} < \kappa$ and Y is ω-closed. Let $k : R \overset{\sim}{\longleftrightarrow} Y$, where R is transitive. Set: $S = k^{-1}(S_h)$, $\delta = k^{-1}(\delta')$, $\vec{\rho} = k^{-1}(\vec{\rho}^{\,h})$. Then $\langle S, R, \delta, \vec{\rho} \rangle \in V_\kappa^{R_h} = V_\kappa^N$. Moreover $V^N_{\tilde{\lambda}(\bar{\beta})+1} = V^R_{\tilde{\lambda}(\bar{\beta})+1} = V^{R_h}_{\tilde{\lambda}(\bar{\beta})+1}$.

For $\alpha < \lambda_i$ we have: $\delta'(\alpha) = \pi \delta_i^*(\alpha)$. But $\delta_i^*(\alpha) < \lambda_i^* = \pi(\tilde{\lambda})(\beta)$. Hence $\langle \delta_i^*(\alpha), \beta \rangle \in F(Z)$, where $Z = \{\langle \eta, \tau \rangle \mid \eta < \tilde{\lambda}(\tau)\} \in W_2$. Hence $\langle \pi\delta_i^*(\alpha), \bar{\beta} \rangle \in Z$. Hence $\delta'(\alpha) = \pi \delta_i^*(\alpha) < \tilde{\lambda}(\bar{\beta})$. Clearly $k \upharpoonright \tilde{\lambda}(\bar{\beta}) = \mathrm{id}$. Hence $\delta'(\alpha) = k^{-1}\delta'(\alpha) = \delta(\alpha)$. Hence $\delta \upharpoonright \lambda_i = \delta' \upharpoonright \lambda_i$. But $\delta'(\lambda_i) = \delta_h(\kappa_i) = \delta_h^*(\kappa_i) = \kappa$ by (8), (9).

Hence $\delta(\lambda_i) = k\delta'(\lambda_i) = k(\kappa) > \tilde{\lambda}(\beta)$

For $\alpha < \lambda_i$, $\gamma = \delta'(\alpha) = \tau\delta_i^*(\alpha)$ we have:

$$\langle \gamma, \alpha \rangle \in \tilde{\delta}(\bar{\beta}) \longleftrightarrow \langle \delta_i^*(\alpha), \alpha \rangle \in \pi(\tilde{\delta})(\beta)$$
$$\longleftrightarrow \quad " \quad \in \delta_i^* \upharpoonright \lambda$$
$$\longleftrightarrow \delta_i^*(\alpha) = \delta_i^*(\alpha).$$

Hence $\tilde{\delta}(\bar{\beta}) = \delta' \upharpoonright \lambda_i = \delta \upharpoonright \lambda_i$.

Now set $U = k^{-1}(U_h)$. Then $U = U(\delta \circ \pi_{0,i+1}, S, \vec{\rho})$ is well founded and has $|\langle i+1, \delta \rangle|_U$-many cutoff pts, since the corresponding statement holds of U_h & $\langle i+1, \delta' \rangle$ in $V_{\gamma'}^{R_h}$.

We have thus shown that $\bar{\beta} \in X$.

Hence $X \in F_\beta$. Contr! QED (Claim)

In N we can then choose a function $\xi \longrightarrow \langle R(\xi), S(\xi), \delta(\xi), \vec{\rho}(\xi) \rangle$ defined for $\xi \in X$ with the above properties.

Set $R_{i+1} = \pi(R)(\beta)$, $S_{i+1} = \pi(S)(\beta)$, $\delta_{i+1} = \pi(\delta)(\beta)$, $\vec{\rho}^{\,i+1} = \pi(\vec{\rho})(\beta)$.

The verifications are straightforward. QED (Thm 1)

<u>Note</u> A slight modification of the proof of Theorem 1 gives: Let Θ be inaccessible in an inner model V and let Q be a normally $\Theta+1$-iterable one small premouse in V. Let $\mathbb{P} \in V_\Theta$ be a set of conditions and let G be $\mathbb{P}$-generic over V. Then Q is a weak mouse in $V[G]$. The main change in the proof is to require $\kappa > 2^{\bar{\bar{\mathbb{P}}}}$ as an additional condition for placing an extender with critical point κ on the sequence. If we have placed F with $\mathrm{crit}(F) = \kappa$, and $\langle N, F^* \rangle$ is an adequate background certificate in V, then it extends to a background certificate $\langle N[G], F^{**} \rangle$ in $V[G]$. This enables us to carry out virtually the same proof as before in $V[G]$.

We now prove Lemma 3 by induction on μ

Case 1 $\mu = 0$

$\mathbb{E}_0 = \langle R, \langle \delta \rangle, \langle \vec{\rho} \rangle \rangle$ is a realization of $\mathcal{Y}|1$ where $\delta : P \xrightarrow[\Sigma^*]{} N_\gamma$ $\min(\vec{\rho})$ is as assumed and $R = \langle \nabla_{\theta + \eta}, \in, \theta \rangle$, η = the $\omega \cdot |U| + c(0,0)$-th η s.t. $\langle \nabla_{\theta+\eta}, \in, \theta \rangle$ is a coarse premouse.

Case 2 $\mu = j + 1$

Pick $i \leq j$ and let $\mathbb{E}$ realize $\mathcal{Y}|(i+1)$ with room. We first prove (a), assuming i to be a breakpoint at μ. Then either $i = j$ or i is a breakpoint at j. ~~Let~~

$h = T(j+1)$. Suppose first that h survives at $j+1$. Then $i < h \leq j$. By the induction hypothesis, $\mathbb{E}$ extends to a realization $\mathcal{Y}|(h+1)$ with room s.t. $R_h^{\mathbb{E}'} \in R_i^{\mathbb{E}}$.

But h is a breakpoint at j. (Otherwise there is $k \in (h, j]$ s.t. $T(k) \leq h$ and $T(k)$ survives at k. Thus $n(k) =$
$= n(T(k)) > n(h)$, since h survives at $j+1$ and $\neg(h \leq_T k \leq_T j+1)$. Hence $T(k) < h$ and $\neg(T(k) \leq_T h \leq_T k)$, where $n(h) < n(T(k$
Hence $T(k)$ does not survive at k!)

Thus by the induction hypothesis $\mathbb{E}'$ extends to an $\mathbb{E}''$ realizing $\mathcal{T}|(j+1)$. By Lemma 3.3 there is a realization $\mathbb{F}$ for $\mathcal{T}|j+2$ s.t. $R^{\mathbb{F}}_{j+1} = R^{\mathbb{E}''}_h = R^{\mathbb{E}'}_h \in R^{\mathbb{E}}_i$ and $\mathbb{F}\restriction h = \mathbb{E}'\restriction h$ (hence $\mathbb{F}\restriction i = \mathbb{E}$).

Now suppose h does not survive at $j+1$. By the induction hyp. $\mathbb{E}$ extends to $\mathbb{E}'$ realizing $\mathcal{T}|(j+1)$ s.t. $R^{\mathbb{E}'}_j \in R^{\mathbb{E}}_i$ if $i < j$. By Lemma 3.2, $\mathbb{E}'$ extends to $\mathbb{F}$ realizing $\mathcal{T}|(j+2)$ s.t. $R^{\mathbb{F}}_{j+1} \in R^{\mathbb{E}'}_j$. Hence $R^{\mathbb{F}}_{j+1} \in R^{\mathbb{E}}_i$.

QED(a)

We now prove (b). Let $k \leq i$ be maximal s.t. k survives at $j+1$. Assume w.l.o.g. that $k = T(j+1)$. (Otherwise let $h = T(j+1)$. By the ind. hyp. there is $\mathbb{E}'$ realizing $\mathcal{T}|(h+1)$ s.t. $\mathbb{E}'\restriction k = \mathbb{E}\restriction k$, $R^{\mathbb{F}}_h = R^{\mathbb{E}}_k$; $\gamma^{\mathbb{F}}_h \leq \gamma^{\mathbb{E}}_k$ where $\gamma^{\mathbb{F}}_h < \gamma^{\mathbb{F}}_k$ if π_{kh} is not total and otherwise $\sigma^{\mathbb{F}}_h \circ \pi_{kh} = \sigma^{\mathbb{E}}_k$, $\gamma^{\mathbb{F}}_h = \gamma^{\mathbb{E}}_k$. It then suffices to prove the theorem with $h, h, \mathbb{E}'$ in place of $k, i, \mathbb{E}$.) By the usual argument

k is a breakpoint at j. (Otherwise there is l s.t. $k < l \leq j$, $T(l) \leq k$, $T(l)$ survives at l. But then $\neg(k \leq_T l \leq_T j+1)$. Hence $n(l) > n(k)$. Hence $T(l)$ does not survive at l.) Hence $\mathbb{E}$ extends to $\mathbb{E}'$ realizing $\mathbb{J}|(j+1)$. By Lemma 3.3, $\mathbb{E}'$ extends to $\mathbb{F}$ realizing $\mathbb{J}|(j+2)$ with the required properties. QED (Case 2)

<u>Case 3</u> Lim(μ).

We first show that $\mathbb{J}|(\mu+1)$ possesses a realization with room. Choose $\langle i_n \mid n < \omega \rangle$ monotone s.t. i_n survives at μ, $\mu = \sup_n i_n$, and each $\pi_{i_n, \mu}$ is total. By the induction hypothesis we can successively choose $\mathbb{F}_n$ realizing $\mathbb{J}|(i_n+1)$ s.t. $\mathbb{F}_{n+1} \upharpoonright i_n =$ $= \mathbb{F}_n \upharpoonright i_n$, $R_{i_{n+1}}^{\mathbb{F}_{n+1}} = R_{i_n}^{\mathbb{F}_n}$, $\gamma_{i_{n+1}} = \gamma_{i_n}$,

$$\delta_{i_{n+1}}^{\mathbb{F}_{n+1}} \pi_{i_n, i_{n+1}} = \delta_{i_n}, \quad \rho^{i_{n+1}\,\mathbb{F}_{n+1}} = \rho^{i_n\,\mathbb{F}_n},$$

$$\delta_j^{i_{n+1}} = \delta_j^{i_n} \text{ for } j < i_n, \quad \bar{\rho}^{i_{n+1},0} = \rho^{i_n,0}.$$

We use this to define a realization $\mathbb{F}$ for $\mathbb{J}|\mu+1$. We set:

$\mathbb{F} \upharpoonright i_n = \mathbb{F}_n \upharpoonright i_n$. $\mathbb{F}_\mu = \langle R, \vec{\delta}^\mu, \vec{\rho}^{\mu,0} \rangle$ where $R = R_{i_n}^{\mathbb{F}_n}$, $\vec{\rho}^{\mu,0} = \vec{\rho}^{i_n,0}$, ~~and~~ $\delta_j^\mu = \delta_j^{i_n}$ for $j < i_n$, and $\delta_\mu = \delta_\mu^\mu$ is defined by: $\delta_{\mu\pi_{i_n,\mu}} = \delta_{i_n}$ for $n < \omega$. (Note Clearly we will have $\vec{\rho}^\mu = \vec{\rho}^{i_n}$ for $n < \omega$.) $\mathbb{F}$ is easily seen to have room.

We now prove (b). Let $\mathbb{E}$ be a realizatio with room for $\mathcal{Y}|(i+1)$ and let $k \le i$ be largest s.t. k survives at μ. Choose the above sequence with $i_0 > i$. Choose $\mathbb{F}_0$ s.t. $\mathbb{F}_0 \upharpoonright k = \mathbb{E} \upharpoonright k$ and $\mathbb{F}_0$ has the right properties wrt. $\mathbb{E}$. Then $\mathbb{F}$ will have the right properties wrt. $\mathbb{E}$. To prove (a), let $i < \mu$ be a break point and let $\mathbb{E}$ realize $\mathcal{Y}|(i+1)$. Choose $\mathbb{F}_0$ as an extension of $\mathbb{E}$ s.t. $R_{i_0}^{\mathbb{F}_0} \in R_i^{\mathbb{E}}$. Then $R_\mu^{\mathbb{F}} \in R_i^{\mathbb{E}}$.

QED (Lemma 3)