Smooth Iterations

Introduction

By a smooth iteration (as defined in NFS) we mean a (finite or infinite) chain of successive normal iterations. (Steel calls this a "stack" iteration.) In [NFS] and [ANFS] we were able to use a fairly strong background condition to show that, in some cases, mice are not only normally iterable but also smoothly iterable. Later we discovered the weaker background condition of robustness. This enabled the construction of K^c (and eventually K itself), assuming only ZFC + "no inner model with a Woodin". At some point I realized, however, that I did not know how to derive smooth iterability from this weak background condition. In fact, I could not even show that iterates of a normally iterable mouse are themselves normally iterable. (Not even when the original mouse M is uniquely normally iterable — i.e.

satisfies UBH in addition to being normally iterable.) In 2015 an approach dawned on me which might eventually show that uniquely normally iterable mice are smoothly iterable. The approach is easiest to see if we consider the problem of showing that a normal iterate N of a uniquely normally iterable M is normally iterable. There is then a normal iteration I leading from M to N. Given an iteration on N, we then perform a sequence of operations on I which "mirror" the iteration of N. In the case of a one step iteration, where we apply an extender E_ν^N to N, getting ultrapower N′, we modify I to a new iteration I′ by applying E_ν^N at the appropriate point in the iteration, and then copying what I had done after that point. This gives an embedding σ from N′ to the final model M′ of I′.

This process can be continued. If $N_0, \ldots, N_i, \ldots$ is the sequence given by the iteration of N, we get a sequence $I_0, \ldots$ of normal iterations of M and embeddings $\sigma_i : N_i \to M_i$ into the final model of I_i. Each of the I_i is viable by unique iterability. The N_i are connected by a sequence of maps $\bar{\pi}_{ij}$ indexed by a tree T. The I_i are connected by a sequence by maps e_{ij} called insertions of I_i to I_j, indexed by the same tree. We call the sequence:

$\langle \langle I_i \rangle, \langle \nu_i \rangle, \langle e_{ij} \rangle, T \rangle$

an <u>insertion iteration</u> or <u>inseration</u> for short. (The precise definition is given in §2.) ($\nu_i = \sigma_i(\bar{\nu}_i)$ indexes the extender used to get I_{i+1}, where $\bar{\nu}_i$ is the extender used to get N_{i+1}.) If we can show that inserations of M are continuable by the unique strategy, then it follows easily that there is a derived strategy for iterating N.

In the summer of 2015 at a fine structure meeting in Muenster, I spoke with John Steel and discovered that he had essentially the same idea. For both of us, the work was at an early stage. Neither of us knew how to prove "unique iterability". There were also some differences in our approach. I was working with Σ^*-iterable mice, whereas Steel was working with n-bounded iteration of n-iterable mice. Justifying smooth iterability was a minor concern to him, since in that setting, any reasonable background condition will give smooth n-iterability.

After the 2015 meeting I put this approach aside in favor of methods I thought were more likely to solve my problem. Happily, though, John Steel continued with his work. In 2017 I decided to return to this approach, since all else had failed. I then discovered Steel's paper [S]. In it, he had carried the approach very far, finding applications to many fields. He had also essentially proven "unique iterability" and used it to prove smooth iterability for <u>finite</u> sequences $I_1, \ldots, I_n$ of normal iterations. All that remained for me to do was to redo his proofs in terms of Σ^*-iterations. This involved one new twist: The embeddings $\sigma_i : N_i \to M_i$ mentioned above, whereby the iteration of I "mirrors" the iteration of N, are not always Σ^*-preserving, but are Σ^*-preserving modulo <u>pseudo projecta</u>. My recasting of Steel's work is contained in §1-§4.

There still remained the problem of proving smooth iterability of M for infinite chains of normal iterations. The natural way to do this is to show that M be uniquely smoothly iterable. After I had written §1 – §4 of this paper, I wrote both to Steel and his former student Farmer Schlutzenberg and asked whether they had any suggestions. In reply I received a remarkable set of notes from Schlutzenberg. In it, he essentially proved unique iterability of M, using his ingenious new concept of "inflation". In §5 – §7 I work this out in the context of Σ^* - iterations.

In [5] Steel shows that the assumption of unique iterability for M can be replaced with a weaker assumption that I call insertion invariance (which we define ~~at~~ the end of §1). (Steel calls it "pseudo hull condensation".)

In this paper I maintain the assumption that M is uniquely normally iterable, but, as I point out from time to time, the results go through under a weaker assumption. As an example: Let Σ be a normal iteration strategy. Call an insertation Σ-conforming iff all of its component normal iterations are Σ-conforming. If Σ is an insertion invariant successful strategy for M, then the unique strategy works for Σ-conforming insertations based on M. The proofs remain virtually unchanged.

(Note I have since learned from Steel that, at the Muenster meeting of 2015, he also had conversations with Farmer Schlutzenberg on these topics. Working independently, Schlutzenberg had arrived at the same basic approach (embedding an iterate of an iterate into the result of an inseration). Moreover, he had already defined the notion of inflation. Like Steel, Schlutzenberg worked in the context of n-bounded n-iterations.)

(Note I apologize for the awkward term "inseration", which is not an english word and is easily confused with "iteration". In the future I shall use another designation, though I decided not to try to change it in this paper. Since an inseration is, in a very real sense, an iteration of an iteration, the term "reiteration" seems appropriate.)