§ 3 $\mathcal{L}$ - Forcing

A major step in our proof will be to show that $\mathbb{N}^*$ is equivalent to an $\mathcal{L}$-forcing. By "$\mathcal{L}$-forcing" I refer to a body of related forcings, all of which have in common that the conditions represent certain statements in an infinitary language. The criterion for being a condition is then that the statement represented is consistent. Beyond that, however, I have not had much success in stating a general definition, even of those $\mathcal{L}$-forcings which do not add reals (which is the relevant case here). The definition I give here is closer to that given in [LF] rather than the more abstruse definition in [Sing]. I shall state some lemmas without proof, but it should be possible to reconstruct their proofs from [LF].

We first introduce some notation.

Set: $L_\beta^{A_1,\ldots,A_n} =: \langle M, \in, A_1 \cap M, \ldots, A_n \cap M \rangle$ where $M = L_\beta[A_1,\ldots,A_n]$.

If $M = L_\beta^{A_1, \ldots, A_n}$, we set:

$$M^{B_1, \ldots, B_m} =: L_\beta^{A_1, \ldots, A_n, B_1, \ldots, B_m}$$

Finally we note the following trivial:

<u>Fact</u> Let $\bar{M} = L_{\bar{\beta}}^{\bar{A}}$, $M = L_\beta^A$. Let $\pi : \bar{M} \prec M$. Given classes $\bar{B}_1, \ldots, \bar{B}_n$ and $B_1, \ldots, B_n$, there is at most one extension $\pi^* \supset \pi$ s.t. $\pi^* : \bar{M}^{\bar{B}_1, \ldots, \bar{B}_n} \prec M^{B_1, \ldots, B_n}$.

Now fix $M = L_\beta^A$ where $\beta > 2^\omega$ is a cardinal. Let A be so chosen that $\mathbb{P}(\omega) \subset L_\beta^A$ and the property of being a cardinal is absolute in M for all ordinals $\alpha < \beta$.

Let $N = \langle H_{\beta^+}, \ldots \rangle$ be a structure of countable type. (In our specific example we take $\beta = \omega_2$ and $N = \langle H_{\omega_3}, \in, < \rangle$ where $<$ is a well ordering of H_{ω_3}.) (We shall usually include a well ording of H_{β^+} among the predicates of N.).

N is then an admissible set and we can apply Barwise Theory to it. (For a brief account of Barwise Theory see [Sing].) Consider an infinitary language $\mathcal{L}$ on N with;

<u>Predicate</u>: $\dot{\in}$

<u>Constants</u>: $\dot{M}, \dot{\pi}, \dot{B}, \underline{x}$ $(x \in N)$

<u>Axioms</u>;

- ZFC$^-$ (meaning the finite axioms of ZFC$^-$)
- $\bigwedge v \left(v \in \underline{x} \longleftrightarrow \bigvee\!\!\!\bigvee_{z \in x} v = \underline{z} \right)$
- $\dot{M} = \langle \dot{M}_i \mid i \leq \underline{\omega_1} \rangle$, $\dot{\pi} = \langle \dot{\pi}_{ij} \mid i \leq j \leq \omega_1 \rangle$
- $\dot{\pi}$ is a commutative sequence of elementary embeddings $\dot{\pi}_{ij} : \dot{M}_i \prec \dot{M}_j$.
- $\dot{M}_{\underline{\omega_1}} = \underline{M}$; $\bigwedge i < \underline{\omega_1}$ ($\dot{M}_i$ is countable and transitive)
- $\dot{\alpha}_i = \mathrm{crit}(\dot{\pi}_{ij})$, where $\dot{\alpha}_i = (\omega_\eta)^{\dot{M}_i}$
- $\dot{\beta}_i < \dot{\alpha}_{i+1}$, where $\dot{\beta}_i = \omega_1^{\dot{M}_i}$
- $\dot{B} \subset \dot{M}$
- $H_{\omega_1} = \underline{H_{\omega_1}}$.

We refer to these axioms as the *basic axioms*. $\mathcal{L}$ might, of course, contain further axioms (but no further predicates and constants). We then use $\mathcal{L}$ to define a set $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ of forcing conditions.

Before proceeding to that, however, we recall some definitions:

Def Let $\mathfrak{A} = \langle |\mathfrak{A}|, \in^{\mathfrak{A}}, \ldots \rangle$ be a model of ZFC$^-$. By the *well founded core* of $\mathfrak{A}$ ($\mathrm{wfc}(\mathfrak{A})$) we mean the set of $x \in \mathfrak{A}$ s.t. $\in^{\mathfrak{A}} \cap\, e(x)^2$ is well founded, where $e(x) =:$ the closure of $\{x\}$ under $\in^{\mathfrak{A}}$.

Def $\mathfrak{A}$ is *grounded* iff $\mathrm{wfc}(\mathfrak{A})$ is transitive and $x \in y \leftrightarrow x \in^{\mathfrak{A}} y$ for $x, y \in \mathrm{wfc}(\mathfrak{A})$.

(*Note* In [LF] we called such $\mathfrak{A}$ "solid".)

It is then easy to see that:

Fact Let $\mathfrak{A}$ be a grounded model of the basic axioms. Then:

- $\underset{\sim}{x}^{\mho} = x$ for $x \in N$
- $H_{\omega_1}^{\mho} = H_{\omega_1}$
- Let $\tilde{M} = \dot{M}^{\mho}$, Then $\tilde{M} = \langle M_i \mid i \leq \omega_1 \rangle$ where $M_i = L_{\beta_i}^{A_i}$ is transitive + countable and $M_{\omega_1} = M$.
- Let $\tilde{\pi} = \dot{\pi}^{\mho}$. Then $\tilde{\pi} = \langle \pi_{ij} \mid i \leq \omega_1 \rangle$ is a continuous, commutative sequence of elementary embeddings $\pi_{ij} : M_i \prec M_j$. Moreover $\mathrm{crit}(\pi_{ij}) = \alpha_i =: \omega_1^{M_i}$ for $i < j$.
- $\beta_i < \alpha_{i+1}$ for $i < \omega_1$
- Let $B = \dot{B}^{\mho}$, $B_i = \pi_{i,\omega_1}^{-1}$ "B, Then $B_i \in H_{\omega_1}$ and $\pi_{ij} : \langle M_i, B_i \rangle \longrightarrow \langle M_j, B_j \rangle$ is a structure preserving map $(i \leq j \leq \omega_1)$

We now turn to the definition of the conditions $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$. We first define a set of <u>preconditions</u> $\tilde{\mathbb{P}}$:

<u>Def</u> $\tilde{\mathbb{P}}$ is the set of pairs $p = \langle p_0, p_1 \rangle$ s.t.

(a) $p_0 = \langle M^p, \pi^p, b^p \rangle \in H_{\omega_1}$, where:

- $M^p = \langle M_i^p \mid i \leq \gamma \rangle$, $\pi^p = \langle \pi_{ij}^p \mid i \leq j \leq \gamma \rangle$ where $\pi_{ij}^p : M_i^p \prec M_j^p$ is a continuous commutative system of elementary embeddings. (Set $|p| =: \gamma$)
- $b^p \subset M_\gamma^p$

(b) p_1 is a countable set of pairs $\langle a, \bar{a} \rangle$ s.t. $a \subset M$ and $\bar{a} \subset \bar{M}$.

We then define:

<u>Def</u> Let $p \in \tilde{\mathbb{P}}$. φ_p is the formula:

$$\underline{M}^p = \langle \dot{M}_i \mid i \leq \underline{|p|} \rangle \wedge \underline{\pi}^p = \langle \dot{\pi}_{ij} \mid i \leq j \leq \underline{|p|} \rangle$$

$$\wedge \bigwedge_{\langle a, \bar{a} \rangle \in p_1} \dot{\pi}_{\underline{|p|}, \underline{\omega_1}} : \underline{\langle M_{|p|}^p, \bar{a} \rangle} \prec \underline{\langle M, a \rangle} \;{}^{*)}$$

$$\wedge \; \underline{b}^p = \dot{\pi}^{-1}_{\underline{|p|}, \underline{\omega_1}} {}'' \mathring{B}.$$

<u>Def</u> $\mathbb{P} = \mathbb{P}_{\mathcal{L}} =:$ the set of $p \in \tilde{\mathbb{P}}$ s.t. $\mathcal{L}(p) =: \mathcal{L} + \varphi_p$ is consistent.

*) $\langle M, a \rangle$ + hence $\langle M_{|p|}^p, \bar{a} \rangle$ need <u>not</u> be amenable!

Def Let $p \in \mathbb{P}$. We set:

$F^p = p_1$, $R^p = \text{rng}(F^p)$, $D^p = \text{dom}(F^p)$.

We partially order $\mathbb{P}$ by:

Def $p \le q \longleftrightarrow: (|p| \ge |q| \wedge M^q = M^p \restriction (|q|+1)$
$\wedge\, \pi^q = \pi^p \restriction (|q|+1)^2 \wedge R^q \subset R^p$.

Lemma 0.1 $(F^p)^{-1}$ is a function

proof

Let $\mathfrak{A}$ be a grounded model of $\mathcal{L}(p)$.

Then $\langle a, \bar{a} \rangle \in F^p \longleftrightarrow \bar{a} = (\dot{\pi}^{\mathfrak{A}}_{|p|,\omega_1})^{-1} {}'' a$.

QED (0.1)

Lemma 0.2 Let R^p be closed under set difference. Then F^p injects D^p onto R^p.

proof

Let $\langle a, \bar{a} \rangle, \langle b, \bar{b} \rangle \in F^p$. It suffices to show:

Claim $\bar{a} \subset \bar{b} \to a \subset b$.

Set $\bar{c} = \bar{b} \setminus \bar{a}$, $c = b \setminus a$. Let $\mathfrak{A}$ be a grounded model of $\mathcal{L}(p)$. Let $\pi = \dot{\pi}^{\mathfrak{A}}_{|p|,\omega_1}$. Then:

$$F^{p-1}(c) = \pi^{-1}{}''(b \setminus c) = \pi^{-1}(b) \setminus \pi^{-1}(c) = \bar{b} \setminus \bar{a} = \bar{c},$$

Hence $\bar{b} \subset \bar{a} \to \bar{c} = \emptyset \to c = \emptyset \to b \subset a$,

since $\pi : \langle M^p_{|p|}, \bar{c} \rangle \prec \langle M, c \rangle$.

QED (0.2)

<u>Lemma 0.3</u> Let $p \leq q$:

(i) If $\langle \bar{a}, a \rangle \in F^q$ and $\langle a', a \rangle \in F^p$, then $\pi^p_{|q|,|p|} : \langle M^q_{|q|}, \bar{a} \rangle \prec \langle M^p_{|p|}, a' \rangle$.

(ii) $B^q = (\pi^p_{|q|,|p|})^{-1} {}'' B^p$.

proof:
Let $\mathfrak{A}$ be a grounded model of $\mathcal{L}(p)$.
Let $\bar{\pi} = \pi^{\mathfrak{A}}_{|q|,\omega_1}$, $\pi = \pi^{\mathfrak{A}}_{|p|,\omega_1}$. Then
$\pi^p_{|q|,|p|} = \pi^{-1} \circ \bar{\pi}$. QED (0.3)

<u>Def</u> $\pi^p_i = \pi^p_{i,\omega_1} =: F^p \circ \pi^p_{i,|p|}$ for $i \leq p$.

We obviously have:

<u>Lemma 0.4</u> π^p_i is a partial injection of M^p_i into M. Moreover, if $p \leq q$, then $\pi^q_i \subset \pi^p_i$ for $i \leq |q|$.

<u>Def</u> Let $p, q \in \mathbb{P}$

$p \| q \leftrightarrow :$ p, q are compatible in $\mathbb{P}$

$p \perp q \leftrightarrow : \neg(p \| q)$.

We then get:

<u>Lemma 1.1</u> $p \parallel q \leftrightarrow \mathcal{L}(p) \cup \mathcal{L}(q)$ is consistent

proof

$(\rightarrow)$ Let $r \leq p, q$. Then $\mathcal{L}(r) \vdash \mathcal{L}(p) \cup \mathcal{L}(q)$

$(\leftarrow)$ Let $\mathfrak{A}$ be a grounded model of $\mathcal{L}(p) \cup \mathcal{L}(q)$. Then ω_1 is regular in $\mathfrak{A}$ and $R^p \cup R^q$ is countable in $\mathfrak{A}$. There is then an $\alpha < \omega_1$ s.t. $\alpha \geq |p|, |q|$ and for all $a \in R^p \cup R^q$ we have:

$$\pi^{\mathfrak{A}}_{\alpha, \omega_1} : \langle \dot{M}^{\mathfrak{A}}_{\alpha}, a^* \rangle \prec \langle M, a \rangle$$

where $a^* = (\dot{\pi}^{\mathfrak{A}}_{\alpha, \omega_1})^{-1} {}'' a$. (Hence $a^* \in H_{\omega_1}$)

This clearly exists, since $\mathfrak{A}$ models ZFC^- and H_{ω_1} is absolute in $\mathfrak{A}$.

Define $r \in \tilde{\mathbb{P}}$ by:

$M^r = \langle \dot{M}^{\mathfrak{A}}_i \mid i \leq \alpha \rangle$, $\pi^r = \langle \dot{\pi}^{\mathfrak{A}}_{ij} \mid i \leq j \leq \alpha \rangle$,

$b^r = (\dot{\pi}^{\mathfrak{A}}_{i, \omega_1})^{-1} {}'' \dot{B}^{\mathfrak{A}}$

$$F^r = \{ \langle a, a^* \rangle \mid a \in R^p \cup R^q \}.$$

Then $\mathfrak{A} \models \mathcal{L}(r)$. Hence $r \in \mathbb{P}$. But $r \leq p, q$ in $\mathbb{P}$. QED (Lemma 1.1)

The proof of $(\leftarrow)$ in Lemma 1.1 is a model for many similar proofs. Using it, we get the <u>extension lemmas</u>:

<u>Lemma 1.2</u> Let $p \in \mathbb{P}$. Let u be any countable collection of subsets of M. Then there is $q \leq p$ s.t. $u \subset R^q$.

(The proof turns on the fact that if $\mathfrak{A}$ is a grounded model of $\mathcal{L}(p)$, then u is countable in $\mathfrak{A}$, since $u \in N = H_{\omega_3} \subset \mathfrak{A}$.)

<u>Corollary 1.3</u> Let $p \in \mathbb{P}$, $u \subset M$ be countable. Then there is $q \leq p$ s.t. $u \subset \mathrm{rng}(\pi^q_{|q|})$

<u>Lemma 1.4</u> Let $p \in \mathbb{P}$, let $u \subset M_i^p$ be finite. There is $q \leq p$ s.t. $u \subset \mathrm{dom}(\pi_i^q)$

<u>Lemma 1.5</u> Let $p \in \mathbb{P}$, $|p| \leq \alpha < \omega_1$. There is $q \leq p$ s.t. $|q| \geq \alpha$.

Fuller proofs of these lemmas can be found in [LF].

Using these lemmas we get:

<u>Lemma 2</u> Let G be $\mathbb{P}$-generic over V. Define $M^G = \langle M_i^G \mid i \leq \omega_1 \rangle$, $\pi^G = \langle \pi_{ij}^G \mid i \leq j \leq \omega_1 \rangle$, $b^G = \langle b_i^G \mid i \leq \omega_1 \rangle$ as follows: (Let $\omega_1 =: \omega_1^V$)

$M_i^G =: M_i^p$, $\pi_{ij}^G =: \pi_{ij}^p$, $b_i^G =: b_i^p$ for $p \in G$, $i \leq |p|$,

$\pi_{i,\omega_1}^G =: \bigcup_{\substack{p \in G \\ i < |p|}} \pi_{i,\omega_1}^p$, $B^G = b_{\omega_1}^G =: \bigcup \pi_{i,\omega_1}^G{}'' b_i^G$.

$M_{\omega_1}^G =: M$, $\pi_{\omega_1, \omega_1}^G = \mathrm{id} \upharpoonright M$.

Then:

(A) M^G, π^G is a continuous commutative sequence of elementary embeddings.

(B) $\omega_1 \in \mathrm{rng}(\pi_{i,\omega_1}^G)$. Moreover:
$\alpha_i^G = \mathrm{crit}(\pi_{i,\omega_1}^G)$ where $\alpha_i^G = (\pi_{i,\omega_1}^G)^{-1}(\omega_1)$.

(C) $\beta_i^G < \alpha_{i+1}$ where $\beta_i^G = \mathrm{On}_{M_i^G}$.

(D) $b_i^p = (\pi_{i,\omega_1}^G)^{-1}{}'' B^G$ for all $p \in G$

(E) If $p \in G$, $\alpha = \omega_1^V$, and $\langle a, \bar{a} \rangle \in F^p$, then $\pi_{|p|,\alpha}^G : \langle M_{|p|}^p, \bar{a} \rangle \prec \langle M, a \rangle$

<u>Note</u> G is recoverable from M^G, π^G, B^G by:
$p \in G \leftrightarrow \bigvee j < \omega_1 (p_0 = \langle M^G \upharpoonright (j+1), \pi^G \upharpoonright (j+1)^2, \pi_{j\omega_1}^{G\,-1}{}'' B^G \rangle$
$\wedge \bigwedge_{\langle a, \bar{a} \rangle \in p_1} \pi_{j,\omega_1}^G : \langle M_j^G, \bar{a} \rangle \prec \langle M, a \rangle)$.

We now introduce an important concept.

<u>Def</u> Let $W = \langle H_\gamma, M, <, \ldots \rangle$ be a model of countable or finite type, where $\gamma \geq \beta^+$ is a cardinal and $<$ well orders H_γ.

p <u>conforms to</u> W iff $p \in \mathbb{P}$, W is as above, and whenever $a_1, \ldots, a_n \in R^p$ and $b \subset M$ is W-definable in $a_1, \ldots, a_n$, then $b \in R^p$.

(This holds for $n = 0$ as well, meaning that any W-definable $b \subset M$ is in R^p. By Lemma 0.2 we then have: F^p bijects D^p onto R^p.)

By the extension lemmas it follows easily that $\{p \mid p \text{ conforms to } W\}$ is dense in $\mathbb{P}$.

In [LF] we prove the following deep lemma:

<u>Lemma 3.1</u> Let p conform to W. There is a unique $\overline{W}$ s.t.

(i) $\overline{W}$ is transitive and of the same type as W.

(ii) If $a_1, \ldots, a_n \in R^p$ and $b \subset M$ is W-definable from $a_1, \ldots, a_n$, then $\bar{a}_1^p, \ldots, \bar{a}_m^p \in \overline{W}$ and $\bar{b}^p$ is $\overline{W}$ definable from $\bar{a}_1^p, \ldots, \bar{a}_n^p$ by the same definition.

(iii) Each $x \in \overline{W}$ is $\overline{W}$-definable from parameters in $M^p_{|p|} \cup D^p$.

<u>Def</u> We call $\overline{W}$ the <u>pulldown</u> of W by p and denote it by $PD(p, W)$.

Lemma 3.2 Let $\bar{W} = PD(p, W)$. If $\mathfrak{A}$ is any grounded model of $\mathcal{L}(p)$, Then $\hat{\pi}^{\mathfrak{A}}_{|p|, \omega_1} \cup F^p$ extends uniquely to a $\pi : \bar{W} \prec W$.

Lemma 3.3 Let $\bar{W} = PD(p, W)$. Let G be $\mathbb{P}$-generic over V s.t. $p \in G$. Then $\pi^{G}_{|p|, \omega_1} \cup F^p$ extends uniquely to $\pi : \bar{W} \prec W$.

. . . .

We note that by Lemma 2:

Lemma 4.1 Let G be $\mathbb{P}$-generic over V. Then $p \in G \longleftrightarrow V[G] \models \varphi_p$.

Lemma 4.2 Let G be $\mathbb{P}$-generic over V. If $V[G]$ has no new reals, then $V[G] \models \psi$ for all basic axioms ψ.

Of course $\mathcal{L}$ may contain further axioms. Since, however, we design $\mathcal{L}$ with a view to defining the set of conditions $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$, we are unlikely to adopt an axiom which we do not expect to be forced by $\mathbb{P}$. Thus, in the best case $\mathcal{L}$ will be good in the following sense:

<u>Def</u> $\mathcal{L}$ is <u>good</u> iff whenever G is $\mathbb{P}$-generic over V, then $V[G] \models \psi$ for all axioms ψ.

In general, the axioms of $\mathcal{L}$ will be so chosen, that goodness will follow as soon as we have shown that the forcing $\mathbb{P}$ adds no reals, the verification of the other axioms being trivial.

Obviously we have:

<u>Lemma 4.3</u> Let $\mathcal{L}$ be good. If $p \in G$, G is $\mathbb{P}$-generic over V, then

$$\vdash_{\mathcal{L}} (\varphi_p \to \psi) \quad \longrightarrow \quad V[G] \models \psi.$$