§ 1 s-iterability

As mentioned in [ANFS], the mice used in [MS] differ from ours not only in having an ostensibly weaker iterability requirement but also in the way the extenders are indexed. This indexing in turn affects the structure of an iteration. It is very hard to compare the two types of mice directly. We can, however, define for our premice a new type of iteration, whose pattern will be like that of the iterations in [MS]. The trick is to assign to each extender E_ν its "natural length" $s(\nu)$ (in the sense of [MS]) as an "iterability index". We then define the notion of a "normal s-iteration" by employing the indices $s(\nu)$ as we previously employed $\lambda(\nu) = lh(E_\nu)$ – i.e. $T(i+1)$ is the least h s.t. $\kappa_i < s(\nu_h)$ in M_h, where $\mathcal{I} = \langle\langle M_i\rangle, \ldots, T\rangle$ is the iteration.

This leads, of course, to new notions of "smooth" and "good" iteration, as well as "coiteration". The theorems in [NFS] then have clear analogues for these new notions. (Alternatively we can use the MS version of s-iterability and straightforwardly transcribe the theorems of [MS], since our iteration patterns are now the same. The construction of an inner model with a Woodin cardinal in [MS]§11 will go through without difficulty.) Since the notions of iterability and s-iterability now involve the same premice, one can ask whether they are equivalent. This section gives a positive answer under the assumptions A1, A2.

We first give precise definitions and develop the basic properties of s-iterations.

<u>Def</u> Let $M = \langle J_\alpha^E, E_{\omega\alpha} \rangle$ be a premouse. Let $E_\nu \neq \emptyset$. $s(\nu) = s(\nu)^M =_{Df}$ the least $s \geq \kappa^{+J_\nu^E}$ s.t. each $\alpha < \lambda$ has the form $\alpha = \pi(f)(\xi_1, \dots, \xi_n)$ for $\xi_1, \dots, \xi_n < s$ (where $\kappa = \mathrm{crit}(E_\nu)$, $\lambda = \mathrm{lh}(E_\nu)$, and $\pi : J_\tau^E \xrightarrow[E_\nu]{} J_\nu^E$, where $\tau = \kappa^{+J_\nu^E}$).

<u>Def</u> Let M, ν, λ etc. be as above. $\gamma < \lambda$ is a <u>generator</u> of E_ν iff $\gamma \neq \pi(f)(\xi_1, \dots, \xi_n)$ for all $\xi_1, \dots, \xi_n < \gamma$, $f \in M$, $f : \kappa \to \kappa$.

It is easily seen that:

<u>Lemma 7.1</u> $s(\nu) =$ the least upper bound of all generators if $s(\nu) > \kappa^{+J_\nu^E}$.

<u>Lemma 7.2</u> $\kappa < s(\nu) \leq \lambda$. Moreover, if M is a sound 1-small mouse, then $s(\nu) < \lambda$.

<u>Def</u> A premouse $M = \langle J_\lambda^E, E_{\omega\alpha} \rangle$ is <u>modest</u> iff $s(\nu) < \lambda = \mathrm{lh}(E_\nu)$ whenever $E_\nu \neq \emptyset$.

We shall mainly be concerned with modest premice.

<u>Lemma 7.3</u> If $\sigma : \bar{M} \longrightarrow_{\Sigma_1} M$, then $\sigma'' s(\nu)^{\bar{M}} \subset s(\sigma(\nu))^M$ whenever $E_\nu^M \neq \emptyset$.

(Here $\sigma(\mathrm{On} \cap \bar{M}) =_{df} \mathrm{On} \cap M$.)

proof.

The fact that γ is a generator of $F = E_\nu$ is expressed by the Π_1 statement:

$\bigwedge f : \kappa \to \kappa \; \bigwedge x \, (x = \langle \delta_1, \ldots, \delta_n \rangle$ where $\delta_i < \gamma$ for all i

$\to \{ \langle \alpha, \beta \rangle \mid \alpha \neq f(\beta) \} \in F_{\langle \gamma, \langle \delta_1, \ldots, \delta_n \rangle \rangle}$.

QED (Lemma 7.3)

<u>Lemma 7.4</u> If $\sigma : \bar{M} \longrightarrow_{\Sigma_2} M$, then $\sigma(s(\nu)^{\bar{M}}) = s(\sigma(\nu))^M$.

prf.

"s = the lub of the generators" is Π_2. QE

<u>Lemma 7.5</u> Let $\sigma : \bar{M} \longrightarrow_G M$ where $\bar{M} = \langle J_{\bar{\nu}}^{\bar{E}}, \bar{F} \rangle$, $M = \langle J_\nu^E, F \rangle$, $\mathrm{crit}(\sigma) < s(\bar{\nu})^{\bar{M}}$.

Then $s(\nu)^M = \mathrm{lub}\, \sigma'' s(\bar{\nu})^{\bar{M}}$.

proof

Set $\bar{s} = s(\bar{\nu})^M$, $s = s(\nu)^M$. Let $\tilde{\kappa} = \mathrm{crit}(G)$, $\bar{\kappa} = \mathrm{crit}(\bar{F})$, $\kappa = \mathrm{crit}(F)$.

Then $\sigma \bar{\pi}(X) = \pi \sigma(X)$ for $X \in \mathcal{P}(\tilde{\kappa}) \cap \bar{M}$, since $Y = \bar{F}(X) \longleftrightarrow \sigma(Y) = F(\sigma(X))$. Let $\xi \geq \mu = \text{lub}\, \sigma''\iota(\bar{\nu})^{\bar{M}}$.

<u>Claim</u> ξ is not a generator.

Let $\xi = \sigma(g)(\alpha)$, $g \in \bar{M}$, $g: \tilde{\kappa} \to \tilde{\kappa}$, $\alpha < \text{lh}(($

(Hence $\alpha < \sigma(\tilde{\kappa}) < \mu$ by Lemma 7.3.)

Let $\bar{\tau} = \bar{\kappa}^{+J_{\bar{\nu}}^{E^{\bar{M}}}}$, $\pi : J_{\bar{\tau}}^{\bar{E}} \xrightarrow[\bar{F}]{} J_{\bar{\nu}}^{\bar{E}}$.

Then $g = \pi(f)(\vec{\delta})$ where $f \in J_{\bar{\tau}}^{\bar{E}}$, $f: \bar{\kappa}^n \to \bar{\kappa}$, $\delta_1, \ldots, \delta_n < \bar{\tau}$. Hence

$\xi = \sigma(\bar{\pi}(f)(\vec{\delta}))(\alpha) = \sigma\bar{\pi}(f)(\sigma(\vec{\delta}))(\alpha) =$

$= \pi\sigma(f)(\sigma(\vec{\delta}))(\alpha) = \pi(h)(\sigma(\vec{\delta}), \alpha)$,

where $\sigma(\delta_i), \alpha < \mu$. QED (Lemma 7.5)

<u>Lemma 7.6</u> Let $\sigma : \bar{M} \xrightarrow[\Sigma]{} M$, $\tilde{\kappa} = \text{crit}(\sigma)$, $\bar{M} = \langle J_{\bar{\nu}}^{\bar{E}}, \bar{F}\rangle$, $M = \langle J_{\nu}^{E}, F\rangle$, $\bar{F} \neq \emptyset$, and $\tilde{\kappa}$ is a cardinal in M. Then $\tilde{\kappa} < \iota(\nu)^M$

(<u>Note</u> It follows that a modest mouse can become immodest in the course of a normal iteration. This will not be possible in s-normal iterations.)

proof. Suppose not

Then $\tilde{\kappa} > \kappa = \text{crit}(\bar{F})$. Hence $\tilde{\kappa} > \bar{\tau} =$

$= \bar{\tau}^{+\bar{M}}$. Hence $\tilde{\kappa} > \bar{\tau}$, since $\bar{\tau}$ is a successor cardinal in $\bar{M}$.

Then $\kappa = \mathrm{crit}(F)$, $\tau = \kappa^{+M}$, $J_\tau^{\bar{E}} = J_\tau^E$. Let $\bar\pi : J_\tau^{\bar E} \to_{\bar F} J_{\bar\nu}^{\bar E}$, $\pi : J_\tau^E \to_F J_\nu^E$. Then $\sigma\bar\pi \restriction \mathbb{P}(\kappa) = \pi\sigma \restriction \mathbb{P}(\kappa)$, since $Y = \bar F(X) \longleftrightarrow$

$\longleftrightarrow \sigma(Y) = F(\sigma(X))$. $\tilde\kappa$ is not a generator in M. Hence $\tilde\kappa = \pi(f)(\vec\delta)$, where $\delta_1, \ldots, \delta_n < \tilde\kappa$ and $f \in M$, $f : \kappa^n \to \kappa$. (Note: $\kappa = \mathrm{crit}(F)$, since $\tilde\kappa > \kappa$.) Then τ is a cardinal in M and $f \in J_\tau^{E^M} = J_\tau^{E^{\bar M}}$. Hence $\tilde\kappa = \pi(f)(\vec\delta) = \pi\sigma(f)(\sigma(\vec\delta)) =$
$= \sigma\bar\pi(f)(\sigma(\vec\delta)) = \sigma(\bar\pi(f)(\vec\delta)) \in \mathrm{rng}(\sigma)$.
Contr!

QED (Lemma 7.6)

<u>Lemma 7.7</u> $\omega\rho^1_{M\|\nu} \leq \lambda(\nu)^M$ for $E_\nu^M \neq \emptyset$.

proof. (Let $M = M\|\nu$)
Then $M \subset h_M(\lambda(\nu))$, since each ξ has the form: $\xi =$ that ordinal ξ s.t. $\{\langle\alpha,\beta\rangle \mid \alpha = f(\beta)\} \in F_{\langle\xi, \langle\delta_1,\ldots,\delta_n\rangle\rangle}$ for some $\delta_1, \ldots, \delta_n < \lambda(\nu)$, where $F = E_\nu^M$.

QED (Lemma 7.7)

<u>Def</u> $s^+(\nu) = s(\nu)^{+J_\nu^E}$ for $E_\nu \neq \emptyset$.

(Hence $s(\nu) < s^+(\nu) \leq \nu$).

<u>Lemma 7.8</u> $\nu \neq \nu' \longrightarrow s^+(\nu) \neq s^+(\nu')$.

proof. Suppose not.

Let e.g. $\nu < \nu'$, where $s^+(\nu) = s^+(\nu')$.

Then $s(\nu) < s^+(\nu') \leq \nu$ and $\omega\rho^1_{M\|\nu} \leq s(\nu)$.

Hence $s(\nu')$ is not a cardinal in $J^E_{\nu'}$. Contr! QED (Lemma 7.8).

(Note: It follows that $\nu \neq \nu' \longrightarrow s(\nu) \neq s(\nu$

(<u>Note</u>: It is possible to have: $s^+(\nu) < s^+(\nu') < \nu' < \nu$. We shall define the notion of s-iteration so as to exclude the application of an $E_{\nu'}$ with this property. Having applied E_ν, however, we have $E_\nu = \emptyset$ at later stages and hence may be free to apply $E_{\nu'}$ then.)

<u>Lemma 8.1</u> Let $M = \langle J_\nu^E, F\rangle$ be a premouse, $F \neq \emptyset$. Let $\sigma : M \xrightarrow[G]{} M_0 = \langle J_{\nu_0}^{E_0}, F_0\rangle$, where G is weakly amenable. Let $\kappa = \mathrm{crit}(F) < \tilde{\kappa} = \mathrm{crit}(G)$. Let N be a premouse s.t. $J_\tau^E = J_\tau^{E^N}$ where $\tau = \kappa^{+M} = \tilde{\kappa}^{+N}$. (Hence F is an extender on N.) Let $\pi_0 : N \xrightarrow[F]{} N_0$ (Thus $\nu = \pi_0(\kappa)$ is a cardinal in N_0 and $J_\nu^E = J_\nu^{E^N}$, so G is an extender on N_0.) Let $\pi_1 : N_0' \xrightarrow[G]{} N_1'$. Then $\pi_1\pi_0 : N \xrightarrow[F_0]{} N_1$.

proof.

(1) $\sigma = \pi_1 \restriction M$, since ν is regular in N_0

(2) $F_0 = \pi_1\pi_0 \restriction \mathbb{P}(\kappa)$

proof.

$Y = F(X) \leftrightarrow \sigma(Y) = F^0(X)$, hence

$\pi_1\pi_0(X) = \sigma F(X) = F^0(X)$.

(3) $N_1 = $ the $\Sigma_0(N_1)$-closure of $\mathrm{rng}(\pi_1\pi_0) \cup \lambda_0$

(where $\lambda_0 = \mathrm{lh}(F_0) = \pi_1\pi_0(\kappa)$).

prf. Let $x \in N_1$. Then $x = \pi_1(f)(\mu)$, $\mu < \mathrm{lh}(G)$, $f \in N_0$. Let $f = \pi_0(g)(\eta)$, $\eta < \lambda = \mathrm{lh}(F)$. Then $x = \pi_1(\pi_0(g)(\eta))(\mu) = (\pi_1\pi_0(g)(\pi_1(\eta)))(\mu)$. But $\pi_1(\eta) < \pi_1(\lambda) = \pi_1\pi_0(\kappa) = \lambda_0$; $\mu < \mathrm{lh}(G) \leq \pi_1(\tilde{\kappa}) \leq \pi_1(\lambda) = \pi_1\pi_0(\kappa) = \lambda_0$.

QED (Lemma 8.1

Lemma 8.2 Let M, M_0, G be as above. Let $\tilde{\kappa} = \text{crit}(G) \geq \lambda = \lambda(\nu)^M$. Then $\lambda(\nu_0)^{M_0} \leq lh(G)$.

proof.

Let $\tau = \kappa^+$, $N = J_\tau^E$, $N_0 = J_\nu^E$, $N_1 = J_{\nu_0}^{E_0}$, with $\pi_0 : N \xrightarrow[F]{} N_0$, $\sigma = \pi_1 : N_0 \xrightarrow[G]{} N_1$.

Then $\sigma\pi : N \xrightarrow[F_0]{} N_1$ by Lemma 8.1.

Claim $N_1 =$ the $\Sigma_0(N_1)$-closure of $\text{rng}(\sigma\pi) \cup lh(G)$.

proof.

Let $x \in N_1$, $x = \sigma(f)(\mu)$, $f \in N_0$, $f : \tilde{\kappa} \to N_0$, $\mu < lh(G) = \tilde{\lambda}$. Then $f = \pi(g)(\vec{\eta})$, where $\eta_1, \ldots, \eta_n < \lambda$, $g \in M$, $g : \kappa^n \to M$. Hence
$x = \sigma(\pi(g)(\vec{\eta}))(\mu) = (\sigma\pi(g))(\vec{\eta}))(\mu)$
(since $\sigma \restriction \lambda = id$), where $\mu < \tilde{\lambda}$ and $\eta_1, \ldots, \eta_n <$
$< \bar{\lambda} \leq \tilde{\kappa} < \tilde{\lambda}$. QED (Lemma 8.2.

Lemma 8.3 Let M, M_0, G, σ, N be as above where G is close to N. Let $\pi_0 : N \xrightarrow[F]{*} N_0$, $\pi_1 : N_0 \xrightarrow[G]{*} N_1$. Then $\pi_1\pi_0 : N \xrightarrow[F_0]{*} N_1$.

prf.

We imitate the proof of 9.1.

(1) $\sigma = \pi_1 \restriction M$, since ν is regular in N_0 and $\nu < \omega\rho^n_{N_0}$ whenever $\kappa < \omega\rho^n_N$.

(2) $F_0 = \pi_1\pi_0 \restriction \mathcal{P}(\kappa)$ is proven as before.

Since $\pi_1\pi_0 : N \longrightarrow_{\Sigma^*} N_1$ we need only:

(3) N_1 = the closure of $\mathrm{rng}(\pi_1\pi_0) \cup \lambda_0$ under $\Sigma_0^{(n)}$ fcns for $\omega\rho^n_N > \kappa$.

proof.

Let $x \in N_1$. Then $x = \pi_1(f)(\mu)$, $f \in \Gamma^*(\tilde{\kappa}, N_0)$, $\mu < \tilde{\lambda} = lh(G)$. If $f \in N_0$, let $f = \pi_0(g)(\eta)$, $g \in \Gamma^*(\kappa, N)$, $\eta < \lambda = lh(F)$.

Then $x = \pi_1(\pi_0(g)(\eta))(\mu) = (\pi_1\pi_0(g)(\pi_1(\eta)))(\mu)$, where $\mu, \pi_1(\eta) < \lambda_0$ as before;

Otherwise f is a good $\Sigma_1^{(n)}$ function in a parameter p, where $\omega\rho^{n+1}_{N_0} > \kappa$.

Let $f(\xi) \simeq f'(p, \xi)$, where f' is $\Sigma_1^{(n)}(N_0)$.

Let f_1 have the same functionally absolute definition over N_1. Let $p = \pi_0(g)(\eta)$, $g \in \Gamma^*(\kappa, N)$, $\eta < \lambda$. Then

$x = f_1(\pi_1(p), \mu) = f_1(\pi_1(\pi_0(g)(\eta)), \mu) =$

$= f_1(\pi_1\pi_0(g)(\pi_1(\eta)), \mu)$, where $\pi_1(\eta), \mu < \lambda_0$.

QED (Lemma 8.3)

<u>Def</u> Let $\mathcal{J} = \langle\langle M_i\rangle, \langle \nu_i \mid i\in D\rangle, \langle \eta_i\rangle, \langle \pi_{ij}\rangle, T\rangle$ be an iteration. Set: $s_i = s(\nu_i)^{M_i}$, $s_i^+ = s_i^+(\nu_i)^{M_i}$. $\mathcal{J}$ is called a <u>normal</u> <u>s-iteration</u> iff

(a) $\mathcal{J}$ is standard

(b) $\nu_i > s_h^+$ for $i \in D$, $h \in D \cap i$

(c) $T(i+1) =$ the least $\xi \in D$ s.t. $\kappa_i < s_\xi$ for $i \in D$.

(d) If $i \in D$, there is no $\nu > \nu_i$ s.t. $E_\nu^{M_i} \neq \emptyset$ and $s^+(\nu) < s_i^+$ in M_i.

<u>Remark</u> A heuristic justification for (d) is the following: If M_i were a mouse in the sense of [MS], we would have $s^+(\nu) = \nu$ and ν_i would not appear in $J_\nu^{E^{M_i}}$, hence not in M_i. It might, however, appear in N, where $\pi : J_\tau^{E^{M_i}} \xrightarrow[E_\nu]{} N$, $\tau = \kappa^{+J_\nu^{E^{M_i}}}$, $\kappa = \mathrm{crit}(E_\nu^{M_i})$. Thus we will not be free to apply ν_i until we have first applied ν.

In the following let $\mathcal{I} = \langle\langle M_i \rangle, \dots, T\rangle$ be a normal s-iteration of a premouse M. As usual, we set: $\kappa_i = \mathrm{crit}(E^{M_i}_{\nu_i})$, $\tau_i = \kappa_i^{+J^{E^M}_{\nu_i}}$, $\lambda_i = lh(E^M_{\nu_i})$, $s_i = s(\nu_i)^{M_i}$, $s_i^+ = s^+(\nu_i)^{M_i}$.

<u>Lemma 9.1</u> Let $j \in D \cap i$. Then $J^{E^{M_i}}_{s_j^+} = J^{M_i}_{s_j^+}$ and s_j^+ is a cardinal in M_i.

prf. Ind. on i.

The condition for continuing an iteration "one further step" is given by:

<u>Lemma 9.2</u> Let $i \in D$, $\xi = T(i+1)$. Then $\tau_i \le s_\xi^+$ (hence $J^{E^{M_i}}_{\tau_i} = J^{E^{M_\xi}}_{\tau_i}$).

proof,
$s_\xi^+ > \kappa_i$ is a cardinal in $J^{E^{M_i}}_{\nu_i}$ or $s_\xi^+ = \nu_i$.
For $i = \xi$ this is immediate. Otherwise s_ξ^+ is a cardinal in M_i. QED (9.2)

<u>Lemma 9.3</u> Let $i \le_T j+1$, where $i, j \in D$. Then $s_\xi \le \kappa_j$ for all $\xi < i$ s.t. $\xi \in D$.
prf. trivial.

<u>Lemma 9.4</u> Let $\nu = ht(M_{i+1})$, $E_\nu^{M_{i+1}} \ne \emptyset$, $i \in D$. Let $\xi = T(i+1)$, $\pi_{\xi, i+1}(\bar\nu) = \nu$ (i.e. $\bar\nu = \eta_i$). Then $\tau_i \le s(\bar\nu)^{M_\xi}$.

proof of Lemma 9.4. Suppose not,

(1) $s^+_\xi \leq s^+(\bar\nu)$ in M_ξ, since otherwise $s^+(\bar\nu) < s^+_\xi \leq \nu_\xi < \bar\nu$, contradicting (d) in the def. of "normal s-iteration".

By Lemma 9.2:

(2) $s(\bar\nu) < \tau_i \leq s^+_\xi \leq s^+(\bar\nu)$ in M_ξ, where $\tau_i = \kappa_i^{+M_\xi\|\bar\nu}$ is a cardinal in $J_{\bar\nu}^{E^{M_\xi}}$.

Hence:

(3) $\tau_i = s^+(\bar\nu) = s^+_\xi$ in M_ξ.

$\nu_\xi \leq \bar\nu$, since $\tau_i = s^+_\xi$ is not a cardinal in $J_\eta^{E^{M_\xi}}$ for any $\eta > \bar\nu$. Hence

(4) $\nu_\xi < \bar\nu$,

since otherwise $\nu_\xi = \bar\nu$ and $\tau_i \leq$ . . ,

But τ_i is a cardinal in $J_{\bar\nu}^{E^{M_\xi}}$; hence $\tau_i \leq \omega\rho^1_{M_\xi\|\nu_\xi} \leq s_\xi < s^+_\xi = \tau_i$. Contr!

Corollary 9.4.1 Let $i \in D$, $\nu = ht(M_{i+1})$, $E_\nu^{M_{i+1}} \neq \emptyset$. Then $\nu_i \leq s(\nu)^{M_{i+1}}$.

proof.

$\nu_i = \sup \pi_{\xi,i+1}{}''\tau_i \leq \sup \pi_{\xi,i+1}{}''s(\bar\nu) \leq s(\nu)$, where ξ, ν are as above. QED(9.4.1)

<u>Lemma 9.5</u> Let $j, i \in D$, $i < j$. Then $s_j^+ < s_i^+$.

proof. Suppose not.

Let $\mathcal{I}$ be a counterexample with $lh(\mathcal{I})$ minimal. Then $lh(\mathcal{I}) = i+2$, where $i = j+1$ and $s_i^+ \leq s_j^+$, since otherwise we could shorten the counterexample. Hence

$\omega\rho^1_{M_i \| \nu_i} \leq s_i < s_j^+ \leq \nu_j$. Hence $\nu_i = ht(M_i)$, since s_j^+ is a cardinal in M_i and $\nu_j \in M_i$.

Hence $s_i^+ \leq s_j^+ \leq \nu_j \leq s_i < s_i^+$ by Corollary 9.4.1. Contr! QED (9.5)

We are mainly interested in s-iterations of modest mice. We show that modesty is preserved.

<u>Lemma 9.6</u> Let $i \in D$, $\xi = T(i+1)$, $\pi_{\xi, i+1}(\bar{\nu}) = \nu$, $E_{\bar{\nu}}^{M_\xi} \neq \emptyset$. Then

$s(\nu)^{M_{i+1}} \leq \pi_{\xi, i+1}(s(\bar{\nu})^{M_\xi})$.

(Here $\pi_{\xi, i+1}(\eta_i) =_{\text{Df}} ht(M_{i+1})$).

proof.

Let $M^* = M_i^* = M_\xi \| \eta_i$. For $\bar{\nu} < ht(M^*)$ the claim is trivial, since then:

(1) $\pi_{\xi,i+1}(s(\bar\nu)^{M^*}) = s(\nu)^{M_{i+1}}$.

Now let $\bar\nu = \eta_i = ht(M^*)$. At $\omega\rho^1_{M^*} > \kappa_i$, we still have (1) by Lemma 7.4. Now let $\omega\rho^1_{M^*} \le \kappa_i$. Then $\pi_{\xi,i+1} : M^* \longrightarrow_{E^{M_i}_{\nu_i}} M_{i+1}$ is a Σ_0 ultrapower and Lemma 7.5 gives us:

(2) $s(\nu)^{M_{i+1}} = \text{lub}\, \pi_{\xi,i+1}{}'' s(\bar\nu)^{M^*}$.

QED (Lemma 9.6)

<u>Korollary 9.7</u> Let M be modest. Then each M_i is modest.

proof. Ind. on i.

. . . . . .

<u>Def</u> Let $\mathcal{I} = \langle\langle M_i\rangle, \langle \nu_i\rangle, \langle \eta_i\rangle, \langle \pi_{ij}\rangle, T\rangle$ be a normal s-iteration. For $i \in D$ set:

$\tilde\lambda_i =_{\text{df}} \{\lambda_j \mid i \le j \in D\}$

By a straightforward proof we get:

<u>Lemma 9.8</u> Let $i \in D$, $i < j$. Then $J^{E^{M_i}}_{\tilde\lambda_i} = J^{E^{M_j}}_{\tilde\lambda_i}$ and $\tilde\lambda_i$ is a limit cardinal in M_j.

Moreover, $\tilde\lambda_i > s^+_i$ (by Lemma 9.6).

In trying to imitate our proofs about ordinary normal iterations it is often useful to use $\tilde{\lambda}_i$ in place of λ_i. An example is the proof of the following theorem (which we omit):

__Lemma 10__ Let $i \in D$, $\xi = T(i+1)$, $M^* = M_\xi \| \eta_i$. Then $E_{\nu_i}^{M_i}$ is close to M^*. (Hence $\pi_{\xi, i+1} : M^* \xrightarrow[\Sigma^*]{} M_{i+1}$.)

This can be verified by a close imitation of the proof of [NFS] §4 Lemma 1, using $\tilde{\lambda}_j$ in place of λ_j. We can similarly imitate the "copying lemmas" in [NFS] §5. (Thus showing e.g. that if $\sigma : \bar{M} \xrightarrow[\Sigma^*]{} M$ and M has an s-iteration strategy (normal s-iteration strategy, smooth s-iteration strategy), then so does $\bar{M}$. (The notions "good s-iteration", "smooth s-iteration" are defined in the obvious way.) We can also carry through the development in [NFS] §9 and show that M is s-iterable if it is smoothly s-iterable.

Using the copying theorems we can prove the Dodd-Jensen and Neeman-Steel lemmas. We also believe (but have not checked) that the proofs in [NFS] §6 go through as before. This means that can again apply the "economical strategy" when s-normally iterating a 1-round premouse M (i.e. at a limit point λ, the branch $\{i \mid i T \lambda\}$ is chosen - if possible - be "economical" in the sense that if $\delta = \sup\{\kappa_i \mid i < \lambda\}$, then there is no $\nu \geq \delta$ s.t. $E_\nu^{M_\lambda} \neq \emptyset$.) The iteration will then have at most one economical branch and all other branches will be unique.

Before we can imitate the proofs in [NFS] §7, §8, we must define the appropriate notion of comparison iteration. We set:

Def $\tilde{E}_s^M = \begin{cases} E_\nu^M & \text{if } s = s^+(\nu)^M \, ; \\ \emptyset & \text{if no such } \nu \text{ exists} \end{cases}$

<u>Def</u> $s(M,M') \simeq$ the least s s.t. $\tilde{E}^M_s \neq \tilde{E}^{M'}_s$.

(Hence $s(M,M') \leq ht(M), ht(M')$).

<u>Def</u> The s-<u>coiteration</u> of M^0, M^1 is a pair $\mathcal{J} = \langle \mathcal{J}^0, \mathcal{J}^1 \rangle$ of iterations
$\mathcal{J}^h = \langle \langle M^h_i \rangle, \langle \nu^h_i \mid i \in D^h \rangle, \langle \eta^h_i \rangle, \langle \pi^h_{i,l} \rangle, T^h \rangle$
with <u>indices</u> $\langle s_i \mid i+1 < lh(\mathcal{J}^h) \rangle$ $(h = 0,1)$
s.t.

(a) $\mathcal{J}^h$ is standard

(b) $s_i \simeq s(M^0_i, M^1_i)$

(c) $\nu^h_i \simeq$ that ν s.t. $s_i = s^+(\nu)^{M^h_i}$

(d) $T^h(i+1) =$ the least ξ s.t. $\xi \in D^h$ and $\kappa_i < s_\xi$, if $i \in D^h$.

Then $\mathcal{J}^0, \mathcal{J}^1$ are easily seen to be s-normal. If M^0, M^1 are normally s-iterable, it follows as before that the coiteration terminates in a comparable pair M'^0, M'^1. If a truncation occurs on both sides on the main branch, then $M'^0 = M'^1$ as before. Finally, it follows as before that it

M^0 is presolid (i.e. $M^0 \| \alpha$ is solid for $\alpha < ht(M^0)$). and a truncation occurs in the main branch on the M^0-side, then there is no truncation on the M^1-side, and M'^1 is a segment of M'^0. We believe that the whole of [NSF] §7, §8 can be carried out for s-iterations, though, again, we haven't checked it

. . . . .

Our main concern here is to establish connections between iterability and s-iterability. We first define:

<u>Def</u> A <u>mixed</u> <u>iteration</u> is an iteration $\mathcal{I}$ which composes into a sequence $\langle \mathcal{I}_i \mid i < \Gamma \rangle$ of successive iterations, each of which is either a normal iteration or a normal s-iteration (with truncations allowed between the $\mathcal{I}_i$'s). If there are no truncations between the $\mathcal{I}_i$'s, we call $\mathcal{I}$ a <u>smooth</u> <u>mixed</u> <u>iteration</u>.

(This can be made more precise by imitating the def. of <u>good</u> <u>iteration</u> in [NFS].)

<u>Def</u> M is called (<u>smoothly</u>) <u>mixed</u> <u>iter-able</u> iff every (smooth) mixed iteration of M can be continued.

The methods of [NFS] §9 should show that every smoothly mixed iterable premouse is mixed iterable. We call M a <u>mixed mouse</u> iff M is mixed iterable. M is a <u>weak</u> <u>mixed</u> <u>mouse</u> iff whenever $\sigma : \bar{M} \longrightarrow_{\Sigma^*} M$ and $\bar{M}$ is countable, then $\bar{M}$ is countably mixed iterable. We now strengthen Theorem 1 to:

<u>Thm 11</u> Assume (+). Let $Q \in V_\theta$ be a 1-small premouse which is $\theta+1$-normally iterable. Then M is a weak mixed mouse.

proof of Lemma 11

The proof is a slight reworking of the proof of Thm 1. We define $\vec{Q}, \vec{N}, \vec{\nu}$ exactly as before. Let $\sigma: P \xrightarrow[\Sigma^*]{} N_\gamma$, where γ is countable. Then $\sigma: P \xrightarrow[\Sigma^*]{} N_\gamma \min(\vec{\rho})$ for appropriate $\vec{\rho}$ and it suffices to Lemma 5.3, where $\mathcal{Y}$ is now a countable putative normal s-iteration of P. We again sketch the proof for the case that $lh(\mathcal{Y}) = \omega$ and $\mathcal{Y}$ is without truncations. (Thus we must show that there is a cofinal branch b and a $\sigma': P_b \xrightarrow[\Sigma^*]{} N_\gamma \min(\vec{\rho})$ s.t. $\sigma' \pi_b = \sigma$.) The tree $\mathcal{U} = \mathcal{U}(\sigma, N_\gamma, \vec{\rho})$ is defined as before. γ is again taken as the least counterexample, so $\mathcal{U}$ is well founded. We define $\langle \sigma_i, s_i, R_i, \vec{\rho}^{\,i} \rangle$ satisfying the same properties as before except that (5) now becomes:

(5′) Let $h < i$. Then $s_h^* < s_i^*$,

$$V^{R_h}_{s_h^*+2} = V^{R_i}_{s_h^*+2} \text{ and } \sigma_i \upharpoonright s_h = \sigma_h \sigma_h \upharpoonright s_h$$

where $s_i^* = \sigma_i \sigma_i (s_i)$

(8)–(10) go through as before except that in the proof of (9) we use $\delta_i(s_h)$ in place of $\delta_i(\lambda_h)$. (11) needs a new proof, since we have changed the order of iteration, but this should be unproblematical. Case 1 in the definition of $\langle R_i, \delta_i, S_i, \vec{p}^i \rangle$ is unchanged. In Case 2 define $F, F', \langle N, F^* \rangle, \delta_i^*$ and select $\tilde{\lambda}, \tilde{\delta}, \beta$ exactly as before. The Claim which we then made is proven exactly as before. $\langle \delta_{i+1}, S_{i+1}, R_{i+1}, \vec{p}^{\,i+1} \rangle$ is then defined as before. This gives us (1)–(4), (6), (7). In place of (5) we have:

(1) $\lambda_i^* < \delta_{i+1}(\lambda_i)$ and $V^{R_i}_{\lambda_i^*+2} = V^{R_{i+1}}_{\lambda_i^*+2}$,

$$\delta_{i+1} \restriction \lambda_i = \sigma_i \delta_i \restriction \lambda_i ,$$

where $s_i \leq \lambda_i$. It remains only to show:

(2) $s_{i+1}^* > s_i^*$.

We in fact show: $\delta_{i+1}(s_{i+1}) > s_i^*$.

Let $\nu_{i+1} = ht(P_{i+1})$. Then $\lambda_i \leq s_{i+1}$; hence $\delta_{i+1}(s_{i+1}) \geq \delta_{i+1}(\lambda_i) > \lambda_i^* \geq s_i^*$.

Otherwise s_i^+ is a cardinal in P_{i+1} and hence $s_i^+ \le \omega\rho^1_{M_{i+1}\|\nu_{i+1}} \le s_{i+1}$. Hence

$$\delta_{i+1}(s_{i+1}) \ge \delta_{i+1}(s_i^+) > \delta_{i+1}(s_i) > \lambda_i^* \ge s_i^* .$$

QED (Thm 11).

Remark A modification of the proof of Thm 1 shows that if Q is a 1-small premouse which is s-iterable up to $\theta+1$ (and if the appropriate version of (+) holds), then Q is a weak s-mouse. We are unable to prove from these assumptions that Q is a weak mou

. . . . .

Def An inner model $W = J^E_\infty$ is s-universal iff W is normally s-iterable and the s-coiterati with any 1-small premouse terminates below ∞.

Imitating the proof of Corollary 2, using Thm 11 in place of Thm 1 we get:

Corollary 12 Assume A1 + A2. Then K^c is mixed iterable. Moreover K^c is s-universal.

Corollary 12.1 Assume A1+A2. Let Q be normally iterable or normally s-iterable in V_θ, where Q is a 1-small premouse. Then Q is mixed iterable in V_θ.

Modifying the proof of Cor 12.1 the way the proof of [ANFS] §3 Cor 6.1 was modified to get [ANFS] §3 Thm 7, we get:

Lemma 13 Assume A1+A2. Let Q be a 1-small countable premouse which is either countably normally iterable or countably normally s-iterable. Then Q is mixed iterable in V_θ.

Corollary 13.1 Assume A1+A2. Let $Q \in V_\theta$ be weakly normally iterable or weakly normally s-iterable. Then Q is weakly mixed iterable in V_θ.