§ 6 Non-Unique Iterations

(by Martin Zeman)

In the following suppose M to be a premouse and $\mathcal{T}$ to be a direct normal iteration of M of limit length λ. Let b, b' be distinct branches cofinal in λ s.t. $M_b, M_{b'}$ both are well founded (hence can be taken as transitive), where

$$\langle M_b, \langle \pi_{ib} \rangle \rangle = \lim_{i <_T j \in b} (M_i, \pi_{ij})$$

and similarly for $M_{b'}, \pi_{ib'}$.

Clearly, if $\xi \in b \cap b'$, then $b \cap (\xi+1) = b' \cap (\xi + 1)$, so there is an $\alpha < \lambda$ s.t. $(b - \alpha) \cap (b' - \alpha) = \emptyset$.

We may assume w.l.o.g. that $b - \alpha$, $b' - \alpha$ have no truncation point.

We now recall some definitions

Def Let N be transitive + p.r. closed.

$\kappa \in N$ is strong in N iff there are arbitrary large $\lambda \in N$ s.t.

a) λ is a cardinal in N and $H^N_\lambda \in N$

b) There is an extender $F \in N$ at κ, λ s.t. $\pi : N \xrightarrow[F]{} N'$ exists and $H^N_\lambda = H^{N'}_\lambda$

Def Let N be as above, where $\langle N, A \rangle$ is amenable, $A \subset On \cap$ Let F be an extender at κ, λ.

F is coherent wrt A over N iff $\pi : N \xrightarrow[F]{} N'$ exists and $F(A \cap \kappa) = A \cap \lambda$

Def Let N, A be as above. $\kappa \in N$ is A-strong in N iff there are arbitrarily large $\lambda \in N$ s.t. some $F \in N$ is at κ, λ and is coherent wrt A.

<u>Note</u> Let $N = J_\alpha^A$ be acceptable, where A need not be a set of ordinals. We define the coherency of F wrt A over N as before, except that we require $\pi(A \cap |J_\kappa^A|) = A \cap |J_\lambda^A|$ instead of $F(A \cap \kappa) = A \cap \lambda$.

(<u>Note</u> Let $N = J_\alpha^A$ be acceptable s.t. there are arbitrarily large $\tau < \alpha$ s.t. τ is a cardinal in N. If κ is A-strong in N, then κ is strong in N.)

<u>Def</u> Let N be as above, where $N = H_\alpha^M$, $\alpha \in M$. α is <u>Woodin in</u> M iff for every $A \in M$, $A \subset \alpha$, there is $\kappa \in N$ which is A-strong in N.

(<u>Note</u> Assuming that N has a well-ordering in M, this is equivalent to the usual definition of Woodinness. In particular $\langle N, A \rangle \models ZFC$ whenever $A \in M$, $A \subset N$.)

. . . .

Returning to our structures $M, \mathcal{Y}$ we define:

$$\xi_0 := \min(b - (\alpha + 1))$$

$$\xi_{2i+1} = \min(b' - \xi_{2i})$$

$$\xi_{2i+2} := \min(b - \xi_{2i+1})$$

Then $\langle \xi_i \mid i < \omega \rangle$ is monotone and:

$$T(\xi_{2i+1}) < \xi_{2i} < \xi_{2i+1}$$

$$T(\xi_{2i+2}) < \xi_{2i+1} < \xi_{2i+2}$$. Hence:

(1) $\quad T(\xi_{i+1}) < \xi_i < \xi_{i+1}$

Clearly each ξ_i is a successor ordinal since it is a successor in its branch. Set:

<u>Def</u> $\quad \gamma_n := \xi_n - 1 \qquad \xi_n^* := T(\xi_n)$

By (1) we conclude:

(2) $\quad \kappa_{\gamma_{n+1}} < \lambda_{\xi^*_{n+1}} \leq \lambda_{\gamma_n} \leq \kappa_{\gamma_{n+2}}$

$(\lambda_{\gamma_n} \leq \kappa_{\gamma_{n+2}}$, since $(\gamma_n + 1)\ T\ (\gamma_{n+2} + 1)\quad)$

We have.

(3) $\quad \pi_{\xi_n^* \xi_n} : M_{\xi_n^*} \xrightarrow[E^{M_{\gamma_n}}_{\nu_{\gamma_n}}]{*} M_{\xi_n^*}$,

since ξ_n is not a truncation point.

(4) $\quad \sup_n \xi_n = \lambda$,

since otherwise $\sup_n \xi_n \in (b \cap b') - \alpha$.

(5) $\quad \tau_{\gamma_n} = \kappa_{\gamma_n}^{+M_{\gamma_n} \| \nu_{\gamma_n}}$ is a cardinal in $M_{\xi_n^*}$,

since otherwise ξ_n would be a truncation point.

But $\tau_{\gamma_n} < \lambda_{\xi_n^*}$ and $\lambda_{\xi_n^*}$ is a limit cardinal in M_η for $\eta > \lambda_{\xi_n^*}$. Hence:

(6) $\quad \tau_{\gamma_n}$ is a cardinal in M_η for $\eta \geq \xi_n^*$

<u>Def</u> $\quad \delta := \sup_n \kappa_{\gamma_n} = \sup_n \nu_{\gamma_n}$

$N := J^E_\delta = \bigcup_n J^{E^{M_n}}_{\kappa_{\gamma_n}} = \bigcup_n J^{E^{M_n}}_{\nu_{\gamma_n}}$

Lemma 1 There are arbitrarily large τ which are strong in N.

Proof

For all but finitely many $i<\omega$ we construct a sequence of extenders $\langle G_i^n \mid n<\omega \rangle$ s.t.

(a) $G_i^n \in N$ and N is extendible by G_i^n

(b) $\mathrm{cr}(G_i^n) = \kappa_i^*$, where $\kappa_{\gamma_{i+1}} \leq \kappa_i^* < \kappa_{\gamma_{i+2}}$

(c) $\mathrm{lh}(G_i^n) = \kappa_{\gamma_{i+n+2}}$

(d) $G_i^m = G_i^n \mid \kappa_{\gamma_{i+m+2}}$ for $m \leq n$

(e) G_i^n is coherent wrt E

This will prove the lemma.

We first specify the set ND on which $\langle G_i^n \rangle$ is not defined.

Def ND := the set of those $i<\omega$ for which

$$\mathrm{cr}(E_{top}^{M_{\xi_{i+2}}}) \in [\kappa_{\gamma_{i+1}}, \kappa_{\gamma_{i+2}})$$

Claim 1.1 ND is finite.

Proof Suppose not. Then there are $i, j \in$ ND s.t. $i<j$ and i,j are both odd or even.

Then $\xi_{i+2} <_T \xi_{j+2}$, $\pi_{\xi_{i+2}, \xi_{j+2}} : M_{\xi_{i+2}} \longrightarrow M_{j+2}$,

and $\mathrm{cr}(\pi_{\xi_{i+2}, \xi_{j+2}}) \geq \lambda_{\gamma_{i+2}} > \kappa_{\gamma_{i+2}} > \bar{\kappa} = \mathrm{cr}(E_{top}^{M_{\xi_{i+2}}})$

But then $\bar{\kappa} = \mathrm{cr}(E_{top}^{M_{\xi_{j+2}}})$, thus $\mathrm{cr}(E_{top}^{M_{\xi_{j+2}}}) < \kappa_{\gamma_{i+2}} < \kappa_{\gamma_{j+}}$

which means that $j \notin$ ND. Contradiction

$\boxtimes$ Claim 1.1.

Before proceeding further we prove some general lemmas on normal iterations $\mathcal{I} = \langle \langle M_i \rangle, \ldots T \rangle$.

<u>Claim 1.2.</u> Let $\bar{\kappa} = \mathrm{cr}(E^{M_\gamma}_{\nu_\gamma}) < \kappa < \lambda_\gamma$, $E^{M_\gamma}_{\nu_\gamma}|\kappa \notin M_\gamma$, κ card in J

a) $\nu_\gamma = \mathrm{lh}(M_\gamma)$

b) $\rho^1_{M_\gamma} \leq \kappa$

<u>Proof</u> a) is immediate

b) $\{\langle x, \alpha\rangle;\ x \subset \bar{\kappa}\ \&\ \alpha < \kappa\ \&\ \alpha \in E^{M_\gamma}_{\nu_\gamma}(x)\}$

is a $\Sigma_1(M_\gamma)$ subset of $J^{E^{M_\gamma}}_\kappa$.

☒ Claim 1.2

<u>Claim 1.3</u> Let $\gamma, \bar{\kappa}, \kappa$ be as above. Let $\beta+1 \leq_T \gamma$ s.t.

$\kappa < \lambda_\beta$. Then

a) $\mathrm{cr}(\pi_{\beta+1,\gamma}) > \kappa$ for $\beta+1 \neq \gamma$

b) $\pi_{\beta+1,\gamma}$ is total

<u>Proof</u> a) Let δ be s.t. $\beta+1 = T(\delta+1)$. Then

$\beta+1$ is the least ξ with $\lambda_\xi > \kappa_\delta = \mathrm{cr}(\pi_{\beta+1,\gamma})$.

Thus, $\kappa < \lambda_\beta \leq \lambda_\delta \leq \mathrm{cr}(\pi_{\beta+1,\gamma})$.

b) Suppose not. Let ξ be the last point of truncation.

Then $\pi_{\xi,\gamma} : M^*_\xi \xrightarrow[\Sigma^*]{} M_\gamma$ with $\mathrm{cr}(\pi_{\xi,\gamma}) > \kappa$, [1]

Thus, $\rho^1_{M^*_\xi} \leq \kappa$ since $E^{M^*_\xi}_{top}|\kappa = E^{M_\gamma}_{top}|\kappa$. But λ_β

$\in M^*_\xi$, thus it is collapsed in M_ξ; but λ_β must

be a cardinal in M_ξ. Contradiction

☒ Claim 1.3

[1] M^*_ξ is the truncate of M_ξ, i.e. $M^*_\xi = M_\xi \| \eta_\delta$, where $\xi = T(\delta$

Def Let γ, κ be as above. We define $\gamma^* = \gamma^*(\kappa) < \gamma$ as follows:
$\gamma^* \simeq$ that β s.t. $\beta+1 \leq_T \gamma$ and $\kappa_\beta < \kappa < \lambda_\beta$.
If there is not such β or γ, κ are not as in Claim 2 then γ^* is undefined.

Def Let γ, κ be as above. We define $\gamma^h = \gamma^h(\kappa)$ inductively by:
$\gamma^0 := \gamma$, $\gamma^{h+1} \simeq (\gamma^h)^*$
(Hence γ^h, κ are as in Claim 1.2 if γ^{h+1} is defined)
Let $n = n(\gamma, \kappa)$ be the largest h s.t. γ^h is defined.

Def Let γ, κ be as above. Set
$\delta = \delta(\kappa) :=$ the least δ s.t. $\kappa < \lambda_\delta$

Claim 1.4 Let γ, κ be as above. Let $n := n(\gamma, \kappa)$, $\bar\gamma := \gamma^n(\kappa)$.
Then one of the following holds:

a) $E^{M_{\bar\gamma}}_{\nu_{\bar\gamma}} | \kappa \in M_{\bar\gamma}$

b) $\delta = \delta(\kappa) \leq_T \bar\gamma$

Proof Suppose not. Thus both a), b) fail. In particular, $\delta(\kappa) \not\leq_T \bar\gamma$, thus $\delta(\kappa) < \bar\gamma$.

Let β be least s.t. $\beta+1 \leq_T \bar\gamma$ and $\kappa < \lambda_\beta$. Let $\xi := T(\beta+1)$.
Then $\kappa \leq \kappa_\beta$ since $\bar\gamma^*$ is not defined. But then $\kappa < \lambda_\xi$ since $\kappa_\beta < \lambda_\xi$. Thus $\delta(\kappa) \leq \xi$, in fact $\delta(\kappa) < \xi$ by the failure of b)
Then either $\xi = \varsigma + 1$
or ξ is limit, in which case we pick any $\varsigma \in [\delta, \xi)$ s.t. $\varsigma + 1 \leq_T \xi$

In both cases $\kappa < \lambda_\varsigma$, $\varsigma + 1 \leq_T \bar\gamma$ and $\varsigma < \beta$, which is a contradiction with the definition of β.

$\boxtimes$ Claim 1.4

Claim 1.5 Let $i \notin ND$. Set $\gamma := \gamma_{i+1}$, $\kappa := \kappa_{\gamma_{i+2}}$. Define $\gamma^h = \gamma^h(\kappa)$ as above. Let $n := n(\gamma, \kappa)$, $\bar\gamma := \gamma^n$. Then $E^{M_{\bar\gamma}}_{r_{\bar\gamma}} | \kappa \in M_{\bar\gamma}$.

It is enough to prove

Claim 1.6 We take the assumptions of Claim 1.5. Let γ^h be defined. Set $\kappa_{\gamma^h} := cr(E^{M_{\gamma^h}}_{r_{\gamma^h}})$. If $E^{M_{\gamma^h}}_{r_{\gamma^h}} | \kappa \notin M_{\gamma^h}$, then

a) γ^{h+1} is defined

b) $\kappa_{\gamma^{h+1}} > \kappa_{\gamma^h}$.

Proof By induction. Suppose γ^{h+1} is not defined. Then $\delta(\kappa) \leq_T \gamma^h$ by Claim 4.

Case 1 $\delta(\kappa) = \gamma^h$

But $\delta(\kappa) = \xi^*_{i+2}$. Thus $\kappa = \kappa_{\gamma_{i+2}} = cr(\pi_{\xi^*_{i+2}, \xi_{i+2}})$

and $\kappa > \kappa_{\gamma^h} > \kappa_{\gamma^{h-1}} > \cdots > \kappa_{\gamma^0} = \kappa_{\gamma_{i+1}}$ $(*)$

and $\kappa_{\gamma^h} = cr(E^{M_{\xi^*_{i+2}}}_{top})$. Thus, $\kappa_{\gamma^h} = cr(E^{M_{\xi_{i+2}}}_{top})$, thus $i \in ND$ by $(*)$. Contradiction.

☒ Case 1

Case 2 $\delta(\kappa) <_T \gamma^h$.

Let β be s.t. $\xi^*_{i+2} = \delta(\kappa) = T(\beta+1)$ and $\beta+1 \leq_T \gamma^h$.

By Claim 3, $\pi_{\beta+1, \gamma^h}$ is total and $cr(\pi_{\beta+1,\gamma^h}) > \lambda_\xi$.

Thus, $\kappa_{\gamma^h} = cr(E^{M_{\beta+1}}_{top})$ and $E^{M_{\beta+1}}_{top} | \kappa \notin M_{\beta+1}$.

Now $\kappa \leq \kappa_\beta$ since we assume that γ^{h+1} is not defined. Thus, $\kappa_{\gamma^h} = cr(E^{M_{\xi^*_{i+2}} \| \eta_\beta}_{top})$ and $E^{M_{\xi^*_{i+2}} \| \eta_\beta}_{top} | \kappa \notin M_{\xi^*_{i+2}} \| \eta_\beta$ i.e. $\kappa^{+}_{M_{\xi^*_{i+2}} \| \eta_\beta} \leq \kappa$. But $P(\kappa) \cap M_{\xi^*_{i+2}} = P(\kappa) \cap M_{\xi_{i+2}} =$

$= P(\kappa) \cap J^{E^{M_{\xi_{i+2}}}}_{\lambda_\beta} = P(\kappa) \cap J^{E^{M_\beta}}_{\lambda_\beta} \subset M_\beta$, thus $\eta_\beta = ht(M_{\xi^*_{i+2}})$, thus $\kappa_{\gamma^h} = cr(E^{M_{\xi^*_{i+2}}}_{top}) = cr(E^{M_{\xi_{i+2}}}_{top})$. Thus $i \in ND$ as in Case 1. Contradiction.

☒ Case 2

☒ a)

Proof of b) Set $\beta = \gamma^{h+1}$, $\xi := T(\beta+1)$. Then again by Claim we have $\pi_{\beta+1,\gamma^h}$ is total + $cr(\pi_{\beta+1,\gamma^h}) > \lambda_\beta > \kappa > \kappa_{\gamma^h}$. Thus $\kappa_{\gamma^h} = cr(E_{top}^{M_{\beta+1}})$, thus $\kappa_{\gamma^h} \in rng(\pi_{\xi,\beta+1})$. But $\kappa_{\gamma^h} < \lambda_\beta = \pi_{\xi,\beta+1}(\kappa_\beta)$, where $\kappa_\beta = cr(\pi_{\xi,\beta+1})$.
Thus, $\kappa_{\gamma^h} < \kappa_\beta = \kappa_{\gamma^{h+1}}$

⊠ b)

⊠ Claim 1.6

⊠ Claim 1.5

We now define the sequence $\langle G_i^n ; i\in\omega\rangle$ for $i \notin ND$.

Def Let $i \notin ND$, $\gamma := \gamma_{i+1}$, $\kappa := \kappa_{\gamma_{i+2}}$, $n := n(\gamma,\kappa)$ and $\bar\gamma := \gamma^n$. Set

$G_i^0 = E_{\nu_{\bar\gamma}}^{M_{\bar\gamma}} | \kappa$ $\qquad \kappa_i^* := cr(G_i^0)$

Claim 1.7 $G_i^0 \in M_{\xi_{i+2}^*}$.
In fact, $G_i^0 \in J_{\lambda_{\xi_{i+2}^*}}^{E^{M_{\xi_{i+2}^*}}}$ and $J_{\lambda_{\xi_{i+2}^*}}^{E^{M_{\xi_{i+2}^*}}} \models (G_i^0$ is ω-complete$)$

Proof We know $G_i^0 \in M_{\bar\gamma}$ and $\xi_{i+2}^* = \delta(\kappa) \leq \bar\gamma$ (since $\kappa < \lambda_{\bar\gamma}$)

Case 1 $\xi_{i+2}^* = \delta(\kappa) = \bar\gamma$

Then $G_i^0 \in M_{\xi_{i+2}^*}$ follows immediately. Now, we know that G_i^0 is codable by a subset of κ, $\lambda_{\xi_{i+2}^*}$ is a limit cardinal in $M_{\xi_{i+2}}$ and $\mathcal{P}(\kappa) \cap M_{\xi_{i+2}^*} = \mathcal{P}(\kappa) \cap M_{\xi_{i+2}}$. Thus $\kappa^{+M_{\xi_{i+2}^*}} = \kappa^{+M_{\xi_{i+2}}} < \lambda_{\xi_{i+2}^*}$, so $G_i^0 \in J_{\lambda_{\xi_{i+2}^*}}^{E^{M_{\xi_{i+2}^*}}}$.

Case 2 $\xi_{i+2}^* = \delta(\kappa) < \bar\gamma$.

Thus, $\lambda_{\xi_{i+2}^*}$ is a cardinal in $M_{\bar\gamma}$ and $G_i^0 \in M_{\bar\gamma}$. By acceptability, $G_i^0 \in J_{\lambda_{\xi_{i+2}^*}}^{E^{M_{\xi_{i+2}^*}}}$.

As to ω-completeness, we know that $J_{\nu_{\bar\gamma}}^{E^{M_{\bar\gamma}}}$ is extendable by $E_{\nu_{\bar\gamma}}^{M_{\bar\gamma}}$, so $J_{\lambda_{\xi_{i+2}^*}}^{E^{M_{\xi_{i+2}^*}}}$ is, since $\lambda_{\xi_{i+2}^*} < \nu_{\bar\gamma}$ and $\xi_{i+2}^* \leq \bar\gamma$.

But then $J_{\lambda_{\xi^*_{i+2}}}^{E^{M_{\xi^*_{i+2}}}}$ is expandible by G_i^0, which is an initial segment of $E^{M_{\bar\gamma}}_{\nu_F}$. Since $G_i^0 \in J_{\lambda_{\xi^*_{i+2}}}^{E^{M_{\xi^*_{i+2}}}}$ and $J_{\lambda_{\xi^*_{i+2}}}^{E^{M_{\xi^*_{i+2}}}} \models ZFC$, the rest of the claim follows

$\boxtimes$ Claim 1.7

<u>Claim 1.8</u> G_i^0 has the properties (a) – (e)

<u>Proof</u> (a) We know $E^{M_{\xi^*_{i+2}}} \restriction \lambda_{\xi^*_{i+2}} = E \restriction \lambda_{\xi^*_{i+2}}$, so $G_i^0 \in N$.

Since $\lambda_{\xi^*_{i+2}}$ is a cardinal in N, we get $N \models (G_i^0$ is ω-complete$)$, so N is extendible by G_i^0.

(b) – (d) are trivial.

Proof of (e): Let $\pi : J_{\tau_i^*}^{E^{M_{\bar\gamma}}} \xrightarrow[G_i^0]{} N'$, $\tilde\pi : J_{\tau_i^*}^{E^{M_{\bar\gamma}}} \xrightarrow[E^{M_{\bar\gamma}}_{\nu_{\bar\gamma}}]{} N''$,

where $\tau_i^* = (\kappa_i^*)^{+M_{\bar\gamma}\|\nu_{\bar\gamma}}$. Then

$\pi(a) \cap J_\kappa^{E^{M_{\bar\gamma}}} = \tilde\pi(a) \cap J_\kappa^{E^{M_{\bar\gamma}}}$ for any $a \in J_{\tau_i^*}^{E^{M_{\bar\gamma}}}$, $a \in J_{\kappa_i^*}^{E^{M_{\bar\gamma}}}$

by a standard argument. In particular,

$\pi(E^{M_{\bar\gamma}} \restriction \kappa_i^*) \cap J_\kappa^{E^{M_{\bar\gamma}}} = \tilde\pi(E^{M_{\bar\gamma}} \restriction \kappa_i^*) \cap J_\kappa^{E^{M_{\bar\gamma}}} = E^{M_{\bar\gamma}} \restriction \kappa$.

The rest follows from the fact that $E^{M_{\bar\gamma}} \restriction \kappa = E \restriction \kappa$.

$\boxtimes$ Claim 1.8

We now define G_i^n for $n > 0$. Suppose G_i^n is defined.

Then $G_i^n \in M_{\xi^*_{i+n+2}}$

<u>Def</u> $\tilde G_i^n := \pi_{\xi^*_{i+n+2}, \xi_{i+n+2}}(G_i^n)$

$G_i^{n+1} := \tilde G_i^n \mid \kappa_{\gamma_{i+n+3}}$

<u>Claim 1.9</u> G_i^{n+1} has the properties (a) – (e).

Moreover, $G_i^{n+1} \in J_{\lambda_{\xi^*_{i+n+3}}}^{E^{M_{\xi^*_{i+n+3}}}}$ and is ω-complete inside it.

<u>Proof</u> (a) we know $G_i^{n+1} \in M_{\xi_{i+n+2}}$ and is codable there by a subset of $\kappa_{i+n+3} < \lambda_{\xi^*_{i+n+3}} \leq \lambda_{\gamma_{i+n+2}}$. Both the latter are cardinals in $M_{\xi_{i+n+2}}$, so $G_i^{n+1} \in J^{E^{M_{\xi_{i+n+2}}}}_{\lambda_{\xi^*_{i+n+3}}}$.
Then $G_i^{n+1} \in N$ follows from: $E^{M_{\xi_{i+n+2}}} \upharpoonright \lambda_{\gamma_{i+n+2}} = E \upharpoonright \lambda_{\gamma_{i+n+2}}$.

The second part of Claim 9 follows from the fact, that

i) $J^{E^{M_{\xi_{i+n+2}}}}_{\lambda'_{\xi^*_{i+n+2}}} \models (\tilde{G}_i^n$ is ω-complete$)$,

where $\lambda'_{\xi^*_{i+n+2}} = \pi_{\xi^*_{i+n+2}, \xi_{i+n+2}}(\lambda_{\xi^*_{i+n+2}})$

ii) $E^{M_{\xi_{i+n+2}}} \upharpoonright \lambda_{\xi^*_{i+n+3}} = E^{M_{\xi^*_{i+n+3}}} \upharpoonright \lambda_{\xi^*_{i+n+3}}$

and from acceptability.
Using all of this we get $N \models (G_i^{n+1}$ is ω-complete$)$, since $E^{M_{\xi_{i+n+2}}} \upharpoonright \lambda_{\xi^*_{i+n+3}} = E \upharpoonright \lambda_{\xi^*_{i+n+3}}$ and $\lambda_{\xi^*_{i+n+3}}$ is a cardinal in N, so N is extendible by G_i^{n+1}.

(e) We recall the following two facts, which we will use in the proof:

$$(7) \begin{cases} E \upharpoonright \kappa_{\gamma_{i+n+2}} = E^{M_{\xi^*_{i+n+2}}} \upharpoonright \kappa_{\gamma_{i+n+2}} \\ E \upharpoonright \lambda_{\gamma_{i+n+2}} = E^{M_{\xi_{i+n+2}}} \upharpoonright \lambda_{\gamma_{i+n+2}} \end{cases}$$

By the induction hypothesis, if π_i^n is defined by
$\pi_i^n : J^E_{\tau_i^*} \xrightarrow[G_i^n]{} N_i^n$, then $E^{N_i^n} \upharpoonright \kappa_{\gamma_{i+n+2}} = E \upharpoonright \kappa_{\gamma_{i+n+2}}$

Let $\tilde{\pi}_i^n : J^E_{\tau_i^*} \xrightarrow[\tilde{G}_i^n]{} \tilde{N}_i^n$. Then $E^{N_i^n} \upharpoonright \lambda_{\gamma_{i+n+2}} = E \upharpoonright \lambda_{\gamma_{i+n+2}}$

Thus, as in Claim 8, $E^{N_i^{n+1}} \upharpoonright \kappa_{\gamma_{i+n+3}} = E^{\tilde{N}_i^n} \upharpoonright \kappa_{\gamma_{i+n+3}} = E \upharpoonright \kappa_{\gamma_{i+n}}$
and $\pi_i^{n+1}(a) \cap J^E_{\kappa_{\gamma_{i+n+3}}} = \tilde{\pi}_i^n(a) \cap J^E_{\kappa_{\gamma_{i+n+3}}}$

for $a \in J^E_{\tau_i^*}$, $a \subset J^E_{\kappa_i^*}$.

Then:

$$\pi_i^{n+1}(E\restriction\kappa_i^*) \cap J^E_{\kappa_{\gamma_{i+n+3}}} = \tilde{\pi}_i^{n}(E\restriction\kappa_i^*) \cap J^E_{\kappa_{\gamma_{i+n+3}}} =$$

$$= \tilde{\pi}_i^{n}(E\restriction\kappa_i^*) \cap J^E_{\lambda_{\gamma_{i+n+2}}} \cap J^E_{\kappa_{\gamma_{i+n+3}}} =$$

$$= \pi_{\xi^*_{i+n+2},\,\xi_{i+n+2}}\left(\pi_i^{n}(E\restriction\kappa_i^*) \cap J^E_{\kappa_{\gamma_{i+n+2}}}\right) \cap J^E_{\kappa_{\gamma_{i+n+3}}} =$$

$$= \pi_{\xi^*_{i+n+2},\,\xi_{i+n+2}}\left(E\restriction\kappa_{\gamma_{i+n+2}}\right) \cap J^E_{\kappa_{\gamma_{i+n+3}}} = E\restriction\kappa_{\gamma_{i+n+3}}$$

We used the identities (¥) to derive the third and the last equality.

$\boxtimes$ Claim 1.9

This proves Lemma 1.

$\boxtimes$ Lemma 1

In what follows we shall make extensive use of the sequences $\langle G_i^n ; n\in\omega\rangle$, $i\notin ND$ defined above.

<u>Def</u> $\alpha := \min(ht(M_b), ht(M_{b'}))$

$Q := J^E_\alpha$

(Note: $E^N \subset N = J^{E^N}_\delta$, hence $E^Q_{\omega\nu} = \emptyset$ for $\delta \le \nu < \alpha$)

<u>Lemma 2</u> If $\delta<\alpha$ ~~$\alpha<\delta$~~, then δ is Woodin in Q.

<u>Proof</u> $\delta = \sup\{\kappa_{\gamma_i};\ i\in\omega\}$, thus δ is a cardinal in Q and $N = H^Q_\delta$ by acceptability.

We prove, that if $A\in Q$, $A\subset\delta$, then all but finitely many κ_i^* are A-strong in N.

Let $A\in Q$, $A\subset\delta$. Pick i big enough so that $A\in rng(\pi_{\eta,b})\cap rng(\pi_{\eta',b'})$ for $\eta\ge\xi^*_{i+2}$, $\eta'\ge\xi^*_{i+1}$.

For $\eta\in b\cup b'$ let A_η be s.t. $\pi_{\eta,b}(A_\eta)=A$, resp $\pi_{\eta,b'}(A_\eta)=A$.

Claim 2.1 G_i^0 is coherent wrt A.

Proof We remind the definition of G_i^0: $G_i^0 = E_{r_{\bar\gamma}}^{M_{\bar\gamma}} | \kappa_{\gamma_{i+2}}$, where $\bar\gamma = \gamma^n$ and $n = n(\gamma_{i+1}, \kappa_{\gamma_{i+2}})$. We prove the claim by induction on h, $0 \le h \le n$, for $E_{r_{\gamma^h}}^{M_{\gamma^h}} | \kappa_{\gamma_{i+2}}$.

Case 1 $h = 0$.

Then $\gamma^0 = \gamma_{i+1}$, thus $\left(E_{r_{\gamma^0}}^{M_{\gamma^0}} | \kappa_{\gamma_{i+2}}\right)(A \cap \kappa_{\gamma^0}) =$

$$= \left(E_{r_{\gamma_{i+1}}}^{M_{\gamma_{i+1}}} | \kappa_{\gamma_{i+2}}\right)(A \cap \kappa_{\gamma_{i+1}}) = \pi_{\xi_{i+1}^*, \xi_{i+1}}(A_{\xi_{i+1}^*} \cap \kappa_{\gamma_{i+1}}) \cap \kappa_\gamma$$

$$= (A_{\xi_{i+1}} \cap \lambda_{\gamma_{i+1}}) \cap \kappa_{\gamma_{i+2}} = (A \cap \lambda_{\gamma_{i+1}}) \cap \kappa_{\gamma_{i+2}} = A \cap \kappa_{\gamma_{i+2}}$$

$\boxtimes$ Case 1

Case 2 Suppose that the claim holds for $E_{r_{\gamma^h}}^{M_{\gamma^h}} | \kappa_{\gamma_{i+2}}$ and that γ^{h+1} is defined. Set $\beta := \gamma^{h+1}$, $\xi := T(\beta+1)$, $M_\xi^* := M_\xi \| \eta_\beta$. Then $\pi_{\beta+1, \gamma^h}$ is total and $cr(\pi_{\beta+1,\gamma^h}) > \lambda_\beta$ by Claim 3, so we have:

$$(8) \quad E_{top}^{M_{\beta+1}}(A \cap \kappa_{\gamma^h}) \cap \kappa_{\gamma_{i+2}} = E_{top}^{M_{\gamma^h}}(A \cap \kappa_{\gamma^h}) \cap \kappa_{\gamma_{i+2}} = A \cap \kappa_{\gamma_{i+2}}$$

Moreover, $\pi_{\xi, \beta+1} : M_\xi^* \xrightarrow[E_{r_\beta}^{M_\beta}]{*} M_{\beta+1}$ and $\kappa_\beta = cr(E_{r_\beta}^{M_\beta}) > \kappa_{\gamma^h}$

thus

$$(9) \quad E_{top}^{M_\xi^*}(A \cap \kappa_{\gamma^h}) \cap \kappa_\beta = E_{top}^{M_{\beta+1}}(A \cap \kappa_{\gamma^h}) \cap \kappa_\beta = A \cap \kappa_\beta .$$

Putting all of this together we get

$$\left(E_{r_\beta}^{M_\beta} | \kappa_{\gamma_{i+2}}\right)(A \cap \kappa_\beta) = \pi_{\xi, \beta+1}(A \cap \kappa_\beta) \cap \kappa_{\gamma_{i+2}} =$$

$$= \pi_{\xi, \beta+1}\left(E_{top}^{M_\xi^*}(A \cap \kappa_{\gamma^h}) \cap \kappa_\beta\right) \cap \kappa_{\gamma_{i+2}} =$$

$$= E_{top}^{M_{\beta+1}}(A \cap \kappa_{\gamma^h}) \cap \lambda_\beta \cap \kappa_{\gamma_{i+2}} = A \cap \kappa_{\gamma_{i+2}} .$$

We used (8) to derive the second and (9) to derive the third equality. This proves Claim 2.1. $\boxtimes$ Case 2

Claim 2.2 G_i^n is coherent wrt A over N

Proof By induction on n.

$n = 0$ This was done in Claim 2.1.

$n = m+1$ We know that $G_i^m \in M_{\xi^*_{i+n+2}}$ and

$$(10)\ \begin{cases} A \cap \kappa_{\gamma_{i+m+2}} = A_{\xi^*_{i+n+2}} \cap \kappa_{\gamma_{i+n+2}} \\ A \cap \lambda_{\gamma_{i+n+2}} = A_{\xi_{i+n+2}} \cap \lambda_{\gamma_{i+n+2}} \end{cases}$$

Then $G_i^{m+1}(A \cap \kappa_i^*) = \tilde{G}_i^n(A \cap \kappa_i^*) \cap \kappa_{\gamma_{i+m+3}} =$

$= \pi_{\xi^*_{i+m+2}\,\xi_{i+n+2}}(G_i^n(A \cap \kappa_i^*)) \cap \kappa_{\gamma_{i+m+3}} =$

$= \pi_{\xi^*_{i+n+2}\,\xi_{i+m+2}}(A \cap \kappa_{\gamma_{i+m+2}}) \cap \kappa_{\gamma_{i+n+3}} =$

$= \pi_{\xi^*_{i+n+2}\,\xi_{i+n+2}}(A_{\xi^*_{i+m+2}} \cap \kappa_{\gamma_{i+m+2}}) \cap \kappa_{\gamma_{i+n+3}} =$

$= A_{\xi_{i+m+2}} \cap \lambda_{\gamma_{i+n+2}} \cap \kappa_{\gamma_{i+n+3}} = A \cap \kappa_{\gamma_{i+n+3}}$.

We used the identities (10) to derive the fifth and the last equalities

$\boxtimes$ Claim 2.2.

This proves Lemma 2

$\boxtimes$ Lemma 2

Def Let M be a pm. We define $\delta(M)$ by

$\delta(M) := \text{lub.}\{\gamma ; E_\gamma^M \neq 0\}$

(Hence, if M is active, i.e. $E^M_{On \cap M} \neq 0$, then

$\delta(M) = On_M + 1$)

Fact Let $\mathcal{I} = \langle\langle M_i \rangle, \quad T\rangle$ be a ~~good~~ smooth iteration.

Let $\xi := T(i+1)$. Then $\kappa_i < \delta(M_\xi \| \gamma_i)$.

(This holds since $\kappa_i < \lambda_\xi$ and $E^{M_\xi}_{\gamma_\xi} \neq 0$)

Def A pm M is called basic (or "one-small") iff there is no $\mu \leq ht(M)$

s.t. for some $\delta < \mu$ the following holds:

(*) $\quad E^M_\mu \neq \emptyset$ and $J^E_\mu \models \delta$ is Woodin. *)

Clearly any good iterate of a basic mouse is basic.

Lemma 3 Let $M, \mathcal{I}, N, Q, b, b'$ be as above, where M is basic.

Then $Q = M_b$ or $Q = M_{b'}$.

Moreover, if $ht(M_b) = \alpha = ht(M_{b'})$ then

$M_b = Q = M_{b'}$.

Proof We prove the first part of the lemma; the second part follows easily from this proof.

So suppose $M_b \neq Q \neq M_{b'}$ and w.l.o.g. $\alpha = ht(M_b)$.

Then there is a γ s.t. $\delta \leq \gamma \leq \alpha$ and $E^{M_b}_\gamma \neq 0$. Since δ is a limit cardinal in M_b, $\delta < \gamma$. Pick γ to be the least possible. Then $J^{E^M}_\mu \models \delta$ is Woodin, Contr! QED (Lemma 3)

*) It follows from (*) that $J^{E^M}_\mu \models V\delta \leq \kappa$ δ is Woodin, where $\kappa = crit(E^M_\mu)$. To see this, let $\tau = \kappa^{+J^{E^M}_\mu}$ and $\pi : J^{E^M}_\tau \xrightarrow[E_\mu]{} J^{E^M}_\mu$. Then $\delta \leq \pi(\tau)$. But then $J^{E^M}_\mu \models V\delta < \lambda$ δ is Woodin for $\lambda = \pi(\tau)$ and by the same argument: $J^{E^M}_\mu \models V\delta < \kappa$ δ is Woodin.

We now prove an analogue of lemma 2 which is a generalization in the sense that we allow sets A which are not elements of Q but are in certain sense definable over Q.

<u>Lemma 4</u> Let M be basic. Let $A \subset \delta$ be $\Sigma_0^{(n)}(Q)$ where $\rho_Q^n \geq \delta$. Then κ_i^* is A-strong in N for sufficiently large $i \in \omega$.

<u>Proof</u> Then $\langle N, A \rangle$ is amenable.

By lemma 3, we can w.l.o.g. assume $M_b = Q$. Let A be $\Sigma_0^{(n)}(M_b)$ in p. If $lt(M_{b'}) > \alpha = lt(M_b)$, then $A \in M_{b'}$. Otherwise, again by lemma 3, $M_{b'} = Q = M_b$ so A is $\Sigma_0^{(n)}(M_{b'})$ in p. Thus, let $p' \in M_{b'}$ be s.t.

$$p' := A \text{ if } A \in M_{b'} \quad ; \quad A \text{ is } \Sigma_0^{(n)}(M_{b'}) \text{ in } p'$$

Now pick $i \notin ND$ big enough s.t.

$$p \in \operatorname{rng}(\pi_{\xi_{i+2}^*, b}) \text{ and } p' \in \operatorname{rng}(\pi_{\xi_{i+1}^*, b'})$$

<u>Claim</u> G_i^n is coherent wrt A.

<u>Proof</u> The proof is like the proof of lemma 2. As before we define the sets A_η, which now need not be elements of M_η, but are $\Sigma_0^{(n)}(M_\eta)$ in p_η resp $p_{\eta'}$,

where $\pi_{\eta,b}(p_\eta) = p$, resp $\pi_{\eta,b'}(p'_\eta) = p'$. As before we observe that

$$A_{\xi^*_{i+2}} \cap \kappa_{\gamma_{i+2}} = A \cap \kappa_{\gamma_{i+2}}$$
$$A_{\xi_{i+2}} \cap \lambda_{\gamma_{i+2}} = A \cap \lambda_{\gamma_{i+2}} .$$

The amendments to the proof of lemma 2 are then straightforwa

$\boxtimes$ lemma 4

Recapitulating:

<u>Corollary 4.1</u> Let M be basic. Let A be $\Sigma_0^{(n)}(Q)$ in p, where $\omega\rho_Q^n \geq \delta$. Let $i \notin ND$ be s.t. for all $\eta, \eta' > \xi^*_{i+1}$ we have $p \in \mathrm{rng}(\pi_{\eta,b}) \cap \mathrm{rng}(\pi_{\eta',b'})$ and $A \in \mathrm{rng}(\pi_{\eta,b'})$ if $M_{b'} \neq Q$. Then G_i^n is coherent wrt A for $n < \omega$.

$\boxtimes$ Cor 4.1.

Using this we prove:

<u>Corollary 4.2</u> Let M be basic and let F be a $\Sigma_1^{(n)}(Q)$ partial map to δ. Then $F''\kappa_{\gamma_i} \subset \kappa_{\gamma_i}$ for sufficiently large $i < \omega$.

<u>Proof</u> Pick i big enough s.t. $i \notin ND$ and for all $\eta, \eta' > \xi^*_{i+1}$:

- $p \in \mathrm{rng}(\pi_{\eta,b}) \cap \mathrm{rng}(\pi_{\eta',b'})$ if $M_b = Q = M_{b'}$
- $p \in \mathrm{rng}(\pi_{\eta,b})$ and $F \in \mathrm{rng}(\pi_{\eta',b'})$,

where F is $\Sigma_1^{(n)}(Q)$ in p.

<u>Claim</u> $F''\kappa_{\gamma_{i+1}} \subset \kappa_i^*$

<u>Proof</u> Suppose not. Let $\xi < \kappa_{\gamma_{i+1}}$ be s.t. $\nu = F(\xi) > \kappa_i^*$. Let $H(z^n, u, x, y)$ be a $\Sigma_0^{(n)}(Q)$ relation s.t. $y = F(x) \longleftrightarrow (\exists z^n) H(z^n, p, x, y)$. Let β be the least s.t. $(\exists z^n, \zeta \in S_\beta^E) H(z^n, p, \xi, \zeta)$. Then $\beta \in \mathrm{rng}(\pi_{\eta,b})$ for $\eta > \xi^*_{i+1}$ and $\beta \in \mathrm{rng}(\pi_{\eta',b'})$ for $\eta' > \xi^*_{i+1}$, if $M_{b'} = Q$

Then $\{\nu\} = \{\zeta < \delta \mid (\exists z^n \in X) H(z^n, p, \xi, \zeta)\}$ is $\Sigma_0^{(n)}(Q)$ in $\langle X, p, \xi \rangle$ and the conditions of Cor. 3.1. are satisfied, thus $\{\nu\}$ is coherent wrt. σ_i^n for $n > \omega$. Pick n big enough s.t. $\nu < \kappa_{\gamma_{i+n+2}}$. Then

$\phi = \{\nu\} \cap \kappa_i^* = \sigma_i^n(\{\nu\} \cap \kappa_i^*) = \{\nu\} \cap \kappa_{\gamma_{i+n+2}} = \{\nu\}$

Contradiction!

☒ Claim

Thus, let $F_{\xi^*_{i+1}}$ be a $\Sigma_1^{(n)}(M_{\xi^*_{i+1}})$ map in $p_{\xi^*_{i+1}}$ by the same definition, where $\pi_{\xi^*_{i+1}, b'}(p_{\xi^*_{i+1}}) = p$ or $F_{\xi^*_{i+1}} \in M_{\xi^*_{i+1}}$

and $\pi_{\xi^*_{i+1}, b'}(F_{\xi^*_{i+1}}) = F$, if $F \in M_{b'}$. Then $\nu = F(\xi) \in \operatorname{rng}(\pi_{\xi^*_{i+1}, b'})$

and $\nu < \kappa_i^* < \kappa_{\gamma_{i+2}} < \lambda_{\gamma_{i+1}} = \pi_{\xi^*_{i+1}, \xi_{i+1}}(\kappa_{\gamma_{i+1}}) \leq \pi_{\xi^*_{i+1}, b'}(\kappa_{\gamma_{i+1}})$.

But $\kappa_{\gamma_{i+1}} = \operatorname{cr}(\pi_{\xi^*_{i+1}, b'})$. Thus, $\nu < \kappa_{\gamma_{i+1}}$.

☒ Cor 4.2.

Recall that a <u>mouse</u> is an iterable premouse.
We then get:

<u>Lemma 5</u> Let each M_i be iterable. Then:

a) $\rho^\omega_Q \geq \delta$

b) δ is Σ^*-regular in Q (i.e. if f is a Σ^*-partial map to δ, then $\sup(f''\gamma) < \delta$ for $\gamma < \delta$)

<u>Proof</u> It is enough to prove a). b) then follows from Cor. 4.2.

So suppose a) fails. Let n be s.t. $\rho^n_Q \geq \delta$ and $\rho^{n+1}_Q < \delta$. Pick $p \in P^n_Q$ and A which is $\Sigma_1^{(n)}(Q)$ in p s.t. $A \cap \rho^{n+1}_Q \notin Q$. Let $h = h_{Q^{n,p}}$. Then h is $\Sigma_1^{(n)}(Q)$, so there is $i \in \omega$ s.t. $\rho^{n+1}_Q < \kappa_{\gamma_i}$ and $h(\kappa_{\gamma_i}) \cap \delta = \kappa_{\gamma_i}$. Let $X = h(\kappa_{\gamma_i})$ and $\bar\sigma : \bar{Q}' \xleftrightarrow{\sim} X$, $\bar{Q}'$ transitive. Then $\bar\sigma : \bar{Q}' \xrightarrow[\Sigma_1]{} Q^{n,p}$.

Using downward extensions of embeddings lemma lift $\bar\sigma$

to $\sigma : Q' \xrightarrow[\Sigma_1^{(n)}]{} Q$ where $\sigma(p') = p$, $p' \in R^n_{Q'}$. Then $\kappa_{\gamma_i} \in Q'$ since otherwise $Q' = J^E_{\kappa_{\gamma_i}}$ and A is $\underline{\Sigma}_1^{(n)}(Q')$, thus $A \in Q$, a contradiction. But then $\kappa_{\gamma_i} = cr(\sigma)$ and $\sigma(\kappa_{\gamma_i}) \geq \delta$, so $Q' = J^{E'}_{\alpha'}$ where $E' = E \restriction \kappa_{\gamma_i} \in Q$, $\alpha' \leq \alpha$. But $\sup\{\nu;\ E^Q_\nu \neq 0\} = \delta$, so pick $\nu > \kappa_{\gamma_i}$ s.t. $E^Q_\nu \neq 0$ and iterate J^E_ν by E^Q_ν long enough s.t. the height of the resulting structure $\tilde{Q}$ is greater than α'. $^{*)}$ Since A is $\underline{\Sigma}_1^{(n)}(Q')$, $A \in \tilde{Q}$. Since ν is a cardinal on $\tilde{Q}$, $A \in J^{E^{\tilde{Q}}}_\nu \subset Q$. Contradiction.

$^{*)}$ This possible since $J^E_\nu = J^{E^{M_j}}_\nu$ for some $j \in b$.

$\boxtimes$ Lemma 5

Thus, by the previous lemma, if $\mathcal{I}$ is a ~~good~~ smooth iteration of a basic mouse above the ultimate projectum, then $\mathcal{I}$ has at most one cofinal branch. More generally:

<u>Corollary 6</u> Let M be a basic mouse s.t. $u\rho^\omega_M \leq \nu$ for a ν with $E^M_\nu \neq \emptyset$. Let $\mathcal{I}$ be a normal iteration of M of limit length. Then $\mathcal{I}$ has at most one cofinal well-founded branch.

<u>Proof</u> Let $\mathcal{I}$ be a counterexample of minimal length. Then $\mathcal{I}$ is an iteration by the unique strategy. Hence each M_i is a mouse. Let b, b' be distinct branches with $M_b = Q$. Then $\omega\rho^\omega_Q \geq$,

<u>Case 1</u> There is a truncation point $i+1 \in b$. Suppose it is the last one, i.e. $\pi_{i+1,b}$ is total. Let $\kappa_i = cr(E^{M_i}_{\gamma_i})$. Then
$u\rho^\omega_Q = u\rho^\omega_{M_\xi \| \gamma_i} \leq \kappa_i < \delta$. Contradiction.

<u>Case 2</u> π_{MQ} is total. Since π_{MQ} is a Σ^*-map, $u\rho^\omega_Q \leq \pi(\nu) < \delta$, since $E^Q_{\pi(\nu)} \neq \emptyset$. Contradiction.

$\boxtimes$ Cor 6.

<u>Corollary 6.1</u> If M is a basic mouse s.t. $u\rho^\omega_M \leq \nu$ for a ν with $E^M_\nu \neq \emptyset$, then M is uniquely smoothly iterable.

$\boxtimes$ Cor 6.1.

<u>Note</u> Cor. 6.1. holds if $\nu = On \cap M$, hence $M = \langle J^E_\nu, F\rangle$ with $F \neq \emptyset$ is always iterable if it is basic + iterable.

Note It follows that the Dodd - Jensen lemma holds for ...

Corollary 6.2 Let M be a basic mouse s.t. $\omega\rho^{\omega}_{M} < On_M$ and no $\gamma \in M$ is Woodin in M. Let $\mathcal{T}$ be a normal iteration of M of limit length. Then $\mathcal{T}$ has at most one cofinal well founded branch.

proof.
Again take $\mathcal{T}$ as being of minimal length. Let b, b' be distinct branches with $M_b = Q$. Then $\omega\rho^{\omega}_{Q} \geq \delta$. In Case 1 we get a contradiction exactly as before. In Case 2 we have $\delta \geq \omega\rho^{\omega}_{Q} < On_Q$. Hence $\delta \in Q$ + δ is not Woodin in Q. Contr! by Lemma 2. QED (6.1)

The Dodd–Jensen Lemma for M as in Cor 6.1 says: If N is a smooth iterate of M, then:

(a) There is no $\eta < ht(N)$ s.t. $\sigma : M \xrightarrow[\Sigma^*]{} N \| \eta$.

(b) Suppose $\sigma : M \xrightarrow[\Sigma^*]{} N$. Then N is a simple iterate of M and $\pi(\xi) \leq \sigma(\xi)$ for $\xi \in M$, where π is a smooth iteration map from M to M. (Hence $\pi = \pi_{M,N}$ is the unique smooth iteration map.)

This will suffice for all our applications (in virtually all applications N will in fact be a normal iterate of M). Nonetheless it would be nice to have the same lemma with "good" in place of "smooth". This in fact holds, since if $\sigma : M \xrightarrow[\Sigma^*]{} N\|\eta$, then $N\|\eta$ is as in Cor 6.1. This fact can be used to turn a good iteration ~~of~~ from M to $N\|\eta$ into a good iteration by a good sequence $\langle\langle M_i\rangle, \langle \nu_i\rangle, \langle \pi_{ij}\rangle\rangle$ with the property:

$\mathcal{T}_i$ is a normal iteration of $M_i \| \eta_i$ where $M_i \| \eta_i$ is as in Cor 6.1. All branches in such an iteration will be unique and hence the proof of Dodd - Jensen can be carried out. We refrain from giving the proof here, since this strong form of Dodd - Jensen will not be needed.

(Note Since writing this we have been able to show that smoothly iterable premice are, in fact, iterable. This will be proven in §9. The proof makes no use of the theorems in this section.)

Σ_0-Iterations

Now suppose $\mathcal{J}$ to be a direct normal Σ_0-iteration of M with distinct branches b, b' cofinal in $\lambda = \text{length}(\mathcal{J})$ s.t. $M_b, M_{b'}$ are well-founded.

We define ξ_i, γ_i, ξ_i^* exactly as before.

Lemmas 1,2,3 go through exactly as before. Lemma 4 goes through in the following form:

Lemma 4′ Let M be basic, let $A \subset \delta$ be $\underline{\Sigma}_0(Q)$ where $\omega\rho_Q^1 \geq \delta$. Then κ_i^* is A-strong in N for sufficiently large $i < \omega$. (Moreover the full Lemma 4 holds if b has a truncation point and $M_{b'} \neq Q$)

$\boxtimes$ Lemma 4′

A similar version of Corollary 4.1 holds. We then get:

Corollary 4.2′ Let M be basic and let F be a $\underline{\Sigma}_1(Q)$ partial map to δ. Then $F''\kappa_{\gamma_i} \subset \kappa_{\gamma_i}$ for sufficiently large $i < \omega$. (Moreover, the full Cor. 4.2 holds if b has a truncation point and $M_{b'} \neq Q$)

$\boxtimes$ Cor 4.2′

Hence:

Lemma 5′ Let M be a basic Σ_0-mouse. Then

a) $\omega\rho_Q^1 \geq \delta$

b) δ is Σ_1-regular in Q.

(Moreover, the full Lemma 5 holds if b has a truncation point and $M_{b'} \neq Q$)

$\boxtimes$ Lemma 5′

Corollary 6 goes through in the following form:

<u>Corollary 6'</u> Let M be basic s.t. $\wp^1_M \leq \nu$ for a ν with $E^M_\nu \neq 0$. Let $\mathcal{I}$ be a ~~good~~ smooth Σ_0-iteration of M of limit length. Then $\mathcal{I}$ has at most one well-founded cofinal branch.

$\boxtimes$ Cor 6'

<u>Corollary 6.1'</u> If M is basic and Σ_0-iterable s.t. $\wp^1_M \leq \nu$ for a ν with $E^M_\nu \neq 0$, then M is uniquely smoothly Σ_0-iterable.

$\boxtimes$ Cor 6.1'

* * * * * * * * *

We close this section with a useful property of basic premice.

<u>Lemma 7</u>. Let M be a basic premouse, $M = \langle J^E_\alpha, F\rangle$ with $F \neq 0$. Then $\wp^1_M < \lambda$, where λ is the largest cardinal in M.

<u>Proof</u> Suppose not. Let $\kappa = cr(F)$. We prove that κ is Woodin in M. Let $\tau = \kappa^{+M}$. If $\nu < \lambda$, then $F|\nu \in J^E_\lambda$, since $F|\nu$ is a ~~subset~~ of $J^E_{\gamma'}$, where $\gamma' = \gamma^{+M} < \lambda$ and is $\Sigma_1(M)$ in ν.

$\langle x, y\rangle \in F|\nu \longleftrightarrow (\exists z)(\langle x, z\rangle \in F \;\&\; y = z \cap \nu)$

<u>Claim</u> J^E_λ is extendible by $F|\nu$.

<u>Proof</u> Suppose not. Let $\mathbb{D}$ be the ultrapower and let But since $J^E_\lambda \models ZFC$ and $F|\nu \in J^E_\lambda$, J^E_λ thinks it is not extendible by $F|\nu$, so there is an $\in_{\mathbb{D}}$-decreasing sequence $\langle [\alpha_i, f_i], i \in \omega\rangle \in J^E_\lambda$, where $\alpha_i < \nu$ and $f_i : \kappa \longrightarrow J^E_\lambda$. Pick $\lambda' < \lambda$ s.t. $f_i \in J^E_{\lambda'}$ and let $X := \Sigma_\omega$-hull of $\{f_i, i \in \omega\} \cup (\kappa+1)$ in $J^E_{\lambda'}$ and $\sigma : W \stackrel{\sim}{\longleftrightarrow} X$, where W is transitive. Then $X, \sigma, W \in J^E_\lambda$ and $\overline{\overline{W}}^{J^E_\lambda} = \kappa$, thus $W \in J^E_\tau$. Let $\sigma(\bar{f}_i) = f_i$ for $i \in \omega$.

Then $\mathrm{dom}(\bar{f}_i) = \kappa$. Now, since the sequence $\langle [\alpha_i, f_i] \rangle_{i \in \omega}$ is $\in_{\mathbb{D}}$-decreasing, we have:

$\langle \alpha_{i+1}, f_{i+1} \rangle \in_{\mathbb{D}} \langle \alpha_i, f_i \rangle \leftrightarrow$

$\leftrightarrow \{ \langle \eta, \zeta \rangle \in \kappa \,;\, f_{i+1}(\eta) \in f_i(\zeta) \} \in F_{\langle \alpha_{i+1}, \alpha_i \rangle} \leftrightarrow$

$\leftrightarrow \{ \langle \eta, \zeta \rangle \in \kappa \,;\, \bar{f}_{i+1}(\eta) \in \bar{f}_i(\zeta) \} \in F_{\langle \alpha_{i+1}, \alpha_i \rangle} \leftrightarrow$

$\leftrightarrow \langle \alpha_{i+1}, \bar{f}_{i+1} \rangle \in_{\mathbb{D}} \langle \alpha_i, \bar{f}_i \rangle$,

since the sets on the left sides of the second and third proposition are equal by the elementarity of σ.
This means that J^E_τ is not extendible by $F|\nu$.
Contradiction, since J^E_τ is extendible by F.

$\boxtimes$ Claim

Now let $\pi : J^E_\tau \xrightarrow[F]{} J^E_\alpha$. If $a \in P(\kappa) \cap J^E_\tau$,
then $J^E_\lambda \models (F|\nu$ is coherent wrt $\pi(a))$ for all $\nu < \lambda$:
$(F|\nu)(\pi(a) \cap \kappa) = F(\pi(a) \cap \kappa) \cap \nu = \pi(\pi(a) \cap \kappa) \cap \nu = \pi(a) \cap$
Thus $J^E_\lambda \models (\exists \mu)(\mu$ is $\pi(a)$-strong$)$
But then $J^E_\kappa \models (\exists \mu)(\mu$ is a-strong$)$, so κ is Woodin in J^E_τ and thus $J^E_\lambda \models (\mathrm{ZFC} + \kappa$ is Woodin$)$. Contradiction.

$\boxtimes$ Lemma 7.

Hence the initial segment condition is vacuously true for basic premice:

<u>Corollary 7.1</u> Let $M = \langle J^E_\alpha, F \rangle$ be as above. There is no $\bar{\alpha} < \lambda$ s.t. $\langle J^E_{\bar{\alpha}}, F|\bar{\alpha} \rangle$ is a premouse.

<u>Proof</u> Suppose not. Let $\bar{\alpha} < \lambda$ be s.t. $\bar{M} = \langle J^E_{\bar{\alpha}}, F|\bar{\alpha} \rangle$ is a pm.
Let $\bar{\lambda}$ be the largest cardinal in $J^E_{\bar{\alpha}}$. Then $E^M_{\bar{\alpha}} \neq 0$ thus $\rho^1_{M||\alpha} < \bar{\lambda}$ thus $\bar{\lambda}$ is collapsed in M. Let μ be the size of $\bar{\lambda}$ in M. Then $\mu < \bar{\lambda} < \mu^+$.

But $\langle \mu, \mu^{+\bar{M}} \rangle \in \bar{F}(\{\langle \eta, \zeta \rangle ; \eta^{+\bar{M}} = \zeta\}) =$

$$= F(\{\langle \eta, \zeta \rangle ; \eta^{+J^E_\tau} = \zeta\}) \cap \bar{\lambda}, \quad [\tau = \kappa^{+M}]$$

which means $\mu^{+M} = \mu^{+\bar{M}} < \bar{\lambda}$. Contradiction.

$\boxtimes$ Cor 7.1