On some problems of Mitchell, Welch and Vickers

Phillip Welch asked whether K_{ω_1} can be an initial segment of an iterate of a countable mouse. More specifically, Mitchell asked the question:

Assume $K = L^U$, where U is a measure on ω_1 (hence $U = E_\nu$ where $\nu = \kappa^{+K}$, $\kappa = \omega_1$). Can $\langle J_\nu^E, E_\nu \rangle$ be an iterate of a countable mouse?

Welch stated a more difficult version of his question:

Assume that

(*) O^s does not exist but the reals are closed under $\#$,

can there be a countable mouse M and a real a s.t.

(a) K_{ω_1} is an initial segment of an iterate of M

(b) M is an initial segment of $K^{L[a]}$

Welch's student John Vickers asked whether, on the assumption (*), there can be a real a s.t.

$$K_{\omega_1} = K_{\omega_1}^{L[a]}.$$

The answer to all these questions is yes.

We use the core model theory of our notes "Non Overlapping Extenders", though the argument should be quite comprehensible to those who know our notes "Measures of Order Zero". We make virtually no direct use of fine structure.

Note The forcing method used in the proof of Thm 1 appears to have more general applications. Fr. ins. forcing over an arbitrary ZFC model we can use it to give ω_2 cofinality ω while preserving ω_1, thus providing an alternative to Bukovski - Namba forcing.

Thm 1 Assume $V = K$. Let $E_\nu = \emptyset$, $\mathrm{crit}(E) = \kappa$, where $\nu \geq \kappa^+$. Let $M = K_\beta$ where $\omega\beta \geq \nu$. Assume that ρ^ω_M is a cardinal of cofinality $> \kappa$. There is a set of conditions $\mathbb{P}$ s.t. if G is $\mathbb{P}$-generic, then in $K[G]$ we have: κ is regular and there are M_0, ν_0 s.t.

(*) M_0 is a mouse and $\nu_0 \leq \mathrm{On} \cap M_0$. Let $\langle M_i \mid i \leq \kappa \rangle$ be the simple iteration with indices $\nu_i = \pi_{M_0 M_i}(\nu_0)$. Then $M_\kappa = M$, $\nu_\kappa = \nu$.

The proof stretches over several lemmas.

Let $\theta = \beta^{++}$. Let $\mathcal{L}$ be an infinitary language on K_θ with predicate $\in$, constants $\underline{M}_0, \underline{\nu}_0$ and $\underline{x}$ $(x \in K_\theta)$, and axioms:

$ZFC^- + \underline{\kappa} = \omega_1 +$ (*) (with $\underline{M}, \underline{\nu}$ in place of M, ν) and

$$y \in \underline{x} \longleftrightarrow \bigvee_{z \in x} y = \underline{z} \quad (x \in K_\theta).$$

(Note $\mathcal{L} \vdash K_{\nu_i}^{M_{i+1}} = K_{\nu_i}^{\overline{M}}$)

Lemma 1 $\mathcal{L}$ is consistent.

proof.

Let $\tau = \theta^+$. Let K_i ($i \leq \tau$) be the iteration of K with maps π_{ij} + indices $\nu_i = \pi_{0i}(\nu)$. Set: $M_i = \pi_{0i}(M)$, $\rho_i = \pi_{0i}(\rho)$, where $\rho = \rho^{\omega}_{M}$. Then $\pi_{i,i+1} \upharpoonright J^{E^{M_i}}_{\rho_i} : J^{E^{M_i}}_{\rho_i} \longrightarrow_{E^{M_i}_{\nu_i}} J^{E^{M_{i+1}}}_{\rho_{i+1}}$

since ρ is regular, $\mathrm{cf}(\rho) > \omega$.

But $\rho_i = \rho^{\omega}_{M_i}$ + M_i is sound, hence M_i = the closure of ρ_i under good Σ^* functions.

It follows that $\pi_{i,i+1} \upharpoonright M_i : M_i \xrightarrow{*}_{E_{\nu_i}} M_i$

Hence $\langle M_i \rangle$ is the iteration of M to $\pi_{0\tau}(M)$ with indices $\nu_i = \pi_{M M_i}(\nu)$. Now add $F \subset K_\tau$ which generically collapses each $\xi < \tau$ to ω. Then $\langle K_\tau + [F], M \rangle$ models $\pi_{0\tau}(\mathcal{L})$. Hence $\pi_{0\tau}(\mathcal{L})$ is consistent. Hence so is $\mathcal{L}$.

QED (Lemma 1)

Def Let $p = \langle p_0, p_1, p_2 \rangle$ s.t.

(a) p_0 is a finite partial map $\kappa \to \kappa$

(b) $\mathrm{dom}(p_1) = \mathrm{dom}(p_0)$ and each $p_1(i)$ is a finite partial map $\kappa \to \beta$

(c) $\mathrm{dom}(p_2) \subset \mathrm{dom}(p_0)$ and $p_2(i) \subset$

Set: $\mathrm{dom}(p) =_{Df} \mathrm{dom}(p_0)$;

$\kappa_i^p = p_0(i)$, $\pi_i^p = p_1(i)$, $a_i^p = p_2(i)$

$\mathcal{L}(p)$ is $\mathcal{L}$ enhanced by the axioms:

(i) $\underline{\kappa_i^p} = \kappa_{\underline{i}}$ $(i \in \mathrm{dom}(p))$

(ii) $\underline{\pi_i^p} \subset \pi_{\underline{i}}$ ("), where

$\pi_{ij} =_{Df} \pi_{M_i M_j}$, $\pi_i =_{Df} \pi_{i\underline{\kappa}}$

(iii) $\bigvee \bar{a}\ \pi_{\underline{i}} : \langle M_{\underline{i}}, \bar{a} \rangle \prec \langle \underline{M}, \underline{a_i^p} \rangle$ for $i \in \mathrm{dom}(p_2)$.

p is good iff $\mathcal{L}(p)$ is consistent

Def Let $i, j \in \mathrm{dom}(p_2)$, $i < j$.

i is neat in j wrt p iff a_i is $\langle M, a_j(n) \rangle$-definable from parameters in $\mathrm{rng}(\pi_j^p)$ for some $n < \omega$, where $a_j(n) =_{Df} \{z \mid \langle n, z \rangle \in a_j\}$

p is neat iff i is neat in j for all $i, j \in \mathrm{dom}(p_2)$, $i < j$.

Def $\mathbb{P}$ = the set of good, neat p, ordered by:

$p \leq q \longleftrightarrow p_0 \supset q_0,\ p_2 \supset q_2$ and

$\pi_i^p \supset \pi_i^q$ for $i \in \mathrm{dom}(q)$.

We now state some lemmas on the possibility of extending conditions.

Lemma 2.1 Let $p \in \mathbb{P}$. Let u be finite s.t. $\mathrm{dom}(p) \subset u \subset \alpha$. There is $p' \leq p$ s.t. $u \subset \mathrm{dom}(p')$.

proof.

Let $K[F]$ be a generic extension which makes Θ countable. In $K[F]$ there is a model $\mathfrak{A}$ for $\mathcal{L}(p)$ which we may take to be solid in the sense that its well founded core $\overline{\mathfrak{A}}$ is transitive. Then $x = \underline{x}^{\mathfrak{A}} \in \overline{\mathfrak{A}}$ for $x \in K_\Theta$ and $x \in \overline{\mathfrak{A}}$ whenever $\mathrm{sn}(x)^{\mathfrak{A}} = \alpha \in \overline{\mathfrak{A}}$. In particular

$L_{\beta^+} \in \mathfrak{M}$, $\langle \kappa_i \mid i \leq n \rangle = \langle \kappa_i \mid i \leq \underline{n} \rangle^{\mathfrak{M}} \in \mathfrak{M}$,

$\langle \pi_{ij} \mid i \leq j \leq n \rangle = \langle \pi_{ij} \mid i \leq j \leq \underline{n} \rangle^{\mathfrak{M}} \in \mathfrak{M}$ etc.

Set: $p' = \langle p_0', p_1', p_2 \rangle$, where

$p_0' = \langle \kappa_i \mid i \in u \rangle$, ..

$$p_1'(i) = \begin{cases} p_1(i) & \text{if defined} \\ \emptyset & \text{if not.} \end{cases}$$

p' is neat. But p' is good since $\mathfrak{M}$ models $\mathcal{L}(p')$. Hence $p' \leq p$.

QED (Lemma 2.1)

By entirely similar proofs:

<u>Lemma 2.2</u> Let $p \in \mathbb{P}$, $i \in \mathrm{dom}(p)$, $\xi \leq \nu_i$. There is $p' \leq p$ s.t. $\xi \in \mathrm{dom}(\pi_i^{p'})$.

(where $\nu_i^p = \sigma(\kappa_i^p)$).

<u>Lemma 2.3</u> Let $p \in \mathbb{P}$, $i \in \mathrm{dom}(p)$, $\xi \in \mathrm{dom}(\pi_i^p)$, $\zeta < \xi$. There is $p' \leq p$ s.t. $\zeta \in \mathrm{dom}(\pi_i^{p'})$

<u>Lemma 2.4</u> Let $p \in \mathbb{P}$, $i \in \mathrm{dom}(p)$, $\xi \in M$. There is $p' \leq p$ s.t. $\zeta \in \mathrm{rng}(\pi_i^{p'})$ for a $\zeta \geq \xi$.

Lemma 2.5 Let $p \in \mathbb{P}$, $\xi \in M$. There is $p' \leq p$ s.t. $\xi \in \mathrm{rng}(\pi_i^{p'})$ for an $i \in \mathrm{dom}(p')$.

Lemma 2.6 Let $p \in \mathbb{P}$, $\lambda \in \mathrm{dom}(p)$, $\overline{\xi} \leq \xi \in \mathrm{dom}(p_\lambda)$. There is $p' \leq p$ s.t. $\overline{\xi} \in \mathrm{rng}((\pi_\lambda^{p'})^{-1} \circ \pi_i^{p'})$ for an $i \in \mathrm{dom}(p')$.

Using $\mathcal{L} \vdash \pi_{i,i+1} : M_i \xrightarrow[E_{\nu_i}]{*} M_{i+1}$, we get:

Lemma 2.7 Let $p \in \mathbb{P}$, $i, i+1 \in \mathrm{dom}(p)$. Let $\xi \in \mathrm{rng}(\pi_{i+1}^p)$. There is $p' \leq p$ s.t. $\xi = f(\pi_{i+1}^{p'}(\zeta_1), \ldots, \pi_{i+1}^{p'}(\zeta_n))$ where $\zeta_1, \ldots, \zeta_n < \nu_i$ and f is M-definable in parameters from $\mathrm{rng}(\pi_i^{p'})$.

[Note By the form of M, $\pi : M_i \xrightarrow[E_{\nu_i}]{*} M_{i+1}$ simply says that M_{i+1} is the ultrafilter of M_i by $\Sigma_\omega(M_i)$ functions.]

<u>Lemma 2.8</u> Let $p \in \mathbb{P}$, $i \in dom(p)$ and let $\bar{a} \in [\nu_i]^{<\omega}$, $\bar{a} \subset dom(\pi_i^p)$, and $a = \pi_i^{p\,\prime\prime}\bar{a}$. Let $X \subset [\kappa]^{|a|}$ be M-definable in parameters from $rng(\pi_i^p)$. Then:

$$X \in E_{\nu a} \longleftrightarrow \bar{a} \in X.$$

We also have:

<u>Lemma 2.9</u> Let $p \in \mathbb{P}$, $i \in dom(p)$ + let ξ be M definable from parameters in $rng(\pi_i^p)$. There is $p' \leq p$ s.t. $\xi \in rng(\pi_i^{p'})$.

<u>Lemma 2.10</u> Let $p \in \mathbb{P}$, $i \in dom(p)$, $j \in dom(p)$, $j > i$, $\xi \in dom(\pi_i^p)$. There is $p' \leq p$ s.t. $\xi \in dom($

Now let G be $\mathbb{P}$-generic over K.
By the extension lemma we can set:

$$\langle \nu_i \mid i < n \rangle = \bigcup_{p \in G} p_0$$

$$\pi_i = \bigcup_{\substack{p \in G \\ i \in \mathrm{dom}(p)}} \pi_i^p \quad , \ \beta_i = \mathrm{dom}(\pi_i),$$

Then $\beta_i \geq \nu_i$ and $\pi_i : \beta_i \to \beta$ is order preserving. $\pi_i(\nu_i) = \nu$ if $\nu < \beta$.

Set: X_i = the smallest $X \prec M$ s.t. $\mathrm{rng}(\pi_i) \subset X$.

Then $X_i \cap \beta = \mathrm{rng}(\pi_i)$ by Lemma 2.9. Set: $\tilde{\pi}_i : M_i \stackrel{\sim}{\longleftrightarrow} X_i$.

Then $\tilde{\pi}_i : M_i \prec M$ and $\mathrm{rng}(\tilde{\pi}_i) \subset \mathrm{rng}(\tilde{\pi}_j)$ for $i \leq j$ by Lemma 2.

Set: $\tilde{\pi}_{ij} = \tilde{\pi}_j^{-1} \circ \tilde{\pi}_i$, $\tilde{\pi}_{in} = \tilde{\pi}_i$,

$\nu_n = \nu$, $M_n = M$. By the extension lemma it follows easily that

<u>Lemma 3</u> $\langle M_i \mid i \leq \kappa \rangle$ is a simple iteration of M_0 with maps $\tilde{\pi}_{ij}$ and indices $\nu_i = \tilde{\pi}_{0i}(\nu_0)$.

Thus it remains only to show:

<u>Lemma 4</u> κ is regular in $K[G]$.

prf.

Let $\delta < \kappa$, $\Vdash \mathring{f} : \check{\delta} \to \check{\kappa}$. Let $p \in \mathbb{P}$

<u>Claim</u> There is $p' \leq p$ s.t.

$p' \Vdash \mathrm{rng}(\mathring{f}) \subset \check{\alpha}$ for an $\alpha < \kappa$.

Let $X =$ the smallest $X \prec \langle K_{\theta^+}, M, p \ldots$ s.t. $\beta \subset X$. Let $\sigma : N \overset{\sim}{\longleftrightarrow} X$, where N is transitive. Then $\sigma(M, p) = M, p$. Let $\tilde{f} = \sigma^{-1}(\mathring{f})$.

Let $\langle \varphi_i(v_1, \ldots, v_{q_i}) \mid i < \omega \rangle$ be a recursive enumeration of the N-functions + set:

$b_i = \{ \vec{z} \mid N \models \varphi_i[\vec{z}] \text{ and } \vec{z} \in M \}$.

Set: $b = \{\langle i, z\rangle \mid z \in b_i\}$. Then $a \in \mathcal{P}(\beta) \cap N = \mathcal{P}(\beta) \cap X$ iff a is $\langle M, b_i\rangle$ definable in some parameter for an $i < \omega$.

Go to $K[F]$, where F generically collapses θ to ω and let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$. Since κ is regular in $\mathfrak{A}$, there must be $\alpha < \kappa$ s.t. $\alpha = \kappa_\alpha \supset \mathrm{dom}(p)$ and $\pi_\alpha : \langle M_\alpha, \bar{b}\rangle \prec \langle M, b\rangle$ for some $\bar{b}$. Set: $p' = \langle p_0', p_1', p_2'\rangle$ where $p_0' = p_0 \cup \{\langle \alpha, \alpha\rangle\}$, $p_1' = p_1 \cup \{\langle \emptyset, \alpha\rangle\}$, $p_2' = p_2 \cup \{\langle b, \alpha\rangle\}$, p' is good, since $\mathfrak{A}$ models $\mathcal{L}(p')$. p' is neat since each a_i^p ($i \in \mathrm{dom}(p)$) is $\langle M, b_\alpha\rangle$-definable in no parameters. Hence $p' \leq p$

<u>Claim</u> $p' \Vdash \mathrm{rng}(\mathring{f}) \subset \check{\alpha}$.

Suppose not. Let $q \leq p'$ s.t.

(1) $q \Vdash \mathring{f}(\check{\xi}) \geq \check{\alpha}$, where $\xi < \delta$.

Pick solid $\mathfrak{A}$ which models $\mathcal{L}(q)$. Again there is $\bar{b} \in \mathfrak{A}$ s.t. $\pi_\alpha : \langle M_\alpha, \bar{b} \rangle \prec \langle M, b \rangle$. Set: Y = the smallest $Y \prec N$ s.t. $\mathrm{rng}(\pi_\alpha) \subset$ It follows easily that $Y \cap M = \mathrm{rng}(\pi_\alpha)$. Set: $\pi : \bar{N} \leftrightarrow Y$, where $\bar{N}$ is transitive. Then $\pi : \bar{N} \prec N$ and $\pi \supset \pi_\alpha$. It is easily seen that $\pi(M_\alpha) = M$. Set: $\bar{p}, \bar{f} = \pi^{-1}(p, \tilde{f})$. If we define $\mathbb{P}_{\bar{N}}, \mathbb{P}_N$ in $\bar{N}, N$ resp. as $\mathbb{P}$ was defined in $N' = \langle K_{\theta^+}, M, p, \mathring{f} \rangle$, then $\mathbb{P}_N = \mathbb{P} \cap N$.

Now set: $q' = \langle q_0 \restriction \alpha, q_1 \restriction \alpha, q_2 \restriction \alpha \rangle$.

Then $q \leq q' \leq p$. But

(2) $q' \in \mathbb{P}_N$

since if $i \in \mathrm{dom}(q_2')$, then $a = a_i^{q'}$ is $\langle M, b_n \rangle$-definable in parameters from $\mathrm{rng}(\pi_i^{q'})$ + hence $a \in N$.

Let $\bar{q}' = \pi^{-1}(q')$. Since $\Vdash^{\bar{N}}_{\mathbb{P}_{\bar{N}}} \bar{f} : \check{\delta} \to \check{\alpha}$, there is $\bar{q}'' \leq \bar{q}'$ in $\mathbb{P}_{\bar{N}}$ s.t.

(3) $\bar{q}'' \Vdash^{\bar{N}}_{\mathbb{P}_{\bar{N}}} \bar{f}(\check{\xi}) = \check{\nu}$, where $\nu < \alpha$

+ ξ is as in (1). Hence

(4) $q'' \Vdash \mathring{f}(\check{\xi}) = \check{\nu}$,

since $\sigma\pi : \bar{N} \prec N'$. Hence

(5) q'', q are incompatible.

We derive a contradiction by constructing $q^* \leq q'', q$.

Set $\bar{\mathcal{L}} = (\pi \circ \sigma)^{-1}(\mathcal{L})$. Then $\bar{\mathcal{L}}(\bar{q}'')$ is consistent theory on $K_{\bar{\theta}} = (\pi\circ\sigma)^{-1}(K$

But $\bar{\theta}$ is countable in $\mathfrak{A}$. Hence there is a solid $\bar{\mathfrak{A}} \in \mathfrak{A}$ which models $\bar{\mathcal{L}}(\bar{q}'')$, giving $\langle \bar{M}_i \mid i \leq \alpha \rangle$, $\langle \bar{\pi}_{ij} \mid i \leq j \leq \alpha \rangle$ with indices $\bar{\nu}_i$ s.t. $\bar{\nu}_i = \bar{\pi}_{0i}(\bar{\nu}_0)$ and $\bar{M}_\alpha, \bar{\nu}_\alpha = M_\alpha, \nu_\alpha$. But then $\bar{M}_0, \bar{\nu}_0$ iterate up to M, ν, the iteration being identical to that of M_0, ν_0 from α onward. It follows easily that $\tilde{\mathfrak{A}} = \langle |\mathfrak{A}|, \bar{M}_0, \bar{\nu}_0$ models $\mathcal{L}(q'')$ and $\mathcal{L}(q)$.

Thus we may assume w.l.o.g.:

(6) $\mathfrak{A}$ models $\mathcal{L}(q'')$, $\mathcal{L}(q)$.

Set: $q^* = \langle q_0^*, q_1^*, q_2^* \rangle$ where $q_0^* = q_0 \cup q_0''$, $q_2^* = q_2 \cup q_2''$ and $q_1^* \supset q_1 \cup q_1''$ is defined as follows:

$$\pi_i^{q^*} = \pi_i^{q''} \text{ for } i < \alpha.$$

Since $\bar{a}_i = a_i^{\bar{q}''} \in \bar{N}$ for $i \in \mathrm{dom}(\bar{q}'')$, there are $n < \omega$ and a finite $u \subset \beta_\alpha$ s.t. $\bar{a}_i$ is $\langle M_\alpha, \bar{b}_n \rangle$-definable in parameters from u. Hence $a_i = a_i^{q''}$ is $\langle M, b_n \rangle$-definable in parameters from π_α "u.

Assume w.l.o.g. that

$\pi_{i\alpha}$ "$\mathrm{dom}(\pi_i) \subset u$ for $i \in \mathrm{dom}(q'')$

For $j \in \mathrm{dom}(q)$, $j \geq \alpha$ set:

$$\pi_j^{q^*} = \pi_j \restriction (\pi_{\alpha j}{}''u \cup \mathrm{dom}(\pi_j^q)).$$

This definition guarantees neatness. But q^* is good, since $\mathfrak{A}$ models $\mathcal{L}(q^*)$. Hence $q^* \leq q, q''$.

Contr! QED (Thm 1)

We recall the defining property of K

(1) At $K = J_\infty^E$ and $\langle J_\nu^E, F \rangle$ is a strong mouse s.t. $F \neq \emptyset$, then $F = E_\nu$

[A mouse $N = \langle J_\alpha^{E^N}, E_{\omega\alpha}^N \rangle$ is called <u>strong</u> iff

(2) Whenever M is a premouse s.t. $N = M \| \alpha$ and M is iterable beyond $\omega\alpha$ + (i.e on extenders of index $> \omega\alpha$), then M is a mouse and $N = \mathrm{core}(M) \|$

This is equivalent to:

(3) $N = W \| \alpha$, where W is a universal weasel.]

To obtain the full force of Thm 1, we need:

<u>Lemma 5</u> Let W be a set generic extension of K. Then $K^W = K$.

prf.

Suppose not. Then there is a set of conditions s.t.

If $K_\nu \neq \check{K}_\nu$ for some ν. Let ν be least s.t. such a $\mathbb{P}$ exists. Let G be $\mathbb{P}$-generic, $W = K[G]$. Then letting $K' = K^W = J_\infty^{E'}$, we have $J_\nu^E = J_\nu^{E'}$ and $E \neq E'$. But K is universal in W since successors of sufficiently large singular cardinals are preserved. Hence $E_{\omega\nu} = \emptyset$ and $E'_{\omega\nu} \neq \emptyset$, since otherwise $\langle J_\nu^{E'}, E_{\omega\nu}\rangle$ is strong + hence $E_{\omega\nu} = E'_{\omega\nu}$. Now let $G \times G'$ be $\mathbb{P} \times \mathbb{P}$-generic. Set $W' = K[G']$, $K'' = J_\infty^{E''} = K^{W'}$. Then $K''_\nu = \langle J_\nu^E, E''\rangle$. K', K'' are universal in $K[G \times G']$, since sufficiently large singular cardinals are preserved. Hence $\langle J_\nu^E, E'_{\omega\nu}\rangle, \langle J_\nu^E, E''_{\omega\nu}\rangle$ are strong in $K[G \times G']$. Hence $E'_{\omega\nu} = E''_{\omega\nu} = E_{\omega\nu}^{K^*}$, where $K^* = K^{K[G \times G']}$, But then $E'_{\omega\nu} \in K$, since $E'_{\omega\nu} \subseteq K$ and $E'_\nu \in K[G] \cap K[G']$, where $G \times G'$ is $\mathbb{P} \times \mathbb{P}$-generic. Hence $\langle J^E_\nu, E'_{\omega\nu} \rangle$ is strong in K. Hence $E'_{\omega\nu} = E_{\omega\nu}$. Contra!

QED (Lemma 5)

<u>Corollary 6</u> Let $W = K[G]$ be the model of Theorem 1. Then $\kappa = \omega_1{}^W$ and $K = K^W$.

prf.

$K = K^W$ is immediate. Since $E_{\omega\nu_i} = \emptyset$ for $i < \kappa$, $\langle J^E_{\nu_i}, E^{M_i}_{\omega\nu_i} \rangle$ is not strong for any $i < \kappa$. Hence $E^{M_i}_{\omega\nu_i}$ is not ω complete. Hence $cf(\lambda) = \omega$ for limit $\lambda < \kappa$, since otherwise $E^{M_\lambda}_{\omega\nu_\lambda}$ is ω-complete. Hence $\kappa = \omega_1$. QED (Cor 6)

This takes care of Mitchell's problem.

Note The question whether such anomalies can occur between ω_1 and ω_2 seems to be more difficult. Fr. ins. let $M = \langle J_\alpha^E, E_\alpha \rangle$ be a mouse with $\mathrm{crit}(E_\alpha) = \kappa$ = the largest cardin in M. Let $M_i = \langle J_{\alpha_i}^{E^i}, E^i_{\alpha_i} \rangle$ be the iteration of M by the top measure. Set: $\beta = \beta_M \simeq$ nf that β s.t. $M_\beta = \langle J_{\alpha_\beta}^{E^\kappa}, E^\kappa_{\alpha_\beta} \rangle$ and $\alpha_\beta = \kappa^{+K}$. Then $\beta \leq \omega_1$ if it exists. We can show that β can take any finite value, but do not know if $\beta = \omega$ is possible. (We conjecture that β can have any value $\leq \omega_1$.)

We now give the solution of Welch's problem. Assume $V = K$ and $E_\nu \neq \emptyset$, where $\nu = \kappa^+$, $\mathrm{crit}(E_\nu) = \kappa$. Assume furthermore that there are arbitrarily large ξ with $E_\xi \neq \emptyset$. Hence every set of ordinals has a #. Let β = the least $\beta > \nu$ s.t. $E_\beta \neq \emptyset$. Set $M = K_\beta = \langle J_\beta^E, E_\beta \rangle$. (Then M has the constructibility degree of $A^\#$ where $A \subset \nu$ codes $\langle J_\nu^E, E_\nu \rangle$.)

Let $W = K[G]$ be the extension of Thm 1 and let M_i, ν_i be as in Thm 1.

Then $M_\kappa = M$, $\nu_\kappa = \nu$. Note that if $N_i = M_i \| \nu_i$, then N_i is the iteration of N_0 to $M \| \nu = \langle J_\nu^E, E_\nu \rangle$ by the same indices.

Now let $Q_i = \langle J^{\bar{E}^i}_{\bar{\beta}_i}, \bar{E}_{\bar{\beta}_i} \rangle$ be the iteration of $Q_0 = M_0$ by the indices $\bar{\beta}_i$ (i.e. only the top measure is moved). Letting $\bar{\tau}_i = \mathrm{crit}(\bar{E}_{\beta_i})$, we have:

$$\bigcup_i Q_i \| \bar{\tau}_i = J^{\tilde{E}}_{\infty}\,,$$ where $\tilde{E} = (E^0 \upharpoonright (\nu_0+1))$. [Thus $|J^{\tilde{E}}_{\infty}| = L[N_0]$]. Set: $\tilde{K} = J^{\tilde{E}}_{\infty}$. Since $\nu_0^{+\tilde{K}}$ is countable in W, there is an $F \in W$ which is generic over $\tilde{K}$ by the condition for collapsing ν_0 to ω. Set: $\tilde{W} = \tilde{K}[F]$. Then $\tilde{W} = \tilde{K}[a] = L[a]$ for an $a \subset \omega$ which codes N_0, F. Hence $\tilde{K} \doteq K^{\tilde{W}}$ and N_0 is an initial segment of $\tilde{K}$. But K_{ω_1} is an initial segment of $N_\kappa = \langle J^E_\nu, E_\nu \rangle$, which is an iterate of N_0..

Finally we note that every set of ordinals has a sharp in W, since W is a set generic extension of K. This gives a positive solution to Welch's problem.

To solve Vickers' problem, we use a theorem on coding which is stated on p. 308 of Coding the Universe (Beller, Jensen, Welch):

(**) Let M be an inner model of ZFC, U an ultrafilter on $\mathcal{P}(\kappa) \cap M$ s.t.

(a) $\langle H^M_{\kappa^+}, U \rangle$ is amenable ($\kappa^+ =_{df} \kappa^{+M}$) and U is normal on κ in $\langle H^M_{\kappa^+}, U \rangle$.

(b) M is iterable by U

(c) κ^+ is countable.

Then there is $a \subset \omega$ s.t.

(i) $M \subset L[a]$

(ii) Cardinals and cofinalities of M are preserved in $L[a]$.

The proof shows that a looks "locally" like a real which is added to M by the forcing for coding the universe. In particul

(iii) If α is a cardinal in M, then
$L[a] = L[b][\bar{G}]$, where
$L[b]^{\alpha} \subset M$ and $\bar{G}$ is set
generic over $L[b]$, (and M is a definable inner model of $L[b]$).

Now assume that in K we have
$E_{\kappa^+} \neq \emptyset$, $\kappa = \mathrm{crit}(E_{\kappa^+})$, $\nu = \kappa^+$.
Set $M = \langle J_{\nu}^{E}, E_{\nu} \rangle$. Let $W = K[G$
be as in Thm 1, giving M_0, ν_0.
Set $\bar{K} = \bigcap_{i<\infty} M_i = \bigcup_{i<\infty} (M_i \| \kappa_i)$,
where $M_i = \langle J_{\nu_i}^{E^i}, U_i \rangle$ are the
iterates of M_0 with indices ν_i.
Set $\bar{U} = U_0$, $\bar{\kappa} = \kappa_0$. Then $\bar{K}, \bar{\kappa}, \bar{U}$
satisfy the assumptions of $(**)$
in place of M, κ, U. Let $L[a]$
be as in (i), (ii), (iii). Recall that
$\kappa = \omega_1$ in W. Thus it suffices
to show:

<u>Claim</u> $K_{\kappa} = K_{\kappa}^{L[a]}$.

Since $K_\kappa = \bar{K}_\kappa$, this follows by:

Claim $K^{L[a]} = \bar{K}$

Set: $K' = K^{L[a]}$. Let α be regular in $\bar{K}$. Let $L[a] = L[b][\bar{G}]$, where $\bar{G}$ is set generic and $L[b]^\alpha \subset \bar{K}$. b will code $\bar{K}$ as an inner model of $L[b]$. $K' = K^{L[b]}$ by Lemma 5.

Claim $K'_\xi = \bar{K}_\xi$ for $\xi < \alpha$.

Suppose not. Let ξ be the least counterexample. Let $\bar{K}_\xi = \langle J_\xi^{\bar{E}}, F\rangle$, $K'_\xi = \langle J_\xi^{\bar{E}}, F'\rangle$. $\bar{K}$ is universal in $L[b]$, since cardinals are preserved. Thus if $F \neq \emptyset$, we have $F' = F$, since $\bar{K}_\xi$ is strong in $L[b]$. If $F' \neq \emptyset$, then $F' \in \bar{K}$ and K'_ξ is strong in $L[b]$, hence in $\bar{K}$. Hence $F' = F$. Contradiction!

QED