§7 Mice and Solidity

From now on we consider only *basic* premice M (i.e. there is no $\mu \leq ht(M)$ s.t. $J^{E^M}_\mu \models (ZFC + V\delta\ \delta \text{ is Woodin}))$. We have seen that if a basic M is iterable above $\omega\rho^\omega_M$, then it is uniquely iterable above $\omega\rho^\omega_M$.

Def M is a *mouse* (above ν, beyond ν) iff M possesses a good iteration strategy (above ν, beyond ν).

By our earlier definition, a mouse M is a *uniqueness mouse* iff its strategy is the uniqueness strategy. Thus, by basicness, M is always a uniqueness mouse above $\omega\rho^\omega_M$.

Def Let M be a pm. The <u>standard parameter</u> p_M is defined to be the $<_*$ - least $p \in P^*_N$, where $<_*$ is the well ordering of $[On]^{<\omega}$ defined by:

$$a <_* b \longleftrightarrow \bigvee \nu \, (a \setminus (\nu+1) = b \setminus (\nu+1) \wedge \\ \wedge \; \nu \in b \setminus a)$$

Set: $p^i_M = p_M \cap [\omega\rho^{i+1}_N , \omega\rho^i_N) \quad (i<\omega)$

$$p_M \restriction i = \langle p^0_M , \ldots , p^{i-1}_M \rangle \quad (i<\omega)$$

(also $p_M \restriction 0 = \emptyset$). Then $p^i_M \restriction i \in P^i_M$ and $p^i_M \in P_{M^{i}, p_M \restriction i}$.

We intend to show that $\pi_{MN}(p_M) = p_N$ whenever N is a simple iterate of a mouse M. A major tool in showing this will be the <u>standard witness</u>: For each $\nu \in p_M$ we define a <u>standard witness</u> W^ν_M to the fact that $\nu \in p_M$:

Def Let $\omega\rho_M^{i+1} \leq \nu < \omega\rho_M^i$ $(i < \omega)$.
$W = W_M^\nu$ is defined as follows:
Let $X = h_{M^{i,p^i_M}}(\nu \cup (p_M^i \setminus (\nu+1)))$

$\bar{W} \simeq (M^{i,p^i_M} | X)$, where $\bar{W}$ is transitive. $W, \bar{p}$ are then defined to be the unique pair s.t. $W^{i\bar{p}} = \bar{W}$ and $\bar{p} \in R_W^i$. (Hence $h_{\bar{W}}(\nu \cup (p_M^i \setminus (\nu+1)) = \bar{W}$ so $\omega\rho_W^{i+1} \leq \nu$.)

If $W = W_M^\nu \in M$, then W "witnesses" the fact that $\nu \in p$, as shown by:

<u>Lemma 1.1</u> Let $W_M^\nu \in M$. Then $\nu \in p_M$.

proof.
Otherwise there is $A \notin M$, $A \subset \omega\rho_M^{i+1}$ s.t. A is $\Sigma_1(M^{i,p^i})$ in parameters from $\nu \cup (p \setminus (\nu+1))$. But then $A \in \underline{\Sigma}^*(W) \in N$. Contr! QED(1.1)

We can also generalize the notion of witness as follows:

<u>Def</u> Let $\omega\rho_M^{i+1} \le \nu < \omega\rho_M^i$ $(i<\omega)$

$\langle W, q, r\rangle$ is a <u>generalized</u> <u>witness</u> <u>to</u> ν iff

(a) W is a J-model, $q \in P_W^i$ and $r \in \overline{W} = W^{i,q}$

(b) For all $\vec{\xi} < \nu$ and all Σ_1 fmla φ we have:

$M^{i,p_M^{r\,i}} \models \varphi(\vec{\xi}, p_M^i \setminus (\nu+1)) \to \overline{W} \models \varphi(\vec{\xi}, q$

Hence the property of being a generalized witness is uniformly Π_1 $(M^{i,p_M^{r\,i}})$ in $\nu, p_M^i \setminus (\nu+1)$.

We call W a generalized witness if $\langle W, q, r\rangle$ is a generalized witness for some $q, r \in W$. Clearly W_M^ν is a generalized witness to ν.

The importance of generalized witnesses lies in:

<u>Lemma 1.2</u> Let $\omega\rho_M^{i+1} \leq \nu < \omega\rho_M^i$ $(i<\omega)$.
If $W \in M$ is a generalized witness,
then $W_M^\nu \in M$. (Hence $\nu \in P_M$).

proof.

Let $\langle W, q, r\rangle \in M$ be a gen. witness
Set: $U = W^{i,q}$. Define:

$\omega\gamma$ = the sup of the $h_U(\vec{\xi}, r)$ s.t.
$\vec{\xi} < \nu$ and $h_{M^{i,p,r_i}}(\vec{\xi}, p')$ exists

where $p' = p^i \setminus (\nu+1)$, $p = P_M$.

Then, letting $U' = U|\gamma = \langle J_\gamma^{E^W}, A_W^{i,q} \cap J_\gamma^{E^W}\rangle$,
we have:

(*) $U' \models \varphi(\vec{\xi}, r) \longleftrightarrow M^{i,p,r_i} \models \varphi(\vec{\xi}, p')$

for $\vec{\xi} < \nu$ + Σ_1 fmlae φ.

Hence $M^{i,p,r_i}|X \simeq U'|X'$, where
$X = h_{M^{i,p,r_i}}(\nu \cup p')$, $X' = h_{U'}(\nu \cup q)$.
But $U'|X' \in M$. It follows that
$\nu \in X'$ and
$\nu \notin X$, since otherwise $\nu \in X'$ and
if A is $\Sigma_1(M^{i,p,r_i})$ in parameters
from $p^i \cup \nu$, then A is $\Sigma_1(M^{i,p,r_i}|X$
in parameters from $p' \cup \nu$; ~~hence~~ co

A is $\Sigma_1(U'|X')$ in parameters from $q \cup \nu$; hence $A \in M$. Contr!

If $W_M^\nu \cap On \subset \nu$, then $W_M^\nu = M^{i,p}|X = U'|X' \in M$ (and $i=0$, $M^{i,p} = M$).

Otherwise let $\sigma : \bar{W} \xleftrightarrow{\sim} M^{i,p}|X$, where $\bar{W}$ is transitive and let $\tilde{W}^{i,\bar{p}} = \bar{W}$, where $\bar{p} \in R^i_{\tilde{W}}$. Let $\sigma' \supset \sigma$ s.t. $\sigma' : \tilde{W} \longrightarrow_{\Sigma_1^{(i)}} M$ and $\sigma'(\bar{p}) = p$ $\upharpoonright i$. Then $\nu = \mathrm{crit}(\sigma')$ and $\nu' = \sigma(\nu) > \nu$ is a regular cardinal in M.

Clearly there is $a \subset \nu$ definable from $U'|X'$ coding $\bar{W}$. But $\bar{W}$ can be decoded from a in every ZF^- model — in particular in $J_{\nu'}^{E^M}$. But then $\tilde{W}$ can be recovered from $\bar{W}$ in $J_{\nu'}^{E^M}$. Hence $\tilde{W} = W_M^\nu \in J_{\nu'}^{E^M} \subset M$.

QED(1.2)

Since "$\langle W, q, r\rangle$ is a generalized witness to ν" is uniformly $\Pi_1^{(i)}(M^{i,p,r^i})$ in ν, $p^i \setminus (\nu+1)$, we have:

Lemma 1.3 If $\pi : \overline{M} \longrightarrow_{\Sigma_1^{(i)}} M$, $\omega\rho_{\overline{M}}^{i+1} \leq \bar{\nu} < \omega\rho_{\overline{M}}^{i}$ and $\overline{W} \in \overline{M}$ is a generalized witness to $\bar{\nu}$, where $\pi(\bar{\nu}) = \nu$, $\pi(p_{\overline{M}} \setminus (\bar{\nu}+1)) = p_M \setminus (\nu+1)$, Then $W = \pi(\overline{W})$ is a generalized witness to ν in M. (Hence $\nu \in p_M$, $\pi(p_{\overline{M}} \setminus \bar{\nu}) = p_M \setminus \nu$.)

Lemma 1.4 Assume $\pi : \overline{M} \longrightarrow_{\Sigma_1^{(i)}} M$; $\omega\rho_M^{i+1} \leq \nu < \omega\rho_M^{i}$; $\langle W, q, r\rangle$ is a generalized witness to ν; $\pi(\langle\bar{\nu}, \overline{W}, \bar{q}, \bar{r}\rangle) = \langle \nu, W, q, r\rangle$; $\pi(p_{\overline{M}} \setminus (\bar{\nu}+1)) = p_M \setminus (\nu+1)$. Then $\langle \overline{W}, \bar{q}, \bar{r}\rangle$ is a generalized witness to $\bar{\nu}$ in $\overline{M}$. (Hence $\bar{\nu} \in p_{\overline{M}}$, $\pi(p_{\overline{M}} \setminus \bar{\nu}) = p_M \setminus \nu$.)

(Note There are canonical $q = q_M^{\nu}$, $r = r_M^{\nu}$ uniformly definable in ν over W_M^{ν} s.t. $\langle W_M^{\nu}, q, r\rangle$ is a generalized witness to ν.)

Def Let M be a pm. M is solid (solid above $\xi \in M$) iff $W_M^\nu \in M$ for all $\nu \in P_M$ (all $\nu \in P_M \setminus \xi$).

Lemma 2.1 Let $\pi : \bar{M} \longrightarrow_{\Sigma_1^{(i)}} M$, where $\pi '' P_{\bar{M}}^* \subset P_M^*$. Let $\bar{M}$ be solid above $\bar{\xi} \geq \omega\rho_{\bar{M}}^{i+1}$. Then M is solid above $\xi = \pi(\bar{\xi})$ and $\pi(P_{\bar{M}} \setminus \bar{\xi}) = P_M \setminus \xi$.

proof.

Suppose not. Since $p = \pi(P_{\bar{M}}) \in P_M^*$, then $P_M <_* p$ and $\nu \geq \xi$, where $p \setminus (\nu+1) = P_M \setminus (\nu+1)$, $\nu \in p \setminus P_M$.

Let $\nu = \pi(\bar{\nu})$, $W = \pi(W_{\bar{M}}^{\bar{\nu}})$. Then $\pi(P_{\bar{M}} \setminus (\bar{\nu}+1)) = P_M \setminus (\nu+1)$ and W is a generalized witness to ν. Hence $\nu \in P_M$. Contr! QED(Lemma

Cor 2.2 Let $\pi : \bar{M} \xrightarrow{*}_F M$, where F is weakly amenable and Σ_1-amenable. Let $\bar{M}$ be solid above $\bar{\xi}$, where $\pi(\bar{\xi}) = \xi$. Then M is solid above ξ and $\pi(P_{\bar{M}} \setminus \bar{\xi}) = P_M \setminus \xi$.

<u>Cor 2.3</u> Let M be a simple good iterate of $\overline{M}$ with iteration map π. Then the conclusion of Lemma 2.1 holds.

More generally:

<u>Cor 2.4</u> Let $\mathcal{I}$ be a good iteration. Let $i = T(k+1) \leq_T j$ s.t. $\mathrm{dom}(\pi_{ij}) = M_i \| \eta_k$. Let $\overline{M} = M_i \| \eta_k$ be solid above $\overline{\xi}$, where $\pi_{ij}(\overline{\xi}) = \xi$. Then M_j is solid above ξ and $\pi_{ij}(p_{\overline{M}} \setminus \overline{\xi}) = p_M \setminus \xi$.

<u>Lemma 2.5</u> Let $\pi : \overline{M} \xrightarrow[\Sigma_1^{(i)}]{} M$, $\overline{\xi} \geq \omega\rho_{\overline{M}}^{i+1}$, where M is solid above $\xi = \pi(\overline{\xi})$. Assume $p_M \in \mathrm{rng}(\pi)$ and that each $\nu \in p_M \setminus \xi$ has a generalized witness $\langle W, q, r \rangle \in \mathrm{rng}(\pi)$. Then $\overline{M}$ is solid above $\overline{\xi}$ and $\pi(p_{\overline{M}} \setminus \overline{\xi}) = p_M \setminus \xi$.

prf. As above using Lemma 1.4.

<u>Def</u> M is <u>presolid</u> iff M is a pm and $M \| \alpha$ is solid for $\alpha < ht(M)$.

Def Let M be a pm. The core of M $\bar{M} = \mathrm{core}(M)$ and the canonical core map $\sigma : \bar{M} \to M$ are defined as follows:

Let $\omega\rho^n_M = \omega\rho^\omega_M$. Let $\bar{M}, \tilde{p}$ be s.t. $\bar{M}^{n,\tilde{p}} = M^{n,p_M \restriction n}$, $\tilde{p} \in R^n_{\bar{M}}$.

$\sigma : \bar{M} \to_{\Sigma_1^{(n)}} M$ s.t. $\sigma \restriction \bar{M}^{n,\tilde{p}} = id$ and $\sigma(\tilde{p}) = p_M \restriction n$. Set $\bar{p} = \pi^{-1}(p_M)$

It is easily seen that the choice of n does not affect the definition and hence that $\sigma : \bar{M} \to_{\Sigma^*} M$.

If M is a mouse, then so is $\bar{M}$.

Clearly $\bar{p} \in P^*_{\bar{M}}$ iff $\sigma(\bar{p}) = p_M$.

The interesting case is: $\bar{p} = p_{\bar{M}}$.

Lemma 3.1 Let $\bar{M} = \mathrm{core}(M)$ with core map σ. Let $\sigma(\bar{p}) = p_M$.

$\bar{M}$ is sound iff $\bar{p} = p_{\bar{M}}$.

prf. of Lemma 3.1

($\rightarrow$) Suppose not. Then $p_{\bar{M}} <_* \bar{p}$ and $p_{\bar{M}} \in R^*_{\bar{M}}$. Hence $\bar{p} = f(\vec{\xi}, p_{\bar{M}})$ for a good $\Sigma_1^{(n)}$ fcn f and $\vec{\xi} < \omega\rho^n = \omega\rho^\omega$. Hence $p_M = f(\vec{\xi}, \sigma(p_{\bar{M}}))$ in M. Hence $\sigma(p_{\bar{M}}) <_* p_M$, $\sigma(p_M) \in P^*_M$. Contr!

($\leftarrow$) Suppose not. Let q be $<_*$-minimal in $P^*_{\bar{M}} \setminus R^*_{\bar{M}}$. Then $\bar{p} = p_{\bar{M}} <_* q$ and $q = f(\vec{\xi}, \bar{p})$ for a good $\Sigma_1^{(n)}$ fcn f and $\vec{\xi} < \omega\rho^n = \omega\rho^\omega$. Set $N^{n,\tilde{q}} = \bar{M}^{n, q \upharpoonright n}$, $\tilde{q} \in R^n_N$. Let $\sigma' : N \rightarrow_{\Sigma_1^{(n)}} \bar{M}$ s.t. $\sigma' \upharpoonright N^{n,\tilde{q}} = id$ and $\sigma'(\tilde{q}) = q \upharpoonright n$. As before, σ' is Σ^*-preserving. Let $\sigma'(\bar{q}) = q$. Then $N \models \bigvee r <_* \bar{q}\ \ \bar{q} = f(\vec{\xi}, \bar{r})$. Let $r \in N$ have this property. Then $\sigma'(r) <_* q = f(\vec{\xi}, \sigma'(r))$ in $\bar{M}$. Hence $\sigma'(r) \in P^*_{\bar{M}}$ and $\sigma'(r) \in R^*_{\bar{M}}$ by the minimality of q. Hence $N = \bar{M}$, $\sigma' = id$, since $\sigma'(r) \in rng(\sigma')$. Hence $q \in R^*_{\bar{M}}$. Contr! QED(3.1)

Corollary 3.2 If $\bar{M} = \mathrm{core}(M)$ is sound, then $\mathrm{core}(\bar{M}) = \bar{M}$.

By Corollary 2.4;

Lemma 3.3 Let $\mathcal{J} = \langle\langle M_i\rangle, \ldots, T\rangle$ be a good iteration. Let $i = T(k+1) \leq_T j$ s.t. π_{ij} is defined on $\bar{M} = M_i \| \eta_k$. If $\bar{M}$ is sound + solid, then $\bar{M} = \mathrm{core}(M_j)$ and π_{ij} is the core map.

If M_0 is presolid, so is M_i. Hence $M_i \| \eta_k$ is both sound + solid if $\eta_k < ht(M_i)$. Hence;

Corollary 3.3 Let $\mathcal{J}, i, k, j, \bar{M}$ be as above. If M_0 is presolid + $\eta_k < ht(M_i)$, then $\bar{M} = \mathrm{core}(M_j)$ + π_{ij} is the core map.

This is our main tool in proving;

Lemma 4 Let M^0 be a presolid mouse. Let M^1 be coiterable with M^0 and let $M^{0\prime}, M^{1\prime}$ be the coiterates. Either $M^{0\prime}$ is a simple iterate of M^0 or M^1 is a simple iterate of M^0.

proof of Lemma 4.

Suppose not. Then $M^{0\prime} = M^{1\prime} = M'$. Let $\langle \mathcal{I}^0, \mathcal{I}^1 \rangle$ be the coiteration, where $\mathcal{I}^h = \langle \langle M_i^h \rangle, \dots, T^h \rangle$ and $|\mathcal{I}^h| = \theta$ (h=0,

Let $l^h, j_h, \bar{M}_h$ be defined by:

l^h = the least $l \leq_{T^h} \theta$ s.t. $\pi^h_{l\theta}$ is to

on $\bar{M}^h$, where $l = T(j_h + 1) \leq_{T^h} \theta$ and

$\bar{M}^h = M^h_l \| \nu^h_{j_h}$.

Then:

(1) $\bar{M}^0 = \bar{M}^1 = \bar{M} = \text{core}(M')$

$\pi^0_{l_0,\theta} = \pi^1_{l_1,\theta} = \bar{\pi}$ = the core map.

Set $F^h = E^{M^h_{j_h}}_{\nu_{j_h}}$ $(h=0,1)$. Then

(2) $\text{crit}(F^0) = \text{crit}(F^1) = \kappa =_{df} \text{crit}(\bar{\pi})$.

(3) $\alpha \in F^h(X) \longleftrightarrow \alpha \in \bar{\pi}(X)$

for $\alpha < \nu_{j_h}$.

Hence $j_0 \neq j_1$, since otherwise $E^{M^0_j}_{\nu_j} = E^{M^1_j}_{\nu_j}$ $(j = j_h)$. Let e.g. $j_0 < j_1$.

Then: $E^{M^1_{j_1}}_{\nu_{j_1}} | \nu_{j_0} = E^{M^0}_{\nu_{j_0}}$ and

$\langle J^{E^{M^1}_{j_1}}_{\nu_{j_0}}, (E^{M^1}_{\nu_{j_1}} {}_{j_1} | \nu_{j_0}) \rangle = M^0 \| \nu_{j_0}$ in a p.m.

Hence $E^{M^1}_{\nu_{j_0}} {}_{j_1} \neq \emptyset$. But then $E^{M^0}_{\nu_{j_0}} {}_{j_1} =$ $E^{M^1}_{\nu_{j_0}} {}_{j_1} = \emptyset$, since $j_0 + 1 \leq j_1$, $E^{M^0}_{\nu_{j_0}} {}_{j_0+1} = \emptyset$ and $M^0_{j_1} | \lambda_{j_0+1} = M^0_{j_0+1} | \lambda_{j_0+1}$.

Contr! QED (Lemma 4)

Clearly the same result holds for Σ_0 coiteration and for mixed co-iteration (in which the Σ_0 side can be either the M^0 or M^1 side)

Double rooted iterations

In order proceed further we must modify our general notion of iteration slightly so as to allow "double rooted" iterations on a pair of premice $\langle N, M \rangle$. At stages i where $E^{M_i}_{\nu_i}$ was previously applied to the "root model" M_0 to get M_{i+1}, we permit ourselves to apply $E^{M_i}_{\nu_i}$ to either of the root models N, M. We take $M_0 = M$ however and set: $M_{-1} = N$. Thus we can return to -1 as well as 0 if we have an applicable extender.

We first define:

<u>Def</u> T is a <u>double rooted iteration tree</u> iff $T \subset [-1, \theta)^2$ s.t.

(a) T is a tree with exactly two roots $-1, 0$.

(b) $\nu + 1 \in [0, \theta)$ immediately succeeds a pt. $T(\nu+1) \in [-1, \nu]$ in T

(c) If $\lim(\lambda)$, $\lambda < \theta$, then λ is a limit pt. of T and $\sup T''\{\lambda\} = \lambda$.

(Note $\xi \in [-1, \theta)$ is called a root of T iff
iff $\eta \not<_T \xi$ for $\eta \in [-1, \theta)$.)

The notion of a generalized double rooted iteration of length θ is defined exactly as before (in §4) except that:

(a) T is double rooted;

(b) M_i is defined for $i \in [-1, \theta)$.

ν_i is again defined on a set of points $D \subset \theta = [0, \theta)$ and η_i is defined for *) $i+1 \in \theta = [0, \theta)$.

The notions "truncation point" and "simple" are defined exactly as before, as is the notion "direct". The clause "$T(i+1) \in D$" in the notion of "standard" must be modified to: $T(i+1) \in D \cup \{-1\}$.

Our intention is to use a point $\lambda \in N \cap M$ to decide whether or not $T(i+1) = -1$.

If $\kappa_i = \mathrm{crit}(E^{M_i}_{\nu_i}) < \lambda$, we set $T(i+1) = -1$ and otherwise $T(i+1) \geq 0$.

In a normal iteration we ensure that $\lambda_i \geq \lambda$ for $i \geq 0$. Thus λ can be regarded as λ_{-1}.

*) We thus write: $i < \theta$ for $i \in [0, \theta)$.

Def $\mathcal{I}$ is a <u>normal</u> <u>iteration</u> <u>on</u> $\langle N, M, \lambda \rangle$ iff

(a) $\lambda \in N \cap M$ and $J_\lambda^{E^N} = J_\lambda^{E^M}$

(b) $\mathcal{I}$ is a standard iteration with
$M_{-1} = N$, $M_0 = M$

(c) Let $i \in D$. Then:

(i) $\lambda_i \geq \lambda$ and $\lambda_i > \lambda_h$ for $h \in D \cap i$

(ii) $T(i+1) =$ the least ξ s.t.
$(\xi \in D \wedge \kappa_i < \lambda_\xi) \vee (\xi = -1 \wedge \kappa_i < \lambda)$.

In dealing with such iterations we shall write: $\lambda_{-1} = \lambda$, which simplifies the formulation of (c)(ii).

In order that this works, we impose a condition on $\langle N, M, \lambda \rangle$:

<u>Def</u> $\langle N, M, \lambda \rangle$ is a <u>good</u> <u>triple</u> iff
iff N, M are premice, $\lambda \in N \cap M$, $J_\lambda^{E^N} = J_\lambda^{E^M}$,
λ is a cardinal in M, and every $\beta < \lambda$ which is a cardinal in M is also a cardinal in N, and:
$\mathbb{P}(\lambda) \cap M \subset N$, $\mathbb{P}(\lambda) \cap \underset{\sim}{\Sigma}_1(M) \subset \underset{\sim}{\Sigma}_1(N)$?

<u>Note</u> If $\mathcal{I}$ is a normal iteration on a good triple $\langle N, M, \lambda \rangle$, then the situation: $T(i+1) = -1$ and $\eta_i < ht(N)$ can only occur if κ_i is the largest cardinal $< \lambda$ in M, since otherwise $\tau_i = \kappa_i^{+J_{\nu_i}^{E^{M_i}}} < \lambda$ is a cardinal in M, hence in N. We refer to such $i+1$ as an <u>anomaly</u>. Clearly, only one anomaly can occur in a branch.

<u>Note</u> In a normal iteration we have: λ_i is a cardinal in M_j for $-1 \le i < j$: $\lambda_i = \lambda_j$ $(i<j)$ can only occur if $i = -1$, $j = 0$, and $\lambda = \lambda_0$ is a limit cardinal in M (hence no anomaly is possible.) We always have $\lambda_i \le \lambda_j$ for $-1 \le i < j$.

Moreover $J_{\lambda_i}^{E^{M_i}} = J_{\lambda_i}^{E^{M_j}}$. If $0 \le i < j$, then $J_{\nu_i} = J_{\nu_i}$ and ν_i is a cardinal in M_j

We obtain a slightly weaker form of §4 Lemma 1;

<u>Lemma 5.1</u> Let $\langle N, M, \lambda\rangle$ be a good triple. Let $\mathcal{I} = \langle\langle M_i\rangle, \ldots, T\rangle$ be a normal iteration of $\langle N, M, \lambda\rangle$. If $i \in D$ and $i+1$ is not an anomaly, then $E^M_{\nu_i}$ is Σ_1-amenable wrt $M_{T(i+1)} \| \eta_i$. (Moreover, if $i+1$ is an anomaly, we still have: $(E^M_{\nu_i})_\alpha \in N$ for $\alpha < \lambda_i$.)

This implies:

<u>Corollary 5.1.1</u> Let $\mathcal{I}$ be as above. If $i \leq_T j$ and π_{ij} is defined on M_i, then $\pi_{ij} : M_i \longrightarrow_{\Sigma^*} M_j$.

In fact;

<u>Corollary 5.1.2</u> Let $\mathcal{I}$ be as above. If $h \leq_T i$ + ξ is least s.t. $h <_T \xi+1 \leq_T i$, π_{hi} is defined on $M_h \| \eta_\xi$ and $\xi+1$ is not an anomaly, then

$$\pi_{hi} : M_h \| \eta_\xi \longrightarrow_{\Sigma^*} M_i.$$

The derivations of 5.1.1 + 5.1.2 from 5.1 are as before. We prove Lemma 5.1

As before, assume w.l.o.g. that $\mathcal{I}$ is direct. As before set:

<u>Def</u> Let $E^{M_i}_{On \wedge M_i} \neq \emptyset$, $i \geq 0$.

$\bar{\kappa}_i = crit(E^{M_i}_{On\wedge M_i})$, $\bar{\tau}_i = \bar{\kappa}_i^{+M_i}$

δ_i = the least $\delta \geq -1$ s.t. $\delta = i$ or $\bar{\kappa}_i < \lambda_\delta$

$\bar{\eta}_i$ = the max $\eta \leq ht(M_{\delta_i})$ s.t.
$$\bar{\tau}_i = \kappa^{+M_{\delta_i}\|\eta}$$

(hence $\bar{\eta}_i \geq \lambda_{\delta_i}$, since $\bar{\kappa}_i < \lambda_{\delta_i}$ and λ_{δ_i} is a cardinal in M_i if $\delta_i < i$)

Corresponding to §4 Lemma 1.2 we have:

<u>Lemma 5.2</u> Let $i \geq 0$ s.t. δ_i exists. Then

(a) $\mathbb{P}(\bar{\tau}_i) \cap \underline{\Sigma}_1(M_i) \subset \underline{\Sigma}_1(M_{\delta_i}\|\bar{\eta}_i)$ if $\delta_i \geq 0$

(b) $\mathbb{P}(\bar{\tau}_i) \cap \underline{\Sigma}_1(M_i) \subset \underline{\Sigma}_1(N)$ if $\delta_i = -1$.

Call i a δ-<u>anomaly</u> iff $\delta_i = -1$ and $\bar{\eta}_i < ht(N)$. Then (a) holds iff i is not a δ-anomaly.

The proof of Lemma 5.2 is a virtual repetition of §4 Lemma 1.2. Suppose not. Let i be the least counterexample. Then $\delta_i < i$. Moreover, $i > 0$, since $\mathbb{P}(\kappa) \cap \Sigma_1(M) \subset \Sigma_1(N)$. Thus $i = h+1$, by minimality. Set: $\xi = T(i)$. Set: $\kappa = \bar{\kappa}_i$, $\tau = \bar{\tau}_i$, $M^* = M_\xi \| \eta_h$.

(1), (2) follow exactly as before.

In place of (3) we get:

(3') Let $F = F_{\nu_h}^{M_h}$, $\alpha < \lambda_h$. Then

(a) $F_\alpha \in \Sigma_1(M^*)$ if i is not an anomaly

(b) $F_\alpha \in \Sigma_1(N)$ otherwise.

If $\nu_h = On \cap M_h$, the proof is exactly as before. If $\nu_h \in M_h$ and $\xi \geq 0$, the proof is exactly as before. If $\xi = -1$, we show: $F_\alpha \in N$. We have $F_\alpha \in J_{\lambda_h}^{E^{M_h}} \cap \mathbb{P}(\tau) \subset J_{\lambda_0}^{E^{M_h}} \cap \mathbb{P}(\tau) \subset$

$\subset M \cap \mathbb{P}(\tau) \subset N$. QED (3')

(4) follows as before if $\delta_i \geq 0$. If $\delta_i = -1$ we have for $A \in \mathbb{P}(\tau) \cap \Sigma_1(M_i)$:

$A \in \mathbb{P}(\tau) \cap M_i$; hence:

$A \in \mathcal{P}(\tau) \cap J_{\lambda_0}^{E^{M_i}} = \mathcal{P}(\tau) \cap J_{\lambda_0}^{E^M} \subset \mathcal{P}(\tau) \cap M \subset N$, QED (4)

As before we get:

(5′) $\omega\rho^1_{M^*} \leq \tau$ if i is not an anomaly

(6′)(a) $\mathcal{P}(u_h) \cap \Sigma_1(M_i) \subset \Sigma_1(M^*)$ if $\xi \geq 0$

(b) " $\subset \Sigma_1(N)$ if $\xi = -1$

If i is not an anomaly, then (a) follows as before. Otherwise, the proof of §1 Lemma 8 shows:

$$A\xi \longleftrightarrow \bigvee u \in M^* \{\eta \mid P(u, \xi, f(\eta))\} \in F_\alpha.$$

for an $\alpha < \lambda_h$, where P is $\Sigma_0(M^*)$

But $M^* \in N$ and $F_\alpha \in \Sigma_1(N)$.

Hence A is $\Sigma_1(N)$. QED (6′)

(7) $\xi > \delta_i$ is proven as before

(8) then follows as before, since $\xi \geq 0$

But then $\delta_\xi = \delta_i$ and $\bar\eta_\xi = \bar\eta_i$ as before

In particular, ξ is a δ-anomaly iff i is a δ-anomaly. We get a contradiction as before. QED (Lemma 5.2)

The derivation of 5.1 from 5.2 is as before.

The <u>coiteration</u> $\langle \mathcal{I}_0, \mathcal{I}_1 \rangle$ of $\langle N, M, \lambda \rangle$ with a premouse Q is defined in the obvious way. A virtual repetition of the proof of Lemma 4 gives:

<u>Lemma 5.3</u> Let $\langle N, M, \lambda \rangle$ be good, where N, M are presolid. Let $\langle N, M, \lambda \rangle$ be coiterable with Q and let M', Q' be the coiterates. Either M' is a simple iterate of $\langle N, M, \lambda \rangle$ or Q' is a simple iterate of Q. (Moreover, the simple coiterate is a segment of the non simple coiterate.)

(<u>Note</u> M' is a nonsimple iterate of $\langle N, M, \rangle$ if there is an anomaly on the branch to M'.)

<u>Def</u> $\langle N, M, \lambda \rangle$ is <u>witnessed by</u> σ iff

iff (a) N, M are premice

(b) $\lambda \in N \cap M$ is a cardinal in M

(c) $\sigma : M \longrightarrow N$, $\sigma \upharpoonright \lambda = \mathrm{id}$

(d) σ is cardinal preserving and $\Sigma_0^{(m)}$-preserving for $\omega\rho_M^m > \lambda$.

Clearly $\langle N, M, \lambda \rangle$ is a good triple if it is witnessed by some σ.

Now let $\langle N, M, \lambda \rangle$ be witnessed by σ + let $\mathcal{J} = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle \eta_i \rangle, \langle \pi_{ij} \rangle, T \rangle$ be a normal iteration of $\langle N, M, \lambda \rangle$. We attempt to define a "copy" $\mathcal{J}' = \sigma(\mathcal{J}) = \langle \langle N_i \rangle, \langle \nu_i' \rangle, \langle \eta_i' \rangle, \langle \pi_{ij}' \rangle, T' \rangle$ of $\mathcal{J}$ with copying maps

$\sigma_i : M_i \longrightarrow N_{\max(0,i)}$ $(i \geq -1)$.

T' will be the same as T except that the two roots $0, -1$ have been consolidated into one. We have:

(a) $0 T' h \longleftrightarrow (-1 T h \vee 0 T h)$

$j T' h \longleftrightarrow j T h$ for $j > 0$

We set:

(b) $\sigma_{-1} = id$, $\sigma_0 = \sigma$

(c) $D_{y'} = D_y$, $\nu_i' = \sigma_i(\nu_i)$ for $i \in D$

(hence $\kappa_i' = \sigma_i(\kappa_i)$, $\tau_i' = \sigma_i(\tau_i)$).

Let $i \in D$, $\xi = T(i+1)$, $\xi' = T'(i+1) = \max(0, \xi)$. We normally expect that $\eta_i' = \sigma_\xi(\eta_i)$ and, letting $M^* = M_\xi \| \eta_i$, $N^* = N_{\xi'} \| \eta_i'$, that

$\pi_{\xi, i+1} : M^* \xrightarrow[E^{M_i}_{\nu_i}]{} M_{i+1}$, $\pi'_{\xi', i+1} : N^* \xrightarrow[E^{N_i}_{\nu_i'}]{} N_{i+1}$

and $\sigma_{i+1} : M_{i+1} \to N_{i+1}$ is defined by:

(*) $\sigma_{i+1}(\pi_{\xi, i+1}(f)(\alpha)) = \pi'_{\xi', i+1}\, \sigma_\xi(f)\, (\sigma_i(\alpha))$

Where $f \in \Gamma(\kappa_i, M^*)$, $\alpha < \lambda_i$

This works <u>except</u> in the case that $i+1$ is an anomaly. We then have $\eta_i < ht(N)$, since $\tau_i = \lambda$ is not a cardinal in N.

⋮

But $\tau_i' = \sigma_i(\tau_i) = \sigma(\tau_i) = \sigma(\lambda)$ is a cardinal in N; hence $\eta_i' = ht(N$. Thus $M^* = N \| \eta_i \in N$, $N^* = N$. In this case we arrange for

$\sigma_{i+1} : M_{i+1} \longrightarrow \pi_{0,i+1}(M^*)$

to be defined by (*). This can be justified as follows. Let $\mathbb{D} = \mathbb{D}^*(\kappa, M^*)$. Let $\langle \alpha_i, f_i \rangle \in \mathbb{D}$ for $i = 1, \ldots, r$. Set: $\alpha_j' = \sigma_i(\alpha_i)$. Let φ be a $\Sigma_0^{(m)}$ fmla for $\omega\rho^m_{M^*} > \kappa$.

Then:

$\mathbb{D} \models \varphi(\langle \alpha_1, f_1 \rangle, \ldots, \langle \alpha_r, f_r \rangle) \longleftrightarrow$

$\longleftrightarrow \{\vec{\xi} \mid M^* \models \varphi(f_1(\xi_1), \ldots, f_r(\xi_r))\} \in F_{\vec{\alpha}}$

$\longleftrightarrow$ $\qquad$ " $\qquad \in F'_{\vec{\alpha}'}$

$\longleftrightarrow N_{i+1} \models \pi'(M^*) \models \varphi(\pi'(f_1)(\alpha_1), \ldots, \pi'(f_r)(\alpha_r))$

$\longleftrightarrow \pi'(M^*) \models \varphi(\pi'(f_1)(\alpha_1), \ldots, \pi'(f_r)(\alpha_r))$

where $\pi' = \pi'_{0,i+1}$.

Thus $\mathbb{D}$ is well founded.

M_{i+1} is defined and the map σ_{i+1} given by (*) exists and is $\Sigma_1^{(m)}$ preserving for $\omega\rho^m_{M^*} > \kappa_i$. We define:

Def Let σ witness $\langle N, M, \lambda\rangle$. Let $\mathcal{J} = \langle\langle M_i\rangle, \langle \nu_i\rangle, \langle \eta_i\rangle, \langle \pi_{ij}\rangle, T\rangle$ be a normal iteration of $\langle N, M, \lambda\rangle$. $\mathcal{J}' = \sigma(\mathcal{J})$ is the <u>copy of</u> $\mathcal{J}$ <u>onto</u> N with <u>copying maps</u> $\langle \sigma_i\rangle$ iff (a), (b), (c) above hold and:

(d) $\mathcal{J}' = \langle\langle N_i\rangle, \langle \nu_i'\rangle, \langle \eta_i'\rangle, \langle \pi_{ij}'\rangle, T'\rangle$ is a normal iteration of N of the same length

(e) $\sigma_i : M_i \to N_{\max(0,i)}$ s.t.

$$\sigma_i \pi_{hi} = \pi'_{\max(0,h),i}\, \sigma_h$$

(f) If $i = h+1$, $h \notin D$, then $\sigma_i = \sigma_h$

(g) Let $i = h+1$, $h \in D$. If i is not an anomaly, then:

(i) $\sigma^* : M^* \longrightarrow_{\Sigma_0^{(n)}} N^*$ for $\omega\rho^n_{M^*} > \kappa_h$, where σ^*, M^*, N^* are as above.

(ii) $\langle \sigma_i^*, \sigma_h \upharpoonright \lambda_h \rangle : \langle M_i^*, F \rangle \to \langle N_i^*, F' \rangle$

where $F = E^{M_h}_{\nu_h}$, $F' = E^{N_h}_{\nu'_h}$.

(iii) $\sigma_i : M_i \to N_i$ is defined by:

$\sigma_i(\pi_{\xi i}(f)(\alpha)) = \pi'_{\xi i} \sigma_\xi(f)(\sigma_h(\alpha))$

where $\xi = T(i)$, $f \in \Gamma(\kappa_h, M^*)$, $\alpha < \lambda_h$

(h) Let $i = h+1 \in D$ be an anomaly. Then

$\sigma_i : M_i \to \pi'_{0i}(M^*)$ is defined by:

$\sigma_i(\pi_{\xi i}(f)(\alpha)) = \pi'_{0i}(f)(\sigma_h(\alpha))$

where $f \in \Gamma(\kappa_h, M^*)$, $\alpha < \lambda_h$.

Remark Suppose that $i = h+1$ is an anomal and $i = T(j+1)$ where $j+1 \in D$. Then $\lambda_h \leq \kappa_j < \lambda_i$. Hence $\sigma_i(\lambda_h) \leq \kappa'_j < \lambda'_i$, since $\kappa'_j = \sigma_j(\kappa_j) = \sigma_i(\kappa_j)$. But $\sigma_i(\lambda_h) =$
$= \sigma_i(\pi_{-1,i}(\kappa_h)) = \pi'_{0i}(\kappa_h)$. We have:

$\tau'_j = (\kappa'^+_j)^{J^{E^{N_i}}_{\lambda'_\delta}} = (\kappa'^+_j)^{J^{E^{N_i}}_{\lambda'_i}}$

since λ'_i is a cardinal in N_i. But $\pi'_{0i}(M^*) = N_i \| \eta$, where $\eta = \pi'_{0i}(\eta_h) =$
$=$ the max. η s.t. $\pi'_{0i}(\lambda)$ is a cardinal in $N_i \| \eta$. Hence $\omega\eta$ is collapsed to $\pi'_{0i}(\kappa_h)$ in $N_i \| \eta+1$,

since $\pi'_{0i}(\lambda) = \pi'_{0i}(\kappa_h)^+$ in $\pi'_{0i}(M^*)$. But then τ'_j is collapsed to $\pi'_{0i}(\kappa_h) \le \kappa'_j$ in $N_i \| \eta + 1$. Hence $\eta'_j \le \eta$, since τ'_j is a cardinal in $N_i \| \eta'_j$. This means that (g)(ii), (iii) hold for $j+1$ in place of i, hence that $\sigma_{j+1} : M_{j+1} \to N_{j+1}$ can be defined as in (g)(iii).

<u>Remark</u> $\sigma_\xi(\eta_h) = \eta'_h$ ($\xi = T(h+1)$) is obviously false for anomalous $i = h+1$. Now let $i = T(j+1)$. It is possible that $\eta'_j = \mathrm{On} \cap \pi'_{0i}(M^*)$, where $\eta_j = \mathrm{On} \cap M_i$. Our convention is that $f(\mathrm{On} \cap P) = \mathrm{On} \cap Q$ if $f : P \to Q$. Thus $\sigma_i(\eta_j) = \eta'_j$ is true <u>if</u> we regard $\pi'_{0i}(M^*)$ as our target model, rather than N_i. In all other cases $\sigma_\xi(\eta_h) = \eta'_h$ holds unambiguously.

<u>Note</u> We again have $\sigma_{i+1} \restriction \lambda_i = \sigma_i \restriction \lambda_i$ for $i \in \xi - 1 \cup D$. Hence $\sigma_i \restriction \lambda_j = \sigma_j \restriction \lambda_j$ for $-1 \le j < i$, $j \in \xi - 1 \cup D$ by Ind. on i.

Using the fact that $\eta'_j \le ht(\pi'_{0i}(M_i))$ if $i = T(j+1)$ is anomalous, we can repeat the proof of §5 Lemma 1 to show:

<u>Lemma 5.4</u> Let σ witness $\langle N, M, \lambda\rangle$. Let $\mathcal{I}$ be a normal iteration of $\langle N, M, \lambda\rangle$ and $\mathcal{I}' = \sigma(\mathcal{I})$. Then for all i we have $\sigma_i : M_i \xrightarrow{\Sigma_0^{(n)}} \tilde{N}_i$ whenever $\omega\rho^n_{M_i} \geq \sup\limits_{-1 \leq h < i} \lambda_h$.

This shows - just as in §5 - that the copying process "can be continued" as long as we are following an iteration strategy for N at limit points in $\mathcal{I}'$. More precisely:

<u>Def</u> Let σ witness $\langle N, M, \lambda\rangle$. Let S be an iteration strategy. The <u>derived</u> normal iteration strategy $\bar{S}$ for $\langle N, M, \lambda\rangle$ is defined as follows: Let $\mathcal{I}$ be a normal it. of $\langle N, M, \lambda\rangle$ of limit length θ. If $\mathcal{I}' = \sigma(\mathcal{I})$ exists and $b' = S(\mathcal{I}')$ is defined set $\bar{S}(\mathcal{I}) = b$, where b is the branch in $\mathcal{I}$ s.t. $b \setminus 2 = b' \setminus 2$.

<u>Lemma 5.5</u> Let σ witness $\langle N, M, \lambda\rangle$. Let N have normal iteration strategy S + let $\bar{S}$ be the derived strategy. Then $\bar{S}$ is a normal iteration strategy for $\langle N, M, \lambda\rangle$. In particular, if $\mathcal{I}$ is an $\bar{S}$-iteration of $\langle N, M, \lambda\rangle$, then $\mathcal{I}' = \sigma(\mathcal{I})$ exists + is an S-iteration of N.

. . . . . .

Corresponding to §5 Lemma 3.1 we get:

<u>Lemma 5.6</u> Let σ witness $\langle N, M, \lambda\rangle$ + let $\mathcal{I}' = \sigma(\mathcal{I})$, where $\mathcal{I}$ is a normal iteration of $\langle N, M, \lambda\rangle$. Let $i \in D$, $\xi = T(i+1)$, where $i+1$ is not an anomaly. Set:

$M^* = M_\xi \| \eta_i$, $N^* = N_\xi \| \eta'_i$, $F = E^{M_i}_{\nu_i}$, $F' = E^{N_i}_{\nu'_i}$.

Let $\sigma^* = \sigma_\xi \upharpoonright M_\xi$. Then:

$$\langle \sigma^*, \sigma_i \upharpoonright \lambda_i\rangle : \langle M^*, F\rangle \longrightarrow^* \langle N^*, F'\rangle.$$

In order to prove this we assume w.l.o.g. that $\mathcal{I}$ is direct and define:

<u>Def</u> Let $i \geq 0$ s.t. $E^{M_i}_{On \cap M_i} \neq \emptyset$. Set:

$\hat{\kappa}_i = \mathrm{crit}(E^{M_i}_{On \cap M_i})$, $\hat{\tau}_i = (\hat{\kappa}_i)^{+M_i}$

$\delta_i =$ the least $\delta \geq -1$ s.t. $\delta = i$ or $\hat{\kappa}_i < \lambda_{\delta_i}$

$\hat{\eta}_i =$ the maximal $\eta \leq ht(M_{\delta_i})$ s.t.

$\hat{\tau}_i = (\hat{\kappa}_i)^{+M_{\delta_i} \| \eta}$.

$\hat{\kappa}'_i, \hat{\tau}'_i, \delta'_i, \hat{\eta}'_i$ are defined similarly from $\mathcal{I}'$. Then $\hat{\kappa}'_i, \hat{\tau}'_i = \sigma_i(\hat{\kappa}_i, \hat{\tau}_i)$, $\delta_i = \delta'_i$. If $\delta_i \geq 0$, then $\hat{\eta}'_i = \sigma_{\delta_i}(\hat{\eta}_i)$.

A literal repetition of the proof of §5 Lemma 3.2 gives:

<u>Lemma 5.7</u> Let $\delta_i \geq 0$. Let $A \subset \hat{\tau}_i$, $A' \subset \hat{\tau}'_i$ s.t. A is $\Sigma_1(M_i)$ in p and A' is $\Sigma_1(N_i)$ in $p' = \sigma_i(p)$ by the same definition. There is $q \in M_{\delta_i} \| \hat{\eta}_i$ s.t. A is $\Sigma_1(M_{\delta_i} \| \hat{\eta}_i)$ in q and A' is $\Sigma_1(N_{\delta_i} \| \hat{\eta}'_i)$ in $q' = \sigma_{\delta_i}(q)$ by the same definition.

From this we derive Lemma 5.6. If $\xi = T(i+1) \geq 0$, the proof is exactly as before. Now let $\xi = -1$. Since $i+1$ is not an anomaly we have $M^* = N^* = N$, $\sigma^* = \mathrm{id} \restriction N$. Let $\alpha < \lambda_i$. Then $F_\alpha = F'_{\sigma_i(\alpha)}$ & it suffices to observe that F' is Σ_1-amenable wrt. $N = N_0$. Thus $F'_{\sigma(\alpha)} = F_\alpha$ is $\Sigma_1(N)$ in a parameter p and $\sigma^*(p) = p$. QED (Lemma 5.6).

As consequences of Lemma 5.6 we get:

<u>Lemma 5.8</u> Let $\sigma, \mathcal{Y}, \mathcal{Y}'$ be as above.

(a) If $i \geq_T -1$ s.t. no $h \leq_T i$ is anamolous then $\sigma_i : M_i \longrightarrow_{\Sigma^*} N_i$.

(b) If some non anomalous $h+1 \leq_T i$ is a truncation point, then σ_i is Σ^*-preserv.

(c) Let σ be strongly $\Sigma_h^{(n)}$-preserving, where $\omega\rho^n \leq \lambda$. If $i \geq_T 0$ is simple, then σ_i is strongly $\Sigma_h^{(n)}$-preserving.

We are now ready to prove:

<u>Lemma 6</u> Every mouse is solid.

prf.

Let M be a counterexample of minimal height. Then M is presolid. Let $\alpha \in p = p_M$ be maximal s.t. $W^\alpha \notin M$. Set: $W = W^\alpha$ + let $\sigma : W \to M$ be the canonical map. Let $\omega\rho^{n+1} \le \alpha < \omega\rho^n$ in M. Let $\sigma \upharpoonright \bar{W} : \bar{W} \stackrel{\sim}{\longleftrightarrow} h(\alpha \cup (p^n \setminus (\alpha+1)))$, where $h = h_{M^{n, p \upharpoonright n}}$. Then $\bar{h}(\alpha \cup q) = \bar{W} = W^{n \bar{p}}$, where $\sigma(q) = p^n_M \setminus (\alpha + 1)$ and $\sigma(\bar{p}) = p_M \setminus \omega\rho^n_M$ and $\bar{h} = h_{\bar{W}}$. It follows that $\omega\rho^{n+1}_W = \omega\rho_{\bar{W}} \le \alpha$, since $\bar{p} \in R^n_W$.

We may assume $\omega\rho^\omega_M \le \nu$ where $E^M_\nu \neq \emptyset$, since otherwise $W \neq M$ is a segment of M. Hence M is uniquely smoothly iterable.

<u>Case 1</u> $\alpha = ht(W)$ (hence $\sigma = id$)

Then $\alpha \in p \subset M$. Hence $E^W_{\omega\alpha} \neq \emptyset$, since otherwise $W = \langle J^{E^M}_\alpha, \emptyset \rangle \in M$.

Let $F = E^M_{On \cap M}$, $\bar{F} = E^W_{On \cap W}$. Then $\bar{F} = F \cap W$, since $\sigma = id$. Hence $\bar{F}(X) = F(X)$ for $X \in dom(\bar{F})$.

Now let $\kappa = \mathrm{crit}(F)$. Then $\kappa = \mathrm{crit}(\bar{F})$
and $\bar{F}(\kappa) = F(\kappa) < \alpha$.

But then $\kappa^{+W} = \kappa^{+M} <$
$< F(\kappa)$, since $F(\kappa)$ is a limit card.
in M and W. Since $\alpha < On \cap M$, there
is $X \in \mathbb{P}(\kappa) \cap M = \mathbb{P}(\kappa) \cap W$ s.t. $F(X) \notin$
$\notin J_\alpha^{E^M}$. Hence $\bar{F}(X) = F(X) \notin W$.

QED (Case 1]

Contr!

<u>Case 2</u> Case 1 fails. Then $\alpha =$
$= \mathrm{crit}(\sigma)$. Hence α is a regular
cardinal in W and $\sigma(\alpha)$ is regular
in M. Clearly, $\langle M, W, \alpha \rangle$ is a
good triple as witnessed by σ.
Let $\langle \mathcal{J}^W, \mathcal{J}^Q \rangle$ be the coiteration of
$\langle M, W, \alpha \rangle, M$, where:

$\mathcal{J}^W = \langle \langle W_i \rangle, \langle \nu_i^W \rangle, \langle \eta_i^W \rangle, \langle \pi_{ij}^W \rangle, T^W \rangle$

$\mathcal{J}^Q = \langle \langle Q_i \rangle, \langle \nu_i^Q \rangle, \ldots, T^Q \rangle$.

Let $\mathcal{J}^M = \sigma(\mathcal{J}^W) = \langle \langle M_i \rangle, \ldots, T^M \rangle$ be
the copy of $\mathcal{J}^W$ onto M with copying
maps $\langle \sigma_i \rangle$. Let $\theta + 1 = \mathrm{length}(\mathcal{J}^Q)$

Case 2.1 $\theta \geq -1$ in T^W.

In the following we make use of the Dodd-Jensen lemmas §5 4.1 + 4.2,

Case 2.1.1 Q_θ is a proper segment of W_θ. Then $\sigma_\theta \circ \pi^Q_{0\theta} : M \to_{\Sigma^*} \sigma(Q_\theta)$, where $\sigma(Q_\theta)$ is a proper segment of a smooth iterate of M. Contr! by Dodd Jensen (§5 Lemma 4.2)

Case 2.1.2 $W_\theta = Q_\theta$ and there is a non-anomalous truncation pt. $i+1 \leq \theta$ in T^W. Then σ_θ is Σ^*-preserving and hence $\sigma_\theta \pi^Q_{0\theta} : M \to_{\Sigma^*} M_\theta$, where M_θ is a non simple smooth iterate of M. Contr! (§5 Lemma 4.-

Case 2.1.3 W_θ is a proper segment of Q_θ or Q_θ is a non simple iterate of M. Then $\pi^W_{0\theta} : M \to_{\Sigma^*} W_\theta$, where W_θ is a proper segment of a smooth iterate of M. Contr!

Case 2.1.4 The above fail and some $i+1 \leq \theta$ in T^W is an anomaly. Then $\pi^M_{i+1,\theta}$ is defined on M_{i+1}. Let $M^* = M \| \eta_i$. Then $M^* \in M$ is sound and there is n s.t. $\omega\rho^{n+1} \leq \kappa < \omega\rho^n$ in M^*

... $n = n_i$ ...

Hence $\omega\rho^{n+1} = \kappa$ in M^*, since κ is a cardinal in M and $M^* \in M$. We have: $\pi^W_{-1,i+1} : M^* \xrightarrow[F]{*} W_{i+1}$, where $F = E^{W_i}_{\nu_i}$. But by Lemma 5.1 we have $F_\xi \in M$ for $\xi < \lambda^W_i$. Hence the proof of §2 Lemma 5.2 shows: $\Sigma^{(n)}_1(W_{i+1}) \cap \mathbb{P}(\kappa) \subset M$.

Since $E^{W_j}_{\nu_j}$ is Σ_1-amenable w.r.t. $M_{T(j+1)}$ and $\kappa^{W_j}_j \geq \alpha > \kappa$ for $i+1 \leq_T j+1 \leq_T \theta$, it follows easily that:

(1) $\Sigma^{(n)}_1(W_\theta) \cap \mathbb{P}(\kappa) \subset M$.

But if $A \subset \kappa$ is $\Sigma^{(n)}_1(M^*)$ s.t. $A \notin M^*$, then A is $\Sigma^{(n)}_1(W_{i+1})$ and $A \notin W_{i+1}$, since $\mathbb{P}(\kappa) \cap W_{i+1} \subset M^*$. Hence $\omega\rho^{n+1}_{W_{i+1}} \leq \kappa$. Hence: $\omega\rho^{n+1}_{W_{i+1}} = \kappa$ by (1) + $\mathbb{P}(\gamma) \cap W_{i+1} = \mathbb{P}(\gamma) \cap M^* = \mathbb{P}(\gamma) \cap M$ for $\gamma < \kappa$. Since $\kappa^{W_j}_j \geq \alpha$ for $i+1 \leq_T j+1 \leq_T \theta$, we conclude:

(2) $\omega\rho^{n+1}_{W_\theta} = \kappa$.

We know that $Q_\theta = W_\theta$ and that

Q_θ is a simple iterate of M, since W_θ is a non simple iterate of $\langle N, M, \alpha \rangle$. Hence $\omega\rho^{n+1}_{Q_\theta} = \kappa$. Hence $\kappa_j^{Q_i} \geq \kappa$ for $j+1 \leq_{TQ} \theta$, since otherwise:

$\omega\rho^{n+1}_{Q_\theta} \geq \pi_{0\theta}(\kappa) > \kappa$. But then:

(3) $\Sigma_1^{(n)}(M) \cap \mathcal{P}(\kappa) = \Sigma_1^{(n)}(W_\theta) \cap \mathcal{P}(\kappa)$.

By (1), (3) we have: $\Sigma_1^{(n)}(M) \cap \mathcal{P}(\kappa) \subset M$ hence $\omega\rho^{n+1}_M > \kappa$. Contr!

QED (Case 2.1.4

<u>Case 2.1.5</u> The above fail. Then

(1) $\pi^W_{-1,\theta} = \pi^Q_{0,\theta}$.

prf.

$\pi^M_{0\theta}(\xi) \leq \sigma_\theta \pi^Q_{0,\theta}(\xi)$ since $\sigma_\theta \pi_{0\theta} : M \to^* M_\theta$

Hence $\pi^{W_\theta}_{-1,\theta}(\xi) = \sigma_\theta^{-1} \pi^M_{0\theta}(\xi) \leq \pi^Q_{0,\theta}(\xi)$.

But $\pi^Q_{0\theta}(\xi) \leq \pi^W_{-1,\theta}(\xi)$, since $\pi^W_{-1,\theta} : M \to_{\Sigma^*} Q_\theta$. QED (1)

Let $\pi = \pi^W_{-1\theta} = \pi^Q_{0,\theta}$; $\kappa = \mathrm{crit}(\pi)$

Let $i+1$ = the least $i+1 \leq_{T_W} \theta$,

$j+1$ = " " $j+1 \leq_{T_Q} \theta$.

Then: $\kappa_i^W = \kappa_j^Q = \mathrm{crit}(\pi_{0\theta}^Q)$, and:

(2) $X \in E_{\nu_i,\alpha}^{W_i} \longleftrightarrow \alpha \in \pi(X) \quad (\alpha < \lambda_i^W)$

$X \in E_{\nu_j,\alpha}^{Q_j} \longleftrightarrow \alpha \in \pi(X) \quad (\alpha < \lambda_j^W)$.

Hence $i \neq j$, since otherwise:

$E_{\nu_i}^{W_i} = E_{\nu_i}^{Q_i}$. Contr! Let e.g. $i < j$.

Dann $E_{\nu_i}^{W_j} = E_{\nu_i}^{Q_j} = \emptyset$. But $E_{\nu_j}^{Q_j} | \nu_i$

$= E_{\nu_i}^{W_i}$ by (2), hence

$\langle J_{\nu_i}^{E^{Q_j}}, E_{\nu_j}^{Q_j} | \nu_i \rangle = W_i \| \nu_i$ is

a premouse. Hence $E_{\nu_i}^{Q_j} \neq \emptyset$.

Contr! Similarly for $j < i$.

QED (Case 2.1)

Case 2.2 $\theta \geq 0$ in T^W.

Case 2.2.1 Q_θ is a proper segment of W_θ.

Then $\sigma_\theta \pi^Q_{0,\theta} : M \xrightarrow[\Sigma^*]{} \sigma_\theta(Q_\theta)$, where $\sigma_\theta(Q_\theta)$ is a non simple iterate of M. Contr!

Case 2.2.2 $Q_\theta = W_\theta$ is a non simple iterate of $\langle M, W, \alpha \rangle$.

Then $\sigma_\theta \pi^Q_{0\theta} : M \xrightarrow[\Sigma^*]{} M_\theta$, where M_θ is a non simple it. of M. Contr!

Case 2.2.3 $Q_\theta = W_\theta$ + both sides are simple

Then, letting $\omega\rho^{n+1} \leq \alpha < \omega\rho^n$ in M, we have $\omega\rho^{n+1}_W \leq \alpha$. Since $\kappa^W_i \geq \alpha$ for $i+1 \leq \theta$ in T^W, we know that $\omega\rho^{n+1}_{W_\theta} = \omega\rho^{n+1}_W$ and $\Sigma^{(n)}_1(W_\theta) \wedge \omega\rho^{n+1} = \Sigma^{(n)}_1(W) \wedge \omega\rho^{n+1}$. But then $\omega\rho^{n+1}_{Q_\theta} = \omega\rho^{n+1}_{W_\theta} \leq \alpha$. Hence $\kappa^Q_i \geq \omega\rho^{n+1}_M$ for $i+1 \leq \theta$ in T^Q, since otherwise $\omega\rho^{n+1}_{Q_\theta} > \lambda_i \geq \alpha$,

where $i+1$ is the least $i+1 \leq \theta$ in T^{Q}.
Hence $\omega\rho_{W}^{n+1} = \omega\rho_{W_\theta}^{n+1} = \omega\rho_{M}^{n+1}$.

Clearly then $\underline{\Sigma}_1^{(n)}(Q_\theta) \cap \omega\rho^{n+1} =$
$= \underline{\Sigma}_1^{(n)}(M) \cap \omega\rho^{n+1}$. Let $A =$
$= A_M^{n+1, p_M \upharpoonright (n+1)}$. Then $A \in \underline{\Sigma}_1^{(n)}(W)$,

But $\bar{W} = W^{n,\bar{p}} = h_{\bar{W}}(\alpha \cup q)$, where
$\sigma(\bar{p}) = p_M \upharpoonright n$ and $\sigma(q) = p_M^n \setminus (\alpha+1)$.
Hence A is $\Sigma_1(\bar{W})$ in parameters
from $\bar{p} \cup \alpha$. Hence A is $\Sigma_1(M^{n, p_M \upharpoonright n})$
in parameters from $\alpha \cup (p_M^n \setminus (\alpha+1))$.
Let A be $\Sigma_1(M^{n, p_M \upharpoonright n})$ in
$u \cup p_M^n \setminus (\alpha+1)$, where $u \subset \alpha$ is finite.
Set $p' = (p \setminus \{\alpha\}) \cup u$. Then
$p' <_* p$ and $p' \in P_M^*$. Contr!

QED (Case 2.2.3)

Case 2.2.4 W_θ is a proper segment of Q_θ. Let $\omega\rho^{n+1} \leq \alpha < \omega\rho^n$ in W. By the construction of W there is $A \subset \alpha$ s.t. A is $\underline{\Sigma}_1^{(n)}(W)$ and A codes W. Hence A is $\underline{\Sigma}_1^{(n)}(W_\theta)$, since $\kappa_i \geq \alpha$ for $i < \theta$ in T^W. Hence $A \in Q_\theta$.

Claim $A \in M$

Suppose not. Then $M \neq Q_\theta$. Hence $\theta > 0$ + there is a least i s.t. $E_{\nu_i}^{Q_i} \neq \emptyset$. Hence $M = Q_i$. But $\nu_i > \alpha$, since otherwise $i = 0$ and $\nu_0 = \alpha$. This is impossible by §6 Lemma since λ is then a cardinal in M, where $\lambda =$ = the largest cardinal in $J_{\nu_0}^M$. Hence $\omega\rho^\omega_{M\|\nu_0} \geq \lambda$. Contr! Since $\nu_i > \alpha$ is a cardinal in Q_θ, we conclude: $A \in \mathbb{P}(\alpha) \cap Q_\theta \subset J_{\nu_0}^{E^{Q_\theta}} = J_{\nu_0}^{E^M} \subset M$. QED (Claim).

Since $\sigma(\alpha)$ is regular in M and $\alpha < \sigma(\alpha)$, we have: $A \in J_{\sigma(\alpha)}^{E^M}$ and $J_{\sigma(\alpha)}^{E^M} \models ZFC^-$. But then we can decode A in $J_{\sigma(\alpha)}^{E^M}$ to get: $W \in J_{\sigma(\alpha)}^{E^M} \subset M$. Contr!

QED (Case 2.2.4)

Case 2.2.5 The above cases fail.

Then $W_\theta = Q_\theta$ is a non simple iterate of M. Let $i+1 \leq \theta$ in T^Q be maximal s.t. $\eta_i < ht(Q_\xi)$, where $\xi = T^Q(i+1)$. Let n, A be as in Case 2.2.4. Then $A \in \Sigma_1^{(n)}(Q_{i+1})$, since $\kappa_j^Q \geq \lambda_i^Q \geq \alpha$ for $i+1 < j+1 \leq \theta$ in T^Q. Set: $\kappa = \kappa_i^Q$.

Case 2.2.5.1 $\kappa \geq \alpha$.

Then $A \in \Sigma_1^{(n)}(Q_\xi \| \eta_i) \subset Q_\xi$. We repeat the above proof to get: $A \in M$; hence $W \in M$. Contr!

Case 2.2.5.2 $\kappa < \alpha$.

Then $\xi = 0$. Let $M^* = M \| \eta_i$. Then $M^* \in M$. Set $\rho = \omega\rho_W^{n+1}$. Then $\rho = \omega\rho_{W_\theta}^{n+1} = \omega\rho_{Q_\theta}^{n+1} \leq$

$\leq \alpha \leq \lambda_0 \leq \lambda_i \leq \pi_{0\theta}(\kappa)$.

If $\rho < \pi_{0\theta}(\kappa)$, then $\rho = \omega\rho_{M^*}^{n+1} < \kappa$. Hence κ would not be a cardinal in M, since $M^* \in M$ is sound. But $\kappa < \alpha$ is a cardinal in W, hence in M, since σ is cardinal preserving. Contr!

Thus $\rho = \pi_{0\theta}(\kappa) = \alpha$. Hence α is a limit cardinal in $Q_\theta = W_\theta$, hence in W. Hence $\kappa^{+W} = \kappa^{+Q_\theta} = \kappa^{+Q_i}$ is a cardinal in W, hence in M.
But then $\gamma_i = ht(M)$. Contr!

QED (Lemma 6)