A New Fine Structure Theory for Higher Core Models

In these notes we develop a new fine structure theory for constructing inner models such as the model for one Woodin cardinal in the Mitchell-Steel book Fine Structure and Iteration Trees or the models K^c and K of Steel's notes The Core Model Iterability Problem. Our basic fine structure is the Σ^* theory developed in our notes Measures of Order Zero.

An advantage of this approach is that it enables us to define a general notion of fine structural ultraproduct $Ult^*(M, F)$ of an acceptable structure M and an extender F. This is developed in §2 (and already appeared in Measures of Order Zero).

Steel, on the other hand, defines an n-ultrapower for each $n \leq \omega$. His $\mathrm{Ult}_n(M, F)$ can only be formed, however, if M is sound above the n-th projectum ρ_n and $\mathrm{crit}(F) < \rho_n$. This leads him to define "n-iteration" for each $n \leq \omega$. His strongest general notion of iterability is: n-iterability of the n-th core for all $n \leq \omega$. All of this is simplified by using a single notion of fine structural ultrapower. (Our $\mathrm{Ult}^*(M, E)$ is, in fact the same as $\mathrm{Ult}_n(M, E)$ where n is maximal s.t. $\mathrm{crit}(E) < n$ - in those cases where $\mathrm{Ult}_n(E)$ is defined. Our basic notion of iteration is the normal iteration (or "ω-maximal iteration tree" in Steel's terminology). Our iterability requirement is, however, more stringent. We call M an iterable mouse only if, starting with M, we can linearly iterate the process of taking normal iterations (allowing truncation between normal iterations). In particular, every normal iterate of M must itself be normally iterable. Steel could not require this, since a normal iterate might no longer satisfy his soundness requirement for forming an ultrapower. (However, his notion of k-iterability is similar, but is restricted to "k-bounded k-iterations".) One result of this is that he can state only a weak form of the Dodd-Jensen lemma (and the more recently discovered Neeman Steel lemma).

Another respect in which our approach differs from Steel's is the indexing of the extenders in a premouse. A premouse has the form $M = \langle J_\alpha^E, E_{\omega\alpha} \rangle$ where $E_\nu = \emptyset$ or is an extender for $\nu \leq \omega\alpha$. We choose our extenders to be "maximally long" and index E_ν

E_ν by the cardinal successor of its length in the ultrapower. In other words, if $\kappa = \mathrm{crit}(E_\nu)$ and $\tau = \kappa^{+J_\nu^E}$, then J_ν^E is the ultrapower of J_τ^E by E_ν and E_ν has length $\lambda = \pi(\kappa)$, where π is the ultrapower embedding (in symbols: $\pi : J_\tau^E \xrightarrow[E_\nu]{} J_\nu^E$). We adopt the "functional" representation of extenders rather than the more familiar "hypermeasure" representation (e.g. in the present case we identify E_ν with $\pi \restriction \mathcal{P}(\kappa)$ rather than with $\langle E_{\nu,\alpha} \mid \alpha < \lambda \rangle$, where $E_{\nu,\alpha} =_{\mathrm{df}}$ $=_{\mathrm{df}} \{ X \subset \kappa \mid \alpha \in \pi(X) \}$ for $\alpha < \lambda$). This has the advantage that $\langle J_\nu^E, E_\nu \rangle$ is automatically an amenable structure. Moreover, if M is a premouse, we can develop an iteration theory without having to "shorten" M (as in Steel's Case III) or "lengthen" it

(as in our notes Non Overlapping Extenders.)

Thus our approach has two aspects — the use of Σ^* fine structure theory and a stronger iterability notion on the one hand, and the revised indexing on the other — which are more or less independent of each other. Indeed it would be quite possible to apply our fine structure and iterability theory to Steel's premice (including, if one wished, his Case III), We chose not to, because the revised indexing seemed both simpler and more capabable of generalization. We believe that our fine structure and iterability theory brings a clear gain in succinctness, clarity, and strength. The relative advantage of revised indexing is less clear, however. It certainly simplifies the basic theory (e.g. almost no mention of "limits of generators"), but it turns out that we must pay a price. This is already apparent in §6, where we encounter new techical difficulties in proving the uniqueness of branches theorems. A more serious difficulty appears in §10, where we attempt to construct our version of the Mitchell-Steel inner model for a Woodin cardinal. We assume θ is Woodin and construct $E \subset V_\theta$ s.t. J_θ^E is a premouse. Mitchell and Steel then show that $L^E \models$ "There is a Woodin cardinal". The idea of the construction - speaking very roughly - is that an extender E_ν will be added to the sequence iff it is verified by a "background" extender F on V which is <u>strong</u> <u>up to the length of</u> E_ν. Since our extenders are longer than Steel's, the background condition is harder to meet. It is presumably for this reason that we are unable to verify the existence of a Woodin cardinal in our L^E (although we strongly suspect that it is true). However, Steel has an alternative construction (the K^c-construction) in which the background condition is relaxed. (Essentially, there must be background extenders with the strength condition, but which need not be defined on the whole of $\mathbb{P}(\kappa)$ where $\kappa = \mathrm{crit}(E_\nu)$.) We believe — but have not yet verified — that the construction in §10 can be adapted mutatis mutandis to the K^c construction. We assume this in §11 and examine the consequences. We again construct $E \subset V_\theta$, assuming only that θ is inaccessible and V_θ is closed under #. It is then easy to repeat Steel's proof of the "cheap covering lemma" if Θ is measurable. We then show that if Θ is Woodin, then indeed $L^E \models$ "There is a Woodin cardinal". However, the proof is decidedly more convoluted than Steel's original argument.

The plan of the paper is as follows: In §1 we introduce the functional representation of extenders and prove some basic lemmas. In §2 we develop the fine structural ultrapower $Ult^*(M, F)$. We write $\pi : M \xrightarrow{*}_F M'$ to mean: $M' = Ult^*(M, F)$ and π is the ultrapower embedding. The properties of π depend on the extender F. We call F <u>weakly amenable</u> iff $\mathbb{P}(\kappa) \cap M = \mathbb{P}(\kappa) \cap M'$. If each $F_\alpha = \{X \subset \kappa \mid \alpha \in \pi(X)\}$ is $\Sigma_1(M)$ for $\alpha < lh(F)$, we call F <u>Σ_1-amenable</u>.

If both things hold, then F is <u>close</u> to M. Our main lemma says that, if F is close to M, then π is fully Σ^* preserving. In §3 we refine §2 to consider problem of "copying" an ultrapower. Suppose we have $\pi : M \xrightarrow[F]{*} M'$. Suppose also that $\sigma : \bar{M} \xrightarrow[\Sigma^*]{} M$ and that $\bar{F}$ is an extender on $\bar{M}$. Under what conditions can we show that $\bar{\pi} : \bar{M} \xrightarrow[\bar{F}]{*} \bar{M}'$ exists and that there is a map $\sigma' : \bar{M}' \xrightarrow[\Sigma^*]{} M$ with $\sigma'\bar{\pi} = \pi\sigma$?

We suppose $\bar{F}, F$ to be close to their respective structures. What is needed is the existence of an order preserving $g : lh(\bar{F}) \to lh(F)$ satisfying the following conditions:

(a) Let $\alpha_1, \ldots, \alpha_m < lh(\bar{F})$. Then

$\langle \alpha_1, \ldots, \alpha_m \rangle \in \bar{F}(X) \longleftrightarrow \langle \sigma(\alpha_1), \ldots, \sigma(\alpha_m) \rangle \in F(\sigma(X))$

(b) Let $\alpha < lh(\bar{F})$. Then $\bar{F}_\alpha$ is $\Sigma_1(\bar{M})$ in a parameter $\bar{p}$ and $F_{\sigma(\alpha)}$ is $\Sigma_1(M)$ in $p = \sigma(\bar{p})$ by the same definition.

We express this as: $\langle \sigma, g \rangle : \langle \bar{M}, \bar{F} \rangle \xrightarrow{*} \langle M, F \rangle$.

In §4 we introduce the notion of <u>premouse</u> and the fine structural <u>iteration</u> of a premouse. (This is the same as Steel's "iteration tree".) We call an iterate M' of M <u>simple</u> (in the iteration $\mathcal{I}$) iff there is no truncation on the branch from M to M'. We show that if M' is a simple iterate of M, then the iteration map $\pi : M \to M'$ is Σ^*-preserving. This, in turn, follows from the fact that, if M_{i+1} is obtained by applying the extender $E^{M_i}_{\nu_i}$ to M^*, then $E^{M_i}_{\nu_i}$ is close to M^*.

(We fear that our proof of this "closeness lemma" is less elegant than Steel's, which we had not seen at the time.)

We call M *normally iterable* if it has a *normal iteration strategy* (i.e. a partial function S assigning cofinal well founded branches to iterations of limit length s.t. any iteration whose branches are picked according to S can be continued). By a *smooth iteration* we mean a linear progression of normal iterations (which can, of course, be arranged as a single iteration). M is *smoothly iterable* iff it has a smooth iteration strategy. By a *good* iteration we mean a linear progression of normal iterations in which truncations are permitted between the individual normal iterations. We call M _iterable_ or a _mouse_ iff M has a good iteration strategy. (In §9 we shall show that every smoothly iterable premouse is a mouse.)

In §5 we show that if $\sigma : \overline{M} \xrightarrow[\Sigma^*]{} M$ and M is a mouse, then $\overline{M}$ is a mouse. The proof is based on the copying lemma of §3. Let S be an iteration strategy for M. We construct a strategy $\overline{S}$ for $\overline{M}$ as follows. Let $\overline{\mathcal{I}}$ be a good iteration of $\overline{M}$ with iterates $\overline{M}_i$ + maps $\overline{\pi}_{ij}$. Let T be the associated tree (i.e. $\overline{\pi}_{ij}$ is defined for $i \le j$ in T). We attempt to define an iteration $\mathcal{I}$ of M with the same tree and with maps $\sigma_i : \overline{M}_i \xrightarrow[\Sigma^*]{} M_i$ s.t. $\sigma_j \overline{\pi}_{ij} = \pi_{ij} \sigma_i$, where π_{ij} are the iteration maps of $\mathcal{I}$. We proceed as follows:

Suppose $\bar{\pi}_{\xi,i+1} : \bar{M}^* \longrightarrow^*_{E_{\bar{\nu}_i}} M_{i+1}$ ($\bar{M}^*$ being, of course, a segment of $\bar{M}_\xi$). Let $\sigma_i(\bar{\nu}_i) = \nu_i$. We form the corresponding segment M^* of M_ξ and let σ_{i+1} be s.t. $\sigma_{i+1}\bar{\pi}_{\xi,i+1} = \pi_{\xi,i+1}\sigma_\xi \upharpoonright \bar{M}^*$ and σ_{i+1} is Σ^*-preserving. This is possible if

(*) $\langle \sigma_\xi \upharpoonright \bar{M}^*, \sigma_i \upharpoonright \bar{\lambda}_i \rangle : \langle \bar{M}^*, E^{\bar{M}_i}_{\bar{\nu}_i} \rangle \xrightarrow{*} \langle M^*, E^{M_i}_{\nu_i}$

To prove (*), we essentially repeat the proof of the closeness theorem of §4. Now let $\bar{\mathcal{J}}$ be of limit length. If the construction of $\mathcal{J}$ breaks down due to lack of well foundedness, then $\bar{S}(\bar{\mathcal{J}})$ is undefined. Otherwise set: $\bar{S}(\bar{\mathcal{J}}) \simeq S(\mathcal{J})$. Then $\bar{S}$ is a strategy for $\bar{M}$.

In §6 we examine the question of uniqueness for well founded cofinal branches in an iteration. We repeat

The Steel-Martin proof that if a norma[l] iteration $\mathcal{J}$ has distinct cofinal well founded branches b, b' resulting in premice $M_b, M_{b'}$, then a point δ is Woodin in $M_b \cap M_{b'}$. (As mentioned above, the proof involves some new technical difficulties due to the use of longer extenders.) We then introduce the notion of a basic (or one - small) premouse and prove Steel's theorem that if M is a basic mouse and $\omega\rho_M^\omega \leq \nu$ for a ν s.t. $E_\nu^M \neq \emptyset$, then M is uniquely normally iterable (hence uniquely smoothly iterable). We also show that the initial segment condition in our definition of premouse is vacuously true for basic mice.

In §6, which also follows Steel fairly closely, we prove that mice are solid. From this we get the basic facts about the core of a mouse, the coiteration of two mice etc. We use Steel's technique of "double rooted iterations". As usual, our fine structure leads to some simplification of the proof. However, the proof is longer than in the Steel-Mitchell presentation because we deal with a case (the "anomaly") which they forgot. As in Steel's case, we make strong use of the Dodd-Jensen lemma for uniquely smoothly iterable mice. (We fear, however, that the term "uniquely iterable" should often be read as "uniquely smoothly iterable.") After writing this, we became aware of the Neeman-Steel lemma, which can, in fact, replace the Dodd-Jensen lemma everywhere in §7 and enables us to prove the theorems of §7 for <u>all</u> mice rather than just basic ones. This is explained in the appendix to §7. There we also point out that the theorems of §7 hold for basic <u>weak</u> <u>mice</u> under the assumption that $a^{\#}$ exists for all $a \subset \omega_1$. (M is defined to be a weak mouse iff it is a premouse and whenever $\sigma : Q \xrightarrow{\Sigma^*} M$ and Q is countable, then Q is countably iterable.) In §8 we use the methods of §7 to prove condensation lemmas and other structural properties of mice. Again most of these lemmas hold for arbitrary rather than basic ones. We deal with this question in the appendix to §8.

In §9 we develop an important technique which is to be used in the model construction of §10 and use it here to show that every smoothly iterable premouse is a mouse. The basic structure of

The proof is as follows. Suppose that M is smoothly iterable and that e.g. $\bar{M}$ is a truncation of M. How can we show e.g. that $\bar{M}$ is normally iterable? We have a normal iteration strategy S for M and we must use it to define a strategy $\bar{S}$ for $\bar{M}$. Let $\bar{\mathcal{I}}$ be a normal iteration of $\bar{M}$ with iterates $\bar{M}_i$ and iteration maps $\bar{\pi}_{ij}$. We again attempt to "copy" $\bar{\mathcal{I}}$ onto a normal iteration $\mathcal{I}$ of M with iterates M_i and maps π_{ij}. Simultaneously - following the strategy of §5 - we construct embeddings $\sigma_i : \bar{M}_i \to M_i'$, where $M_0' = \bar{M}$, $\sigma_0 = \mathrm{id}$, and M_i' is a segment of M_i. Suppose e.g. that $F = E^{\bar{M}}_{\bar{\nu}_0}$ is an extender in M as well as $\bar{M}$. We then form $\bar{\pi}_{01} : \bar{M} \to_F \bar{M}_1$, $\pi_{01} : M \to_F M_1$ and set $M_1' = \pi_{01}(\bar{M})$. It is not hard to construct $\sigma_1 : \bar{M}_1 \to M_1'$ s.t. $\sigma_1 \bar{\pi}_{01} = \pi_{01} \sigma_0$. However, since M_1' is <u>not</u> the ultrapower of $\bar{M}$ by F, we cannot ~~use~~ the copying lemma of §3 to show that σ_1 is Σ^*-preserving. (In general σ_1 will not have this property.) We get around this by the use of <u>pseudoprojecta</u>. We write $\sigma : \bar{Q} \to_{\Sigma^*} Q \bmod (\vec{\rho})$ to mean that σ is Σ^*-preserving in the sense of the pseudoprojecta $\langle \rho_i \mid i < \omega \rangle$, (meaning that the variables v^i are interpreted in Q as ranging over $H_i^Q(\vec{\rho}) = J_{\rho_i}^{A^Q}$, rather than over H_Q^i.) $\langle \rho_Q^n \mid n < \omega \rangle$ is then just one possible sequence of pseudo projecta. Having made this notion precise, we then show that for any pseudoprojecta $\vec{p}$ there is a <u>minimal</u> sequence $\vec{p}' = \min(\vec{p}) = \min(\sigma, \ldots, \vec{p})$ s.t. $\sigma : \bar{Q} \rightarrow_{\Sigma^*} Q \mod(\vec{p}')$ and $\sigma''p^i_{\bar{Q}} \subset p'_i \leq p_i$ with $Q \models \varphi(\vec{z}, \sigma(x))$ holding $\mod(\vec{p})$ iff it holds $\mod(\vec{p}')$ for $\vec{z} \in H_i(\vec{p}')$ and φ a $\Sigma_1^{(i)}$ formula. We also show: $\vec{p}' = \min(\vec{p}')$ and define $\sigma : \bar{Q} \rightarrow_{\Sigma^*} Q \min(\vec{p})$ to mean that $\sigma : \bar{Q} \rightarrow_{\Sigma^*} Q \mod(\vec{p})$ and $\vec{p} = \min(\vec{p})$. Returning to the above case, we have $\sigma : \bar{M} \rightarrow M$ $\sigma_0 : \bar{M} \rightarrow_{\Sigma^*} \bar{M} \min(\vec{p})$ for some $\vec{p}$ and we get a copying lemma analogous to that of § 3 which gives us $\sigma_1 : \bar{M}_1 \rightarrow_{\Sigma^*} M'_1 \min(\vec{p}^1)$

where $\pi_{01}''\rho_i \subset \rho_i^1 \leq \rho_i$ for $i < \omega$.

Working in this way, we are able to carry out the arguments of §5 and show the iterability of M.

In §10 we immitate Steels construction of a fine structural inner model below an inaccessible cardinal θ. We assume that V_θ is closed under sharps and construct a sequence $\langle N_\nu \mid \nu < \theta \rangle$ of weak mice converging to an $N = J_\theta^E$. The main effort comes in verifying that each N_ν is a weak mice. Steels proof shows only that if $\sigma : Q \xrightarrow{}_{\Sigma^*} N_\nu$ and Q is countable, then Q is countably <u>normally</u> iterable. The difficulty in getting full countable iterability of Q is analogous to the problem in §9 and we again solve it by the use of pseudoprojekta.

Steel is able to show for his model that if Θ is Woodin, then $L^E \models V\delta$ δ is Woodin. As mentioned earlier, our use of long extenders makes it harder to insert new extenders into the models N_ν $(\nu < \Theta)$ and we were thus unable to prove this result. In §11, however, we describe Steel's K^c construction, which again gives a sequence $\langle N_\nu \mid \nu < \Theta \rangle$ converging to $N = K^c = J^E_\theta$. We adapt the definition to our premice and assume without proof that each N_ν is a weak mouse. (We assume that the proof using pseudoprojecta can be adapted mutatis mutandis.) Steel's theorem that N satisfies a "cheap covering lemma" if Θ is measurable then goes through easily. We then show, by a fairly laborious proof, that if Θ is Woodin, then $L^E = L[N]$ thinks there is a Woodin cardinal.

I am grateful to Achim Ditzen and Martin Zeman for listening patiently to a series of talks I gave on the first part of these notes and making valuable comments. Martin Zeman also read the early version of the notes, discovered errors and suggested corrections. His help was invaluable particularly in §6 (which, the reader may notice, is in his handwriting).

These notes are self contained, given a knowledge of basic Σ^* fine structure.