§0 Introduction

The following condition on a poset $\mathbb{P}$ - though more involuted than Shelah's definition - is clearly equivalent to saying that $\mathbb{P}$ is proper:

(*) For sufficiently large cardinals θ we have; let $\mathbb{P} \in H_\theta$ and let $N = \langle L_\tau[A], A, \ldots \rangle$ where $\tau > \theta$ is regular and $H_\theta \subset N$. Let $\sigma : \bar{N} \prec N$ where $\bar{N}$ is transitive. Let $\sigma(\bar{\theta}, \bar{\mathbb{P}}, \bar{q}) = \theta, \mathbb{P}, q$, where $q \in \mathbb{P}$. There is $p \leq q$ which forces that if $G \ni p$ is $\mathbb{P}$-generic, then

$$\bar{G} = \sigma^{-1}{}''G \text{ is } \bar{\mathbb{P}}\text{-generic over } \bar{N}.$$

Note that in $V[G]$, σ extends uniquely to a $\sigma^* : \bar{N}[\bar{G}] \prec N[G]$ s.t. $\sigma^*(\bar{G}) = G$. For this reason a forcing - like Namba forcing - which makes an uncountable regular cardinal κ ω-cofinal cannot be proper, since if $\sigma(\bar{\kappa}) = \kappa$, then $N[\bar{G}]$ contains a cofinal ω-sequence in $\bar{\kappa}$ which must be carried cofinally to κ by σ. But $\sup \sigma''\bar{\kappa} < \kappa$.

One way of countering this difficulty

would be to alter (*) so as to require only that p force the exitence of a $\sigma_0 \in V[G]$ with:

(a) $\sigma_0 : \bar{N} \prec N$

(b) $\bar{G} = \sigma_0^{-1}{}''G$ is $\bar{\mathbb{P}}$-generic over $\bar{N}$.

We could even require that for any previously specified $\bar{s} \in \bar{N}$ with $s = \sigma(\bar{s})$ we have:

(c) $\sigma_0(\bar{s}) = s$.

A first question is whether we obtain a good iteration theorem for such forcings, using revised countable support iterations. In order to obtain such a theorem we, in fact, had to impose a further requirement: If $\bar{\lambda}_1, \ldots, \bar{\lambda}_m \in \bar{N}$ is any (possibly empty) previously selected sequence s.t. $\lambda_i = \sigma(\bar{\lambda}_i) \in (\omega_1, \theta)$ is regular and $\overline{\overline{\mathbb{P}}} < \lambda_i$, then $\sigma_0(\bar{\lambda}_i) = \lambda_i$ and

(d) $\sup \sigma_0{}''\bar{\lambda}_i = \sup \sigma{}''\bar{\lambda}_i \quad (i = 1, \ldots, m)$

It is also useful and natural to require $\sup \sigma_0{}''\bar{\lambda}_0 = \sup \sigma{}''\bar{\lambda}_0$, where $\bar{\lambda}_0 = \mathrm{On} \cap \bar{N}$.

(We needed (d) to handle certain regular limit points in the iteration. The experts on the subject may well be able to modify or eliminate (d).)

A more serious issue is this: Forcings which make an uncountable regular cardinal ω-cofinal still do not satisfy the revised definition: Let $\mathrm{rng}(\sigma)$ consist of the definable points of N. Then σ is, in fact, the <u>only</u> elementary embedding of $\bar{N}$ into N. Thus we perforce have $\sigma_0 = \sigma$. In order to counter this problem we must impose a "fullness" requirement on $\bar{N}$ which guarantees the possibility of many embeddings into N. The following definition is sufficient:

Def Let $\bar{N}$ be a transitive ZFC^- model. $\bar{N}$ is <u>full</u> iff there is $\mu > 0$ s.t. $L_\mu(\bar{N})$ is a ZFC^- model and $\bar{N}$ is regular in $L_\mu(\bar{N})$ (i.e. whenever $f \in L_\mu(\bar{N})$, $x \in \bar{N}$ and $f : x \to \bar{N}$, then $f \in \bar{N}$).

If $\sigma : \bar{N} \prec N$ we also say that σ is full or that $X = \mathrm{rng}(\sigma)$ is full.

Without further ado we can now define:

Def $\mathbb{P}$ is subproper iff for sufficiently large cardinals θ we have:
Let $\mathbb{P} \in H_\theta$ and let $N = L_\tau[A]$, where $\tau > \theta$ is regular and $H_\theta \subset N$. Let $\sigma : \bar{N} \prec N$ be countable and full. Let $\sigma(\bar{\theta}, \bar{\mathbb{P}}, \bar{\tau}, \bar{\lambda}_1, \ldots \bar{\lambda}_m) = \theta, \mathbb{P}, \tau, \lambda_1, \ldots, \lambda_m$ where $\lambda_1, \ldots, \lambda_m$ is a (possibly empty) sequence s.t. $\lambda_i \in (\omega_1, \theta)$ is regular and $\bar{\mathbb{P}} < \lambda_i$. Let $\bar{\lambda}_0 = \mathrm{On} \cap \bar{N}$. Then for any $q \in \bar{\mathbb{P}}$ there is $p \leq \sigma(q)$ in $\mathbb{P}$ which forces that if $G \ni p$ is $\mathbb{P}$-generic, there is $\sigma_0 \in V[G]$ with:

(a) $\sigma_0 : \bar{N} \prec N$

(b) $\bar{G} = \sigma_0^{-1}{}''G$ is $\bar{\mathbb{P}}$-generic over $\bar{N}$

(c) $\sigma_0(\bar{\theta}, \bar{\mathbb{P}}, \bar{\tau}, \bar{\lambda}_i) = \theta, \mathbb{P}, \tau, \lambda_i$ for $i = 1, \ldots, m$

(d) $\sup \sigma_0{}''\bar{\lambda}_i = \sup \sigma''\bar{\lambda}_i$ for $i = 0, \ldots, m$.

Note Subproperness should not be confused with semiproperness. We shall show that there is a subproper forcing which is provably not semiproper (the forcing $\mathbb{P}_A$ below). Presumably the converse is also true.

In [S] V §1 Shelah introduces a notion of <u>complete forcing</u> (actually $\mathcal{E}$-complete with $\mathcal{E} = \{S_{\omega_0}(\overline{\overline{\mathbb{P}}})\}$). The complete forcings add no reals and are closed under countable support iterations. An equivalent definition can be obtained by modifying (*) to:

(**) For sufficiently large cardinals θ, if $N, \sigma; \overline{N} \prec N, \overline{\mathbb{P}}, \overline{\theta}$ are as above and $\overline{G}$ is any set which is $\overline{\mathbb{P}}$-generic over $\overline{N}$, then there is $p \in \mathbb{P}$ which forces that if $G \ni p$ is $\mathbb{P}$-generic, then $\overline{G} = \sigma^{-1}{}''G$.

Modifying this as (*) was modified to get "subproper" we get:

<u>Def</u> $\mathbb{P}$ is <u>subcomplete</u> iff for sufficiently large cardinals θ, if $N, \sigma; \overline{N} \prec N$, $\overline{\theta}, \overline{\mathbb{P}}, \overline{\tau}, \tau, \overline{\lambda}_i, \lambda_i$ are as in the def. of subproper and $\overline{G}$ is $\overline{\mathbb{P}}$-generic over $\overline{N}$, then there is $p \in \mathbb{P}$ which forces that if $G \ni p$ is $\mathbb{P}$-generic, then there is $\sigma_0 \in V[G]$ s.t. (a), (c), (d) hold together with: (b') $\overline{G} = \sigma_0^{-1}{}''G$.

Subcomplete forcings will be the main focus of this paper.

We can also modify the definition of "semi-proper" in the same way, arriving at:

<u>Def</u> $\mathbb{P}$ is <u>semi subproper</u> iff for sufficiently large cardinals θ, if N, $\sigma : \bar{N} \prec N$, $\bar{\theta}$, $\bar{\mathbb{P}}$, $\bar{\tau}$, $\bar{\lambda}_i$, λ_i are as in the def. of subproper and $q \in \bar{\mathbb{P}}$, there is $p \leq \sigma(q)$ in $\mathbb{P}$ which forces that if $G \ni p$ is $\mathbb{P}$-generic, then there is $\sigma_0 \in V[G]$ s.t. (a), (c), (d) hold together with: If $\sigma_0(\bar{t}) = t$ and $\Vdash_{\mathbb{P}} t < \check{\omega}_1$, then $\bar{t}^{\bar{G}} = t^G$, where $\bar{G} = \sigma_0^{-1}$"G.

The iteration theorem we shall prove holds for "subcomplete", "subproper" and "semi subproper". However, we shall not carry out the proof for semi-subproper forcings, since we shall not otherwise deal with that notion in this paper.

An addition to proving an iteration theorem we show that the following forcings are subcomplete:

(a) $\mathbb{P}_A$ where $A \subset \omega_2$ is a stationary set of ω-cofinal points and the forcing adds a closed subset of A of order type ω_1. (The conditions have the form $p : \alpha + 1 \to A$, where $\alpha < \omega_1$ and p is a normal function.)

(b) Prikry forcing

(c) Namba forcing if $2^{\omega}=\omega_1$ and $2^{\omega_1}=\omega_2$,

(d) Some other Namba-like forcings

(e) Various $\mathcal{L}$-forcings in the sense of [J]. In particular, forcings which add iterable structures without adding reals. (e.g. if U is a normal ultrafilter on κ and $\tau>\kappa$ is regular s.t. $2^{\tau}=\tau$, then such a forcing adds an $\bar{N} = \langle \bar{H}, \bar{U} \rangle$ which iterates up to $\langle H_\tau, U \rangle$ in ω_1 many steps).

In [S] Shelah discusses an interesting variant of Namba forcing which he calls Nm'. (The conditions are trees which can have a finite stem of arbitrary length, but above the stem each node has ω_2 many immediate successors.) We attempted to show that Nm' is subcomplete, but were only able to show that it is subproper (hence it can be iterated without collapsing ω_1). It is hard for us to believe that an RCS-iteration of Nm' would add new reals, so there is more work to be done.

<u>Note</u> The requirement $2^{\omega_1} = \omega_2$ for Namba forcing may be unduly strict, but we do not know how to dispense with it. However, we can always make it true by a prior forcing which adds a Cohen generic subset of ω_2 with conditions of size $\leq \omega_1$. This forcing is complete; hence the $*$-product of the two forcings is subcomplete. (Similarly if in (e) $\tau = \kappa^+$, we can make $2^{\kappa} = \tau$ true by a prior complete forcing.)

In §1 we develop some basic conventions. In §2 we prove our iteration theorem. In dealing with abstract iteration theory we prefer to use the Boolean approach to forcing. We also use Donder's version of revised countable support iteration. We prove:

Let $\langle \mathbb{B}_i \mid i \leq \alpha \rangle$ be a tower of complete Boolean algebras s.t. $\mathbb{B}_0 = 2$, $\mathbb{B}_i$ is completely contained in $\mathbb{B}_j$ for $i \leq j$, and $\mathbb{B}_\lambda$ is Donder's revised countable support limit of $\langle \mathbb{B}_i \mid i < \lambda \rangle$ for limit $\lambda \leq \alpha$. Suppose also that $\Vdash_{\mathbb{B}_i} (\check{\mathbb{B}}_{i+1}/\mathring{G}_i$ is subproper$)$ and $\Vdash_{\mathbb{B}_{i+1}} \mathrm{card}(\check{\mathbb{B}}_i) \leq \omega_1$ for all $i < \alpha$. Then $\mathbb{B}_\alpha$ is subproper.

We also show that this holds with "subcomplete" in place of "subproper". (The Theorem also holds with "semisubproper" in place of "subproper", but we do not prove this.) We also prove a more general version which obviates the need to collapse $\mathbb{B}_i$ at stage $i+1$. (but there is, of course, no guarantee that $\mathbb{B}_i$ will not ultimately be collapsed to ω_1).

In §3 we then prove that the above-mentioned forcings are subcomplete or subproper respectively. We begin by developing some consequences of our notion of "fullness". Given these, the verifications for Prikry forcing and the forcing $\mathbb{P}_A$ are straightforward. For the others we need the theory of $\mathcal{L}$-forcing as developed in [J]. In particular we use the fact that Namba forcing is equivalent to an $\mathcal{L}$-forcing.

Bibliography

[J] Jensen $\mathcal{L}$-Forcing (handwritten notes)

[S] Shelah Proper and Improper Forcing
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