§ 2 Iterability (a short sketch)

In this section – following a time honored precedent – we sketch the proof of Thm 1 in the special case that the iteration $\mathcal{I} = \langle\langle P_i\rangle, \langle \nu_i\rangle, \langle \pi_{ij}\rangle, T\rangle$ is of length ω and has no truncation. We assume: $\sigma_0 : P_0 \prec N = N_\theta$, where P_0 is countable. In order to avoid messy definability considerations we also assume that N is a ZFC$^-$ model. We prove:

<u>Main Claim</u> Let $N, \mathcal{I}, \sigma_0, P_0$ be as above. There is a cofinal branch b in $\mathcal{I}$ and a $\sigma : P_b \prec N$ s.t. $\sigma \pi_b = \sigma_0$.

Because the notion "robust" is self contained and $\mathcal{I}$ has no truncation, it turns out that we can handle this case without recourse to the "resurrection sequence" of [MS] and, indeed, without reference to any element of the array other than N itself. To this end we define:

Def Let N be an active premouse. Let $\kappa \leq \gamma \leq \lambda$ (where $\kappa = \kappa_F =_{Df} \mathrm{crit}(F)$, $\lambda = \lambda_F =_{Df} \mathrm{lh}(F)$).
F is _robust_ _up to_ γ _in_ N iff for every pair of countable sets $u \subset \lambda$, $W \subset \mathbb{P}(\kappa) \cap N$ there is a $g : u \longrightarrow \kappa$ s.t.

(a) $\langle g(\vec{\alpha})\rangle \in X \longleftrightarrow \langle\vec{\alpha}\rangle \in F(X)$
for $\alpha_1, \ldots, \alpha_n \in u$, $X \in W$.

(b) Let $c = \mathrm{lub}(u \cap \gamma)$, $\bar{c} = \mathrm{lub}\, g''(u \cap \gamma)$.
Then for all $v \subset u$ and all Σ_1-fmlae φ:
$C^E_{\bar{c},\kappa} \models \varphi(g''v, g''u) \longleftrightarrow C^E_{c,\infty} \models \varphi(v, u)$.

[Note If F is robust up to κ, then
$C^E_{\tau,\kappa} \prec_{\Sigma_1} C^E_{\tau,\infty}$ for all $\tau < \kappa$.]

[Remark If $N = M\|\nu$ and F is robust up to $\omega\rho^\omega_M$, $\bar{M} = \mathrm{core}(M)$, σ is the core map and $\sigma(\bar{\nu}) = \nu$, $\bar{N} = M\|\bar{\nu} = \langle J^E_{\bar{\nu}}, \bar{F}\rangle$, then $\bar{F}$ is also robust up to $\omega\rho^\omega_M$.]

Def A premouse M is _robust_ iff whenever $M\|\nu = \langle J^E_\nu, F\rangle$ is active and $\gamma \in (\kappa_F, \lambda_F]$ is a cardinal in M, then F is robust up to γ in $M\|\nu$.

Using the above remark, it follows by induction on i that N_i is robust for all elements of the robust array.
In particular $N = N_\theta$ is robust.

Def A premouse M is <u>mouselike</u> iff
each of its segments $M\|\xi$ ($\xi < ht(M)$)
is mouselike in M – i.e. $M\|\xi$ is solid
and $M \models$ ($M\|\xi$ satisfies the condensation
lemma)

– [We are referring to [CR] § IV Lemma 4′]

An induction on i shows that each N_i
in the array is mouselike, in particular
$N = N_\theta$.

From now on we assume only that
N is any robust, mouselike premouse
satisfying ZFC⁻. We make no
further reference to the array.
Since $\sigma_0 : P_0 \prec N$, it follows that P_0 is
mouselike. We need the following
fact, whose proof we defer until
later:

<u>Iteration Fact</u> Let $\mathcal{J}$ be a putative
normal iteration of a mouselike P_0.
Let $\eta = T(i+1)$ in $\mathcal{J}$. There is $\nu \leq ht(P_\eta)$
s.t. $\bar{\nu} > \tau_i = \kappa_i^{+U_{\nu_i}^{E^{N_i}}}$ and $crit(E_\nu^{N_\eta}) = \kappa_i$
(where the κ_i are the iteration pts.).

We shall adhere closely to the structure of Steel's proof. We encompass N and the map $\sigma : P \prec N$ in a "world", W, consisting of a chunk of V. We assume that the Main Claim is false, which means that a certain relation definable in W is well founded. Using this, we construct a sequence of increasingly shorter worlds W_i, each with its own N_i. The contradiction comes from the fact that $\tau_{i+1} < \tau_i$, where $\tau_i = On \cap W_i$. Our method of obtaining the W_i is, however, somewhat different from that of Steel. We take $W_0 = W$. W_{i+1} will, however, not be constructed in V, but rather in a generic extension $V[G]$, which collapses W to ω. Our tool for constructing W_{i+1} is the completeness theorem for the infinitary language of a countable admissible set. The admissible sets employed will be of the form:

$C^{E}_{c,\alpha}$, where $E = E^{N_i}$.

We begin by specifying the notion "world" more precisely. Each world will be a model of the theory ZFC* obtained as follows: We recall that ZF^- is ZF without the power set axiom. ZFC^- is ZF^- together with "every set is enumerable by an ordinal". ZFC* is then $ZFC^- + \bigwedge \alpha\ [\alpha]^\omega$ is a set. In ZFC* we can prove: Let κ be regular s.t. $H_\kappa \models ZFC^*$. Then $C^a_{\tau,\kappa} \prec_{\Sigma_1} C^a_{\tau,\infty}$ for all $a \in H_\kappa$ and all $\tau < \kappa$.

<u>Def</u> a <u>world</u> is a transitive model W of ZFC* s.t. $[\tau]^\omega \cap W = [\tau]^\omega \cap V$, where $\tau = On \cap W$.

<u>Note</u> $[\tau]^\omega \cap W \subset V$ is not vacuous, since we shall construct our worlds in $V[G]$.

<u>Note</u> If W, W' are two worlds, $\alpha \in W \cap W'$ and $e \in W \cap W'$, then $(C^e_{\tau\alpha})^W = (C^e_{\tau\alpha})^{W'}$ for all $\tau < \alpha$. We shall make constant use of this fact.

<u>Def</u> Let $\bar{\theta}$ be a strong limit cardinal s.t. $N \in V_{\bar{\theta}}$ and $\overline{\overline{V_{\bar{\theta}}}} = \bar{\theta}$. Set: $\bar{W} = H_{\bar{\theta}^+}$. Then $\bar{W}$ is a world which we shall call the <u>standard world</u>. Let $a \subset \theta^+$ s.t. $[\theta^+]^\omega \subset L_{\bar{\theta}^+}[a]$ and $V_{\bar{\theta}} = L_{\bar{\theta}}[a]$. a is fixed for later reference and is called the <u>standard predicate</u>.

We assume the Main Claim is false. This says that the following relation R is well founded:

<u>Def</u> $D = \{\langle i, \delta\rangle \mid \delta : P_i \prec N \wedge \delta\pi_{0i} = \delta_0\}$

$R \subset D^2$ is then defined by:

$\langle j, \delta'\rangle R \langle i, \delta\rangle \longleftrightarrow (i \leq_T j \wedge \delta'\pi_{ij} = \delta)$.

For $z \in D$ set: $r(z) =$ the rank of z in $R = \text{lub}\{r(w) \mid w R z\}$,

We also set: $r = r(\langle 0, \delta_0\rangle)$.

Clearly $R \in \bar{W}$, $\langle r(z) \mid z \in D\rangle \in \bar{W}$.

<u>Def</u> Let $\theta = \langle \theta_\nu \mid \nu \leq \rho \rangle$ be defined by $\theta_0 = \bar{\theta}$, $\theta_{\nu+1} = 2^{\theta_\nu^+}$, $\theta_\lambda = \sup_{\nu<\lambda} \theta_\nu$ for limit λ.

Set: $W_\nu = H_{\theta_\nu^+}$.

Then $W = \langle W_\rho, \theta, a \rangle$ is called the <u>standard</u> <u>enhanced</u> <u>world</u>.

This, of course suggests the following definition:

<u>Def</u> $\langle W', \theta', a' \rangle$ is an <u>enhanced</u> <u>world</u> iff

- W' is a world
- $\theta' \in W'$; in W' we have: $\theta' = \langle \theta'_\nu \mid \nu \leq p \rangle$, where $\theta'_0 = \bar{\bar{V}}_{\theta'_0}$ and $\theta'_0 > \omega$, $\theta'_{\nu+1} = (2^{\theta'_\nu})^+$, $\theta'_\lambda = \sup_{\nu<\lambda} \theta'_\nu$ for limit λ
- **Set**: $W'_\nu = (H_{(2^{\theta'_\nu})^+})^{W'}$ for $\nu \leq p$.

Hence $(C^E_{\tau,\infty})_{W'_\nu} \prec_{\Sigma_r} (C^E_{\tau,\infty})_{W'}$ for $\tau \in W'_\nu$, $E \in W'_\nu$

(where $H_{\eta^+} =_{Df} \{x \mid \overline{\overline{TC(x)}} \leq \eta\}$)

Then $W' = W'_p$

- Set $\bar{W}' = W'_0$. Then $a \subset \theta_0^+$ s.t. $V_{\theta_0} = L_{\theta_0}[a]$ and $[\theta_0^+]^{(\omega)} \subset L_{\theta_0^+}[a]$ in W'.

If $W' = \langle W', \theta', a' \rangle$ is any enhanced world, we set:

$\theta^{W'} = \theta'$, $a^{W'} = a'$, $p^{W'} = p = \mathrm{dom}(\theta')$, $\overline{W}' = W'_0$.

Now let G be a generic set which collapses the standard enhanced world W to ω. Working in $V[G]$ we construct a sequence:

$$\langle W_i, N_i, \delta_i \rangle \quad (i < \omega)$$

s.t. W_i is an enhanced world and:

(a) $\langle W_0, N_0, \delta_0 \rangle = \langle W, N, \delta_0 \rangle$

(b) $\langle \overline{W}_i, N_i, e \rangle \equiv \langle \overline{W}, N, e \rangle$, where $e \subset \omega$ codes y (e.g. $e = \{\langle n, m \rangle \mid f(n) \in f(m)\}$, where $f : \omega \xrightarrow{\text{onto}} TC(\{y\})$).

(c) $\delta_i \in \overline{W}_i$ s.t. $\delta_i : P_i \prec N_i$ and

- $\langle \overline{W}_i, N_i, \delta_i \pi_{0i}, e \rangle \equiv \langle \overline{W}, N, \delta_0, e \rangle$
- $\langle \overline{W}_i, N_i, \delta_i \pi_{hi}, e \rangle \equiv \langle \overline{W}_h, N_h, \delta_h, e \rangle$

for $h \leq_T i$.

(d) Let R_i, $r_i = \langle r_i(z) \mid z \in D_i \rangle$ be defined in $\overline{W}_i$ from $N_i, \delta_i \pi_{0i}$ as R, r were defined in $\overline{W}$ from N, δ_0.

Then $P_i = p^{W_i} \geq \tilde{r}_i =_{\mathrm{Df}} r_i(\langle i, \delta_i \rangle)$.

(e) Let $\gamma_i =$ the largest $\gamma \leq \lambda_i$ which is a cardinal in P_i. Set: $c_i = \mathrm{lub}\, \delta_i {}'' \gamma_i$.

Then:

$J^{E_i}_{c_h} = J^{E_h}_{c_h}$ for $h \leq i$, where $E_i = E^{N_i}$,

and $\delta_i \upharpoonright \gamma_h = \delta_h \upharpoonright \gamma_h$ for $h \leq i$.

(f) $\tau_i < \tau_h$ for $h < i$, where $\tau_h = On \cap W_h$.

By (f) we will, of course, have a contradiction.

Let $\langle W_i, N_i, \delta_i \rangle$ be given, satisfying (a)-(e). (For $i = 0$ this is trivial.) We construct $\langle W_{i+1}, N_{i+1}, \delta_{i+1} \rangle$. Let $\eta = T(i+1)$.

By the robustness of N_i in W_i there is $g : \lambda_i \to \delta_i(\kappa_i)$ s.t.

(1) (a) $\langle \vec{\alpha} \rangle \in E^{P_i}_{\gamma_i}(X) \leftrightarrow \langle g(\vec{\alpha}) \rangle \in \delta_i(X)$

for $\alpha_1, \ldots, \alpha_n < \lambda_i$, $X \in \mathcal{P}(\kappa_i) \cap P_i$

(b) Let $\upsilon < \gamma_i$, $c_i = \sup \delta_i{}''\gamma_i$, $\bar{c} = \sup g''\gamma_i$.
Then $C^{E_i}_{c_i, \infty} \models \varphi(\delta_i{}''\upsilon, \delta_i{}''\gamma_i) \leftrightarrow C^{E_i}_{\bar{c}, \delta_i(\kappa_i)} \varphi(g''\upsilon, g''\gamma_i)$
in $\overline{W}_i$ for all Σ_1 fmlae, where $E_i = E^{N_i}$.

(Note that $(C^{E_i}_{c,\infty})_{\overline{W}_i} \prec_{\Sigma_1} (C^{E_i}_{c,\infty})_{W_i}$!)

We know:

(2) $J^{E_\eta}_{c_\eta} = J^{E_i}_{c_\eta}$, $\delta_\eta \upharpoonright J^{E^{P_\eta}}_{\gamma_i} = \delta_i \upharpoonright J^{E^{P_i}}_{\gamma_i}$,

where $\kappa_i < \gamma_i$. Hence $\delta_\eta(\kappa_i) = \delta_i(\kappa_i)$, and $\delta_\eta(X) = \delta_i(X)$ for $X \in \mathcal{P}(\kappa_i) \cap P_i = \mathcal{P}(\kappa_i) \cap P_\eta$.

In particular, if $f_1, \ldots, f_m \in P_\eta$ s.t. $f_\ell : \kappa_i \to P_\eta$, and $\alpha_1, \ldots, \alpha_m < \lambda_i$, then setting: $X = \{\langle \vec{\xi} \rangle \mid P_\eta \models \varphi(f_1(\xi_1), \ldots, f_m(\xi_m))\}$, we have:

$P_{i+1} \models \varphi(\pi_{\eta, i+1}(\vec{f})(\vec{\alpha})) \longleftrightarrow$

$\longleftrightarrow \langle \vec{\alpha} \rangle \in E_{\nu_i}^{P_i}(X) \longleftrightarrow \langle g(\vec{\alpha}) \rangle \in \delta_i(X) = \delta_\eta(X)$

$\longleftrightarrow P_\eta \models \varphi(\delta_\eta(\vec{f})(g(\vec{\alpha})))$.

Hence there is $\sigma : P_{i+1} \prec N_\eta$ defined by: $\sigma(\pi_{\eta, i+1}(f)(\alpha)) = \delta_\eta(f)(g(\alpha))$, for $f \in P_\eta$, $f : \kappa_i \to P_\eta$, $\alpha < \lambda_i$.

Clearly $g \in [\delta_\eta(\kappa_i) \times \lambda_i]^\omega \subset W_\eta$.

Since $\delta_\eta \in W_\eta$, we have:

(3) $\sigma \in W_\eta$

Note that $\sigma \upharpoonright \lambda_i = g$, $\sigma \pi_{\eta, i+1} = \delta_\eta$.

Hence:

(4) $\langle i+1, \sigma \rangle R_\eta \langle \eta, \delta_\eta \rangle$.

Set: $\bar{\Omega} = \Omega_\eta(\langle i+1, \sigma \rangle) < \tilde{\Omega}_\eta$.

Since $\bar{c} = \operatorname{lub} g''\gamma_i < \sigma_i(\kappa_i) < c_\eta$, we know that $J_{\bar{c}}^{E_\eta} = J_{\bar{c}}^{E_i}$. Pick $\alpha > (\theta_{\bar{\Omega}}^+)_{W_\eta}$ s.t. $\alpha \in W_\eta$ and

(5) $C_{\bar{c}, \alpha}^{E_\eta}$ is admissible.

Consider the following theory $T = T_{\bar{\tau}, \alpha}$ in the infinitary language of $C^{E_\eta}_{\bar{\sigma}, \alpha}$, where $\tau = (\theta^{+}_{\pi})_{W_\eta}$:

<u>Predicate</u>: $\dot{\in}$

<u>Constants</u>: $\underline{x}$ ($x \in C^{E_\eta}_{\bar{\sigma}, \alpha}$), $\dot{W}, \dot{N}, \dot{\sigma}$

<u>Axioms</u>:

(A) ZFC^*, $\bigwedge v (v \in \underline{x} \longleftrightarrow \bigvee_{z \in x} v = \underline{z})$ for all $\underline{x}$,

$\dot{W} = \langle \dot{W}, \dot{\theta}, \dot{a} \rangle$ is an enhanced world,

$On \cap \dot{W} = \underline{\tau}$, $[\underline{\tau}]^{\omega} \cap \dot{W} = \underline{[\tau]^{\omega}}$

(B) (i) $\dot{N}, \dot{\sigma} \in \bar{\dot{W}} = (\dot{W})_0$ and

$\langle \bar{\dot{W}}, \dot{N}, \dot{\sigma}, \underline{e} \rangle \models \underline{T_0}$, where

T_0 = the complete theory of $\langle \bar{W}_\eta, N_\eta, \sigma, e \rangle$

(ii) $\dot{\sigma} \upharpoonright \underline{\gamma_i} = \underline{g \upharpoonright \gamma_i}$

(iii) $p^{\dot{W}} = \dot{s}(\langle \underline{i+1}, \dot{\sigma} \rangle)$, where $\dot{s}$ is defined in $\bar{\dot{W}}$ from $\dot{N}, \dot{\sigma}\, \underline{\pi_{0,i+1}}$ as s in $\bar{W}$ from N, σ_0.

Then $T = T_{\bar{\tau}, \alpha}$ is consistent, since W_η is a model with $\dot{W}, \dot{N}, \dot{\sigma}$ interpreted by $(W_\eta)_{\bar{\pi}}$, N_η, σ resp.

Thus in W_γ we have:

$\forall\alpha\,\forall\tau\,.(\bar{\tau} < \tau < \alpha \wedge C^{E_\gamma}_{\bar{\tau},\alpha}$ is admissible $\wedge$

$\wedge\, T_{\tau,\alpha}$ is consistent)

But this can be expressed in $(C^{E_\gamma}_{\bar{\tau},\infty})^{W_\gamma}$ as a Σ_1-statement in the parameters $g``\gamma_i$, T_0;

(6) $C^{E_\gamma}_{\bar{\tau},\infty} \models \varphi(g``\gamma_i, T_0, e)$

(where φ is Σ_1).

Note, however, that

(7) $C^{E_\gamma}_{\xi,\delta_\gamma(\kappa_i)} \prec_{\Sigma_1} C^{E_\gamma}_{\xi,\infty}$ in W_γ

for all $\xi < \delta_\gamma(\kappa_i)$

prf.

By the ~~Iteration Fact~~ there is $\nu \leq ht(P_\gamma)$ s.t. $\nu > \tau_i = \kappa_i^{+P_\gamma}$ and $E^{P_\gamma}_\nu = \kappa_i$. Since $\delta_\gamma(\tau_i)$ is a cardinal in N, $E^N_{\delta_\gamma(\nu)}$ is robust up to $\delta_\gamma(\tau_i)$ in W_γ. Hence (7) follows. QED(7)

Hence in W_η we have:

$$C^{E_\eta}_{\bar{c}, \delta_\eta(\kappa_i)} \prec_{\Sigma_1} C^{E_\eta}_{\bar{c}, \infty} .$$

Hence:

$$(9)\ C^{E_\eta}_{\bar{c}, \delta_\eta(\kappa_i)} \models \varphi(g''\gamma_i, T_0, e).$$

Since $C^{E_\eta}_{\bar{c}, \delta_\eta(\kappa_i)} = C^{E_i}_{\bar{c}, \delta_i(\kappa_i)}$ is the same in W_η and W_i, we conclude by (1)(b) that

$$(10)\ C^{E_i}_{c_i, \infty} \models \varphi(\delta_i''\gamma_i, T_0, e).$$

This means that there are τ, α s.t. $c_i < \tau < \alpha$, $C^{E_i}_{c_i, \alpha}$ is admissible, and $\tilde{T}_{\tau,\alpha}$ is consistent, where $\tilde{T}_{\tau,\alpha}$ is a theory in the language of $C^{E_i}_{c_i,\alpha}$, except that (B)(iii) is replaced by: $\dot{\sigma} \upharpoonright \underline{\gamma_i} = \underline{\delta_i \upharpoonright \gamma_i}$.

Since $\tilde{T}_{\tau,\alpha}$ is consistent it has a model $\mathfrak{A}$, which we can assume to be good in the sense that the well founded core ($\mathrm{wfcore}(\mathfrak{A})$) is transitive and $\dot{\in}^{\mathfrak{A}} | \mathrm{wfcore}(\mathfrak{A}) = \in$, By the axioms (A) it follows that $\underline{x}^{\mathfrak{U}} = x$ for all $x \in C^{E_i}_{c_i, \alpha}$ and that $\dot{W}^{\mathfrak{U}}$ is an enhanced world with $On \cap \dot{W}^{\mathfrak{U}} = \bar{\tau}$. ($[\bar{\tau}]^{\omega} \cap \dot{W}^{\mathfrak{U}} = [\bar{\tau}]^{\omega} \cap V$ $= \underline{[\bar{\tau}]^{\omega} \cap V^{\mathfrak{U}}}$ follows easily.)

If we set:

$W_{i+1} = \dot{W}^{\mathfrak{U}}$, $N_{i+1} = \dot{N}^{\mathfrak{U}}$, $\delta_{i+1} = \dot{\sigma}^{\mathfrak{U}}$,

then the conditions (a) – (f) are easily verified.

This proves the Main Claim. QED

~~It~~ remains only to prove the ~~Iterability~~ Fact mentioned above.

Let $\mathcal{I} = \langle\langle P_i \rangle, \langle \nu_i \rangle, \langle \pi_{ij} \rangle, T\rangle$ be a putative normal iteration, where P_0 is mouselike. It follows that each P_i is mouselike. As usual, we set: $\kappa_i = \text{crit}(E_{\nu_i})$ in P_i, $\tau_i = \kappa_i^{+ P_i}_{\nu_i}$, $\lambda_i = lh(E_{\nu_i}) = \pi_{T(i+1), i+1}(\kappa_i)$.

Let $\eta = T(i+1)$.

<u>Claim</u> There is $\nu \leq ht(P_\eta)$ s.t. $\nu > \tau_i$ and $\text{crit}(E_\nu^{P_\eta}) = \kappa_i$

Suppose not. Let y be a counterexample of minimal length. Then $lh(y) = i+2$. Clearly $i > \eta$. Hence ν_η is a cardinal in P_i, where $J_{\nu_\eta}^{EN_i} \models ZFC^-$.

Case 1 $\nu_i \in P_i$

Set: $\tilde{P} = P_i || \nu_i$. Define $X \subset \tilde{P}$ by:

$X = h_{\tilde{P}}(\tau_i)$ if $\tilde{P}$ is of type 1 or 2

$X = h^1_{\tilde{P}}(\tau_i) =$ the smallest $X \prec_{\Sigma_1^{(1)}} \tilde{P}$ with $\tau_i \subset X$,

if $\tilde{P}$ is of type 3. Then $X \in P_i$.

Let $\sigma : \bar{P} \leftrightarrow \tilde{P}|X$. Then $\bar{P}$ is a premouse, $\tau_i = \omega\rho^\omega_{\bar{P}} = \omega\rho^1_{\bar{P}}$ if $\tilde{P}$ is of type 1 or 2 and $\tau_i = \omega\rho^\omega_{\bar{P}} = \omega\rho^2_{\bar{P}}$ if $\tilde{P}$ is of type 3. In the latter case $\omega\rho^1_{\bar{P}} = \nu$, where $\nu = ht(\bar{P})$. Clearly the standard parameter of $\bar{P}$ is $\emptyset$. Clearly $\gamma = crit(\sigma) \geq \tau_i$. Hence the conditions for applying the condensation lemma [CR] §IV Lemma 4' are satisfied. Since, however, it only holds internally, we must show:

Claim $\sigma \in P_i$, $P \in J_{\nu_\eta}^{EP_i} = J_{\nu_\eta}^{EP_\eta}$.

proof of Claim:

Let $h = h_{\tilde{p}} \upharpoonright (\omega \times \tau_i)$ if $\tilde{p}$ is of type 1 or 2 and $h = h^1_{\tilde{p}} \upharpoonright (\omega \times \tau_i)$ if not. Then $X = h''(\omega \times \tau_i)$, where $h \in P_i$. Set:

$d = \text{dom}(h)$, $e = \{\langle z, w\rangle \mid h(z) \in h(w)\}$,

Then $d, e \in J^{EP_i}_{\nu_\eta} = J^{EP_\eta}_{\nu_\eta}$, since ν_η is a cardinal in P_i and $\omega \times \tau_i \in J^{EP_i}_{\nu_\eta}$.

Since $J^{P_\eta}_{\nu_\eta} \models ZFC'$, we have:

$\delta \in J^{P_\eta}_{\nu_\eta}$, where $\delta : \langle d, e\rangle \xrightarrow{\sim} \bar{p}$ (in the obvious sense). Hence:

$\sigma = \{\langle h(z), \delta(z)\rangle \mid z \in d\} \in P_i$ QED

Since $\bar{\bar{X}} \doteq \tau_i$ in P_i, we have: $\gamma = \text{crit}(\sigma) \in$ $\in (\tau_i, \tau_i^+)$ in P_i (hence in $J^{EP_\eta}_{\nu_\eta}$).

Applying the condensation lemma, one of (a)–(d) must hold. (a) says that $\bar{p} = \text{core}(\tilde{p})$. This fails since $\tilde{p}$ is sound & hence we would have: $\bar{p} = \tilde{p}$, $\sigma = \text{id}$.

(c) is false, since otherwise τ_i would be the critical pt. of an extender in P_i. But τ_i is a successor cardinal in P_i.

(d) is false, since otherwise $\gamma = \omega\rho^\omega_{\bar{p}}$. Hence (c) holds and $\bar{p} = P_i \| \nu = P_\eta \| \nu$.

QED (Case 1)

Case 2 $\nu_i = ht(P_i)$.
By the minimal length of $\mathcal{J}$ it is easily seen that $i = h+1$. Let $\mu = T(i)$.

(1) $\kappa_h > \kappa_i$

Suppose not. Let $\bar{\nu} = ht(P_h^*) = \pi_{\mu,i}^{-1}(\nu_i)$, $\bar{\kappa} = crit(E_{\bar{\nu}}^{P_h^*})$. At $\kappa_h \leq \bar{\kappa}$, then $\kappa_i = \pi_{\mu i}(\bar{\kappa}) = \lambda_i > \lambda_\eta$. Contr.

Hence $\kappa_h > \bar{\kappa}$ and $\kappa_i = \pi_{\mu i}(\bar{\kappa}) = \bar{\kappa}$. QED(1)

Hence $\mu \geq \eta$. Define a shorter putative iteration by $\mathcal{J}'|\mu+1 = \mathcal{J}|\mu+1$, $\nu_\mu' = \bar{\nu}$. Then $\mathcal{J}'$ is a counterexample. Contr!

QED.