§5 Copying an Iteration

Def Let $\sigma : \bar{M} \to M$, where $\bar{M}, M$ are premice. Let $\bar{M}$ have a generalized iteration $\bar{\mathcal{J}} = \langle\langle \bar{M}_i \rangle, \langle \bar{\nu}_i \mid i \in D \rangle, \langle \bar{\eta}_i \rangle, \langle \bar{\pi}_{ij} \rangle, T \rangle$ of length θ. Let $\sigma : \bar{M} \to_{\Sigma_0} M$. We say that $\mathcal{J} = \sigma(\bar{\mathcal{J}})$ is the copy of $\bar{\mathcal{J}}$ onto $\bar{M}$ by σ with copying maps $\langle \sigma_i \mid i < \theta \rangle$ iff the following hold:

(a) $\mathcal{J} = \langle\langle M_i \rangle, \langle \nu_i \mid i \in D \rangle, \langle \eta_i \rangle, \langle \pi_{ij} \rangle, T \rangle$ is a generalized iteration of M of length θ
(Note $D^{\bar{\mathcal{J}}} = D^{\mathcal{J}}$, $T^{\bar{\mathcal{J}}} = T^{\mathcal{J}}$).

(b) $\sigma_i : \bar{M}_i \to_{\Sigma_0} M_i$ is a cardinal preserving map; $\sigma_0 = \sigma$;

$$\sigma_i \bar{\pi}_{hi} = \pi_{hi} \sigma_h \quad \text{if } h \leq_T i$$

(c) If $i = h+1$, $h \notin D$, then $\sigma_h(\bar{\eta}_h) = \eta_h$ and $\sigma_i = \sigma_h \restriction \bar{M}_i$.
(Where $\sigma_h(\mathrm{On} \cap \bar{M}_h) =_{\mathrm{df}} \mathrm{On} \cap M_h$).

(d) Let $i = h+1$, $h \in D$. Then

(i) $\sigma_i \restriction \lambda_h = \sigma_h \restriction \lambda_h$

(ii) Let $\xi = T(i)$. Then $\sigma_\xi(\bar\eta_h) = \eta_h$

(iii) Let $\bar M^* = \bar M_\xi \| \bar\eta_h$, $M^* = M_\xi \| \eta_h$, $\sigma^* = \sigma_\xi \upharpoonright \bar M^*$. Then:

$\sigma^* : \bar M^* \xrightarrow{\Sigma_0^{(n)}} M^*$ whenever $\omega\rho^n_{\bar M^*} > \bar\kappa_h$

(iv) Let $\bar F = E^{\bar M_h}_{\bar\nu_h}$, $F = E^{M_h}_{\nu_h}$. Then:

$\langle \sigma^*, \sigma_h \upharpoonright \bar\lambda_h \rangle : \langle \bar M^*, \bar F \rangle \to \langle M^*, F \rangle$

(v) $\sigma_i(\bar\pi_{\xi i}(f)(\alpha)) = \pi_{\xi i}\, \sigma^*(f)(\sigma_h(\alpha))$ for $f \in \Gamma(\kappa_i, M_i)$, $\alpha < \lambda_i$.

(<u>Note</u> "Cardinal preserving" means that if γ is a cardinal in $\bar M_i$, then $\sigma_i(\gamma)$ is a cardinal in M_i. This is needed for $\eta_i = \sigma_\xi(\bar\eta_i)$ $(\xi = T(i+1))$ in case $\bar\eta_i = \mathrm{On} \cap \bar M$.)

(<u>Note</u> (d)(iii), (iv) enable us to define σ_i as in (d)(v) nach §3 Lemma 1.)

(<u>Note</u> It is obvious that $\mathcal{I}$, $\langle \sigma_i \mid i < \theta \rangle$ are unique if they exist.)

(<u>Note</u> If $\mathcal{I}$ is normal, then since $\sigma_{i+1} \upharpoonright \lambda_i = \sigma_i \upharpoonright \lambda_i$, we have $\sigma_i \upharpoonright \lambda_j = \sigma_j \upharpoonright \lambda_j$ [illegible] ind. on i.)

<u>Lemma 1</u> Let $\sigma : \bar{M} \xrightarrow{}_{\Sigma^*} M$. Let $\bar{\mathcal{J}}$ be a normal iteration of $\bar{M}$ s.t. $\mathcal{J} = \sigma(\bar{\mathcal{J}})$, $\langle \sigma_i \rangle$ exist. Then for all $i < |\mathcal{J}|$;
$\sigma_i : \bar{M}_i \xrightarrow{}_{\Sigma_0^{(n)}} M_i$ whenever $\omega\rho^n_{\bar{M}_i} \geq \sup_{h \in D \cap i} \bar{\lambda}_h$.

proof. Ind. on i.
Clearly $\bar{\mathcal{J}}$ can be replaced by a direct $\bar{\mathcal{J}}'$ simply by omitting repetitions. But then $\mathcal{J}$ can be replaced by a direct $\mathcal{J}'$ s.t. $\mathcal{J}' = \sigma(\bar{\mathcal{J}}')$ and the sequence $\langle \sigma_i' \rangle$ of copying maps is the same except for the omission of repetitions. Hence we may assume w.l.o.g. that $\bar{\mathcal{J}}$ is direct.

<u>Case 1</u> $i = 0$ trivial

<u>Case 2</u> $i = \lambda$, Lim(λ).

Let $\omega\rho^n_{\bar{M}_\lambda} \geq \sup_{h < \lambda} \bar{\lambda}_h$. Pick $j \underset{T}{<} \lambda$ s.t. $\bar{\pi}_{j\lambda}$ is total on $\bar{M}_j$ (hence $\pi_{j\lambda}$ is total on M_j). Then $\bar{\pi}_{j\lambda} \upharpoonright \bar{\lambda}_h = \mathrm{id}$ for $h < j$.

But $\overline{\pi}_{i\lambda}$ is Σ^*-preserving. Hence $\overline{\pi}_{i\lambda}(\omega\rho^n_{\overline{M}_i}) \geq \omega\rho^n_{\overline{M}_\lambda}$. Hence $\omega\rho^n_{\overline{M}_i} \geq \sup_{h<i} \overline{\lambda}$. Hence $\sigma_i : \overline{M}_i \xrightarrow{}_{\Sigma_0^{(n)}} M_i$. But this holds for sufficiently large $i \leq_T \lambda$, where $\overline{\pi}_{i\lambda}, \pi_{i\lambda}$ are Σ^*-preserving. It follows easily that $\sigma_\lambda : \overline{M}_\lambda \xrightarrow{}_{\Sigma_0^{(n)}} M_\lambda$.

QED (Case 2)

<u>Case 3</u> $i = h+1$.

Let $\omega\rho^n_{\overline{M}_i} \geq \sup_{h<i} \overline{\lambda}_h$. By §3 Lemma 1 it is enough to show:

<u>Claim</u> (a) $\omega\rho^n_{\overline{M}^*} > \overline{\kappa}_h$

(b) $\sigma^* : \overline{M}^* \xrightarrow{}_{\Sigma_0^{(m)}} M^*$

whenever $\omega\rho^m_{\overline{M}^*} > \overline{\kappa}_h$,

where: $\xi = T(i)$, $\overline{M}^* = \overline{M}_\xi \| \overline{\eta}_h$, $M^* = M_\xi \| \eta_i$ and $\sigma^* = \sigma_\xi \upharpoonright \overline{M}^*$.

To see (a), we note that otherwise $\omega\rho^n_{\bar{M}_i} = \omega\rho^n_{\bar{M}^*} \leq \bar{\pi}_h < \bar{\lambda}_\xi \leq \sup_{h<i} \bar{\lambda}_h$.

If $\bar{\eta}_h < ht(\bar{M}_\xi)$, σ^* is Σ_ω preserving and (b) is trivial. Otherwise $\sigma^* = \sigma_\xi$, $\bar{M}^* = \bar{M}_\xi$ and $\bar{\pi}_{\xi i}$ is total on $\bar{M}_\xi$. Since $\bar{\pi}_h \geq \bar{\lambda}_l$ for $l < \xi$, it follows that $\omega\rho^m_{\bar{M}_\xi} \geq \sup_{l<\xi} \bar{\lambda}_l$ whenever $\omega\rho^m_{\bar{M}} > \bar{\pi}_h$. Hence (b) holds by the ind. hyp.

QED (Lemma 1)

A modification of this proof yields:

<u>Lemma 1.1</u> Let $\sigma : \bar{M} \xrightarrow[\Sigma_0^{(m)}]{} M$ where $\omega\rho^{m+1}_{\bar{M}} < \rho^m_{\bar{M}}$. Let $\bar{\mathcal{J}}$ be an iteration of $\bar{M}$ above $\omega\rho^{m+1}_{\bar{M}}$ & let $\mathcal{J} = \sigma(\bar{\mathcal{J}})$, $\langle \sigma_i \rangle$ exist. Then the conclusion of Lemma 1 holds for $i > 0$.

The proof is essentially as before. If $i = h+1$ is simple in $\bar{\mathcal{I}}$, we use the fact that $\omega\rho^{m+1}_{\bar{M}_i} = \omega\rho^{m+1}_{\bar{M}} < \nu_l$ for $l \leq h$. Hence we need only consider $n \leq m$. If i is not simple, the proof is exactly as before. QED (Lemma 1.1)

. . . . .

Suppose now that S is an iteration strategy. Let $\sigma : \bar{M} \to M$. We can define a <u>derived strategy</u> $\bar{S}$ on iterations of $\bar{M}$ by: Let $\bar{\mathcal{I}}$ be an iteration of $\bar{M}$ of limit length. If $\mathcal{I} = \sigma(\bar{\mathcal{I}})$, set $\bar{S}(\bar{\mathcal{I}}) = S(\mathcal{I})$. Otherwise $\bar{S}(\bar{\mathcal{I}})$ is undefined. If S is a normal iteration strategy for M, it will follow that $\bar{S}$ is a normal iteration strategy for $\bar{M}$ if σ is Σ^*-preserving.

<u>Note</u> It is obvious that if $\sigma(\bar{\mathcal{I}})$ exists with copying maps $\langle \sigma_i \mid i<\theta$ $(\theta=|\bar{\mathcal{I}}|)$, then $\sigma(\bar{\mathcal{I}}|i)=\sigma(\bar{\mathcal{I}})|i$ exists with copying maps $\langle \sigma_j \mid j<i$ for $i\leq\theta$.

<u>Lemma 2.1</u> Let S be a normal iteration strategy for M. Let $\sigma:\bar{M} \xrightarrow{}_{\Sigma^*} M$ and let $\bar{S}$ be the derived strategy. Let $\bar{\mathcal{I}}$ be a (normal) $\bar{S}$-iteration of $\bar{M}$. Then $\mathcal{I}=\sigma(\bar{\mathcal{I}})$ exists and is an S-iteration of M.

prf.

By ind on $j\leq\theta=\underbrace{|\bar{\mathcal{I}}|}_{\text{s.t. } j\geq 1}$, we construct $\mathcal{I}|j=\sigma(\bar{\mathcal{I}}|j)$ and the copying maps $\langle\sigma_i \mid i<j\rangle$, verifying at each stage that (a)–(e) are satisfied. Assume w.l.o.g. that $\bar{\mathcal{I}}$ is direct.

<u>Case 1</u> $j=1$. $\bar{\mathcal{I}}|1=\langle\langle\bar{M}\rangle,\emptyset,\emptyset,\langle id\rangle,\{0\}\rangle$

We set: $\mathcal{I}|1=\langle\langle M\rangle,\emptyset,\emptyset,\langle id\rangle,\{0\}\rangle$, $\sigma_0=\sigma$. All conditions are satisfied.

Case 2 $j = \lambda$, Lim(λ).

$\mathcal{Y}|\lambda = \bigcup_{j<\lambda} \mathcal{Y}|j$ in the obvious sense.

All conditions are satisfied.

Case 3 $j = \lambda + 1$, Lim (λ).

Let $b = \{ i \mid i \leq_T \lambda \}$. Then $b = \bar{S}(\bar{\mathcal{Y}}|\lambda)$. Hence $b = S(\mathcal{Y}|\lambda)$, since $\bar{S}$ is the derived strategy. But then we can form $\mathcal{Y}|\lambda+1$ by setting:

$$M_\lambda, \langle \pi_{i\lambda} \rangle = \lim_{i \leq_T j \in b} (M_i, \pi_{ij}),$$

σ_λ is defined by: $\sigma_\lambda \bar{\pi}_{i\lambda} = \pi_{i\lambda} \sigma_i$.

All conditions are satisfied.

Case 3 $j = h+2$. Let $i = h+1$.

We set: $\nu_h = \sigma_h(\bar{\nu}_h)$. At $\xi = T(i+1)$, we set: $\eta_h = \sigma_\xi(\bar{\eta}_h)$. Then $\kappa_h = \text{crit}(E_{\nu_h}^{M_h}) = \sigma_h(\bar{\kappa}_h)$ + hence ξ is least s.t. $\kappa_h < \lambda_\xi = \sigma_\xi(\bar{\lambda}_\xi) = \sigma_h(\bar{\lambda}_\xi)$. Moreover, η_h is least s.t. $\kappa^{+M_h \| \nu_h} = \kappa^{+M_\xi \| \eta_h}$. (This uses the cardinal preserving character of $\sigma_{\bar\xi}$ if $\bar\eta_h = ht(\bar M_\xi)$.) We of course define $\pi_{\xi i} : M^* \xrightarrow{*}_{F} M_i$, where $\bar M^* = \bar M_\xi \| \bar\eta_\xi$, $M^* = M_\xi \| \eta_\xi$, $\bar F = E^{\bar M_h}_{\bar\nu_h}$, $F = E^{M_h}_{\nu_h}$. In order to define σ_i, we must verify (d)(iii) and (d)(iv). To see (d)(iii) note that if $\bar\eta_h < ht(\bar M_\xi)$, then $\sigma^* : \bar M^* \longrightarrow_{\Sigma_\omega} M^*$. Otherwise the result follows by Lemma 1 for $\bar y|\xi+$ and the fact that $\bar\kappa_h \geq \sup_{k<\xi} \bar\lambda_k$ by the minimal choice of $\xi = T(i+1)$.

We in fact show: $\langle \sigma^*, \sigma_h \restriction \bar\lambda_h \rangle : \langle \bar M^*, \bar Z \rangle \longrightarrow^{\circ} \langle M^*, Z \rangle$

We now prove (d)(iv). Let $\bar X = \langle \bar X_l \mid l < \bar\kappa_h \rangle \in \bar M^*$, $X = \sigma^*(\bar X) = \langle X_l \mid l < \kappa_h \rangle$. Let $\alpha_1, \dots, \alpha_r < \bar\lambda_h$. $\sigma^* \restriction \mathbb{P}(\bar\kappa_i) = \sigma^*_{i+1} \restriction \mathbb{P}(\bar\kappa_i)$ (since $\sigma_{i+1}(z) = \sigma_{i+1}(\bar\pi_{\xi,i+1}(z) \cap \bar\kappa_i) = \pi_{\xi,i+1}\sigma^*(z) \cap \kappa_i$ $= \sigma^*(z)$ for $z \subset \bar\kappa_i$.) Hence:

$$\sigma^*(\{l \mid \vec\alpha \in \bar\pi_{\xi,i+1}(\bar X_l)\}) = \{l \mid \sigma_i(\vec\alpha) \in \pi_{\xi,i+1}(X)_l\}$$

QED(d)(iv)

This enables us to define σ_i as in (d)(v). The remaining verifications are trivial.

QED (Lemma 2.1)

<u>Cor 2.1.2</u> $\langle \sigma, \sigma_i^*, \sigma_i \upharpoonright \bar{\lambda}_i \rangle : \langle \bar{M}^*, \bar{F} \rangle \rightarrow^0 \langle M, F \rangle$.

<u>Cor 2.2</u> Let $\sigma : \bar{M} \rightarrow_{\Sigma^*} M$ and let S be a normal iteration strategy for M. Let $\bar{S}$ be the derived strategy. Then $\bar{S}$ is a normal iteration strategy for $\bar{M}$.

prf.

Let $\bar{I}$ be an $\bar{S}$ iteration of $\bar{M}$. We must show that $\bar{I}$ can be continued. Let $I = \sigma(\bar{I})$ with copying maps $\langle \sigma_i \mid i < |I| \rangle$. Then I is an S-iteration of M.

<u>Case 1</u>: $\lim(I)$.

Since we can continue I using $b = S(I)$, it follows easily that we can continue $\bar{\mathcal{J}}$ with $b = S(\bar{\mathcal{J}})$ (if $\bar{M}^*, \langle \bar{\pi}_i \mid i \in b \rangle$ is the direct limit of $\langle \bar{M}_i \mid i \in b \rangle$, $\langle \bar{\pi}_{ij} \mid i \leq_T j \in b \rangle$ and $M^*, \langle \pi_i \rangle$ is the corresponding limit for $\mathcal{J}$, then $\sigma' : \bar{M}^* \longrightarrow_{\Sigma_0} M^*$ can be defined by $\sigma' \bar{\pi}_i = \pi_i \sigma_i$. Hence $\bar{M}^*$ is well founded.). QED (Case 1)

Case 2 $|\mathcal{J}| = k+1$.

Let $\bar{\nu} \in \bar{M}_k$ s.t. $E_{\bar{\nu}}^{\bar{M}_k} \neq \emptyset$ and $\bar{\nu} > \nu_h$ for $h < k$. Set $\nu = \sigma_k(\bar{\nu})$.

Let ξ = the least ξ s.t. $\bar{\kappa} = \mathrm{crit}(E_{\bar{\nu}}^{\bar{M}_k}) < \bar{\lambda}_\xi$ or $\xi = k$.

Then ξ is least s.t. $\kappa = \mathrm{crit}(E_\nu^{M_k}) < \lambda_\xi$ or $\xi = k$.

Let $\bar{\eta}$ be max s.t. $\bar{\eta} \leq ht(\bar{M}_\xi)$ and $\bar{\kappa}^+ \bar{M}_\xi \| \bar{\eta} = \bar{\kappa}^+ \bar{M}_k \| \bar{\nu}$.

Then $\eta = \sigma_\xi(\bar{\eta})$ has the corresponding def. above.

Then by the iterability of M, $M^* = M_\zeta \| \gamma$ is extendable by $F = E_\nu^{M_h}$. Set: $\bar{M}^* = \bar{M}_\zeta \| \bar{\gamma}$, $\sigma^* = \sigma_\zeta \upharpoonright \bar{M}^*$, $\bar{F} = E_{\bar{\nu}}^{\bar{M}_h}$. We must show that $\bar{M}^*$ is extendable by $\bar{F}$. Exactly as in the verification of (d) (iii), (iv) in Case 3 of Lemma 2.1 we get:

(1) $\sigma^* : \bar{M}^* \longrightarrow_{\Sigma_0^{(n)}} M^*$ whenever

$$\omega\rho^n_{\bar{M}^*} > \bar{\kappa}_h$$

(2) $\langle \sigma^*, \sigma_h \upharpoonright \bar{\lambda} \rangle : \langle \bar{M}^*, \bar{F} \rangle \longrightarrow \langle M^*, F \rangle$, where $\bar{\lambda} = \text{length}(\bar{F})$.

The result follows by § 3 Lemma 1

QED (Cor 2.2.)

__Def__ S is a normal iteration strategy for M __above__ (__beyond__) ν iff every S-iteration of M above (beyond) ν can be continued.

The same proofs show:

__Cor 2.3__ Let S be a normal iteration strategy for M beyond ν. Let $\sigma : \bar{M} \xrightarrow[\Sigma^*]{} M$, $\sigma(\bar{\nu}) = \nu$. Let $\bar{S}$ be the derived strategy. Then $\bar{S}$ is an (normal) iteration strategy for $\bar{M}$ beyond $\bar{\nu}$. Moreover, if $\bar{\mathcal{I}}$ is a normal $\bar{S}$-iteration of $\bar{M}$ beyond $\bar{\nu}$, then $\sigma(\bar{\mathcal{I}})$ exists.

__Cor 2.4__ Let S be a normal iteration strategy for M above ν. Let $\sigma : \bar{M} \xrightarrow[\Sigma_0^{(n)}]{} M$ where $\omega\rho^{n+1}_{\bar{M}} < \omega\rho^{n}_{M}$ and $\nu \leq \sigma(\omega\rho^{n+1}_{\bar{M}})$. Let $\bar{S}$ be the derived strategy. Then $\bar{S}$ is a normal iteration strategy for $\bar{M}$ above $\omega\rho^{n+1}_{\bar{M}}$. Moreover, if $\bar{\mathcal{I}}$ is a normal $\bar{S}$-iteration of $\bar{M}$ above $\omega\rho^{n+1}_{\bar{M}}$, then $\sigma(\bar{\mathcal{I}})$ exists.

<u>Def</u> M has the ~~uniqueness~~ normal uniqueness <u>property</u> iff every (normal) iteration $\mathcal{I}$ of M of limit length θ has at most one branch b which can continue $\mathcal{I}$ - i.e.

(a) b is cofinal in θ

(b) b has at most finitely many truncation pts.

(c) The direct limit of $\langle M_i^{\mathcal{I}} \mid i \in b \rangle$, $\langle \pi_{ij}^{\mathcal{I}} \mid i \leq_T j \in b \rangle$ is well founded.

Recall that the <u>uniqueness</u> <u>strategy</u> assigns the unique branch b satisfying (a)-(c) if it exists & is otherwise undefined.

Clearly M has the (normal) uniqueness property & has a normal iteration strategy S iff the uniqueness strategy U is a normal iteration strategy for M (since then $S(\mathcal{I}) = U(\mathcal{I})$ for iterations $\mathcal{I}$ of M). In this case we call M <u>uniquely</u> <u>normally</u> <u>iterable</u>.

Thus e.g. if M has a normal iteration strategy, $\sigma : \bar{M} \to_{\Sigma^*} M$, and $\bar{M}$ has the normal uniqueness property, then $\bar{M}$ is normally iterable by Cor 2.2. Similar remarks apply to Cor 2.3, Cor 2.4.

. . . .

We can, in fact, prove a much stronger version of Lemma 1:

<u>Lemma 3</u> Let $\sigma : \bar{M} \to_{\Sigma^*} M$. Let $\bar{\mathcal{I}}$ be a normal iteration of $\bar{M}$ and $\mathcal{I} = \sigma(\bar{\mathcal{I}})$ with copying maps $\langle \sigma_i \rangle$. Then $\sigma_i : \bar{M}_i \to_{\Sigma^*} M_i$ for all i

This follows from:

<u>Lemma 3.1</u> Let $\sigma : \bar{M} \to M$. Let $\bar{\mathcal{I}}$ be a normal iteration of $\bar{M}$ and $\mathcal{I} = \sigma(\bar{\mathcal{I}})$ with copying maps $\langle \sigma_i \rangle$. Let $i \in D$, $\xi = T(i+1)$, $\bar{M}^* = \bar{M}_\xi \| \bar{\eta}_\xi$, $M^* = M_\xi \| \eta_i$, $\sigma^* = \sigma_\xi \upharpoonright \bar{M}^*$, $\bar{F} = E^{\bar{M}_i}_{\bar{\nu}_i}$, $F = E^{M_i}_{\nu_i}$. Then:

$\langle \sigma^*, \sigma_i \upharpoonright \bar{\lambda}_i \rangle : \langle \bar{M}^*, \bar{F} \rangle \to^* \langle M^*, F \rangle$.

Lemma 3 follows from Lemma 3.1 by ind. on i using §3 Lemma 2. We now prove Lemma 3.1. The proof will in fact, be a fairly close imitation of that of §4 Lemma 1. We again assume w.l.o.g. that $\bar{J}$ is direct.

Def Let $E^{M_i}_{On \cap M_i} \neq \emptyset$. Set:

$\hat{\kappa}_i = \mathrm{crit}(E^{M_i}_{On \cap M_i})$, $\hat{\tau}_i = (\hat{\kappa}_i)^{+M_i}$;

δ_i = the least δ s.t. $\delta = i$ or $\hat{\kappa}_i < \lambda_\delta$;

$\hat{\eta}_i$ = the maximal $\eta \leq ht(M_{\delta_i}$ s.t. $\hat{\tau}_i = (\hat{\kappa}_i)^{+M_{\delta_i}\|\eta}$.

$\hat{\bar{\kappa}}_i, \hat{\bar{\tau}}_i, \bar{\delta}_i, \hat{\bar{\eta}}_i$ are defined similarly wrt. $\bar{J}$. Clearly $\sigma_i(\hat{\bar{\kappa}}_i, \hat{\bar{\tau}}_i) = \hat{\kappa}_i, \hat{\tau}_i$, $\bar{\delta}_i = \delta_i$ and $\sigma_{\delta_i}(\hat{\bar{\eta}}_i) = \hat{\eta}_i$.

We then prove:

<u>Lemma 3.2</u> Let $\bar{\mathcal{Y}}, \mathcal{Y}, \langle\sigma_i\rangle$ be as above. Let δ_i exist. Let $\bar{A} \subset \hat{\bar{\tau}}_i$, $A \subset \hat{\tau}_i$ s.t. $\bar{A}$ is $\Sigma_1(\bar{M}_i)$ in $\bar{p}$ and A is $\Sigma_1(M_i)$ in $p = \sigma_i(\bar{p})$ by the same definition. Then there is $\bar{q} \in \bar{M}_{\delta_i} \| \hat{\bar{\eta}}_i$ s.t. $\bar{A}$ is $\Sigma_1(\bar{M}_{\delta_i} \| \hat{\bar{\eta}}_i)$ in $\bar{q}$ and A is $\Sigma_1(M_{\delta_i} \| \hat{\eta}_i)$ in $q = \sigma_{\delta_i}(\bar{q})$ by the same definition.

prf. Suppose not.

Let i be the least counterexample. Then $\delta_i < i$ + $i = h+1$ by minimality.

Set: $\zeta = T(i)$. Set:

$\bar{\tau} = \hat{\bar{\tau}}_i$, $\tau = \hat{\tau}_i$, $\bar{\kappa} = \hat{\bar{\kappa}}_i$, $\kappa = \hat{\kappa}_i$,

Set: $\bar{M}^* = \bar{M}_\zeta \| \bar{\eta}_h$, $M^* = M_\zeta \| \eta_h$,

$\sigma^* = \sigma_\zeta \upharpoonright \bar{M}^*$.

(1) $\kappa < \mu_h$

proof: Let $\kappa' = \pi_{\zeta i}^{-1}(\kappa) = \mathrm{crit}(E^{M^*}_{\omega n \wedge M^*})$. Then $\kappa' < \mu_h$, since otherwise $\kappa = \pi_{\zeta i}(\kappa') \geq \pi_{\zeta i}(\mu_h) = \lambda_h$. Hence $\delta_i = i$. Contr! QED (1)

(2) $\delta_i \le \xi$ since $\mu < \mu_h < \lambda_\xi$,

Set: $F = E^{M_h}_{\nu_h}$, $\bar F = E^{\bar M_h}_{\bar\nu_h}$

(3) $\langle \sigma^*, \sigma_h \restriction \bar\lambda_h \rangle : \langle \bar M^*, \bar F \rangle \longrightarrow^* \langle M^*, F \rangle$

proof.

If $\nu_h = On \cap M_h$, then $\delta_h = \xi$ and the conclusion follows by the minimality of i. Otherwise $\bar F \in \bar M_h$ and $\sigma_h(\bar F) = F$. Let $\bar\alpha < \bar\lambda_h$, $\alpha = \sigma_h(\bar\alpha)$. Then $\bar F_{\bar\alpha} \in J^{E^{\bar M_h}}_{\bar\lambda_h} \cap \mathcal{P}(\bar\kappa) \subset$ $\subset J^{E^{\bar M_h}}_{\bar\lambda_\xi} = J^{E^{\bar M_\xi}}_{\bar\lambda_\xi} \subset \bar M^*$ and $\sigma^*(\bar F_{\bar\alpha}) = \sigma_h(\bar F_{\bar\alpha}) = F_\alpha$. QED(3)

(4) If σ_i is $\Sigma_0^{(1)}$-preserving, then $\omega\rho^1_{\bar M_i} \le \bar\tau$.

proof Suppose not.

Let $\bar A \subset \bar\tau$ be $\Sigma_1(\bar M_i)$ in $\bar p$ and $A \subset \tau$ be $\Sigma_1(M_i)$ in $p = \sigma_i(\bar p)$ by the same definition. Then $\bar A \in \mathcal{P}(\tau) \cap M_i \subset (J^E_{\tau^+})^{M_i} \subset J^{E^{M_i}}_{\lambda_{\delta_i}} =$ $= J^{E^{M_{\delta_i}}}_{\lambda_{\delta_i}} \subset M_{\delta_i} \| \hat\eta_i$. Let $\bar A = \bar a \in M_i$

The statement:

$\bigwedge \xi < \bar\tau \, (\xi \in \bar A \leftrightarrow \xi \in \bar a)$

is $\Sigma_0^{(1)}(\bar{M}_i)$ in $\bar{\tau}, \bar{a}, \bar{p}$. Hence the same statement holds of $\tau, p, a = \sigma_i(\bar{a})$ in M_i. Hence $\sigma_{\delta_i}(\bar{A}) = \sigma_i(\bar{A}) = a = A$. Thus $\bar{A}$ is $\Sigma_1(\bar{M}_{\delta_i} \| \hat{\eta}_i)$ in $\bar{a}$ and A is $\Sigma_1(M_{\delta_i} \| \hat{\eta}_i)$ in $a = \sigma_{\delta_i}(\bar{a})$ by the same definition. Contr! QED(4)

(5) $\omega\rho^1_{\bar{M}^*} \leq \bar{\kappa}_h$.

proof. Suppose not.

Then σ^* is $\Sigma_0^{(1)}$-preserving. Hence σ_i is $\Sigma_0^{(1)}$-preserving. Hence $\omega\rho^1_{\bar{M}^*} = \omega\rho^1_{\bar{M}_i} \leq \tau < \bar{\kappa}_h$, since $\pi_{\xi i}$ is Σ^*-preserving. QED(5)

(6) Let $\bar{A} \subset \bar{\kappa}_h$ be $\Sigma_1(\bar{M}_i)$ in $\bar{p}$ and $A \subset \kappa_h$ be $\Sigma_1(M_i)$ in $p = \sigma_i(\bar{p})$ by the same definition. Then $\bar{A}$ is $\Sigma_1(\bar{M}^*)$ in a $\bar{q} \in \bar{M}^*$ and A is $\Sigma_1(M^*)$ in $q = \sigma^*(\bar{q})$ by the same definition.

prf. of (6).

$\bar{A}$ is $\Sigma_1(\bar{M}^*)$ by §1 Lemma 8. We recapitulate the proof of that lemma.

Let $\bar{\pi} = \bar{\pi}_{\xi i}$; $\pi = \pi_{\xi i}$. Let

$\bar{A}\xi \longleftrightarrow \bigvee z\, \bar{R}(z, \xi, \bar{p})$, where $\bar{R}$ is $\Sigma_1(\bar{M}_i)$. Let $\bar{p} = \bar{\pi}(f)(\bar{\alpha})$, where $f \in \bar{M}^*$, $\bar{f} : \kappa_h \to \bar{M}^*$, $\bar{\alpha} < \bar{\lambda}_h$.

Then:

$$\bar{A}\xi \longleftrightarrow \bigvee u \in \bar{M}^* \bigvee z \in \bar{\pi}(u)\, \bar{R}(z, \xi, \bar{\pi}(\bar{f})(\bar{\alpha}))$$

$$\longleftrightarrow \quad " \quad \underbrace{\bar{P}(\bar{\pi}(u), \xi, \bar{\pi}(\bar{f})(\bar{\alpha}))}_{\Sigma_0}$$

$$\longleftrightarrow \bigvee u \in \bar{M}^* \{\eta \mid \bar{P}^*(u, \xi, \bar{f}(\eta))\} \in \bar{F}_{\bar{\alpha}}$$

where $\bar{P}^*$ is $\Sigma_0(\bar{M}^*)$ by the same definition. But $\bar{F}_{\bar{\alpha}}$ is $\Sigma_1(\bar{M}^*)$ in a $\bar{q}$ and F_α is $\Sigma_1(M^*)$ in $q = \sigma^*(\bar{q})$ by the same definition, where $\alpha = \sigma_h(\bar{\alpha})$, by (3).

<u>Fact</u> π is cofinal (i.e. $\sup \pi''On \cap M^i = On \cap M_i$)

proof.

Set: $G^* = E^{M^*}_{On \cap M^*}$, $G = E^{M_i}_{On \cap M_i}$.

For $\eta \leq \tau$ set: $G^*_\eta = G^* \upharpoonright J^{E^{M^*}}_\eta$, $G_\eta = G \upharpoonright J^{E^M}_\eta$

Then $G_\tau \notin M_i$ and $\pi(G^*_\eta) = G_\eta$ for $\eta < \tau$. Let $\xi \in M_i$. Pick $\eta < \tau$ s.t. $G_\eta \notin J^{E^{M_i}}_\xi$. Pick $\xi^* \in M^*$ s.t. $G^*_\eta \in J^{E^{M^*}}_{\xi^*}$. Then $G = \pi(G^*_\eta) \in J^{E^{M_i}}_{\pi(\xi^*)}$

Hence $\xi < \pi(\xi^*)$. QED (Fact)

But then:

$A\xi \longleftrightarrow \bigvee z\, R(z, \xi, p)$, where R is $\Sigma_1(M_i)$ by the same definition. Hence, letting $f = \sigma^*(\bar{f})$ we have $p = \sigma_i(\bar{p}) = \pi(f)(\alpha)$. Hence:

$$A\xi \longleftrightarrow \bigvee u \in M^* \bigvee z \in \pi(u)\, R(z, \xi, \pi(f)(\alpha))$$

$$\longleftrightarrow \quad " \quad \underbrace{P(\pi(u), \xi, \pi(f)(\alpha))}_{\Sigma_0}$$

$$\longleftrightarrow \bigvee u \in M^* \{\eta \mid P^*(u, \xi, f(\eta))\} \in F_\alpha$$

where P^* is $\Sigma_0(M^*)$ by the same definition. Hence $\bar{A}$ is $\Sigma_1(\bar{M}^*)$ in $\bar{f}, \bar{q}$ and A is $\Sigma_1(M^*)$ in f, q by the same definition.

QED (6)

(7) $\xi > \delta_i$

proof. Suppose not. Then $\xi = \delta_i$.

Let $\bar{A} \subset \bar{\tau}$ be $\Sigma_1(\bar{M}_i)$ in $\bar{p}$ and $A \subset \tau$ be $\Sigma_1(M_i)$ in $p = \sigma_i(\bar{p})$ by the same definition.

If $\eta_h = \hat{\eta}_{\delta_i}$, we get a contradiction by (6), since $M^* = M_{\delta_i} \| \hat{\eta}_{\delta_i}$, $\bar{M}^* = \bar{M}_{\delta_i} \| \hat{\bar{\eta}}_{\delta_i}$. If not, then $\eta_h < \hat{\eta}_{\delta_i}$, since $\tau < \kappa_h$. Hence $\sigma_{\delta_i}(\bar{M}^*) = M^*$, where $\bar{M}^* \in \bar{M}_{\delta_i} \| \hat{\bar{\eta}}_{\delta_i}$ and it follows from (6) that $\sigma_{\delta_i}(\bar{A}) = A$, where $\bar{A} \in \bar{M}_{\delta_i} \| \hat{\bar{\eta}}_{\delta_i}$. Contr! QED(7)

(8) $M^* = M_\xi$

proof.
If not, σ^* is Σ^*-preserving. Hence so is σ_i by (3). Hence $\omega\rho^1_{\bar{M}_i} \leq \bar{\tau}$ by (4). But $\pi_{\xi i}$ is Σ^*-preserving; hence $\omega\rho^1_{M^*} \leq \bar{\tau}$.
But then $\bar{\tau} + \bar{M}_\xi > \omega\eta_h$.

But $\bar{\tau} < \bar{\lambda}_{\delta_i}$ + $\bar{\lambda}_{\delta_i} < \bar{\lambda}_{\xi} \leq \omega\bar{\eta}_h$ is a limit cardinal in $\bar{M}_\xi$. Hence $\bar{\tau} + \bar{M}_\xi < \bar{\lambda}_{\delta_i} < \omega\eta_h$. Contr! QED(8)

Now let $\bar{A} \subset \bar{\tau}$ be $\Sigma_1(\bar{M}_i)$ in $\bar{p}$ and $A \subset \tau$ be $\Sigma_1(M_i)$ in $p = \sigma_i(\bar{p})$. By (6) $\bar{A}$ is $\Sigma_1(\bar{M}_\xi)$ in a $\bar{q}$ and A is $\Sigma_1(M_\xi)$ in $q = \sigma_\xi(\bar{q})$ by the same definition. Clearly, however, $\delta_\xi = \delta_i$ and $\hat{\eta}_\xi = \hat{\eta}_i$, since $\kappa = \mathrm{crit}(E^{M_\xi}_{On \wedge M})$ $= \hat{\kappa}_\xi$ and $\tau = \kappa^{+M_\xi} = \hat{\tau}_\xi$. Similarly $\hat{\bar{\eta}}_\xi = \hat{\bar{\eta}}_i$. By the minimality of i, we conclude that $\bar{A}$ is $\Sigma_1(\bar{M}_{\delta_i} \| \hat{\bar{\eta}}_i)$ in an $\bar{r}$ and A is $\Sigma_1(M_{\delta_i} \| \hat{\eta}_i)$ in $r = \sigma_{\delta_i}(\bar{r})$ by the same definition. Thus i is not a counterexample.

Contr! QED (Lemma 3.2)

We now prove Lemma 3.1. Suppose not. Let i be the least counter-example.

Set:

$\bar{F} = E^{\bar{M}_i}_{\bar{\nu}_i}$, $F = E^{M_i}_{\nu_i}$, $\bar{M}^* = \bar{M}_3 \| \bar{\eta}_i$, $M^* = M_3 \| \eta_i$. We derive a contradiction by showing that Lemma 3.1 holds at i — i.e.

$\langle \sigma^*, \sigma_i \upharpoonright \bar{\lambda}_i \rangle : \langle \bar{M}^*, \bar{F} \rangle \longrightarrow^* \langle M^*, F \rangle$.

Case 1 $\bar{\nu}_i \in \bar{M}_i$.

Then $\bar{F} \in \bar{M}_i$, $F = \sigma_i(\bar{F})$ and we can repeat the argument of (3),

<u>Case 2</u> Case 1 fails.

Then $\bar{\nu}_i = \mathrm{On} \cap \bar{M}_i$, $\bar{\kappa}_i = \hat{\bar{\kappa}}_i$, $\bar{\tau}_i = \hat{\bar{\tau}}_i$, $\delta_i = T(i+1)$, $\bar{\eta}_i = \hat{\bar{\eta}}_i$ and similarly for $\nu_i, \kappa_i, \ldots$ etc. Let $\bar{\alpha} < \bar{\lambda}_i$, $\alpha = \sigma_i(\bar{\alpha})$. Then $\bar{F}_{\bar{\alpha}} \subset J_{\bar{\tau}}^{E^{\bar{M}_i}}$ is $\Sigma_1(\bar{M}_i)$ in some $\bar{p}$ and $F_\alpha \subset J_\tau^{E^{M_i}}$ is $\Sigma_1(M_i)$ in $p = \sigma_i(\bar{p})$ by the same definition. By Lemma 3.2, $\bar{F}_\alpha$ is $\Sigma_1(\bar{M}^*)$ in some $\bar{q}$ and F_α is $\Sigma_1(M^*)$ in $q = \sigma(\bar{q})$ by the same definition.

QED (Lemma 3.1)

This proves Lemma 3.

We now apply our techiques to good iteration. If $\bar{\mathfrak{I}}$ is a good iteration with marking sequence $\langle \alpha_i | i \leq \Gamma$ and $\mathfrak{I} = \sigma(\bar{\mathfrak{I}})$, then $\mathfrak{I}$ is a good iteration with the same marking sequence. Using the above lemmas it is then easy to prove:

Cor 3.3 Let $\sigma : \bar{M} \to_{\Sigma^*} M$. Let $\bar{\mathcal{I}}$ be a good iteration of $\bar{M}$ (beyond $\bar{\nu} = \sigma^{-1}(\nu)$).

(a) If M has a good iteration strategy S (beyond ν) and $\bar{S}$ is the derived strategy, then $\bar{S}$ is a good iteration strategy for $\bar{M}$ (beyond $\bar{\nu}$). Moreover, if $\bar{\mathcal{I}}$ is an $\bar{S}$-iteration, then $\mathcal{I} = \sigma(\bar{\mathcal{I}})$ exists and is an S-iteration.

(b) If $\mathcal{I} = \sigma(\bar{\mathcal{I}})$ exists with copying maps $\langle \sigma_i \rangle$, then:

(i) $\sigma_i : \bar{M}_i \to_{\Sigma^*} M_i$

(ii) Let $i \in D$, $\xi = T(i+1)$. Set:

$\bar{M}^* = \bar{M}_\xi \| \bar{\eta}_i$, $M^* = M_\xi \| \eta_i$, $\sigma^* = \sigma_\xi \| \bar{M}^*$,

$\bar{F} = E^{\bar{M}_i}_{\bar{\nu}_i}$, $F = E^{M_i}_{\nu_i}$. Then

$$\langle \sigma^*, \sigma_i \restriction \bar{\lambda}_i \rangle : \langle \bar{M}^*, \bar{F} \rangle \xrightarrow{*} \langle M^*, F \rangle.$$

Cor 3.4 If $\sigma : \bar{M} \to_{\Sigma^*} M$ where M has an iteration strategy and $\bar{M}$ has the uniqueness property, then $\bar{M}$ is uniquely iterable.

Obvious analogues of these lemmas hold for smooth iteration strategies.

We can use Lemma 3.4 to prove another version of Lemma 3:

<u>Lemma 3.5</u> Let $\sigma : \bar{M} \longrightarrow_{\Sigma_h^{(n)}} M$ strongly $(h, n < \omega)$. Let $\bar{\mathcal{J}}$ be a normal iteration of $\bar{M}$. Let $\mathcal{J} = \sigma(\bar{\mathcal{J}})$ with copying maps $\langle \sigma_i \rangle$. Then

(a) If i is simple in $\bar{\mathcal{J}}$ and $\sup \rho^n_{\bar{M}} \leq \text{crit}(i)$ then $\sigma_i : \bar{M}_i \longrightarrow_{\Sigma_h^{(n)}} M_i$ strongly

(b) If i is non simple in $\bar{\mathcal{J}}$, then $\sigma_i : \bar{M} \longrightarrow_{\Sigma^*} M$.

The proof is by ind. on i using §3 Lemma 2.5 at successor i.

<u>Def</u> Let $\mathcal{J} = \langle\langle M_i\rangle, \ldots, \langle \pi_{ij}\rangle, T\rangle$ be an iteration of limit length. A branch b in T is said to be <u>well founded</u> iff M_b = the direct limit of $\langle M_i | i \in b\rangle, \langle \pi_{ij} | i \leq j \in$ is well founded.

<u>Def</u> The <u>uniqueness strategy</u> is defined by: $S(\mathcal{J}) \simeq$ the unique cofinal well founded branch.

<u>Def</u> $\mathcal{J}$ is called <u>uniquely</u> (<u>normally</u>) <u>iterable</u> iff the uniqueness strategy is a (normal) iteration strategy for ?

Clearly, $\mathcal{J}$ is uniquely iterable iff every (good) iteration of limit length can be continued in exactly one fashion.

We refer to uniquely iterable pre-mice as <u>uniqueness mice</u>. In the following (but not later!) we call them simply <u>mice</u>

We now prove the "Dodd-Jensen lemma" for mice, which says among other things that:

(a) N cannot be both a simple and non simple iterate of M;

(b) If N is a simple iterate of M, then the iteration map is unique.

<u>Lemma 4</u> Let M be a mouse & let N be an iterate of M with iteration map π. Let $\sigma : M \to_{\Sigma^*} N$. Then N is a simple iterate of M and $\pi(\xi) \leq \sigma(\xi)$ for $\xi \in M$.

proof.

We first show that N is not a non simple iterate of M. Suppose not. Define a relation R on mice by: NRM iff N is a non simple iterate of M. Then R is well founded. Let M be R-minimal for the property: There is a non-simple iterate N of M and there is $\sigma : M \to_{\Sigma^*} N$. Let N, σ have these properties & let $\mathcal{Y} = \langle\langle M_i \mid i \leq \theta\rangle, \ldots, T\rangle$ be a good iteration of M s.t. $M_\theta = N$ and θ is not simple in $\mathcal{Y}$. Let $\mathcal{Y}' = \sigma(\mathcal{Y})$ be the copy of $\mathcal{Y}$ onto N with copying maps $\langle \sigma_i \mid i \leq \theta\rangle$, ~~let~~ $\mathcal{Y}' = \langle\langle N_i \mid i \leq \theta\rangle, \ldots, T\rangle$. Then $\sigma_\theta : N \to_{\Sigma^*} N' = N_\theta$, where N' is a non simple iterate of N and $N \lhd M$, contradicting the $\lhd$-minimality of M. Contr!

We now show that $\pi(\xi) \leq \sigma(\xi)$ for $\xi \in M$. Suppose not. Define a relation R on the set of pairs $\langle M, \xi\rangle$ s.t. M is a mouse and $\xi \in M$ by:

$\langle N, \zeta\rangle R \langle M, \xi\rangle$ iff N is a simple iterate of M with an iteration map π s.t. $\xi < \pi(\xi)$. Clearly R is well founded. Let $\langle M, \xi \rangle$ be R-minimal for the property: There is a simple iterate N of M with iteration map π and a $\sigma : M \xrightarrow{}_{\Sigma^*} N$ s.t. $\sigma(\xi) < \pi(\xi)$. Let N, π, σ realize this property and let $\mathcal{J} = \langle\langle M_i \mid i \leq \theta \rangle, \ldots, T \rangle$ be the iteration from M to N with $\pi = \pi_{0\theta}$. Let $\mathcal{J}' = \sigma(\mathcal{J}) = \langle\langle N_i \rangle, \ldots, T \rangle$ with copying maps $\langle \sigma_i \mid i \leq \theta \rangle$. Set $N' = N_\theta$, $\sigma' = \sigma_\theta$. Then

$\langle N, \sigma(\xi) \rangle \, R \, \langle M, \xi \rangle$. But N' is a simple iterate of N with iteration map $\pi' = \pi'_{0\theta} : N \to N'$ and $\sigma' : N \xrightarrow{}_{\Sigma^*} N'$ s.t. $\sigma'(\sigma(\xi)) < \sigma'(\pi(\xi)) = \pi'(\sigma(\xi))$, contradicting the R-minimality of $\langle M, \xi \rangle$. Contr!

QED (Lemma 4)

The same proof yields:

<u>Lemma 4.1</u> Let M be uniquely smoothly iterable. Let N be a smooth iterate of M with iteration map π. Let $\sigma : M \xrightarrow[\Sigma^*]{} N$. Then N is a simple iterate of M and $\pi(\xi) \leq \sigma(\xi)$ for $\xi \in M$.

We also get:

<u>Lemma 4.2</u> Let M be iterable and also uniquely smoothly iterable. Let N be a smooth iterate of M. There is no $\sigma : M \xrightarrow[\Sigma^*]{} N \| \eta$ s.t. $\eta < ht(N)$.

proof. Suppose not.

We define N_i, η_i, σ_i $(i < \omega)$ s.t.

(a) N_{i+1} is a smooth iterate of $N_i \| \eta_i$

(b) $\eta_i < ht(N_i)$ and $\sigma_i : M \xrightarrow[\Sigma^*]{} N_i \| \eta_i$

(c) $N_0 = N$, $\eta_0 = \eta$, $\sigma_0 = \sigma$.

This is a contradiction, since then N_{i+1} is a non simple iterate of N_i and N_0 is iterable.

Let $\mathcal{I}$ be the smooth iteration from M to N. Let N_i, η_i, σ_i be given. Let $\mathcal{I}_i = \sigma_i(\mathcal{I})$ be the copy of $\mathcal{I}$ onto $N_i \| \eta_i$. (This exists since $N_i \| \eta_i$ is iterable and M is uniquely smoothly iterable.) This gives us a copying map $\tilde{\sigma} : N \longrightarrow_{\Sigma^*} N_{i+1}$. Set:

$\eta_{i+1} = \tilde{\sigma}(\eta_i)$; $\sigma_{i+1} = \tilde{\sigma}\sigma : M \longrightarrow_{\Sigma^*} N_{i+1} \| \eta_{i+1}$

QED (Lemma 4.2)