§8 Properties of Mice

The proof of §7 Lemma 6 actually establishes a much more general result for mice M:

Lemma 1 Let M be V. Let $\sigma : W \to M$ witness the goodness of $\langle M, W, \alpha\rangle$. Let $\langle \mathcal{I}^W, \mathcal{I}^Q\rangle$ be the coiteration of $\langle M, W, \alpha\rangle$ with M. Let θ be the length of $\mathcal{I}^Q$. Then:

(a) $\theta \geq 0$ in $\mathcal{I}^W$

(b) W_θ is a simple iterate of W in $\mathcal{I}^W$ and a segment of Q_θ.

(c) W is a mouse.

proof.

(c) follows from (a), (b), since then $\pi^W_{0\theta} : W \xrightarrow[\Sigma^*]{} W_\theta$, where W_θ is a mouse.

We prove (a), (b). If Case 1 of §7 Lemma 6 holds, the coiteration has length 0 and W is a segment of M (since $E^W_{\omega\alpha} = \emptyset$ is impossible as before). Case 2.1 is impossib as before as are Cases 2.2.1 + 2.2.2. Hence (a), (b) hold.

QED (Lemma 1)

<u>Lemma 2</u> Let M be a mouse. Let $p \in P_M^n$. Then $\rho_M^{m+n} = \rho_{M^{n,p}}^m$ (i.e. p can be lengthened to a $q \in P_M^{n+1}$.)

Proof. Suppose not.

Let $\bar M, \bar p$ be defined by: $\bar M^{n,\bar p} = M^{n,p}$, $\bar p \in R_{\bar M}^n$. Let $\sigma : \bar M \to_{\Sigma_1^{(n)}} M$ be defined by: $\sigma : \bar M \to_{\Sigma_1^{(n)}} M$, $\sigma \upharpoonright \omega\rho_M^n = \mathrm{id}$, $\sigma(\bar p) = p$. Then $\rho_{\bar M}^{m+n} = \rho_{\bar M^{n,\bar p}}^m = \rho_{M^{n,p}}^m$, since $\bar p \in R_{\bar M}^n$. Hence $\bar M \neq M$. But $\bar M \notin M$, since otherwise $A_M^{n,p} = A_{\bar M}^{n,\bar p} \in M$. Hence $\bar M$ is not a segment of M.

Hence $E_\nu^M \neq \emptyset$ for a $\nu \geq \omega\rho_M^n$. Hence M is uniquely iterable.

Clearly σ witnesses the goodness of $\langle M, \bar M, \alpha \rangle$, where $\alpha = \omega\rho_M^n = \omega\rho_{\bar M}^n$.

Let $\langle \bar I, I \rangle$ be the coiteration of $\langle M, \bar M, \alpha \rangle$ against M, where $\bar I = \langle \langle \bar M_i \rangle, \langle \bar\nu_i \rangle, \ldots, \bar T \rangle$, $I = \langle \langle M_i \rangle, \langle \nu_i \rangle, \ldots, T \rangle$. Let $\theta = |I|$. Then $\theta \geq 0$ in $\bar T$ and $\bar M_\theta$ is a segment of M_θ. Moreover

$\overline{M}_\theta$ is a simple iterate of $\overline{M}$ in $\overline{\mathcal{J}}$. We show

<u>Claim</u> $\overline{M}_\theta = M_\theta$ and M_θ is a simple iterate of M.

Suppose not. We consider two cases.

<u>Case 1</u> $\overline{M}_\theta$ is a proper segment of M_θ.

Since $\bar{\kappa}_i \geq \alpha$ for $i < \theta$ in $\overline{T}$, we have:

$\alpha = \omega\rho^n_{\overline{M}_\theta}$ and $A^{n\bar{p}}_{\overline{M}} = A^{n, \bar{\pi}_{0,\theta}(\bar{p})}_{\overline{M}_\theta} \in M_\theta$.

Then $M \neq M_\theta$ + hence there is a least i s.t. $E^{M_i}_{\nu_i} \neq \emptyset$. Just as before (§7 Lemma 6 Case 2.2.4) we have $\nu_i > \alpha$ and hence $A = A^{n\bar{p}}_{\overline{M}} \in J^{E^{M_\theta}}_{\nu_i} = J^{E^M}_{\nu_i} \subset M$.

Contr! QED (Case 1)

<u>Case 2</u> Case 1 fails.

Then M_θ is a non simple iterate of M and $\overline{M}_\theta = M_\theta$. Let i be maximal s.t. $i+1 \leq$ is a truncation pt. in $\mathcal{J}$. We know: $\alpha = \omega\rho^n_{M_\theta}$, since $M_\theta = \overline{M}_\theta$. Hence $\kappa_i \geq \alpha$, since otherwise $\kappa_i < \alpha \leq \lambda_i$ and hence $\alpha \leq \lambda_i = \pi_{i,i+1}(\kappa_i) < \omega\rho^n_{M_{i+1}} \leq \omega\rho^n_{M_\theta}$. Contr!

Hence $A \in \mathbb{P}(\kappa_i) \cap \Sigma^{(n)}_1(M_3 \| \gamma_i)$, where $\zeta = T(i+1)$. Hence $A \in M_\zeta$. It follows exactly as in Case 1 that $A \in M$. Contr! QED (Claim).

But then $\kappa_i \geq \alpha$ for $i \leq \theta$ in $\bar{\mathcal{Y}}$ & hence $\omega\rho^h_{\bar{M}} = \omega\rho^h_{M_\theta}$ for $h \geq m$, since $\alpha = \omega\rho^n_{\bar{M}}$. But then $\alpha = \omega\rho^n_{M_\theta} = \omega\rho^n_M$. It follows as in Case 2 that $\kappa_i \geq \alpha$ for $i < \theta$ in $\mathcal{Y}$. Hence $\omega\rho^h_{M_\theta} = \omega\rho^h_M$ for $h \geq m$. Hence:

$$\rho^{m+n}_M = \rho^{m+n}_{M_\theta} = \rho^{m+n}_{\bar{M}} = \rho^m_{\bar{M}^{n,\bar{p}}} = \rho^m_{M^{n,p}}.$$

Contr! QED (Lemma 2)

By §7 Lemma 2.1 and Lemma 6 we know that if $\bar{M} = \mathrm{core}(M)$ and σ is the core map, then $\sigma(p_{\bar{M}}) = p_M$. We prove this in a more general form. We define:

Def Let $\omega\rho^{n+1} \leq \alpha \leq \omega\rho^n$ in M. Set $X = h_{M^{n,p_M}}$ "$(\alpha \cup q)$, where $q = p_M^n \setminus \alpha$. Let $\sigma' : \bar{W} \xrightarrow{\sim} W|X$, where $W = \langle M^{n,p_M} \upharpoonright n$. Let $\bar{M}, \bar{p}$ be s.t. $\bar{M}^{n,\bar{p}} = \bar{W}$, $\bar{p} \in R_{\bar{M}}$. Let $\sigma : \bar{M} \xrightarrow{}_{\Sigma_1^{(n)}} M$ s.t. $\sigma \upharpoonright \bar{W} = \sigma'$; $\sigma(\bar{p}) = p_M \upharpoonright n$. We call $\bar{M}$ the α-core of M ($\mathrm{core}_\alpha(M)$) and σ the α-core map. (Note There might be more than one n satisfying the above, but the choice of n is easily seen not to matter. We have $\sigma : \bar{M} \xrightarrow{}_{\Sigma^*} M$.)

Clearly, $\mathrm{core}(M) = \mathrm{core}_{\omega\rho_M}(M)$.

<u>Def</u> N is <u>sound above</u> α iff whenever $p \in P_N^*$, then each $x \in N$ has the form $x = f(\vec{\xi}, p)$, where f is a good $\Sigma^*(N)$ map and $\vec{\xi} < \alpha$. (Hence N is sound iff N is sound above $\omega\rho_N^\omega$.)

A repetition of the proof of §7 Lemma 3.1 gives:

<u>Lemma 3.1</u> Let $\bar{M} = \mathrm{core}_\alpha(M)$. Then $\bar{M}$ is sound above α iff $\sigma(p_{\bar{M}}) = p_M$, where σ is the α-core map.

But:

<u>Lemma 3.2</u> Let M be a mouse. Let $\bar{M} = \mathrm{core}_\alpha(M)$ with core map σ. Then $\sigma(p_{\bar{M}}) = p_M$ and $\sigma'' P_{\bar{M}}^* \subset P_M^*$.

proof

$\sigma(p_{\bar{M}}) = p_M$ by §7 Lemma 2.1. Now let $\bar{q} \in P_{\bar{M}}^*$. Then $p_{\bar{M}} = \bar{f}(\vec{\xi}, \bar{q})$, where $\vec{\xi} < \alpha$ and $\bar{f}$ is a good $\Sigma^*(M)$ fcn. Hence $p_M = \sigma(p_{\bar{M}}) = f(\vec{\xi}, q)$, where $q = \sigma(\bar{q})$ and f has the same functionally absolute definition over M. Hence $q \in P_M^*$.

QED (Lemma 3.2)

Def Let F be an extender at κ, λ on a suitable $U \subset \mathbb{P}(\kappa)$. Let $X \subset \lambda$. F is generated by X iff for every $\alpha < \lambda$ there are $\beta_1, \ldots, \beta_n \in X$ and $f \in ({}^\kappa\kappa) \cap U^{(n+1)}$ s.t.

$$\langle \alpha, \vec{\beta} \rangle \in F(\{\langle \xi, \vec{\zeta} \rangle \mid \xi = f(\vec{\zeta})\}).$$

(Thus if $\pi : M \xrightarrow[F]{} M'$, this means that $M' =$ the Σ_0 - closure of $\mathrm{rng}(\pi) \cup X$.)

We now attempt to generalize to mice the "condensation lemma" for structures J_α. This says that if $\pi : N \xrightarrow[\Sigma_1]{} J_\alpha$ and N is transitive, then $N = J_{\bar{\alpha}}$ is a segment of J_α.

The best version we can obtain for mice is:

<u>Lemma 4</u> Let M be a mouse and $\bar{M}$ a premouse. Let $\sigma : \bar{M} \xrightarrow[\Sigma_0]{} M$. Set: $\nu = \max\{\xi \mid \sigma \upharpoonright \xi = \mathrm{id}\}$ and let σ be $\Sigma_0^{(n)}$-preserving for $\omega\rho^n_{\bar{M}} > \nu$. Then $\bar{M}$ is a mouse. Moreover, if $\omega\rho^\omega_{\bar{M}} \leq \nu$ and $\bar{M}$ is solid above ν, then one of the following holds:

(a) $\bar{M} = \mathrm{core}_\nu(M)$ and σ is the core map

(b) $\bar{M} = M \| \eta$ for an $\eta < ht(M)$.

(c) $\pi : M\|\eta \xrightarrow[E^M_\mu]{*} \bar{M}$, where:

(i) $\nu \leq \eta < ht(M)$ and $\omega\rho^\omega_{M\|\eta} < \nu$

(ii) $\mu \leq \omega\eta$

(iii) $\nu = \kappa^{+M\|\eta}$ where $\kappa = \mathrm{crit}(E^M_\mu)$

(iv) E^M_μ is generated by $\{\kappa\}$.

<u>Remarks</u>

(1) If $\omega\rho^\omega_{\bar{M}} < \nu$ and (b) holds, then $\eta =$ the least $\eta \geq \nu$ s.t. $\omega\rho^\omega_{M\|\eta} < \nu$.

(2) If (c) holds, then η has the same definition and μ is uniquely determined by (iv).

We prove Lemma 4 by R-induction on M where: $\bar{N} R N \longleftrightarrow_{Df}$ $\bar{N}$ is a non simple iterate of the mouse N. R is well founded. Note first that if $\nu = On \cap \bar{M}$, then $\bar{M}$ is a segment of M + there is nothing to prove. (If then $\bar{F} = F_\nu^{\bar{M}} \neq \emptyset$, $F = F^M_{On \cap M}$, then $\sigma(\bar{F}(X)) = F(X)$ for $X \in \mathbb{P}(\kappa) \cap \bar{M} = \mathbb{P}(\kappa) \cap M$, and it follows easily that $\bar{M} = M$.) Hence we assume w.l.o.g. that $\nu = crit(\sigma)$. If $\omega\rho^\omega_{\bar{M}} > \nu$, then $\sigma: \bar{M} \longrightarrow_{\Sigma^*} M$ and hence $\bar{M}$ is a mouse, which is all there was to prove. So let $\omega\rho^\omega_{\bar{M}} \leq \nu$. We can assume w.l.o.g. that:

(*) σ witnesses the goodness of $\langle \bar{M}, M, \nu \rangle$.

To see this, suppose that (*) fails. Then there is $\alpha < \nu$ which is a cardinal in $\bar{M}$ but not in M. Hence $\omega\rho^1_{\bar{M}} \leq \nu$, since otherwise σ is not $\Sigma_0^{(1)}$-preserving. Let $\bar{M} = \langle J_{\bar{\beta}}^{\bar{E}}, \bar{F} \rangle$ and $M = \langle J_\beta^E, F \rangle$.

Claim $\bar{F} = \emptyset$

If not, then, letting $\bar{\kappa} = \mathrm{crit}(\bar{F})$, $\bar{\lambda} = \bar{F}(\bar{\kappa})$, $\lambda = \sigma(\bar{\lambda})$ we have $\lambda = F(\sigma(\bar{\kappa}))! =$ the largest cardinal in M. Hence α is a cardinal in $M\|\lambda = \sigma(\bar{M}\|\bar{\lambda})$. Hence α is a cardinal in M. Contr! QED (Claim)

Now let $\widetilde{\omega\beta} = \sup \sigma''\omega\beta$, $\tilde{M} = \langle J^{E^M}_{\tilde{\beta}}, \emptyset\rangle$.

Then $\sigma : \bar{M} \longrightarrow_{\Sigma_1} \tilde{M}$ cofinally, where $\omega\rho^1_M \leq \nu$ and σ witnesses the goodness of $\langle \tilde{M}, \bar{M}, \nu\rangle$. Thus $\tilde{M}$ is a proper segment of M & hence a sound mouse. It suffices to prove the Theorem with $\tilde{M}$ in place of M, noting that if (a) holds, for $\tilde{M}$, then $\bar{M} = \tilde{M}$ by soundness.

From now on assume (*)

Let $\omega\rho^{n+1} \leq \nu < \omega\rho^n$ in $\bar{M}$, where $\sigma : \bar{M} \longrightarrow_{\Sigma_0^{(n)}} M$.

Case 1 $E^M_\xi = \emptyset$ for $\xi \geq \nu$.

Then $E^{\bar{M}}_\xi = \emptyset$ for $\xi \geq \nu$ and $\bar{M}$ is a segment of M, hence a mouse. If $\bar{M}$ is a proper segment, we are done. Let $\bar{M} = M$. Then $\omega\rho^h_{\bar{M}} = \omega\rho^h_M$ for all h and it follows easily that $\sigma : \bar{M} \to_{\Sigma^*} M$. Hence $\sigma(p_M) = p_M$. But $\sigma \upharpoonright \nu = id$ and M is sound above ν. Hence $\sigma = id$. Contr! QED (Case 1)

Case 2 Case 1 fails.

Then M is uniquely smoothly iterable. Let $\langle \bar{\mathcal{I}}, \mathcal{I} \rangle$ be the coiteration of $\langle \bar{M}, M, \nu \rangle$ against M, where:

$\bar{\mathcal{I}} = \langle \langle \bar{M}_i \rangle, \ldots, \langle \bar{\pi}_{ij} \rangle, \bar{T} \rangle$, $\mathcal{I} = \langle \langle M_i \rangle, \ldots, \langle \pi_{ij} \rangle, T \rangle$

and $|\mathcal{I}| = \theta$. Then $\theta \geq 0$ in $\bar{\mathcal{I}}$ and $\bar{M}_\theta$ is a simple iterate of M above ν in $\bar{\mathcal{I}}$ and a segment of M_θ. Hence $M_{\bar{\theta}}$ is a mouse. But $\bar{\pi}_{0\theta} : \bar{M} \to_{\Sigma^*} M_{\bar{\theta}}$. Hence $\bar{M}$ is a mouse.

From now on assume that $\bar{M}$ is sound above ν. Since $\bar{\pi}_{0\theta} \upharpoonright \nu = \mathrm{id}$, $\bar{\pi}_{0\theta}(p_{\bar{M}}) = p_M$, it follows that $\bar{M} = \mathrm{core}_\nu(\bar{M}_\theta)$ and $\bar{\pi}_{0\theta}$ is the core map. From now on set: $\bar{\pi} = \bar{\pi}_{0,\theta}$, $\pi = \pi_{0\theta}$. We consider three cases:

<u>Case 2.1</u> $M_\theta = \bar{M}_\theta$ is a simple iterate of M in $\mathcal{T}$.

Then $\omega\rho^{n+1}_{\bar{M}} = \omega\rho^{n+1}_{M_\theta} \leq \nu$, since $\bar{\kappa}_i \geq \nu$ for $i < \theta$ in $\bar{T}$, $i \in \bar{D}$. Hence

(1) $\kappa_i \geq \omega\rho^{n+1}_M$ for $i < \theta$ in T, $i \in D$.

Hence:

(2) $\rho^h_{\bar{M}} = \rho^h_M = \rho^h_{M_\theta}$ for $h > n$.

(3) $\sigma : \bar{M} \longrightarrow_{\Sigma_0^{(n)}} M$ cofinally.

proof.

Suppose not. Set $\omega\gamma = \sup \sigma''\omega\rho^n_{\bar{M}}$.

Let $\bar{p} = p_{\bar{M}}$, $p = \sigma(\bar{p})$, $A = A^{n,p,M}$

$Q = (M^{n,p,M} | \gamma) = \langle J^{E^M}_\gamma, A \cap J^{E^M}_\gamma \rangle$.

Then $\sigma \restriction \bar{M}^{n,\bar{p}\restriction n} : \bar{M}^{n,\bar{p}\restriction n} \xrightarrow{}_{\Sigma_0} Q$ cofinally and $\sigma \restriction \omega\rho_M^{n+1} = \mathrm{id}$. Set:

$B = \omega\rho_{\bar{M}}^{n+1} \cap A^{n+1, p_{\bar{M}} \restriction n+1}$. Then

$B \notin \bar{M}$ and B is $\Sigma_1(Q)$. But $Q \in M$. Hence $B \in M$. By (1) and the fact that $\bar{\kappa}_i \geq \nu$ for $i < \theta$ in T, we conclude:

$B \in \mathbb{P}(\omega\rho^{n+1}) \cap M_\theta = \mathbb{P}(\omega\rho^{n+1}) \cap \bar{M}_\theta =$

$= \mathbb{P}(\omega\rho^{n+1}) \cap \bar{M}$. Contr! QED (3).

Hence by (2):

(4) $\sigma : \bar{M} \xrightarrow{}_{\Sigma^*} M$,

Hence $\sigma(p_{\bar{M}}) = p_M$. Since $\bar{M}$ is sound above ν and $\sigma \restriction \nu = \mathrm{id}$, we conclud $\bar{M} = \mathrm{core}_\nu(M)$ and σ is the core map. QED (Case 2.1)

Case 2.2 $\bar{M}_\theta$ is a proper segment of M_θ.

Then $\bar{M}_\theta$ is sound and $\omega\rho^{n+1}_{\bar{M}_\theta} = \omega\rho^{n+1}_{\bar{M}} \leq \nu$.
It follows easily that $\bar{M}_\theta = \bar{M}$ and
$\bar{\pi} = \mathrm{id}$. If $0 < \theta$, then $\nu_0 > \mathrm{On} \cap \bar{M}$,
since otherwise $\nu_0 \leq \mathrm{On} \cap \bar{M}$ is a cardinal
in M_θ + hence $\omega\rho^{n+1}_{\bar{M}} \geq \nu_0$. Hence
$\nu_0 = \nu = \omega\rho^{n+1}_{\bar{M}}$. But then λ_0 is
a cardinal in $J^{E^M}_\nu$, hence in $\bar{M}$,
hence in M. But $\omega\rho^\omega_{M\|\nu_0} < \lambda_0$.
Contr!
But then $\bar{M}$ is a proper segment of
$J^{E^M}_{\nu_0}$, hence of M. QED (Case 2.2

Case 2.3 $M_\theta = \bar{M}_\theta$ is a non simple
iterate of M in $\mathcal{Y}$.

This is the most difficult case.
We shall prove (c).
Let i_0 be minimal in $D = \{i \mid E^{M_i}_{\nu_i} \neq \emptyset\}$
Then $M_{i_0} = M$ and $\lambda_{i_0} = \pi_{i_0, i_0+1}(\kappa_{i_0}) =$
$=$ the largest cardinal in $J^{E^M}_{\nu_{i_0}}$.
Clearly $\nu_{i_0} \geq \nu$. Moreover:

(1) $\lambda_{i_0} \geq \nu$.

proof. Otherwise $\nu = \nu_{i_0}$ and $i_0 = 0$. But then λ_{i_0} is a cardinal in $\overline{M}$, hence in M. But $\omega\rho^{\omega}_{M \| \nu_0} < \lambda_{i_0}$. Contr!

But then $\kappa_i \geq \nu$ for $i \leq \theta$ in T, $i \in D$. Since ν is a cardinal in M_θ, we conclude:

(2) ν is a cardinal in $J^{E^M}_{\nu'_{i_0}}$

Since ν_{i_0} is a cardinal in M_i for $i > i_0$, we conclude:

(3) ν is a cardinal in M_i for $i > i_0$.

Now let i be maximal s.t. $i+1 \leq \theta$ in T and $\eta_i < ht(M_\xi)$, $\xi = T(i+1)$.

(4) $\xi = T(i+1) = i_0$

proof.

Suppose not. Then ν_{i_0} is a cardinal in M_ξ. Hence $\kappa_i \geq \omega\rho^{\omega}_{M_\xi \| \eta_i} \geq \nu_{i_0} > \nu$.

Hence $\omega\rho^{\omega}_{\overline{M}} = \omega\rho^{\omega}_{M_\theta} = \omega\rho^{\omega}_{M_\xi \| \eta_i} > \nu$.

Contr! (QED(4))

(5) $\varkappa_i < \nu$.

Suppose not. Let $x \in \bar{M}$. Then $x = \bar{f}(\xi, p_{\bar{M}})$ where $\bar{f}$ is a good $\Sigma_1^{(n)}$ function and $\xi < \nu$. Then $\pi^{-1}\bar{\pi}(x) = f(\xi, p_{M\|\eta_i}) \in M\|\eta_i$, where f has the same functionally absolute definition over $M\|\eta_i$. Conversely $\bar{\pi}\pi^{-1}(x) \in \bar{M}$ for $x \in M\|\eta_i$ by the same argument. Hence $\bar{M} = M\|\eta_i$, $\pi^{-1}\bar{\pi} = \mathrm{id}$, and Case 2.2 holds. Contr. QED (5)

(Here $\bar{\pi} = \bar{\pi}_{0\theta}$, $\pi = \pi_{0\theta}$).

Now let $\varkappa = \varkappa_i$, $\eta = \eta_i$

(6) $\kappa^{+M\|\eta} = \nu$,

since otherwise $\kappa^{+M\|\eta} = \kappa^{+J^{E^{M_i}}_{\nu_i}}$ is a cardinal in M. Hence $\eta = ht(M)$. Contr!

Hence by definition:

(7) η = the least $\eta \geq \nu$ s.t. $\omega\rho^\omega_{M\|\eta} < \nu$.

(8) $\kappa = \omega\rho^\omega_{M\|\eta}$

prf.

($\leq$) $\omega\rho^\omega_{M\|\eta} < \nu$

($\geq$) κ_i is a cardinal in M QED (8)

Since $\mathrm{crit}(\bar{\pi}) \geq \nu$, $\mathrm{crit}(\pi) = \kappa < \nu$, $\omega\rho^{n+1} < \nu \leq \omega\rho^n$ in $\bar{M}$, $\pi, \bar{\pi}$ are Σ^*-preserving and $\nu = \kappa^{+\bar{M}\|\eta}$, we conclude:

(9) $\omega\rho^h_{M\|\eta} = \omega\rho^h_{M_\theta} = \omega\rho^h_{\bar{M}}$ for $h > n$.

Clearly:

(10) $\pi^* : M\|\eta \longrightarrow_{\Sigma^*} \bar{M}$ where $\pi^* = \bar{\pi}^{-1}\pi$,

and $\kappa = \mathrm{crit}(\pi^*)$.

(11) $\sup \pi^{*}{}''\omega\rho^n_{M\|\eta} = \omega\rho^n_{\bar{M}}$

prf.

$\sup \pi''\omega\rho^n_{M\|\eta} = \omega\rho^n_{M_\theta}$ since $\kappa_i \geq \kappa$ for $i+1 \leq \theta$ in T

$\sup \bar{\pi}''\omega\rho^n_{\bar{M}} = \omega\rho^n_{M_\theta}$ " $\bar{\kappa}_i \geq \nu$ for $i+1 \leq \theta$ in $\bar{T}$.

Def F at κ, ν on $M\|\eta$ is defined by:

$F(X) = \pi^*(X) \cap \nu$.

(Hence $F(X) = \pi(X) \cap \nu$, since $\pi \restriction \nu = \mathrm{id}$.)

(12) $(\pi^* \restriction H^n_{M\|\eta}) : H^n_{M\|\eta} \longrightarrow_F H^n_{\bar{M}}$

prf.

We must show: $H^n_{\bar{M}}$ = the Σ_0 closure of $\nu \cup \mathrm{rng}(\pi^* \restriction H^n_{M\|\eta})$. Let $x \in H^n_{\bar{M}}$, $x = f(\xi, p_{\bar{M}})$, where $\xi < \nu$ and f is a good $\Sigma_1^{(n)}$ fcn. Let:

$y = f(z, w) \longleftrightarrow \bigvee u^n P(u^n, y, z, w)$,

where P is $\Sigma_0^{(n)}$ + this def. is functionally absolute. Let $\gamma < \omega\rho^n_{M\|\eta}$ s.t. $\bigvee u \in J^{E^{\bar{M}}}_{\pi^*(\gamma)}\ P(u, x, \xi, p_{\bar{M}})$. Let $\bar{P}$ have the same $\Sigma_0^{(n)}$ def. over $M\|\eta$ + set:

$$g(\xi) = \begin{cases} x \text{ if } \bigvee u \in J^{E^M}_\gamma\ \bar{P}(u, x, \xi, p_{M\|\eta}) \\ 0 \text{ if not} \end{cases}$$

for $\xi < \kappa$. Then $g \in H^n_{M\|\eta}$ and $x = \pi^*(g)(\xi)$. QED (12)

(13) $\pi^* : M\|\eta \longrightarrow^*_F \bar{M}$,

since $\bar{M}$ = the closure of $H^n_{\bar{M}}$ under good $\Sigma_0^{(n-1)}$ fcns if $n > 0$, by soundness above $\nu \leq \omega\rho^n_{\bar{M}}$.

Now set: $\lambda^* = \pi^*(\kappa)$, $\nu^* = \pi^*(\nu)$. Define an extender F^* at κ, λ^* on $M\|\eta$ by: $F^*(X) = \pi^*(X)$. Then $F = F^* | \nu$ and by (13):

(14) $\pi^* : M\|\eta \longrightarrow^*_{F^*} \overline{M}$.

We also note:

(15) F^* is generated by $\{\kappa\}$.

prf.

Let $\alpha < \lambda^*$. Then $\alpha = \pi^*(f)(\beta)$ for a $\beta < \nu$, $f \in ({}^{\kappa}\kappa)^{M\|\eta}$ by (12).

Let $a \subset \kappa^2$, $a \in M\|\eta$ s.t. $\beta = \overline{\mathrm{tp}}(a)$.

For $\xi < \kappa$ set: $g(\xi) = f(\overline{\mathrm{tp}}(a \cap \xi^2))$.

Then $\beta = \overline{\mathrm{tp}}(a)$ ~~...~~ $= \overline{\mathrm{tp}}(\pi^*(a) \cap \kappa^2$

and $\alpha = \pi^*(g)(\kappa)$. QED (15)

(16) $\overline{N} = \langle J^{E^{\overline{M}}}_{\nu^*}, F^* \rangle$ is amenable, by §1 Lemma 6.

The Theorem will be proven when we show:

<u>**Claim**</u> $\nu^* \leq \omega\eta$ and $F^* = E^M_{\nu^*}$.

We proceed by cases as follows:

Case 2.3.1 $\nu_i \in M_i$.

Then $N = \langle J^{E^{M_i}}_{\nu_i}, E^{M_i}_{\nu_i} \rangle$ is sound.

(17) $E^{M_i}_{\nu_i} | \nu = F$

prf.

Let $\alpha < \nu$. Then $\pi(\alpha) = \alpha$ and:

$\alpha \in E^{M_i}_{\nu_i}(X) \leftrightarrow \alpha \in \pi_{0,i+1}(X) \leftrightarrow$

$\leftrightarrow \alpha \in \pi(X) \leftrightarrow \alpha \in \pi^{-1}\pi(X) = \pi^*(X)$

$\leftrightarrow \alpha \in F(X)$. QED (17)

Since $\pi^* : M\|\eta \xrightarrow{*}_{F} \bar{M}$, $\pi_{0,i+1} : M\|\eta \xrightarrow{*}_{E^{M_i}_{\nu_i}} M_{i+1}$

and $\langle id, id \rangle : \langle M\|\eta, F \rangle \longrightarrow^{*} \langle M\|\eta, E^{M_i}_{\nu_i} \rangle$,

we can define $k : \bar{M} \xrightarrow{}_{\Sigma^*} M_{i+1}$ by:

$k(\pi^*(f)(\alpha)) = \pi_{0,i+1}(f)(\alpha)$ for

$f \in \Gamma^*(\kappa, M\|\eta)$. Clearly:

(18) $k \upharpoonright (\nu + 1) = id$

Set: $\bar{k} = k \upharpoonright \bar{N}$. Then:

(19) $\bar{k} : \bar{N} \xrightarrow{}_{\Sigma_1} N$

prf.

$k\pi^* = \pi_{0,i+1}$. Hence $k(\nu^*) = \nu_i$ and

$\bar{k} : J^{E^{\bar{N}}}_{\nu^*} \xrightarrow{}_{\Sigma_\omega} J^{E^{M_i}}_{\nu_i}$. Moreover, $\bar{k}$ is cofinal in $J^{E^{M_i}}_{\nu_i}$, since $\nu^* = \sup \pi^{*}{}''\nu$ and $\nu_i = \sup \pi_{0,i+1}{}''\nu$. Thus it suffices to show: <u>Claim</u> $\bar{k}(u \cap F^*) = \bar{k}(u) \cap F^N$ for $u \in \bar{N}$. This follows from:

<u>Claim</u> $k(F^* \upharpoonright u) = F^N \upharpoonright u$ for $u \in \bar{N}$.

Assume w.l.o.g. $u \cap \mathcal{P}(\kappa) \neq \emptyset$. Let $\mathcal{P}(\kappa) \cap u = \{A_i \mid i < \kappa\}$, where $A = \langle A_i \mid i < \kappa \rangle \in \bar{N}$. Then for $A^* = \pi^*(A)$, $A' = \pi_{0,i+1}(A)$, we have $\bar{k}(A^*) = A'$ and:

$$\bar{F} \upharpoonright u = \{\langle A^*_j \cap \kappa, A^*_j \rangle \mid j < \kappa\}$$

$$F^N \upharpoonright u = \{\langle A'_j \cap \kappa, A'_j \rangle \mid j < \kappa\}.$$

Hence $\bar{k}(\bar{F} \upharpoonright u) = F^N \upharpoonright u$. QED (19)

(20) $\omega\rho^1_{\bar{N}} \leq \nu$ and $\bar{N}$ is sound above ν with $p_{\bar{N}} \setminus \nu = \emptyset$.

proof.

<u>Claim</u> $h_{\bar{N}}(\nu) = \bar{N}$.

(a) $\lambda^* \subset h_{\bar{N}}(\nu)$, since if $\alpha < \lambda^*$, there is $f \in ({}^{\kappa}\kappa)^{\bar{N}}$, $\xi < \nu$ s.t. $\alpha = \pi^*(f)(\xi)$. But then f = the ξ-th element of $J^{E^{\bar{N}}}_{\nu^*}$ for a $\xi < \nu$ and α = that α s.t. $\langle \delta, \alpha \rangle \in \bar{F}(\{\langle \delta, \beta \rangle < \kappa \mid f(\delta) = \beta\})$

QED (a)

(b) $\nu^* \subset h_{\bar N}(\nu)$.

It suffices to show: $\pi^{*}{}''\nu \subset h_{\bar N}(\nu)$, since then $\pi^{*}{}''\nu$ is cofinal in ν^* and for each $\lambda \leq \xi < \nu^*$ there is $f \in \bar N$ mapping λ^* onto ξ, where $\lambda^* = \pi^*(\bar\kappa) \in h_{\bar N}(\nu)$. Let $\xi = \pi^*(\bar\xi)$. Let $\bar a \in \bar N$, $\bar a \subset \kappa^2$ s.t. $\mathrm{otp}(\bar a) = \xi$. Then $\bar a \in h_{\bar N}(\nu)$. Hence $a = F(\bar a) \in h_{\bar N}(\nu)$. Hence $\xi = \mathrm{otp}(a) \in h_{\bar N}(\nu)$.

QED (20)

We are arguing by induction on the relation R and it is obvious that $\bar N$ R M. Hence, by (20) and the ind. hyp. one of the following holds:

(a) $\bar N = \mathrm{core}_\delta(N)$ + π^* is the core map, where $\delta = \mathrm{crit}(\pi^*)$

(b) $\bar N$ is a proper segment of N.

(c) $\pi' : N\|\eta' \longrightarrow^{*}_{E^N_\xi} \bar N$, where $\xi \leq \omega\eta$ and η' is least s.t. $\omega\rho_{N\|\eta'} < \delta$ and $\delta = \mathrm{crit}(\pi^*) = \kappa'^{+N\|\eta'}$, where $\kappa' = \mathrm{crit}(E^N_\xi)$.

If (a) holds, then $\bar{N} = N$, since N is sound.

If (b) holds, then $\bar{N}$ is a segment of N.

(c) cannot hold, since then $\kappa' \geq \nu'$ and hence $\bar{N}$ would not be sound above ν. Contr!

Thus $\bar{N}$ is a segment of N, which is a proper segment of M_i.

If $i = i_0$, then $M_i = M$ and $\nu^* \leq \nu_{i_0} < \eta$.

Now let $i > i_0$. Then ν_{i_0} is a cardinal in M_i. But $\omega\rho^1_{\bar{N}} \leq \nu < \nu_{i_0}$. Hence $\bar{N}$ is a proper segment of $J^{E^M}_{\nu_{i_0}}$, hence of $M \| \eta$. QED (Case 2.3.1)

Case 2.3.2

<u>Case 2.3.2</u> Case 2.3.1 fails and M_i is a non simple iterate of M in $\mathcal{J}$.

Then $\nu_i = On \cap M_i$. (17) $(E^{M_i}_{\nu_i} | \nu = F)$ follows exactly as before.

Let j = the maximal j s.t. $j+1 \leq i$ in T and $j+1$ is a truncation point. Let $\bar{\xi} = T(j+1)$. Since $\nu_j \geq \nu_{i_0} > \nu$, we know that:

(21) $(E^{M_{j+1}}_{On \cap M_{j+1}} | \nu) = (E^{M_i}_{\nu_i} | \nu) = F$.

Set: $F' = E^{M_\xi \| \eta_j}_{\omega \eta_j}$. Then

$crit(F') \leq crit(E^{M_{j+1}}_{On \cap M_{j+1}})$. But

(22) $crit(E^{M_{j+1}}_{On \cap M_{j+1}}) = \kappa$,

since $\kappa_h \geq \lambda_j \geq \lambda_{i_0}$ for $h > j$.

Hence:

(23) $\kappa_j > \nu$,

since otherwise $crit(F') < \kappa_j$, in which case $crit(E^{M_{j+1}}_{On \cap M_{j+1}}) = crit(F') < \kappa_j \leq \kappa$, or else $crit(F') \geq \kappa_j$ and

$crit(E^{M_{j+1}}_{On \cap M_{j+1}}) \geq \pi_{\xi, j+1}(\kappa_j) = \lambda_j > \kappa$,

Contr! QED (23)

Hence:

(24) $\mathrm{crit}(F') = \kappa$, and

$$F' | \nu = E^{M_{l+1}}_{\mathrm{On} \cap M^c_{l+1}} | \nu = F.$$

Since $\eta_\xi < \mathrm{ht}(M_\xi)$, we know that $M_\xi \| \eta_\xi = \langle J^{E^{M_\xi}}_{\eta_\xi}, F' \rangle$ is sound.

We repeat the above argument with $M_\xi \| \eta_\xi$ in place of N to show that $\bar{N}$ is a segment of $M_\xi \| \eta_\xi$.

If $\xi = 0$, then $\kappa_\xi < \lambda_{i_0} \leq \eta$. Hence

$$\kappa_\xi^{+M_\xi \| \eta_\xi} < \eta \quad \text{and} \quad \eta_\xi \leq \eta.$$

Thus $\bar{N}$ is a segment of $M \| \eta$.

Now let $\xi > 0$. Then $\xi \geq i_0 + 1$.

Hence ν_{i_0} is a cardinal in M_ξ.

Since $\omega\rho^1_{\bar{N}} \leq \nu < \nu_{i_0}$, it follows that $\bar{N}$ is a proper segment of $J^{E^M}_{\nu_{i_0}}$, hence of $M \| \eta$.

QED (Case 2.3.2)

<u>Case 2.3.3</u> The above cases fail.

We show that this cannot occur.

We have: $\nu_i = On \cap M_i$ and M_i is a simple iterate of M in $\mathcal{I}$, $i \neq i_0$, since $\nu_{i_0} < \eta < ht(M)$. Let j be minimal s.t. $j+1 \leq i$ in T. Then $T(j+1) = 0$. Then $\kappa_j < \eta$, since otherwise $M \| \eta$ is a segment of M_{j+1}^* hence of M_{i+1}, since $\kappa_h > \kappa_j$ for $h > j$. Hence $M \| \eta$ is a non simple iterate of itself. Contr! But then $\kappa_j \leq$ $\leq \kappa =$ the $M\|\eta$ - predecessor of ν, since κ_j is a cardinal in M. Let $\bar{\kappa} = \mathrm{crit}(E_{On \cap M})$. If $\bar{\kappa} < \kappa_j$, then $\kappa = \mathrm{crit}(E_{On \cap M_{j+1}}) = \bar{\kappa} < \kappa_j \leq \kappa$. Contr! If $\kappa_j \leq \bar{\kappa}$, then $\kappa \geq$ $\geq \mathrm{crit}(E_{On \cap M_{j+1}}) \geq \pi_{0,j+1}(\kappa_j) = \lambda_j >$ $> \nu > \kappa$. Contr!

QED (Lemma 4)

<u>Lemma 4.1</u> Let $M = \langle J_\nu^E, F\rangle$ be a mouse with $F \neq \emptyset$, $\lambda = \text{length}(F)$. Let $\kappa = \text{crit}(F)$. Then $\lambda = \sup\{k(f)(\kappa) \mid f \in ({}^\kappa\kappa)^M\}$ where $k : M \xrightarrow[F]{*} N$.

proof.

Suppose not. Let $\bar\lambda = \sup\{k(f)(\kappa) \mid f \in ({}^\kappa\kappa)^M\}$, $\bar\lambda < \lambda$. It is easily seen that:

(1) $\bar\lambda$ is a limit cardinal in M.

(2) $(\alpha < \bar\lambda \wedge f \in ({}^\kappa\kappa)^M) \rightarrow k(f)(\alpha) < \bar\lambda$.

prf.

Let $\pi(g)(\kappa) > \alpha$, $g \in ({}^\kappa\kappa)^M$. Set:

$h(\xi) = \text{lub}\{f(\zeta) \mid \zeta < g(\xi)\}$ for $\xi < \kappa$. Then $h \in ({}^\kappa\kappa)^M$ and $k(f)(\alpha) < k(h)(\kappa) < \bar\lambda$. QED(2).

Let $\bar F = F|\bar\lambda$. Then

$\langle id, id\restriction\bar\lambda\rangle : \langle M, \bar F\rangle \xrightarrow{*} \langle M, F\rangle$.

Hence there is $\bar k : M \xrightarrow[\bar F]{*} \bar N$ and $\sigma : \bar N \xrightarrow[\Sigma^*]{} N$ defined by:

$\sigma(\bar k(f)(\alpha)) = k(f)(\alpha)$. Clearly $\sigma\restriction\bar\lambda = id$ and $\sigma(\bar\lambda) = \sigma\bar k(\kappa) = k(\kappa) = \lambda$.

Hence σ witnesses the goodness of $\langle N, \bar{N}, \bar{\lambda} \rangle$. Let $\langle \bar{\mathcal{J}}, \mathcal{J} \rangle$ be the coiteration of $\langle N, \bar{N}, \bar{\lambda} \rangle$ against N with $\bar{\mathcal{J}} = \langle \langle \bar{N}_i \rangle, \ldots, \langle \bar{\pi}_{ij} \rangle, \bar{T} \rangle$, $\mathcal{J} = \langle \langle N_i \rangle, \ldots, \langle \pi_{ij} \rangle, T$

Then $\bar{N}_\theta$ is a simple iterate of $\bar{N}$ in $\bar{\mathcal{J}}$ and a segment of N_θ. Hence $\bar{N}_\theta$ is a mouse. Hence so is $\bar{N}$. Let $\bar{\tau} = \bar{\lambda}^{+\bar{N}}$. $\bar{\pi}_i \geq \bar{\lambda}$ for $i+1 \leq \theta$ in $\bar{\mathcal{J}}$. Hence $\nu_i \geq \bar{\tau}$ for $i+1 \leq \theta$ in $\bar{\mathcal{J}}$, since otherwise $\bar{\pi}_i^{+J_{\nu_i}^{E^{M_i}}}$ is not a cardinal in $\bar{N}_{T(i+1)}$ + $\bar{N}_\theta$ is a non simple iterate of $\bar{N}$. Contr! Hence $J_{\bar{\tau}}^{E^{\bar{N}}} = J_{\bar{\tau}}^{E^{\bar{N}_\theta}} = J_{\bar{\tau}}^{E^{N_\theta}}$. Let i_0 be minimal s.t. $E_{\nu_{i_0}}^{M} \neq \emptyset$ (if there is such). Then $\nu_{i_0} \geq \bar{\tau}$, since otherwise $\bar{\lambda} < \nu_{i_0} < \bar{\tau}$ and ν_{i_0} is a cardinal in N_θ, hence in $\bar{N}_\theta$. Contr! Thus $J_{\bar{\tau}}^{E^{\bar{N}}} = J_{\bar{\tau}}^{E^{N}}$. Thus $\bar{\tau} \leq \bar{\lambda}^{+N}$, since $\bar{\lambda}$ is the largest cardinal in $J_{\bar{\tau}}^{E^{N}}$.

Hence $\bar\tau < \lambda$ + hence $J^{E^{\bar N}}_{\bar\tau} = J^{E^M}_{\bar\tau}$.

Clearly $\langle J^{E^M}_{\bar\tau}, \bar F \rangle = \langle J^{E^{\bar N}}_{\bar\tau}, \bar F \rangle$ is a premouse, contradicting §6 Cor 7.1. QED (Lemma 4.1)

As a corollary of Lemma 4:

<u>Cor 4.2</u> Let $M = \langle J^E_\nu, F \rangle$ as above be sound. Let $\beta < \lambda$. Either β generates F or $F|\beta \in M$.

Proof.

Let (w.l.o.g.) $\kappa < \beta$. Let

$\bar k : (J^E_{\kappa^+})^M \xrightarrow[F|\beta]{} J^{\bar E}_{\bar\nu}$

$k : (J^E_{\kappa^+})^M \xrightarrow[F]{} J^E_\nu$.

Then $M = \langle J^E_\nu, F \rangle$. Set:

$\bar M = \langle J^{\bar E}_{\bar\nu}, \bar F \rangle$. Then $\bar M$ is amenab

(where $\bar F(X) = \bar k(X)$ for $X \subset \kappa$.)

Let $\sigma : J^{\bar E}_{\bar\nu} \longrightarrow J^E_\nu$ be defined by $\sigma(\bar k(f)(\alpha)) = k(f)(\alpha)$. Then $\sigma : \bar M \xrightarrow[\Sigma_0]{} M$ cofinally by the proof in Lemma 4.1. Moreover

$\omega\rho^1_{\bar{M}} \leq \beta \leq \mathrm{crit}(\sigma)$,
since β generates $\bar{F}$.

Hence σ witnesses the goodness of $\langle M, \bar{M}, \delta \rangle$, where $\delta = \mathrm{crit}(\sigma)$. Hence one of the following holds:

(a) $\bar{M} = \mathrm{core}_\delta(M)$

(b) $\bar{M}$ is a proper segment of M

(c) $\pi : M\|\eta \xrightarrow[E^M_\tau]{} \bar{M}$, where $\eta < ht(M)$ and $\tau \leq \eta$.

In case (a), $\bar{M} = M$ & β generates $F = \bar{F}$, since M is sound. In the other two cases $\bar{M} \in M$; hence $F|\beta = \bar{F}|\beta \in M$. QED (Cor 4.2)

<u>Cor 4.3</u> Let M be as above. There is $\beta < \lambda$ which generates F.

proof.

Set: $D_\beta = \{\pi(f)(\alpha) \mid f \in M, f : \kappa \to \kappa^{+M}, \alpha < \beta\}$ for $\beta \leq \lambda$. Then $M \subset D_\lambda$. Pick $\beta < \lambda$ with $\omega\rho^1_M \leq \beta$ and

$p_M \subset D_\beta$. Let $\bar{M}$ be as above with $F|\beta$, ~~F~~ Let $\sigma : \bar{M} \to M$ be as above, $\bar{p} = \sigma^{-1}(p_M)$. Then $A^{1,\bar{p}}_{\bar{M}} = A^{1,p_M}_M$. Hence $\bar{M} \notin M$, since otherwise $A^{1,p_M}_M \in M$. Hence $\bar{M} = M$. QED (Cor 4.3)

As a corollary of the proof of Lemma 4:

<u>Lemma 5</u> Let $\rho = \omega\rho^n_M \in M$, $\tau = \rho^{+M}$. Let $\bar{M} = \mathrm{core}_\rho(M)$. $\tau = \rho^{+\bar{M}}$ and $J^{E^{\bar{M}}}_\tau = J^{E^M}_\tau$.

proof.
If $\bar{M} = M$, there is nothing to prove, so let $\bar{M} \neq M$. Let σ be the core map. Let ν be as in Lemma 4. Then $\rho \leq \nu$, $\nu \neq \mathrm{On} \cap \bar{M}$, since $\bar{M}$ is not a segment of M. Hence $\nu = \mathrm{crit}(\sigma)$. Clearly (a) holds. Case 1 fails, since $\bar{M}$ is not a segment of M. Thus Case 2 holds. Hence Case 2.1 holds, since otherwise (a) fails. Then $\rho = \omega\rho^n_{\bar{M}} = \omega\rho^n_{M_\theta}$ since $\mathrm{crit}(\pi) \geq \nu \geq \rho$. Hence $\kappa_i \geq \rho$ for $i < \theta$ in $\mathcal{I}$. Hence ...

$\rho = \omega\rho^{n}_{M_\theta} = \omega\rho^{n}_{M}$. But $\pi, \bar{\pi}$ are iteration maps, and $\rho \leq \mathrm{crit}(\pi), \mathrm{crit}(\bar{\pi})$. Hence $\tau = \rho^{+\bar{M}} = \rho^{+M} = \rho^{+M_\theta}$ and $J_\tau^{E^{\bar{M}}} = J_\tau^{E^M} = J_\tau^{E^{M_\theta}}$. QED (Lemma 5)

(<u>**Note**</u> Since $\pi \restriction \rho = \mathrm{id}$, we have $\mathrm{rng}(\bar{\pi}) \subset \mathrm{rng}(\pi)$, since $\bar{M} = \mathrm{core}_\rho(M_\theta)$, $\bar{\pi} =$ the core map. Since $\bar{\pi}(p_{\bar{M}}) = \pi(p_M) = p_{M_\theta}$, it follows that $\sigma = \pi^{-1}\bar{\pi}$.)