§ 2.4 Some Consequences

We recall that by § 2.1 Lemma 3, no M can be both a simple and non simple iterate of a mouse $\overline{M}$.

Def Let M, N be mice.

$M \sim_* N$ iff M, N have a common non simple iterate.

Lemma 1 $\sim_*$ is an equivalence relation on mice.

proof.

$M \sim_* M$, $M \sim_* N \rightarrow N \sim_* M$ are trivial. We prove transitivity.

Let $M \sim_* N \sim_* Q$. Let M' be a common simple iterate of M, N and Q' a common simple iterate of N, Q. By coiteration there is a common iterate P of M', Q' which is a simple iterate of one of them. Suppose e.g. that P is not a simple iterate of M'. Then it is a non simple iterate of N. But it is a simple iterate of Q', hence of N. Contradiction!

QED (Lemma 1.1)

<u>Def</u> $M <_* N$ iff there is a common iterate Q which is a simple iterate of M but not of N.

<u>Lemma 2</u> $<_*$ is a linear ordering of mice modulo the congruence relation $\sim_*$.

The proof stretches over a few sublemmas:

<u>Lemma 2.1</u> $M <_* N \vee M \sim_* N \vee N <_* M$

prf. By coiteration.

<u>Lemma 2.2</u> $M \sim_* N \rightarrow M \not<_* N$

prf. Suppose not.

Let Q be a common simple iterate of M, N. Let M' be a common iterate of M, N which is simple of M but not of N. Let P be a common iterate of Q, M' which is simple of one of them.

If P is a simple iterate of Q, then it is a simple iterate of N, which is impossible since P is an iterate of M'. But then P is a simple iterate of M', hence of M, and P is a non simple iterate of Q, hence of M. Contr!

QED (Lemma 2.2)

<u>Lemma 2.3</u> $M <_* N \longrightarrow N \not<_* M$

prf. Suppose not.

Let M' be an iterate of M, N which is simple of M but not of N. Let N' bear the same relation to the pair N, M. Let Q be an iterate of N', M' which is simple of one of them. Q is not simple of M, since it is an iterate of N'. Hence Q is not simple of M'. Similarly, Q is not simple of N'. Contr!

QED (Lemma 2.3)

<u>Lemma 2.4</u> $M <_* N <_* Q \rightarrow Q \not<_* M$

prf. Suppose not.

Let M' be an iterate of M, N which is simple of M but not N. Let N' bear the same relation to N, Q and Q' to Q, M. By coiterations there is P which is a common iterate of M', N', Q' and a simple iterate of one of them. This is easily seen to be false. Contr!

QED (Lemma 2.4)

<u>Lemma 2.5</u> $M <_* N <_* Q \to M <_* Q$.

prf. Suppose not.

Then $M <_* N <_* Q \sim_* M$ by Lemmas 2.1, 2.5. Let M' be an iterate of M, N which is simple of M + not of N. Let N' bear the same relation to N, Q. Then $M' \sim_* M \sim_* Q$ and M', Q have a common simple iterate Q'. Let P be common iterate of N', Q'. An easy argument again shows that P is not a simple iterate of N' or Q'. Contradiction! QED (Lemma 2.5)

This proves Lemma 2

<u>Note</u> The relation $<_*$ is well founded, as can be seen by the nonexistence of degenerate iterations (§2.1 [illegible])