§4 Premice

<u>Def</u> $M = \langle J_\alpha, E_{\omega\alpha}\rangle$ is a <u>prepremouse</u> (ppm)

iff (a) M is acceptable

(b) $E = \{\langle \nu, \xi, X\rangle \mid \xi \le \nu \le \omega\alpha \wedge \xi \in E_\nu(X)\}$, where $E_\nu = \emptyset$ or E_ν is a whole extender on J_ν^E and $\langle J_\nu^E, E_\nu\rangle$ is coherent. (Hence length$(E_\nu) = \lambda$, where λ = the largest cardinal in the sense of J_ν^E.)

(c) If $\pi : J_\nu^E \xrightarrow[E_\nu]{} N$, then $E_\nu^N = \emptyset$

(d) $M \| \gamma =_{\mathrm{Df}} \langle J_\gamma^E, E_{\omega\gamma}\rangle$ is sound for $\gamma < \alpha$.

<u>Note</u> The model N in (c) need not be well founded, but we take its well founded core as transitive.

<u>Def</u> Let $M = \langle J_\nu^E, F\rangle$ be coherent. Let $\kappa = \mathrm{crit}(F)$ and $\kappa^{+M} \le \bar\nu \le \nu$. We define an extender $F\|\bar\nu$ with $\mathrm{dom}(F\|\bar\nu) = \mathrm{dom}(F)$ by:

$$(F\|\bar\nu)(X) = \begin{cases} \bar\nu \cap F(X) \text{ if } \bar\nu = \kappa^{+M} \text{ or is a limit cardinal in } M; \\ \alpha \cap F(X) \text{ otherwise, where } \alpha = \text{the cardinal predecessor of } \bar\nu \text{ in the sense of } M. \end{cases}$$

Def $M = \langle J_\alpha^E, E_{\omega\alpha} \rangle$ is a premouse (pm) iff

iff M is a ppm and:

(e) If $E_\nu \neq \emptyset$, $\kappa = \mathrm{crit}(E_\nu)$, $\kappa^{+M} \leq \bar\nu \leq \nu$ s.t. $\langle J_{\bar\nu}^E, E \| \bar\nu \rangle$ is a ppm, then $E_{\bar\nu} \neq \emptyset$.

((e) is called the initial segment condition.)

Note that if $M = \langle J_\alpha^E, E_{\omega\alpha} \rangle$ is a premouse (or ppm), then E_ν is always weakly amenable and Σ_1-amenable wrt. $M \| \nu = \langle J_\nu^E, E_\nu \rangle$ if $E_\nu \neq \emptyset$.

Def Let $0 < \theta \leq \infty$. $T \subset \theta^2$ is an iteration tree iff

(a) T is a tree with initial point 0

(b) $\nu+1$ immediately succeeds a point $T(\nu+1) \leq \nu$ in T

(c) If $\mathrm{Lim}(\lambda)$, $\lambda < \theta$, then λ is a limit pt. of T and $\sup T``\{\lambda\} = \lambda$.

Note It follows that $\xi T \zeta \to \xi < \zeta$ and that $T``\{\rho\}$ is closed in ρ for $\zeta < \theta$.

Def $\mathcal{I} = \langle\langle M_i\rangle, \langle \nu_i\rangle, \langle\eta_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ is a generalized iteration of length θ $(0<\theta\leq\infty)$ with iterates $\langle M_i \mid i<\theta\rangle$, indices $\langle \nu_i \mid i\in D\rangle$, $\langle\eta_i \mid i+1<\theta\rangle$, tree T and iteration maps $\langle\pi_{ij} \mid i\leq_T j\rangle$ iff

(a) T is an iteration tree

(b) π_{ij} is a partial map from M_i to M_j and the π_{ij} commute

(c) M_i is a pm.

(d) $\eta_i \leq ht(M_{T(i+1)})$ and $\{i \mid i+1 \leq_T j \wedge \eta_i < ht(M_{T(i+1)})\}$ is finite for $j<\theta$.

(e) If $i\notin D$, $i+1<\theta$, then $i = T(i+1)$ and $M_{i+1} = M_i \| \eta_i$, $\pi_{i,i+1} = id$

(f) Let $i\in D$. Then $i+1<\theta$ and $E_{\nu_i}^{M_i} \neq \emptyset$. Let $\kappa_i = crit(E_{\nu_i}^{M_i})$, $\tau_i = \kappa^{+M_i\|\nu_i}$. Then $\tau_i = \kappa^{+M_\xi\|\eta_i}$, $J_{\tau_i}^{E^{M_i}} = J_{\tau_i}^{E^{M_\xi}}$ and $\pi_{\xi,i+1} : M_\xi\|\eta_i \xrightarrow{*}_{E_{\nu_i}} M_{i+1}$, where $\xi = T(i+1)$

(g) If $lim(\lambda)$, then M_λ, $\langle\pi_{i\lambda} \mid i T \lambda\rangle =$ = the dir. limit of $\langle M_i \mid i<\lambda\rangle, \langle\pi_{ij} \mid i\leq_T j\leq_T\lambda\rangle$.

Def For $i\in D$ set: $\kappa_i = crit(E_{\nu_i}^{M_i})$, $\tau_i =$ $= \kappa_i^{+M_i\|\nu_i}$, $\lambda_i =$ the largest cardinal in $M_i\|\nu_i = \pi_{i,i+1}(\kappa_i)$.

<u>Note</u> π_{ij} is a partial map of M_i to M_j. By (d), if $\lim(\lambda)$, then $\pi_{i\lambda}$ is total for sufficiently large $i \, T \, \lambda$. Hence the limit in (g) is defined.

<u>Def</u> $i+1$ is a <u>truncation point</u> in $\mathcal{T}$ iff $i+1 < \theta$ and $\eta_i < ht(M_{T(i+1)})$.

<u>Def</u> A branch b in T is <u>simple in</u> $\mathcal{T}$ iff b has no truncation pt. $i < \theta$ is <u>simple</u> in $\mathcal{T}$ iff $\{j \mid j \leq_T i\}$ is simple in $\mathcal{T}$.

<u>Def</u> $\mathcal{T}$ is <u>direct</u> iff $i \in D$ for $i+1 < \theta$.

<u>Def</u> $\mathcal{T}$ is <u>standard</u> iff for all $i+1 < \theta$

(a) If $i \notin D$, then $\eta_i = ht(M_i)$

(b) If $i \in D$, then $\xi = T(i+1) \in D$ and $\eta_i =$ the maximal $\eta \leq ht(M_\xi)$ s.t.

$$\varkappa_i^{+M_\xi \| \eta_i} = \varkappa_i^{+M_i \| \nu_i}.$$

<u>Def</u> $\mathcal{T}$ is <u>normal</u> iff $\mathcal{T}$ is standard and for all $i \in D$:

(a) $\nu_i > \nu_h$ for $h \in D \cap i$

(b) $T(i+1) =$ the least $\xi \in D$ s.t. $\varkappa_i < \lambda_\xi$

<u>Note</u> Any standard it. $\mathcal{I}$ can be replaced by a direct standard it $\mathcal{I}'$ simply by omitting repetition. If $\mathcal{I}$ is normal, so is $\mathcal{I}'$.

<u>Note</u> If $\mathcal{I}$ is normal, then $J_{\nu_i}^{E^{M_i}} = J_{\nu_i}^{E^{M_j}}$ for $i \leq j$, $i \in D$. If $i < j$, then ν_i is a cardinal in M_j (but not in M_i). If $\xi = T(i+1)$, then $J_{\nu_\xi}^{E^{M_\xi}} = J_{\nu_\xi}^{E^{M_i}}$.

But then, since $\kappa_i < \lambda_\xi$, M_ξ, M_i coincide up to κ_i^+ $J_{\nu_i}^{E^{M_i}}$ + we can define η_i + apply $E_{\nu_i}^{M_i}$ to $M_\xi \| \eta_i$.

<u>Note</u> If $\mathcal{I}$ is normal, $i \leq_T j+1$ and $i, j \in D$, then $\lambda_h \leq \kappa_j$ for $h < i$ by the def of $T(j+1)$.

<u>Def</u> A <u>loose normal iteration</u> is defined as before except that we drop the requirement of standardness but still require: $M_{i+1} = M_i$ if $i \notin D$.
(We shall, in fact, make no use of this notion.)

<u>Lemma 1</u> Let $\mathcal{I} = \langle\langle M_i \rangle, \ldots\rangle$ be a normal iteration of length θ. If $i \in D$ then $E^{M_i}_{\nu_i}$ is Σ_1 - amenable wrt. $M_{T(i+1)} \| \eta_i$.

This implies:

<u>Cor 1.1</u> If $h \leq_T i$ and π_{hi} is a total fcn. on M_h, then $\pi_{hi} : M_h \longrightarrow_{\Sigma^*} M_i$.

prf.
By Lemma 1 it holds for $h = T(i)$, since then $i = \xi + 1$ where $E^{M_\xi}_{\nu_\xi}$ is Σ_1 - amenable wrt. M_h. The + weakly amenable wrt. M_h. The result follows by ind. on i.

QED (Cor 1.1).

<u>Note</u> This proof actually shows:

<u>Cor 1.1.1</u> If $h\, T\, i$, ξ = the least ξ s.t. $h\, T\, \xi + 1 \leq_T i$, $N = M_h \| \eta_\xi$ and π_{hi} is total on N, then

$\pi_{hi} : N \longrightarrow_{\Sigma^*} M_i$.

We now prove Lemma 1. We assume w.l.o.g. that $\mathcal{Y}$ is direct.

<u>Def</u> Let $E^{M_i}_{On \cap M_i} \neq \emptyset$. Set:

$\bar{\kappa}_i = \mathrm{crit}(E^{M_i}_{On \cap M_i})$, $\bar{\tau}_i = \bar{\kappa}_i^{+M_i}$,

$\delta_i =$ the least δ s.t. $\delta = i$ or $\bar{\kappa}_i < \lambda_\delta$. (Hence $\delta_i \leq i$).

$\bar{\eta}_i =$ the maximal $\eta \leq ht(M_{\delta_i})$ s.t. $\bar{\tau}_i = \bar{\kappa}_i^{+M_{\delta_i} \| \eta}$.

Our main tool in proving Lemma 1 will be the following sublemma:

<u>Lemma 1.2</u> Let $i < \theta$ s.t. δ_i exists. Then $\mathbb{P}(\bar{\tau}_i) \cap \Sigma_1(M_i) \subset \Sigma_1(M_{\delta_i} \| \bar{\eta}_i)$.

proof.
Suppose not. Let i be the least counterexample. Then $\delta_i < i$. Set: $\kappa = \bar{\kappa}_i$, $\tau = \bar{\tau}_i$. By the minimality of i we have $i = h+1$. Set: $\xi = T(i)$. Set: $M^* = M_\xi \| \eta_h$.

(1) $\kappa < \kappa_h$ (hence $\pi_{\xi i} \restriction \tau^{+M^*} = \mathrm{id}$)

prf.

Let $\kappa' = \pi_{\xi i}^{-1}(\kappa) = \mathrm{crit}(E^{M^*}_{On \cap M^*})$. Then $\kappa' < \kappa_h$, since otherwise we would have: $\kappa = \pi_{\xi i}(\kappa') \geq \pi_{\xi i}(\kappa_h) = \lambda_h$. Hence $\delta_i = i$. Contr! Hence $\kappa = \pi_{\xi i}(\kappa') = \kappa' < \kappa_h$.

QED (1)

(2) $\delta_i \leq \xi$ since $\kappa < \kappa_h < \lambda_\xi$.

(3) $F = E^{M_h}_{\nu_h}$ is Σ_1 amenable wrt M^*

prf. (Assume w.l.o.g. $\xi < h$)

If $\nu_h = On \cap M_h$, then $\delta_h = \xi$, $\bar{\eta}_h = \eta_h$, since $\bar{\kappa}_h = \kappa_h$, $\bar{\tau}_h = \tau_h$. For $\alpha < \lambda_h$, we then have $F_\alpha \in \Sigma_1(M_\xi \| \bar{\eta}_h)$ by the minimality of i. Now let $\nu_h \in M_h$. Then $F \in M_h$. For $\alpha < \lambda_h$, we have:

$$F_\alpha \in J^{E^{M_h}}_{\lambda_\xi} = J^{E^{M_\xi}}_{\lambda_\xi} \subset M^*,$$

since λ_ξ is a limit cardinal in M_h, and $\tau_h < \lambda_\xi$. QED (3)

(4) $\omega\rho^1_{M_i} \leq \tau$.

prf.

Suppose not. Let $A \subset \tau$ be $\Sigma_1(M_i)$

Then $A \in \mathcal{P}(\tau) \cap M_i \subset (J^E_{\tau^+})^{M_i} \subset J^{E^{M_i}}_{\lambda_{\delta_i}} =$

$= J^{E^{M_{\delta_i}}}_{\lambda_{\delta_i}} \subset M_{\delta_i} \| \bar{\eta}_i \subset \underline{\Sigma}_1 (M_{\delta_i} \| \bar{\eta}_i)$.

Contr! QED (4)

(5) $\omega\rho^1_{M^*} \leq \tau$

proof. By (3), $\pi_{\xi i}$ is Σ^*-preserving. The conclusion follows by (4) and $\pi_{\xi i}(\tau) = \tau$. QED (5)

By §2 Cor 6.4

(6) $\mathcal{P}(\kappa_h) \cap \underline{\Sigma}_1 (M_i) \subset \underline{\Sigma}_1 (M^*)$

prf.
$\pi_{\xi i} : M^* \xrightarrow[F]{} M$ by (5) and (1). The conclusion follows by (3) and §1 Lemma 8. QED(6)

(7) $\xi > \delta_i$

prf. Suppose not. Then $\xi = \delta_i$ by (2). Then $\eta_h \leq \bar{\eta}_i$, since $\tau < \kappa_h$. But then $\mathcal{P}(\tau) \cap \underline{\Sigma}_1 (M_i) \subset \underline{\Sigma}_1 (M_\xi \| \eta_h) \subset$ $\subset \underline{\Sigma}_1 (M_{\delta_i} \| \bar{\eta}_i)$ by (6). Contr!

QED (7)

(8) $M^* = M_\xi$ (i.e. $\eta_h = ht(M_\xi)$).

prf.
If not, $\tau^{+M_\xi} > \omega\eta_h = On \cap M^*$ by (5)

But $\tau < \lambda_{\delta_i}$, where $\lambda_{\delta_i} < \lambda_\zeta$ is a limit cardinal in M_ζ. Hence $\tau^{+M_\zeta} < \lambda_\zeta \leq \omega_1$.
Contr! QED(8).

But then $\delta_\zeta = \delta_i$ and $\bar{\eta}_\zeta = \bar{\eta}_i$, since $\kappa = \bar{\kappa}_\zeta$, $\tau = \bar{\tau}_\zeta$. Then $\mathbb{P}(\tau) \cap \Sigma_1(M_i) \subset$
$\subset \mathbb{P}(\tau) \cap \Sigma_1(M_\zeta)$ (by (6))
$\subset \Sigma_1(M_{\delta_\zeta} \| \bar{\eta}_\zeta)$ by the minimality of i. Contr!

QED (Lemma 1.2)

We now complete the proof of Lemma 1. Suppose not, let i be the least counterexample. Then $T(i+1) < i$. Let $\zeta = T(i+1)$, $F = E^{M_i}_{\nu_i}$.

We consider two cases:

Case 1 $F \in M_i$

If $i = \xi =_{\mathrm{df}} T(i+1)$, there is nothing to prove. Let $\xi < i$. Then λ_ξ is a limit cardinal in M_i. Then $F_\alpha = \{X \mid \alpha \in F(X)\} \in M_i$. Hence $F_\alpha \in (J_{\kappa_i^{++}})^{M_i} \subset J_{\lambda_\xi}^{E^{M_i}} = J_{\lambda_\xi}^{E^{M_\xi}}$. But $\eta_i \geq \lambda_\xi$. Hence $F_\alpha \in M_\xi \| \eta_i$. Contr!

Case 2 Case 1 fails

Then $\kappa_i = \bar{\kappa}_i$, $T(i+1) = \delta_i$. $F_\alpha \in \underline{\Sigma}_1(M_i)$ and $F_\alpha \subset J_{\tau_i}^{E^{M_i}}$. Hence $F_\alpha \in \underline{\Sigma}_1(M_{\delta_i} \| \eta_i)$ by Lemma 1.2, since $\eta_i = \bar{\eta}_i$. QED (Lemma 1)

[Note A modification of this proof shows that for loose normal iterations:

(a) If $i \in D$ & $\eta_i = ht(M_{T(i+1)})$, then $\Sigma^{M_i}_{\nu_i}$ is Σ_1-amenable wrt $M_{T(i+1)}$

(b) If $i \leq_T j$ and π_{ij} is total on M_h, then π_{ij} is Σ^*-preserving.]

Def If $\mathcal{I} = \langle\langle M_i\rangle, \langle \nu_i \mid i \in D\rangle, \langle \eta_i \rangle, \langle \pi_{ij}\rangle, T\rangle$ is an iteration of length θ, set:
$T^{\mathcal{I}} = T$, $D^{\mathcal{I}} = D$, $M_i^{\mathcal{I}} = M_i$, $\nu_i^{\mathcal{I}} \simeq \nu_i$, $\eta_i^{\mathcal{I}} = \eta_i$, $\pi_{ij}^{\mathcal{I}} = \pi_{ij}$ for $i \leq_T j < \theta$
and $|\mathcal{I}| = \text{length}(\mathcal{I}) = \theta$.

If $\lambda < |\mathcal{I}|$, then $\mathcal{I}|\lambda$ has the obvious meaning.

Def By an <u>iteration strategy</u> we mean a partial function S on iterations $\mathcal{I}$ of limit length s.t. if $b = S(\mathcal{I})$, then b is a branch in $T = T^{\mathcal{I}}$ cofinal in $\theta = |\mathcal{I}|$ containing at most finitely many truncation pts. s.t. the direct limit of $\langle M_i \mid i \in b\rangle$, $\langle \pi_{ij} \mid i \leq_T j \in b\rangle$ is well founded. (In other words, $\mathcal{I}$ extends to $\mathcal{I}'$ s.t. $|\mathcal{I}'| = \theta + 1$, $\mathcal{I} = \mathcal{I}'|\theta$, $T'``\{\theta\} = b$, where $T' = T^{\mathcal{I}'}$.

Def Let S be an iteration strategy. $\mathcal{I}$ is an S-iteration iff $T''\{\lambda\} = S(\mathcal{I}|\lambda)$ for all limit $\lambda < |\mathcal{I}|$.

Def Let S be an iteration strategy. S is a normal iteration strategy for M iff whenever $\mathcal{I}$ is a normal S-iteration of M, then $\mathcal{I}$ can be continued - i.e.

(a) If $\operatorname{Lim}(|\mathcal{I}|)$, then $S(\mathcal{I})$ exists.

(b) If $|\mathcal{I}| = k+1$, $\nu \in M_k$, $\nu > \nu_i$ for $i < k$ and $E_\nu^{M_k} \neq \emptyset$, then $\mathcal{I}$ has an extension $\mathcal{I}'$ of length $k+2$ s.t. $\mathcal{I}'|k+1 = \mathcal{I}$, $k \in D^{\mathcal{I}'}$, $\nu = \nu_k^{\mathcal{I}'}$.

Def M is normally iterable (by S) iff M has a normal iteration strategy S.

Def M is uniquely normally iterable iff iff M is normally iterable by the uniqueness strategy:

$S(\mathcal{T}) \simeq$ the unique branch b cofinal in $\theta = |\mathcal{T}|$ s.t. $\mathcal{T}$ extends to $\mathcal{T}'$ with $|\mathcal{T}'| = \theta + 1$, $\mathcal{T} = \mathcal{T}' | \theta$, $T^{\mathcal{T}'} {}'' \{\theta\} = 1$

Def. $\mathcal{T}$ is an iteration beyond ν iff $\nu_i \geq \nu$ for all $i \in D$.

$\mathcal{T}$ is an iteration above ν iff iff $\kappa_i \geq \nu$ for all $i \in D$

The notion of a normal iteration strategy for M beyond (above) a given ν is defined in the obvious way. M is then called normally iterable beyond (above) ν iff it possesses such a strategy.

Def Let M, N be premice.

A coiteration of M, N with coiteration indices $\langle \nu_i \mid i < \theta \rangle$ is a pair of normal iterations of length $\theta \leq \infty$

$\mathcal{I}_M = \langle \langle M_i \rangle, \langle \nu_i \mid i \in D_M \rangle, \ldots, T_M \rangle$

$\mathcal{I}_N = \langle \langle N_i \rangle, \langle \nu_i \mid i \in D_N \rangle, \ldots, T_N \rangle$ s.t.

(a) If $i < \theta$, then

$\nu_i \simeq$ the least ν s.t. $E_\nu^{M_i} \neq E_\nu^{N_i}$

(b) If $\lambda \leq \theta$, $\lim(\lambda)$, then $\lambda < \theta$

(c) $D_M = \{ i \mid E_{\nu_i}^{M_i} \neq \emptyset \}$, $D_N = \{ i \mid E_{\nu_i}^{N_i} \neq \emptyset \}$

(Hence $D_M \cup D_N = \theta$).

It is clear that if M, N are normally iterable, then a coiteration exists. In fact, if S, S' are strategies for M, N resp. then there is a unique coiteration $\langle \mathcal{I}, \mathcal{I}' \rangle$ s.t. $\mathcal{I}$ is an S-iteration of M and $\mathcal{I}'$ is an S'-iteration of N.

We show that every coiteration terminates below ∞;

<u>Lemma 2</u> Let M, N be premice which are iterable beyond ν where $J_\nu^{E^M} = J_\nu^{E^N}$, Let $\langle \mathcal{I}_M, \mathcal{I}_N \rangle$ be a coiteration. Then $\mathrm{lngth}(\mathcal{I}_M, \mathcal{I}_N) < \max(\bar{\bar{M}}^+, \bar{\bar{N}}^+)$,

proof.

Suppose not. Set $\theta = \max(\bar{\bar{M}}^+, \bar{\bar{N}}^+)$ $\mathcal{I}^0 = \mathcal{I}_M | \theta+1$, $\mathcal{I}^1 = \mathcal{I}_N | \theta+1$ + let
$\mathcal{I}^h = \langle \langle M_i^h \rangle, \langle \nu_i \rangle, \langle \pi_{ij}^h \rangle, T^h \rangle$ $(h = 0, 1)$
Let τ be regular s.t. $\mathcal{I}^0, \mathcal{I}^1 \in H_\tau$. Let $\sigma : \bar{H} \prec H_\tau$ s.t. $\sigma(\bar{\mathcal{I}}^h) = \mathcal{I}^h$ and $\sigma(\bar{\theta}) = \theta$, where $\mathrm{card}(\bar{H}) < \theta$ and $\sigma \upharpoonright \bar{\theta} = \mathrm{id}$. It follows easily that $\bar{\mathcal{I}}^h | \bar{\theta} = \mathcal{I}^h | \theta$. Moreover,
$\sigma(M_i^{h'}) = M_i^h$, $\sigma(\bar{\pi}_{ij}^h) = \pi_{ij}^h$ for $i \leq j < \bar{\theta}$,

Let $\bar{\mathcal{Y}}^h = \langle\langle \bar{M}_i^h \rangle, \langle \nu_i \rangle, \langle \bar{\pi}_{ij}^h \rangle, \bar{T}^h \rangle$.
Then $\bar{T}^h \wedge \bar{\theta}^2 = T^h \wedge \theta^2$ and
$\bar{T}^{h\,\prime\prime}\{\bar{\theta}\} = \bar{\theta} \wedge T''\{\theta\}$. But then $\bar{\theta}$
is a limit pt of the branch $T''\{\theta\}$.
Hence $\bar{\theta}\, T\, \theta$. But

(1) $\bar{M}_{\bar{\theta}}^h, \langle \bar{\pi}_{i\bar{\theta}}^h \rangle = \lim\limits_{i \leq_T j <_T \bar{\theta}} (M_i^h, \pi_{ij}^h)$.

Hence:
(2) $\bar{M}_{\bar{\theta}}^h = M_{\bar{\theta}}^h$ and $\bar{\pi}_{i\bar{\theta}}^h = \pi_{i\bar{\theta}}^h$ for $i \leq_T \bar{\theta}$.
Now let $x \in M_{\bar{\theta}}^h$. Then $x = \pi_{i\bar{\theta}}^h(x')$
for an $i \leq_T \bar{\theta}$. Hence:
$\sigma(x) = \sigma(\pi_{i\bar{\theta}}^h)(x') = \pi_{i\theta}^h(x') = \pi_{\bar{\theta}\theta}^h \pi_{i\bar{\theta}}^h(x') = \sigma(x$
Hence:
(3) $\sigma \upharpoonright \bar{M}_{\bar{\theta}}^h = \pi_{\bar{\theta}\theta}^h$.
Now let $\xi = \xi_h =_{\mathrm{df}}$ the least ξ s.t.
$\bar{\theta} \leq_T \xi + 1 <_T \theta$ and $\xi \in D^h$. Then
(4) $\kappa_{\bar{\theta}} = \mathrm{crit}(\pi_{\bar{\theta}\xi}^h) = \mathrm{crit}(\pi_{\bar{\theta}\theta}^h) = \bar{\theta}$ by (3)
(5) $\alpha \in E_{\nu_\xi}^{M_\xi}(X) \longleftrightarrow \alpha \in \pi_{\bar{\theta}\xi}^h(X)$
$\longleftrightarrow \alpha \in \sigma(X)$
for $\alpha < \nu_\xi$, $X \in \mathbb{P}(\bar{\theta}) \cap M_{\bar{\theta}}$, since

$\sigma(X) = \pi^{h}_{\xi\theta}\pi^{h}_{\theta\xi}(X)$ and crit$(\pi_{\xi\theta}) > \nu_\xi$.

Thus:

(6) $\xi_0 \neq \xi_1$, since otherwise $E^{M^0_\xi}_{\nu_\xi} = E^{M^0_\xi}_{\nu_\xi}$ for $\xi = \xi_0 = \xi_1$. Contr!

Let $\xi_0 < \xi_1$. Then

(7) $E^{M^0_{\xi_0}}_{\nu_{\xi_0}} = E^{M^1_{\xi_1}}_{\nu_{\xi_1}} | \nu_{\xi_0}$ by (5).

Moreover, by the def. of ν_{ξ_0}: $J^{E^{M_0}_{\xi_0}}_{\nu_{\xi_0}} = J^{E^{M}_{\xi}}_{\nu_{\xi_0}} = J^{E^{M_1}_{\xi_1}}_{\nu_{\xi_0}}$. Hence:

(8) $\langle J^{E^{M_1}_{\xi_1}}_{\nu_{\xi_0}}, E^{M^1_{\xi_1}}_{\nu_{\xi_1}} | \nu_{\xi_0} \rangle$ is a p.m.

By the initial segment condition we conclude:

(9) $E^{M_1}_{\nu_{\xi_0}} \neq \emptyset$.

Suppose now that $E^{M^1_{\xi_0}}_{\nu_0} = \emptyset$. Then $M^1_{\xi_0+1} = M^1_{\xi_0} + E^{M^1_{\xi_1}}_{\nu_0} = E^{M^1_{\xi_0+1}}_{\nu_0} = \emptyset$.

Contr! Hence $E^{M^1_{\xi_0}}_{\nu_0} \neq \emptyset$. Hence $E^{M^1_{\xi_0+1}}_{\nu_0} = E^{M^1_{\xi_1}}_{\nu_0} = \emptyset$ as before.

Contr! QED (Lemma 2)

Let $\mathcal{I}^0, \mathcal{I}^1$ be the coiteration of M, N resulting in M', N'. Then clearly, if $ht(M') \leq ht(N')$, then M' is an initial segment of N'. Truncations can occur on the main branch of $\mathcal{I}^h$, however. If the truncations occur on both sides, we may have thrown away too much information for the comparison to be meaningful. We shall show later that this cannot happen if M, N both satisfy a strong iterability criterion. For the moment, however, we content ourselves with showing that if M' is a proper segment of N', then M' is a simple iterate of M. This will follow from:

<u>Lemma 3</u> Let $\mathcal{I}$ be a normal iteration of length $\theta + 1$. If θ is not simple in $\mathcal{I}$, then M_θ is not sound.

proof of Lemma 3

Let $i+1 \leq \theta$ be the maximal truncation point. Set $\xi = T(i+1)$, $M' = M_\xi \| \eta_i$.

Then $\omega\rho^\omega_{M'} \leq \kappa_i$; hence M_{i+1} is not sound and, in fact, if $p \in \Gamma^n_{M'}$, $\omega\rho^{n+1} \leq \kappa_i < \omega\rho^n$ in M',

Then $\pi_{\xi,i+1}(p) \in P^{n+1}_{M_{i+1}}$, but

$\kappa_i \notin h_{M^n_{i+1}, \pi_{\xi,i+1}(p \restriction n)}(\kappa_i \cup \{\pi_{\xi,i+1}(p_n)\})$

Hence $\kappa_i \notin h_{M^n_\theta, \pi_{\xi\theta}(p \restriction n)}(\kappa_i \cup \{\pi_{\xi\theta}(p_n)\})$

where $\kappa_i \geq \omega\rho^{n+1}_{M_\theta} = \omega\rho^{n+1}_{M'}$.

QED (Lemma 3)

<u>Corollary 3.1</u> Let M', N' be the coiterates of M, N by coiteration $\mathcal{I}^0, \mathcal{I}^1$ of length θ (i.e. $M' = M^0_\theta$, $N' = M^1_\theta$). If M' is a proper segment of N', then θ is simple in $\mathcal{I}^0$.

prf. of Cor 3.1 ; M' is round.

Cor 3.2 Let M', N' be as above and suppose θ is simple in neither $\mathcal{T}_0$ nor $\mathcal{T}_1$. Then $M' = N'$.

' ' ' '

As we said, there is a stronger notion of iterability which will guarantee the ultimate pt. of the coiteration is simple on at least one side (hence Cor 3.2 becomes vacuous). Under certain conditions it will also guarantee that an iterate M' of M cannot be both a simple and non simple iterate (by different iterations), as well as that if M' is a simple iterate of M, then the iteration map from M to M' is unique (independently of the iteration chosen).

This notion of iterability requires not only that M be normally iterable, but that the process of taking a normal iterate of a truncate of M be linearly iterable. We can make this precise with the notion of a <u>good sequence</u>:

<u>Def</u> $\langle\langle M_i \mid i<\Gamma\rangle, \langle \mathcal{J}_i \mid i<\Gamma\rangle, \langle \pi_{ij} \mid i\leq j<\Gamma\rangle\rangle$ is a <u>good sequence</u> iff

(a) M_i is a premouse

(b) π_{ij} is a partial map from M_i to M_j + the π_{ij} commute

(c) $\mathcal{J}_i$ is a normal iteration of $M_i \| \eta$ for an $\eta \leq ht(M_i)$

(d) If $i+1<\Gamma$, then $|\mathcal{J}_i| = k+1$, where $M_{i+1} = M_k^{\mathcal{J}_i}$ and $\pi_{i,i+1} = \pi_{0,k}^{\mathcal{J}_i}$.

(e) If $Lim(\lambda)$, $\lambda<\Gamma$, then $\{i<\lambda \mid \pi_{i,i+1}$ is not total on $M_i\}$ is finite and

$$M_\lambda, \langle \pi_{i\lambda}\rangle = \lim_{i\leq j<\lambda} (M_i, \pi_{ij}).$$

Iterability requires that every good sequence which is formed according to an appropriate "strategy" can be continued. However, rather than define a new notion of "strategy" for good sequences, we use the old notion for generalized iterations + observe that every good sequence can be converted into a generalized iteration. Such iterations are called <u>good</u>;

<u>Def</u> $\mathcal{I} = \langle\langle M_i\rangle, \langle \nu_i\rangle, \langle \eta_i\rangle, \langle \pi_{ij}\rangle, T\rangle$ is a <u>good iteration</u> <u>of length</u> θ <u>with a marking sequence</u> $\langle \alpha_i \mid i \leq \Gamma\rangle$ iff the following hold:

(a) $\mathcal{I}$ is a generalized iteration, $|\mathcal{I}| = \theta$

(b) $\langle \alpha_i\rangle$ is normal; $\alpha_0 = 0$; $\alpha_\Gamma = \theta$

(c) $\alpha_i \notin D$ for $i < \Gamma$

(d) If $\alpha_i < j < \alpha_{i+1}$ and $j \notin D$, then

$$\eta_j = ht(M_j)$$

(e) If $\alpha_i < j < \alpha_{i+1}$ and $j \in D$, then

(i) $\nu_j > \nu_h$ for $\alpha_i < h < j$

(ii) $T(j+1)$ = the least $\xi > \alpha_i$ s.t. $\xi \in D$ and $\kappa_j < \lambda_\xi$

(iii) η_j = the max. $\eta \leq ht(M_\xi)$ s.t. $\kappa^{+M_\xi \| \eta} = \kappa^{+M_j \| \nu_j}$, where $\xi = T(j+1)$

Clearly, any good $\mathcal{I}$ can be converted into a good sequence $\langle\langle M_{\alpha_i} \mid i < \Gamma\rangle, \langle \mathcal{I}_i \mid i < \Gamma\rangle, \langle \pi_{\alpha_i+1, \alpha_j} \mid i \leq j < \Gamma\rangle$ where the normal iteration $\mathcal{I}_i$ can be read off from $\mathcal{I}$. Conversely, every good sequence can be converted into a good iteration.

We then define:

<u>Def</u> S is a <u>good iteration strategy for</u> M iff every good S-iteration $\mathcal{I}$ of M can be continued — i.e.

(a) If $\lim(\|\mathcal{I}\|)$, then $S(\mathcal{I})$ exists

(b) If $|\mathcal{I}| = k+1$ and $\langle \alpha_i \mid i \leq h+1 \rangle$ is a marking sequence and $\nu \in M_k$ s.t. $E_\nu^{M_k} \neq \emptyset$ and $\nu > \nu_j$ whenever $\alpha_h < j < k$, $j \in D$, then, setting $\alpha'_i = \alpha_i$ for $i \leq h$, $\alpha'_h = k+2$, then $\mathcal{I}$ extends to $\mathcal{I}'$ with marking sequence $\langle \alpha'_i \rangle$ s.t. $\nu_k^{\mathcal{I}'} = \nu$, $k \in D^{\mathcal{I}'}$.

(<u>Note</u> If $|\mathcal{I}| = h+1$, $\langle \alpha_i \mid i \leq h+1 \rangle$ is a marking sequence and $\eta \leq ht(M_h)$ then, setting: $\alpha''_i = \alpha_i$ for $i \leq h+1$, $\alpha''_{h+2} = k+2$, $\mathcal{I}$ trivially extends to $\mathcal{I}'$ with marking sequence $\langle \alpha''_i \rangle$ s.t. $\eta_k = \eta$.)

Def M is iterable (by S) iff M has a good iteration strategy S.

We call S the uniqueness strategy iff S($\mathcal{I}$) is defined iff $\mathcal{I}$ has a unique well founded cofinal branch b = S($\mathcal{I}$) (for good iterations $\mathcal{I}$ of limit length).

M is uniquely iterable iff it is iterable by the uniqueness strategy (i.e. every good iteration of limit length has a unique cofinal well founded branch.)

, , , ,

By a smooth iteration of M we mean one that can be achieved by a linear sequence of normal iterations, without intermediate truncations. M is smoothly iterable iff it has a strategy which works for smooth iterations. More precisely;

Def $\langle\langle M_i \mid i<\Gamma\rangle, \langle \mathcal{J}_i \mid i<\Gamma\rangle, \langle \pi_{ij} \mid i\leq j<\Gamma\rangle\rangle$ is a smooth sequence iff it is a good sequence and $\mathcal{J}_i$ is a normal iteration of M_i for $i<\Gamma$. (In a good sequence we require only that $\mathcal{J}_i$ be a normal iteration of some $M_i \| \eta$.)

Similarly

Def $\mathcal{J} = \langle\langle M_i\rangle, \langle \nu_i\rangle, \langle \eta_i\rangle, \langle \pi_{ij}\rangle, T\rangle$ is a smooth iteration with marking sequence $\langle \alpha_i \mid i<\Gamma\rangle$ iff $\mathcal{J}$ is a good iteration with this marking sequence and $\eta_{\alpha_i} = ht(M_{\alpha_i})$ for $i<\Gamma$.

(Hence $M_{\alpha_i+1} = M_{\alpha_i}$. Equivalently $\mathcal{J}$ is a smooth iteration iff $\mathcal{J}$ is a good iteration and $M_{i+1} = M_i$ whenever $i \notin D$.)

Finally:

Def S is a smooth iteration strategy for M iff every smooth S-iteration of M can be continued.

A pm M is called <u>smoothly iterable</u> iff it has a smooth iteration strategy. We shall later see that smooth iterability implies good iterability. We call M <u>uniquely smoothly iterable</u> iff every smooth iteration of M of limit length has a unique cofinal well founded branch. In general it is easier to be uniquely smoothly iterable than to be uniquely iterable.

(<u>Remark</u> Since writing this, we have shown that every smoothly iterable premouse is iterable. The proof is in § 9.)

<u>Def</u> A <u>mouse</u> is an iterable premouse

(However, we may from time to time use the term "mouse" in a specified more restrictive sense. Fr. ins. in § 5 when stating the "Dodd-Jensen" lemmas, we use the term to mean "uniquely iterable mouse".)

Σ_0 - iteration

A Σ_0 iteration is like a generalized iteration except that we use Σ_0 ultraproducts until the first truncation point on any branch, after which we use $*$ - ultraproducts.

Def $\mathcal{I}$ is a *generalized Σ_0 iteration* iff it satisfies (a)–(e), (g) in the def. of generalized iteration, as well as:

(f') Let $i \in D$. Then $i+1 < \theta$ and $E_{\nu_i}^{M_i} \neq \emptyset$. Let $\kappa_i = \mathrm{crit}(E_{\nu_i}^{M_i})$, $\tau_i = \kappa^{+M_i \| \nu_i}$. Let $\xi = T(i+1)$. Then $\tau_i = \kappa^{+M_\xi \| \eta_i}$, $J_{\tau_i}^{E^{M_i}} = J_{\tau}^{E^{M_\xi}}$. If $i+1$ is simple (i.e. $\{h \mid h \leq_T i+1\}$ has no truncation point), then $\pi_{\xi, i+1} : M_\xi \| \eta_i \xrightarrow[E_{\nu_i}]{} M_{i+1}$.

Otherwise $\pi_{\xi, i+1} : M_\xi \| \eta_i \xrightarrow[E_{\nu_i}]{*} M_{i+1}$.

The notions *direct*, *standard*, *normal* are then defined exactly as before. The notion of a Σ_0 *iteration strategy* is defined as before, as is the notion: $\mathcal{J}$ is an S-*iteration* where S is such a strategy. Finally the notion: S is a *normal* Σ_0 *iteration strategy* *for* M is defined as before. M is *normally* Σ_0 *iterable* (by S) iff M has a strategy S.

Obviously Cor 1.1 can fail for Σ_0 iterations. However, Lemma 1 still holds:

<u>Lemma 4</u> Let $\mathcal{T} = \langle\langle M_i \rangle, \ldots, T\rangle$ be normal Σ_0 iteration. If $i \in D$, then $E^{M_i}_{\nu_i}$ is Σ_1-amenable wrt $M''_{T(i+1)}$

Hence:

<u>Cor 4.1</u> If $\mathcal{T}$ is as above, $h \leq_T i$, i is nonsimple in $\mathcal{T}$ and π_{hi} is total on M_h, then $\pi_{hi} : M_h \to_{\Sigma^*} M_i$.

Prf. Ind. on i, using the fact that h is nonsimple if π_{hi} is total on M_h.

<u>Cor 4.1.1</u> If $\mathcal{T}$ is as above, $h T i$, i is non simple in $\mathcal{T}$, ξ = the least ξ s.t. $h \leq_T \xi + 1 \leq_T i$, $N = M_h \| \eta_\xi$ and π_{hi} is total on N, then $\pi_{hi} : N \to_{\Sigma^*} M_i$.

Our proof of Lemma 4 will be virtually as before. Assume w.l.o.g. that $\mathcal{T}$ is direct. We first define $\bar{\pi}_i$, $\bar{\tau}_i$, σ_i, $\bar{\eta}_i$ exactly as before and show;

<u>Lemma 4.2</u> Let δ_i exist. Then $\mathcal{P}(\tau_i) \cap \Sigma_1(M_i) \subset \Sigma_1(M_{\delta_i} \| \eta_i)$.

prf.

Suppose not. Let i be the least counterexample. Then $\delta_i < i$. We consider two cases. If i is nonsimple we repeat the proof of Lemma 1.? using the fact that if $\xi T i$, ξ^* = the least ξ s.t. $\xi \leq_T \xi^* + 1 \leq_T i$, and $\pi_{\xi i}$ is total on $M_\xi \| \eta_{\xi^*}$, then $\xi^* + 1$ is nonsimple. This means that $\pi_{\xi\xi^*} : M_\xi \| \eta_{\xi^*} \xrightarrow{*}_F M_{\xi^*}$ where $F = E^{M_{\xi^*}}_{\nu_{\xi^*}}$.

Now let i be simple. Then for each such ξ, $\pi_{\xi\xi^*}$ is given by a Σ_0 ultrapower. Hence (5), (6) no longer go thru, although the proofs of (1)–(4) + (7) still do. (8) still goes thru using §1 Lemma 8 in place of §2 Lemma 5.1.

The rest of the proof is as before.

QED (Lemma 4.2)

The proof of Lemma 4 is then exactly like that of Lemma 1.

QED (Lemma 4).

We can then repeat the proof of Lemma 3 to get:

<u>Lemma 5</u> Let $\mathcal{I}$ be a normal Σ_0 iteration of length $\theta+1$. If θ is not simple in $\mathcal{I}$, then M_θ is not sound.

If M, N are normally Σ_0-iterable beyond ν and $J_\nu^{E^M} = J_\nu^{E^N}$, we can define the Σ_0-<u>coiteration</u> of M, N exactly as before. Lemma goes thru exactly as before, as do Cor 3.1 + Cor 3.2, using Lemma 5. If M is normally Σ_0 iterable + N is normally iterable we can also define a <u>mixed</u> coiteration which is Σ_0 only on the M-side. Lemma 2 and Cor 3.1, 3.2 continue to hold.

A stronger notion of Σ_0 - iterability can be obtained by requiring - as before - not only that M be normally Σ_0 iterable but that the process of taking a normal iterate of a truncation can itself be iterated. This must, however, be formulated with some care. The simplest way is to modify the definition of a good iteration;

<u>Def</u> $\mathcal{J} = \langle\langle M_i\rangle, \ldots, T\rangle$ is a <u>good Σ_0 - iteration</u> of <u>length</u> θ <u>with marking sequence</u> $\langle \alpha_i \mid i \leq \Gamma\rangle$ iff (b) - (e) of the earlier def. hold together with;

(a') $\mathcal{J}$ is a ~~most~~ generalized Σ_0 - iteration of length θ.

Σ_0 - iterability then means that M possesses a good Σ_0 - strategy in the same sense as before.

Smooth Σ_0 - iterability is also defined in the obvious way.