§3.2 Some Properties of Weasels

In §3.1 we introduced the limit Q_∞ of certain weasel iterations $\langle Q_i \rangle$ of length ∞. We begin this section by extending that notion and developing the theory of long iterations more fully.

Def Let Q be a mouse or weasel. By a long iteration of Q we mean an iteration $\langle Q_i \mid i < \infty \rangle$ with indices $\langle \nu_i, \alpha_i \rangle$ s.t.

(*) $E_{\nu_i} \neq \emptyset$ for arbitrarily large $i < \infty$.

Any other iteration is called short.

(Note An iteration of length ∞ which does not satisfy (*) will eventually become constant + hence can be replaced by an iteration of length $< \infty$).

<u>Lemma 1.1</u> Let $\langle Q_i \mid i < \infty \rangle$ be a long iteration with indices $\langle \nu_i, \alpha_i \rangle$ and κ_i. Then

$$\bigwedge \alpha \bigvee \beta \bigwedge i \geq \beta \quad \kappa_i \geq \alpha .$$

prf.
Suppose not. Then there is κ s.t. $\{ i \mid \kappa_i = \kappa \}$ is unbounded in ∞. Let κ be the least such + choose i_0 s.t. $\kappa_j \geq \kappa$ for $j \geq i_0$. Let $i \geq i_0$ s.t. $\kappa_i = \kappa$ + ν_i is minimal for i with this property. Let $j > i$ be least s.t. $\kappa_j = \kappa$. Then $\nu_i = \kappa^{+Q_j}$ and $E^{Q_j}_{\nu_i} = \emptyset$. Hence $\nu_j < \nu_i$. Contr! QED

<u>Def</u> Let $\langle Q_i \mid i < \infty \rangle$ be as above. Set β_α = the least β s.t. $\kappa_i \geq \alpha$ for all $i \geq \beta$. Set:

$$Q_\infty = \bigcup_\alpha J_\alpha^{E^{Q_{\beta_\alpha}}} .$$

Then Q_∞ is a weasel.

[<u>Note</u> $C = \{ \alpha \mid \alpha = \beta_\alpha \}$ is a club class.]

If π_{ij} are the iteration maps we define $\pi_{i\infty}$ $(i < \infty)$ by:

$$\pi_{i\infty}(x) \simeq \begin{cases} \pi_{i\beta_\alpha}(x) & \text{for } \alpha \text{ s.t.} \\ \pi_{i\beta_\alpha}(x) \in J_\alpha^{E^{Q_{\beta_\alpha}}} & , \end{cases}$$

This definition extends our earlier one as shown by:

<u>Lemma 1.2</u> The following are equivalent:

(a) $\bigwedge \xi \bigvee i \bigwedge j \geq i \;\; \pi_{0i}(\xi) < \kappa_i$

(b) $Q_\infty, \langle \pi_{i\infty} \rangle = \lim\limits_{i \leq j} (Q_i, \pi_{ij})$

(c) $\mathrm{dom}(\pi_{0\infty}) = Q$.

(d) $\infty \subset \mathrm{dom}(\pi_{0\infty})$.

The proof is straightforward. If (a)-(d) hold, we call Q_∞ a <u>simple</u> <u>iterate</u> of Q by the iteration $\langle Q_i \rangle$.

Note Q_∞ can fail to be a simple iterate even if $\langle Q_i \mid i<\infty\rangle$ is a simple iteration. The reason is that Q_∞ is really obtained by an "iteration" of length $\infty+1$, where "truncation" can occur at the last stage. One can think of a non simple iterate Q_∞ as follows: Form the direct limit Q' of $\langle Q_i \mid i<\infty\rangle$. Q' properly extends ∞ and we truncate it to get Q_∞. (If ∞ were an inaccessible cardinal we could carry this out explicitely with transitive Q').

The following facts are obvious:

<u>Fact 1</u> Any long iteration results in a weasel.

<u>Fact 2</u> Any simple iterate of a weasel (mouse) is a weasel (mouse).

<u>Fact 3</u> Any short non simple iteration results in a mouse.

<u>Fact 4</u> If Q is a non simple iterate of a weasel W with iteration map π, then there is α s.t. Q is a non simple iterate of $W|\alpha$ with iteration map π.

We now prove:

<u>Lemma 2.1</u> Let W' be a simple iterate of the weasel W with iteration map π and W'' a simple iterate of W' with map π'. Then W'' is a simple iterate of W with iteration map $\pi'\pi$.

proof of Lemma 2.1

If the first iteration is short, the conclusion is trivial, so suppose $W' = W_\infty$, where $\langle W_i \mid i < \infty \rangle$ is a long iteration. Set:

$$C = \{\alpha \mid \beta_\alpha = \alpha \wedge \alpha \text{ is a cardinal in } W\},$$

where $\beta_\alpha =$ the least β s.t. $\kappa_i \geq \alpha$ for $i \geq \beta$ and $|\{i < \beta \mid E_{\nu_i} \neq \emptyset\}| \geq \alpha$.

($|A| =$ order type of A). Now let $\langle W_i' \mid i \leq \theta \rangle$ be the iteration from W' to W''. If $\theta = \infty$, define C' the way C was defined. If $\theta < \infty$, set:

$$C' = \text{the set of cardinals in } W' \text{ s.t. } \alpha > \nu_i' \text{ for } i < \theta.$$

[Here ν_i, κ_i and ν_i', κ_i' are indices of the iterations $\langle W_i \rangle$, $\langle W_i' \rangle$ resp.

Set: $\tilde{C} = C \cap C'$. For $\alpha \in \tilde{C}$: $J_\alpha^{E^{W_\alpha}} = J_\alpha^{E^{W'}}$ is an iterate of $J_\alpha^{E^W}$ with iteration map $\pi \upharpoonright J_\alpha^{E^W}$

and:

$J_\alpha^{E_{W'_\alpha}} = J_\alpha^{E_{W''}}$ is an iterate of $J_\alpha^{E_W}$ with iteration map $\pi' \upharpoonright J_\alpha^{E_{W'}}$.

These iterations are short + hence $J_\alpha^{E_{W''}}$ is a short iterate of $J_\alpha^{E_W}$ with map $\pi'\pi \upharpoonright J_\alpha^{E_W}$. Let

$\langle \tilde{\nu}_i^\alpha \mid i < \alpha \rangle$ be the indices of the <u>normal</u> iteration from $J_\alpha^{E_W}$ to $J_\alpha^{E_{W''}}$. If $\alpha < \beta$, the models coincide up to α, but $\kappa_i^\beta \geq \alpha$ for $i \geq \alpha$. Thus $\tilde{\nu}_i^\alpha = \tilde{\nu}_i^\beta$ for $i < \alpha$. Set:

$\tilde{\nu}_i = \tilde{\nu}_i^\alpha$ for an $\alpha \in C$ s.t. $i < \alpha$.

It follows easily that W'' is an iterate of W by an iteration with indices $\langle \tilde{\nu}_i \rangle$, giving iteration map $\pi'\pi$.

QED (Lemma 2.1)

An easy modification of this proof yields:

<u>Lemma 2.2</u> Let W^0 be a weasel & let W^i be a simple iterate of W^i with map π_{ij} for $i \leq j < \theta \leq \infty$. Set: $W, \langle \pi_i \rangle = \lim_{i \leq j} (W_i, \pi_{ij})$. Then W is a simple iterate of W_i with iteration map π_i ($i < \theta$).

Using Fact 4 above we get:

<u>Lemma 2.3</u> There is no sequence $\langle Q_i \mid i < \omega \rangle$ s.t. Q_{i+1} is a non simple iterate of Q_i for $i < \omega$.

prf. Suppose not.
If Q_0 is a weasel, replace it by a mouse $Q_0' = Q_0 | \alpha_0$ s.t. Q_1 is a non-simple iterate of Q_0'. If Q_1 is a weasel, replace it by $Q_1' = Q_1 | \alpha_1$ s.t. Q_2 is a nonsimple iterate of Q_1'. Then Q_1' is a nonsimple iterate of Q_0'. Proceeding in this way we get a sequence $\langle Q_i' \mid i < \omega \rangle$ of mice s.t. Q_{i+1}' is ~~it~~ a nonsimple iterat of Q_i' for $i < \omega$. Contradiction!

QED (Lemma 2.3)

[Note It is also easy to prove:

Lemma 2.4 Let Q' be a nonsimple iterate of Q & Q'' a nonsimple iterate of Q' with maps π, π' resp. Then Q'' is a nonsimple iterate of Q with map $\pi'\pi$.

However, the corresponding theorem for "mixed" sequences of simple and nonsimple iterates is false.]

. . . . .

Now write $\langle Q, \nu \rangle R \langle \bar{Q}, \bar{\nu} \rangle$ to mean that Q is an iterate (long or short) of $\bar{Q}$ with map π and that either Q is a nonsimple iterate, or $\nu < \pi(\bar{\nu})$. By Lemma 2.3 it is impossible that $\langle Q_{i+1}, \nu_{i+1} \rangle R \langle Q_i, \nu_i \rangle$ for $i < \omega$. But then a virtual repetition of the proof of §2.1 Lemma 3 yields:

<u>Lemma 3</u> Let $W_0, \ldots, W_n$ be weasels + let W_i be an iterate of W_{i-1} with iteration map π_i for $1 \le i \le n$. Set: $\pi = \pi_n \circ \cdots \circ \pi_1$. Let $\sigma : W_0 \xrightarrow[\Sigma_1]{} W_n$. Then W_i is a simple iterate of W_{i-1} $(1 \le i \le n)$ and $\sigma(\xi) \ge \pi(\xi)$ for all ξ.

<u>Cor 3.1</u> If W is a simple iterate of $\overline{W}$, it cannot be a nonsimple iterate of $\overline{W}$. In fact, if $\overline{W} = W_0, \ldots, W_n = W$ is a chain s.t. W_i is an iterate of W_{i-1} for $1 \le i \le n$, then W_i is a simple iterate of W_{i-1} for $1 \le i \le n$.

<u>Cor 3.2</u> If W' is a simple iterate of W, then the iteration map is unique.

<u>Def</u> $\pi_{WW'}$ = the unique iteration map from W to W' if W' is a simple iterate of W.

Def $W = \lim_{i<\theta} W_i$ means:

$$W, \langle \pi_{W_i W} \rangle = \lim_{i \le j < \theta} (W_i, \pi_{W_i W_j}),$$

where W_j is a simple iterate of W_i for $i \le j < \theta \le \infty$.

. . . . .

Thus we can deal with weasels in virtually the same way as mice. We carry this thought further by extending the relations $\sim_*$, $<_*$ of §2.4 to weasels. We there defined $M \sim_* N$ to hold iff M, N possess a common simple iterate. Since we cannot quantify over weasels this definition is unavailable, so we instead define:

Def $W \sim_* W'$ iff W, W' coiterate to a common simple iterate.

The force of the missing quantifier is then given by:

<u>Lemma 4.1</u> Let W, W' have a common simple iterate W''. Then $W \sim_* W'$.

prf. Suppose not.

Let W, W' coiterate to $\tilde{W}$ and suppose e.g. that $\tilde{W}$ is a simple iterate of W but not of W'. Coiterate $W'', \tilde{W}$ to W^*. Since W^* is an iterate of $\tilde{W}$, it is not a simple iterate of W' by Cor 3.1. Hence W^* is not a simple iterate of W''. But then W^* is not a simple iterate of W by Cor 3.1, hence not a simple iterate of $\tilde{W}$. Contr!

QED (Lemma 4.1)

If W is a weasel and Q a mouse, we regard $W \sim_* Q$, $Q \sim_* W$ as false (since they are, indeed, false in the sense of the above definition)

<u>Def</u> Let W be a weasel and Q a mouse or weasel.

$W <_* Q$ iff W, Q coiterate to a W' which is a simple iterate of W but not of Q

Finally:

Def Let Q be a mouse & W a weasel. $Q <_* W$ iff there is a mouse Q' which is a simple iterate of Q and a nonsimple iterate of W.
(In other words: $Q <_* W|\alpha$ for some α)

By a proof very much like that of Lemma 4.1 we get:

Lemma 4.2 Let Q, Q' be mice or weasels. Let Q'' be a simple iterate of Q and let $Q' = Q_0, \ldots, Q_n = Q''$ be s.t. Q_i is an iterate of Q_{i-1} for $1 \leq i \leq n$ and Q_i is a non simple iterate of Q_{i-1} for some i. Then $Q <_* Q'$.

The proof is left to the reader.

Using these lemmas we can virtually repeat the proofs of §2.4 Lemma 1, Lemma 2 to get:

<u>Lemma 4.3</u> $<^*$ is a linear ordering of mice and weasels modulo the congruence relation. In other words:

(a) $Q \sim_* Q$

(b) $Q \sim_* Q' \to Q' \sim_* Q$

(c) $Q \sim_* Q' \sim_* Q'' \to Q \sim_* Q''$

(d) $Q <_* Q' \vee Q \sim_* Q' \vee Q' <_* Q$

(e) $Q \sim_* Q' \to Q \not<_* Q'$

(f) $Q <_* Q' \to Q' \not<_* Q$

(g) $Q <_* Q' <_* Q'' \to Q <_* Q''$

If W is weakly universal, then by definition $W \not<_* Q$ for every mouse Q. But then $W \not<_* W'$ for any weasel W', since otherwise $W <_* W'|\alpha$ for some α. Hence $Q \leq_* W$, whenever Q is a mouse or weasel. If W' is a non weakly universal weasel, then $W' \not<_* W$, since otherwise there is a mouse Q s.t. $W' <_* Q$ and we would have: $W <_* Q$.

Hence:

<u>Lemma 4.4</u> The weakly universal weasels comprise the maximal elements in the ordering $<_*$.

<u>Open Question</u> Is every weakly universal weasel universal?